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A General Framework for Pricing Asian Options Under Markov Processes

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A general framework is proposed for pricing both continuously and discretely monitored Asian options under one-dimensional Markov processes. For each type (continuously monitored or discretely monitored), we derive the double transform of the Asian option price in terms of the unique bounded solution to a related functional equation. In the special case of continuous-time Markov chain (CTMC), the functional equation reduces to a linear system that can be solved analytically via matrix inversion. Thus the Asian option prices under a one-dimensional Markov process can be obtained by first constructing a CTMC to approximate the targeted Markov process model, and then computing the Asian option prices under the approximate CTMC by numerically inverting the double transforms. Numerical experiments indicate that our pricing method is accurate and fast under popular Markov process models, including the CIR model, the CEV model, Merton's jump diffusion model, the double-exponential jump diffusion model, the variance gamma model, and the CGMY model.

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1. Introduction

Asian options, whose payoffs are contingent on the arithmetic average of the underlying asset prices over a prespecified period, are among the most popular path-dependent options that are actively traded in the financial markets. The average of the underlying asset prices can be computed either discretely, for which the average is taken over the asset prices at discrete monitoring time points, or continuously, for which the average is calculated via the integration of asset prices over the monitoring time period. The valuation of Asian options is challenging since the arithmetic average usually does not have a simple distribution. There is a vast literature on the pricing of Asian options under the Black-Scholes model (BSM); we refer to Fu et al. (1999) and Linetsky (2004) for detailed literature reviews.

Results on the valuation of Asian options beyond the Black-Scholes framework are less extensive. Table 1 summarizes some papers on the pricing of either discretely or continuously monitored Asian options under certain special one-dimensional Markov processes, which are closely related to the study of our paper.

It can be seen that the existing literature is usually focused on one type of Asian option (discrete or continuous) and on particular Markov processes. By contrast, our paper provides a unified framework for pricing both continuously and discretely monitored Asian options under general one-dimensional Markov process models. Recently, Novikov and Kordzakhia (2014) propose a unified approach to deriving upper and lower bounds for Asian-type options, including discretely and continuously monitored Asian options and options on volume-weighted average price, under general semimartingale models. Fusai and Kyriakou (2014) develop a unified method to obtain very accurate bounds for both discretely and continuously monitored Asian option prices under a wide range of models, including exponential Lévy models, stochastic volatility models, and the constant elasticity of variance (CEV) model. In comparison, our paper is focused on deriving the prices rather than the bounds, and our method is completely different from theirs.

The contribution of this paper is threefold.

1. Under general one-dimensional Markov processes, we derive the double transforms of the Asian option prices, either discretely or continuously monitored, in terms of the unique bounded solutions to related functional equations; see §§2 and 3.

2. In the special case of continuous-time Markov chain (CTMC), we show that the functional equations reduce to

Table 1. Some literature on Asian option pricing under one-dimensional Markov process models beyond the BSM.

	Discretely monitored	Continuously monitored
One-dimensional diffusions	Fusai et al. (2008) (CIR) Cai et al. (2014) (general)	Fusai et al. (2008) (CIR)
Exponential Lévy processes	Fusai and Meucci (2008) (general) Fusai et al. (2011) (general)	Cai and Kou (2012) (HEM)
One-dimensional Markov processes	This paper (general)	This paper (general)

Notes. Fusai et al. (2008) provide a single transform formula for the discretely monitored Asian option price under the Cox-Ingersoll-Ross (CIR) model. Cai et al. (2014) derive closed-form analytical expansions for the discretely monitored Asian option prices under general one-dimensional diffusion models. Fusai and Meucci (2008) and Fusai et al. (2011) develop recursive algorithms to evaluate discretely monitored Asian options under general exponential Lévy models, by using convolution and maturity randomization, respectively. Cai and Kou (2012) obtain a closed-form double Laplace transform for the continuously monitored Asian option price under the hyperexponential jump diffusion model (HEM).

linear systems that can be solved analytically via matrix inverses, thanks to the Lévy-Desplanques theorem (see, e.g., Horn and Johnson 1985, Corollary 5.6.17); see §4.

3. By first constructing a CTMC to approximate the targeted Markov process via the elegant technique in Mijatović and Pistorius (2013) and then numerically inverting the double transforms related to the constructed CTMC, we can compute the Asian option prices under general one-dimensional Markov process models. Numerical results demonstrate that our method is accurate and fast under popular Markov process models, including the CIR model, the CEV model, Merton's jump diffusion model (MJD), the double-exponential jump diffusion model (DEJD), the variance gamma model (VG), and the Carr-German-Madan-Yor (CGMY) model; see §5.

Pricing Asian options under a CTMC is quite different from pricing other path-dependent options. For example, to price barrier options, one needs to study the first passage times under a CTMC, which have analytical solutions via a matrix obtained by deleting some rows and columns in the transition rate matrix of the CTMC (see Mijatović and Pistorius 2013). Unfortunately, it is not easy to find such simple matrix operations for Asian options. We overcome this difficulty by working on the double transform of the continuously monitored (respectively, discretely monitored) Asian option price with respect to the strike price and the maturity (respectively, the number of monitoring time points), in the spirit of Fu et al. (1999). Then we show that the double transforms are the unique bounded solutions to related functional equations, which can be solved analytically in the case of CTMCs, using the strictly diagonally dominant matrix and the Lévy-Desplanques theorem.

The remainder of this paper is organized as follows. Some preliminary results regarding the double transforms of Asian option prices are given in §2. In §3 we derive the double transforms for Asian option prices under general one-dimensional Markov processes in terms of the unique bounded solutions to related functional equations. Then we apply the general results to the CTMC in §4 and solve the related functional equations explicitly. We summarize our pricing method in two steps and provide numerical results in §5. Error analysis of our pricing method is conducted in §6. For error analysis related to Asian option pricing, we refer to, e.g., Zhang and Oosterlee (2013) about a Fourier cosine expansion method for Asian option pricing under exponential Lévy models.

2. Preliminary Results

For simplicity, we consider only Asian call options, while the put options can be dealt with similarly. Throughout this paper, we shall work under the risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. This implies that for $t \geqslant 0$, $\mathbb{E}^x[S_t] = xe^{(r-d)t}$, where r is the risk-free rate, d is the dividend rate, and $\mathbb{E}^x[\cdot]$ denotes the expectation taken under the risk-neutral measure given the initial price $S_0 = x$. The price of a continuously monitored Asian call option at time 0 is given by

$$V_c(T, K; x) = e^{-rT}\mathbb{E}^x\left[\left(\frac{1}{T}A_T - K\right)^+\right], \quad \text{with } A_T := \int_0^T S_t\, dt,$$

where T is the maturity and K the strike price. Similarly, the price of a discretely monitored Asian call option at time 0 is given by

$$V_d(n, K; x) = e^{-rT}\mathbb{E}^x\left[\left(\frac{1}{n+1}B_n - K\right)^+\right], \quad \text{with } B_n := \sum_{i=0}^{n} S_{i\Delta},$$

where Δ is the length of the monitoring time interval and the $n+1$ monitoring dates are assumed to be equally spaced with $n\Delta = T$. For convenience, define

$$v_c(t, k; x) = \mathbb{E}^x[(A_t - k)^+] \quad \text{and} \\ v_d(n, k; x) = \mathbb{E}^x[(B_n - k)^+]. \tag{1}$$

Note that $V_c(T, K; x) = (e^{-rT}/T)v_c(T, TK; x)$ and $V_d(n, K; x) = (e^{-rT}/(n+1))v_d(n, (n+1)K; x)$. Then the pricing problems of Asian options reduce to the computation of $v_c(t, k; x)$ and $v_d(n, k; x)$.

The following proposition provides a model-free result for the double transforms of $v_c(t, k; x)$ with respect to t

and k and $v_d(n,k;x)$ with respect to n and k. Fu et al. (1999) use the same transform in the pricing of continuously monitored Asian options under the BSM. Here we extend it to general Markov processes for both continuously and discretely monitored Asian options.

PROPOSITION 1 (DOUBLE TRANSFORMS OF ASIAN OPTION PRICES). (i) (DISCRETELY MONITORED ASIAN OPTIONS) *Let $L_d(z,\theta;x)$ be the Z-Laplace transform of $v_d(n,k;x)$ with respect to n and k, i.e., $L_d(z,\theta;x)=\sum_{n=0}^{+\infty} z^n \int_0^{+\infty} e^{-\theta k}\cdot v_d(n,k;x)\,dk$. Then for any complex z and θ such that $|z|<\min\{e^{-(r-d)\Delta},1\}$ and $\mathrm{Re}(\theta)>0$, we have*

$$L_d(z,\theta;x)=\frac{1}{\theta^2}l(z,\theta;x)-\frac{1}{\theta^2(1-z)}+\frac{x}{\theta(1-z)(1-ze^{r\Delta})},$$

where

$$l(z,\theta;x):=\sum_{n=0}^{+\infty} z^n\mathbb{E}^x[e^{-\theta B_n}]. \tag{2}$$

(ii) (CONTINUOUSLY MONITORED ASIAN OPTIONS) *Let $L_c(\mu,\theta;x)$ be the double-Laplace transform of $v_c(t,k;x)$ with respect to t and k, i.e., $L_c(\mu,\theta;x)=\int_0^{+\infty}\int_0^{+\infty} e^{-\mu t}\cdot e^{-\theta k}v_c(t,k;x)\,dk\,dt$. Then for any complex μ and θ such that $\mathrm{Re}(\mu)>\max\{r-d,0\}$ and $\mathrm{Re}(\theta)>0$, we have*

$$L_c(\mu,\theta;x)=\frac{1}{\theta^2}m(\mu,\theta;x)-\frac{1}{\theta^2\mu}+\frac{x}{\theta\mu(\mu-r)},$$

where

$$m(\mu,\theta;x):=\int_0^{+\infty} e^{-\mu t}\mathbb{E}^x[e^{-\theta A_t}]\,dt. \tag{3}$$

PROOF. (i) First, the function $l(z,\theta;x)$ is well defined for any $|z|<1$ and $\mathrm{Re}(\theta)>0$ because $|l(z,\theta;x)|\leqslant\sum_{n=0}^{\infty}|z|^n<\infty$. By Fubini's theorem,

$$\begin{aligned}L_d(z,\theta;x)&=\sum_{n=0}^{\infty} z^n\mathbb{E}^x\left[\int_0^{B_n} e^{-\theta k}(B_n-k)\,dk\right]\\&=\sum_{n=0}^{\infty} z^n\mathbb{E}^x\left[\frac{B_n}{\theta}-\frac{1-e^{-\theta B_n}}{\theta^2}\right]\\&=\frac{1}{\theta^2}\sum_{n=0}^{\infty} z^n(\mathbb{E}^x[e^{-\theta B_n}]+\theta\mathbb{E}^x[B_n]-1)\\&=\frac{1}{\theta^2}l(z,\theta;x)-\frac{1}{\theta^2(1-z)}+\frac{1}{\theta}\sum_{n=0}^{\infty} z^n\mathbb{E}^x[B_n].\end{aligned}$$

Note that under the risk-neutral measure,

$$\mathbb{E}^x[B_n]=x\sum_{i=0}^{n} e^{(r-d)t_i}=x\frac{1-e^{(n+1)(r-d)\Delta}}{1-e^{(r-d)\Delta}}.$$

Then it follows that for any complex z and θ such that $|z|<e^{-(r-d)\Delta}$ and $\mathrm{Re}(\theta)>0$,

$$\begin{aligned}L_d(z,\theta;x)&=\frac{1}{\theta^2}l(z,\theta;x)-\frac{1}{\theta^2(1-z)}\\&\quad+\frac{x}{\theta(1-e^{(r-d)\Delta})}\sum_{n=0}^{\infty} z^n[1-e^{(n+1)(r-d)\Delta}]\\&=\frac{1}{\theta^2}l(z,\theta;x)-\frac{1}{\theta^2(1-z)}\\&\quad+\frac{x}{\theta(1-z)(1-ze^{(r-d)\Delta})}.\end{aligned}$$

(ii) The proof is similar to that for (i) and is thus omitted. □

3. Functional Equations for Markov Processes

According to Proposition 1, to derive the double transforms of discretely and continuously monitored Asian option prices, it suffices to compute the functions $l(z,\theta;x)$ and $m(\mu,\theta;x)$. In this section, we will show that for one-dimensional time-homogeneous Markov processes, the two functions $l(z,\theta;x)$ and $m(\mu,\theta;x)$ are the unique bounded solutions to certain functional equations.

Suppose $\{S_t, t\geqslant 0\}$ is a nonnegative one-dimensional time-homogeneous Markov process. For continuously monitored Asian options, we require $\{S_t\}$ to have the infinitesimal generator G (the existence of infinitesimal generator can be ensured by the Feller property; see, e.g., Ethier and Kurtz 2005, Chap. 4). Infinitesimal generators of some popular models are listed in Table 2.

THEOREM 1 (FUNCTIONAL EQUATIONS: UNIQUENESS AND STOCHASTIC REPRESENTATIONS). (i) (DISCRETELY MONITORED ASIAN OPTIONS) *Given $\Delta>0$, for any z and θ such that $|z|<1$ and $\mathrm{Re}(\theta)>0$, if there exists a function $f(x)$ that solves the functional equation*

$$e^{\theta x}f(x)-z\mathbb{E}^x[f(S_\Delta)]=1, \tag{4}$$

and is bounded, i.e., $\sup_{x\in[0,\infty)}|f(x)|\leqslant C<\infty$ for some constant $C>0$, then we must have $f(x)=l(z,\theta;x)$, where $l(z,\theta;x)$ is defined in (2).

(ii) (CONTINUOUSLY MONITORED ASIAN OPTIONS) *Furthermore, assume that $\{S_t, t\geqslant 0\}$ has right-continuous-and-left-limit paths and the infinitesimal generator G, and $\mathbb{E}^x[S_t^{1+\epsilon}]<\infty$ for some $\epsilon>0$. For any μ and θ such that $\mathrm{Re}(\mu)>0$ and $\mathrm{Re}(\theta)>0$, if there exists a function $f(x)$ that solves the functional equation*

$$(\theta x+\mu-G)f(x)=1, \tag{5}$$

and is bounded, i.e., $\sup_{x\in[0,\infty)}|f(x)|\leqslant C<\infty$ for some constant $C>0$, then we must have $f(x)=m(\mu,\theta;x)$, where $m(\mu,\theta;x)$ is defined in (3).

PROOF. See Appendix A. □

4. Analytical Solutions to the Functional Equations Under Continuous-Time Markov Chains

It turns out that the two functional Equations (4) and (5) in Theorem 1 can be solved analytically for a special family of Markov processes, i.e., the CTMCs. Consider a nonnegative CTMC $\{S_t, t\geqslant 0\}$ with finite state space $\{x_1,\ldots,x_N\}$, whose transition probability matrix $\mathbf{P}(t)=(p_{ij}(t))_{N\times N}$ and transition rate matrix $\mathbf{G}=(q_{ij})_{N\times N}$ are defined as

$$p_{ij}(t)=\mathbb{P}(S_{t+u}=x_j\mid S_u=x_i)\quad\text{and}\quad q_{ij}=p'_{ij}(0),\qquad 1\leqslant i,j\leqslant N,\ t,u\geqslant 0.$$

Table 2. The infinitesimal generators of some popular models.

Model	$Gf(x)$
BSM	$(r-d)xf'(x)+\frac{1}{2}\sigma^2x^2f''(x)$
CIR	$(r-d)xf'(x)+\frac{1}{2}\sigma^2xf''(x)$
CEV	$(r-d)xf'(x)+\frac{1}{2}\sigma^2x^{2(1+\beta)}f''(x)$
DEJD	$(r-d)xf'(x)+\frac{1}{2}\sigma^2x^2f''(x)+\int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu_{de}(dy)$
MJD	$(r-d)xf'(x)+\frac{1}{2}\sigma^2x^2f''(x)+\int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu_{mj}(dy)$
VG	$(r-d)xf'(x)+\int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu_{vg}(dy)$
CGMY	$(r-d)xf'(x)+\int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu_{cgmy}(dy)$

Notes. For the DEJD model, $\nu_{de}(dy)=\lambda(p\eta e^{-\eta y}I_{\{y\geqslant 0\}}+(1-p)\theta e^{\theta y}I_{\{y<0\}})dy$, with $0\leqslant p\leqslant 1$ and $\eta>1,\theta,\lambda>0$; for the MJD model, $\nu_{mj}(dy)=(\lambda/\sqrt{2\pi\delta^2})e^{-((y-\alpha)^2/2\delta^2)}dy$, with $\delta,\lambda>0$; for the VG model, $\nu_{vg}(dy)=(1/(v|y|))\exp((\theta/\delta^2)y-(\sqrt{2/v+\theta^2/\delta^2}/\delta)|y|)dy$, with $\delta, v>0$; for the CGMY model, $\nu_{cgmy}(dy)=(C/|y|^{1+Y})\exp(((G-M)/2)y-((G+M)/2)|y|)dy$, with $C>0$, $G,M\geqslant 0$ and $Y<2$.

Define $\mathbf{x}=(x_1,\ldots,x_N)^{\mathrm{T}}$, $\mathbf{D}=(d_{ij})_{N\times N}$ as a diagonal matrix with $d_{jj}=x_j$ for $j=1,\ldots,N$, $\mathbf{I}=\mathbf{I}_N$ as the identity matrix, and $\mathbf{1}$ as an $N\times 1$ constant vector with all entries equal to 1. Theorem 1 immediately implies the following result.

Corollary 1 (Uniqueness and Stochastic Representation Under CTMCs). (i) (Discretely Monitored Asian Options) *Given $\Delta>0$, for any z and θ such that $|z|<1$ and $\mathrm{Re}(\theta)>0$, if there exists an $N\times 1$ vector $\mathbf{f}$ that solves the linear system*

$$(e^{\theta\mathbf{D}}-z\mathbf{P}(\Delta))\cdot\mathbf{f}=\mathbf{1}, \tag{6}$$

then we have $\mathbf{f}=l(z,\theta;\mathbf{x}):=(l(z,\theta;x_1),\ldots,l(z,\theta;x_N))^{\mathrm{T}}$, where $l(z,\theta;x)$ is defined in (2) for the CTMC $\{S_t\}$.

(ii) (Continuously Monitored Asian Options) *For any μ and θ such that $\mathrm{Re}(\mu)>0$ and $\mathrm{Re}(\theta)>0$, if there exists an $N\times 1$ vector $\mathbf{f}$ that solves the linear system*

$$(\theta\mathbf{D}+\mu\mathbf{I}-\mathbf{G})\cdot\mathbf{f}=\mathbf{1}, \tag{7}$$

then $\mathbf{f}=m(\mu,\theta;\mathbf{x}):=(m(\mu,\theta;x_1),\ldots,m(\mu,\theta;x_N))^{\mathrm{T}}$, where $m(\mu,\theta;x)$ is defined in (3) for the CTMC $\{S_t\}$.

A square matrix $\mathbf{A}=(a_{ij})_{N\times N}$ with complex entries is said to be strictly diagonally dominant, if

$$|a_{jj}|>\sum_{\substack{k=1\\k\neq j}}^{N}|a_{jk}| \quad \text{for all } j=1,\ldots,N.$$

The Lévy-Desplanques theorem (see, e.g., Horn and Johnson 1985, Corollary 5.6.17) states that if $\mathbf{A}$ is strictly diagonally dominant, then it is invertible. Applying the Lévy-Desplanques theorem, we can solve (6) and (7) by inverting the matrices.

Theorem 2 (Analytical Solutions for $l(z,\theta;\mathbf{x})$ and $m(\mu,\theta;\mathbf{x})$ Under CTMCs). (i) (Discretely Monitored Asian Options) *For any z and θ such that $|z|<1$ and $\mathrm{Re}(\theta)>0$, we have $l(z,\theta;\mathbf{x}):=(l(z,\theta;x_1),\ldots,l(z,\theta;x_N))^{\mathrm{T}}=(e^{\theta\mathbf{D}}-z\mathbf{P}(\Delta))^{-1}\cdot\mathbf{1}$.*

(ii) (Continuously Monitored Asian Options) *For any μ and θ such that $\mathrm{Re}(\mu)>0$ and $\mathrm{Re}(\theta)>0$, we have $m(\mu,\theta;\mathbf{x}):=(m(\mu,\theta;x_1),\ldots,m(\mu,\theta;x_N))^{\mathrm{T}}=(\theta\mathbf{D}+\mu\mathbf{I}-\mathbf{G})^{-1}\cdot\mathbf{1}$.*

Proof. The two matrices $e^{\theta\mathbf{D}}-z\mathbf{P}(\Delta)$ and $\theta\mathbf{D}+\mu\mathbf{I}-\mathbf{G}$ are both strictly diagonally dominant because for all $j=1,\ldots,N$, we have

$$|e^{\theta d_{jj}}-zp_{jj}(\Delta)|\geqslant|e^{\theta d_{jj}}|-|z|p_{jj}(\Delta)\geqslant 1-|z|p_{jj}(\Delta)$$
$$>|z|(1-p_{jj}(\Delta))=|z|\sum_{\substack{k=1\\k\neq j}}^{N}p_{jk}(\Delta) \quad \text{and}$$
$$|\theta d_{jj}+\mu-q_{jj}|\geqslant\mathrm{Re}(\theta d_{jj}+\mu-q_{jj})$$
$$=\mathrm{Re}(\theta)d_{jj}+\mathrm{Re}(\mu)-q_{jj}>-q_{jj}=\sum_{\substack{k=1\\k\neq j}}^{N}q_{jk}.$$

Therefore, both of them are invertible thanks to the Lévy-Desplanques theorem. Hence the solutions to (6) and (7) are $\mathbf{f}=(e^{\theta\mathbf{D}}-z\mathbf{P}(\Delta))^{-1}\cdot\mathbf{1}$ and $\mathbf{f}=(\theta\mathbf{D}+\mu\mathbf{I}-\mathbf{G})^{-1}\mathbf{1}$, respectively. By Corollary 1, we complete the proof immediately. □

5. Pricing Asian Options by CTMC-Approximation

According to Proposition 1 and Theorem 2, if we can construct a CTMC properly to approximate a general one-dimensional time-homogeneous Markov process, then the double transform inversion-based analytical solutions to Asian option prices under this constructed CTMC can serve as approximations for Asian option prices under the original Markov process model. Mijatović and Pistorius (2013) develop an elegant technique to construct a sequence of CTMCs that weakly converge to a one-dimensional Markov process with generator G given by

$$Gf(x) = G_D f(x) + G_J f(x),$$

where

$$G_D f(x) = (r-d)xf'(x) + \tfrac{1}{2}\sigma(x)^2 x^2 f''(x) \quad \text{and}$$
$$G_J f(x) = \int_{-1}^{\infty} [f(x(1+y)) - f(x) - f'(x)xy]\nu(x, dy).$$

Here for any $x \geqslant 0$, $\nu(x, dy)$ is a Lévy measure such that $\int_{-1}^{\infty} y^2 \nu(x, dy) < \infty$.

More precisely, based on the technique of Mijatović and Pistorius (2013), we can use a CTMC with state space $\{x_1, \ldots, x_N\}$ and transition rate matrix $\mathbf{G} = \boldsymbol{\Lambda}_D + \boldsymbol{\Lambda}_J$ to approximate the targeted Markov process. Here $\boldsymbol{\Lambda}_J = (\Lambda_{ij}^J)_{N\times N}$ is constructed explicitly by

$$\Lambda_{ij}^J = \begin{cases} 0, & i = 1 \text{ or } N, \\ \displaystyle\int_{\alpha_i(j-1)}^{\alpha_i(j)} \nu(x, dy), & i \neq 1, N \text{ and } i \neq j, \\ \displaystyle -\sum_{\substack{j=1 \\ j\neq i}}^{N} \Lambda_{ij}^J, & i \neq 1, N \text{ and } i = j, \end{cases}$$

where $\alpha_i(0) = -1$, $\alpha_i(N) = +\infty$, and $\alpha_i(j)$ is any number between $x_j/x_i - 1$ and $x_{j+1}/x_i - 1$ for $j = 1, \ldots, N-1$. Moreover, $\boldsymbol{\Lambda}_D = (\Lambda_{ij}^D)_{N\times N}$ is a tridiagonal matrix determined implicitly by

$$\sum_{j=1}^{N} \Lambda_{ij}^D = 0,$$

$$\sum_{j=1}^{N} \Lambda_{ij}^D (x_j - x_i) = (r-d)x_i - \sum_{j=1}^{N} \Lambda_{ij}^J (x_j - x_i),$$

$$\sum_{j=1}^{N} \Lambda_{ij}^D (x_j - x_i)^2 = x_i^2 \left[\sigma(x_i)^2 + \int_{-1}^{\infty} y^2 \nu(x, dy)\right] - \sum_{j=1}^{N} \Lambda_{ij}^J (x_j - x_i)^2,$$

when $i \neq 1, N$, and $\Lambda_{ij}^D = 0$ when $i = 1$ or N. Then $\mathbf{G}$ and $\mathbf{P}(\Delta) = \operatorname{Exp}\{\Delta\mathbf{G}\}$ will be used to calculate $l(z, \theta; \mathbf{x})$ and $m(\mu, \theta; \mathbf{x})$ in Theorem 2, where $\operatorname{Exp}\{\cdot\}$ denotes the matrix exponential. Notice that when the original Markov process is a diffusion, $\mathbf{G} \equiv \mathbf{G}_D$ is simply a tridiagonal matrix. In our numerical implementation, we choose $x_1 = 10^{-3} S_0$ and $x_N = 4S_0$, and generate other states $\{x_2, \ldots, x_{N-1}\}$ based on the procedure in Mijatović and Pistorius (2013). More specifically, we set $x_k = S_0 + \sinh((1 - (k-1)/(N/2-1))\operatorname{arsinh}(x_1 - S_0))$ for $2 \leqslant k \leqslant N/2$, and $x_k = S_0 + \sinh(((k - N/2)/(N/2))\operatorname{arsinh}(x_N - S_0))$ for $N/2 < k \leqslant N-1$. Intuitively, these N states are nonuniformly distributed over $[x_1, x_N]$ and are placed more densely around S_0.

Since the Asian option payoffs are continuous with respect to the sample paths (see, e.g., Prigent 2003, §1.2.2), we can show that the Asian option prices under the constructed CTMC converge to the prices under the targeted Markov process models, thanks to the continuous mapping theorem and the dominated convergence theorem.

In summary, our pricing algorithm consists of two steps:

Step 1. Construct a CTMC to approximate the targeted Markov process model via the technique in Mijatović and Pistorius (2013).

Step 2. Invert the analytical double transforms of Asian option prices under the approximate CTMC numerically to obtain the approximations for Asian option prices under the original Markov process model. For the numerical inversion of the double transforms, we use the algorithms in Choudhury et al. (1994); the details are given in Appendix B.

In the following subsections, numerical results will be provided to illustrate the performance of our method under different models, including the CIR model, the CEV model, the DEJD model, the MJD model, the VG model, and the CGMY model. These models are selected as representatives for different types of Markov process models, namely, diffusion models (CIR and CEV), jump diffusion models (DEJD and MJD), and pure jump models (VG and CGMY). Our method appears to be accurate and fast under these models. All the computations are conducted using MATLAB 7 on a desktop with an Intel Core i7 3.40 GHZ processor.

5.1. The CIR Model

Table 3 provides numerical prices (denoted by "CTMC") for both discretely and continuously monitored Asian option prices under the CIR model obtained via our method. The parameter settings are the same as in Fusai et al. (2008) for the commodity market, and the results obtained from their analytical solution are used as benchmarks (denoted by "Fusai et al."). As shown in Table 3, our method is highly accurate for both discretely monitored Asian options with different monitoring frequencies ($n = 12$, 25, 50, 100 and 250) and continuously monitored Asian options ($n = +\infty$). The average (respectively, the maximum) absolute error is approximately 0.00020 (respectively, 0.00095) and the average (respectively, the maximum) relative error is approximately 0.11% (respectively, 0.44%). Besides, we report that the CPU times to generate one numerical price via our method are about 0.2, 0.4, 0.7, 1.5, and 3.7 seconds for $n = 12$, 25, 50, 100, and 250, respectively, and about 0.12 seconds for continuously monitored Asian options (i.e., $n = +\infty$).

Table 3. Pricing Asian options under the CIR model.

K	Fusai et al.	CTMC	Abs. err.	Rel. err. (%)	Fusai et al.	CTMC	Abs. err.	Rel. err. (%)
		$n=12$				$n=25$		
0.90	0.21279	0.21257	−0.00022	0.10	0.21428	0.21406	−0.00022	0.10
0.95	0.18659	0.18638	−0.00021	0.11	0.18810	0.18789	−0.00021	0.11
1.00	0.16282	0.16264	−0.00018	0.11	0.16432	0.16414	−0.00018	0.11
1.05	0.14140	0.14126	−0.00014	0.10	0.14287	0.14273	−0.00014	0.10
1.10	0.12223	0.12213	−0.00010	0.09	0.12365	0.12355	−0.00010	0.08
		$n=50$				$n=100$		
0.90	0.21501	0.21406	−0.00095	0.44	0.21538	0.21515	−0.00023	0.11
0.95	0.18883	0.18862	−0.00021	0.11	0.18920	0.18899	−0.00021	0.11
1.00	0.16505	0.16487	−0.00018	0.11	0.16542	0.16524	−0.00018	0.11
1.05	0.14359	0.14344	−0.00015	0.10	0.14395	0.14381	−0.00014	0.10
1.10	0.12434	0.12424	−0.00010	0.08	0.12470	0.12460	−0.00010	0.08
		$n=250$				$n=+\infty$		
0.90	0.21560	0.21537	−0.00023	0.10	0.21575	0.21552	−0.00023	0.10
0.95	0.18943	0.18922	−0.00021	0.11	0.18958	0.18937	−0.00021	0.11
1.00	0.16565	0.16547	−0.00018	0.11	0.16580	0.16562	−0.00018	0.11
1.05	0.14418	0.14403	−0.00015	0.10	0.14433	0.14418	−0.00015	0.10
1.10	0.12492	0.12481	−0.00011	0.09	0.12506	0.12496	−0.00010	0.08

Notes. Pricing Asian options under the CIR model via our CTMC approximation with $N=50$. The parameter settings are the same as in Fusai et al. (2008), i.e., $r=0.04$, $d=0$, $\sigma=0.7$, $S_0=1$, and $T=1$. The columns "Fusai et al." are taken from Fusai et al. (2008) and are obtained from their analytical solution. The CPU times to generate one numerical price via our method are about 0.2, 0.4, 0.7, 1.5, and 3.7 seconds for $n=12$, 25, 50, 100, and 250, respectively, and about 0.12 seconds for $n=+\infty$, i.e., for continuously monitored Asian options.

5.2. The CEV Model

Table 4 gives numerical prices (denoted by "CTMC") for both discretely and continuously monitored Asian options under the CEV model obtained via our method (we refer to Davydov and Linetsky 2001 for analytical pricing of other path-dependent options such as continuously monitored lookback and barrier options. Also see Sesana et al. 2014 for a recursive algorithm for the pricing of discretely monitored Asian options under the CEV model). Here we consider three cases with $\beta=0.25$, -0.25 and -0.5,

Table 4. Pricing Asian options under the CEV model.

	(I) Discretely monitored Asian options under the CEV model ($n=250$)				(II) Continuously monitored Asian options under the CEV model				
K	Cai et al.	CTMC	Abs. err.	Rel. err. (%)	MC value	Std. err.	CTMC	Abs. err.	Rel. err. (%)
		$\beta=0.25$					$\beta=0.25$		
80	21.60167	21.60974	0.00807	0.04	21.59408	0.00468	21.61076	0.01667	0.08
90	13.15550	13.15548	−0.00002	0.00	13.15109	0.00425	13.15931	0.00822	0.06
100	6.84034	6.82619	−0.01415	0.21	6.83859	0.00340	6.83128	−0.00731	0.11
110	3.07180	3.05691	−0.01489	0.48	3.07333	0.00239	3.06138	−0.01195	0.39
120	1.22841	1.22497	−0.00344	0.28	1.23175	0.00154	1.22762	−0.00413	0.34
		$\beta=-0.25$					$\beta=-0.25$		
80	21.67122	21.67979	0.00857	0.04	21.66618	0.00464	21.68104	0.01486	0.07
90	13.26903	13.26768	−0.00135	0.01	13.26741	0.00417	13.27147	0.00407	0.03
100	6.84853	6.83407	−0.01446	0.21	6.85150	0.00327	6.83920	−0.01230	0.18
110	2.92962	2.91597	−0.01365	0.47	2.93166	0.00221	2.92049	−0.01116	0.38
120	1.04072	1.04152	0.00080	0.08	1.04453	0.00131	1.04429	−0.00025	0.02
		$\beta=-0.5$					$\beta=-0.5$		
80	21.71428	21.72237	0.00809	0.04	21.71118	0.00465	21.72370	0.01252	0.06
90	13.32877	13.32675	−0.00202	0.02	13.32850	0.00416	13.33052	0.00202	0.02
100	6.85365	6.83904	−0.01461	0.21	6.85984	0.00324	6.84420	−0.01564	0.23
110	2.86119	2.84823	−0.01296	0.45	2.86666	0.00215	2.85276	−0.01390	0.48
120	0.95542	0.95803	0.00261	0.27	0.95995	0.00122	0.96084	0.00089	0.09

Notes. Part (I) compares our numerical prices via CTMC approximation ($N=50$) with asymptotic expansion results in Cai et al. (2014) for discretely monitored Asian options under the CEV model. It takes around 3.6 seconds to generate one price via our method. Part (II) compares our numerical prices via CTMC approximation ($N=50$) with Monte Carlo simulation estimates (denoted by "MC value." The sample size is 1,000,000 and the number of time steps is 10,000) for continuously monitored Asian options under the CEV model. It takes around 0.12 seconds to generate one price via our method. The unvarying parameters are $r=0.05$, $d=0$, $S_0=100$, $T=1$, and $\sigma S_0^{\beta}=0.25$.

respectively. Part (I) in Table 4 provides our numerical results for daily monitored ($n = 250$) Asian option prices as well as the benchmark prices (denoted by "Cai et al.") that are taken from Cai et al. (2014) and obtained from asymptotic expansion. It can be seen that our method is very accurate with average (respectively, maximum) absolute error 0.00798 (respectively, 0.01489) and average (respectively, maximum) relative error 0.19% (respectively, 0.48%). Besides, it takes around 3.6 seconds to generate one price via our method. For continuously monitored Asian options, we construct benchmarks using Monte Carlo simulation with 1,000,000 sample paths and 10,000 time steps. From part (II) of Table 4 we can see that our method is also very accurate with average (respectively, maximum) absolute error 0.00906 (respectively, 0.01667) and average (respectively, maximum) relative error 0.17% (respectively, 0.48%). Besides, it takes around 0.12 seconds to generate one price via our method.

5.3. The Double-Exponential Jump Diffusion Model

Part (I) in Table 5 shows the comparison of discretely monitored Asian option prices obtained via our method (denoted by "CTMC") and those obtained by the recursive algorithm in Fusai and Meucci (2008) under the DEJD model (see Kou 2002). We can see that our method is quite accurate for different monitoring frequencies ($n = 12$, 50, and 250) because the average (respectively, the maximum) absolute error is around 0.00325 (respectively, 0.00458) and the average (respectively, maximum) relative error is around 0.09% (respectively, 0.15%). Besides, it takes about 0.2, 0.7, and 3.6 seconds to generate one price via our method for $n = 12$, 50, and 250, respectively. For continuously monitored Asian options, we compare our numerical results with those obtained via transform-based analytical solution in Cai and Kou (2012); see part (II) in Table 5. It can be seen that the average (respectively, the maximum)

Table 5. Pricing Asian options under the DEJD model.

(I) Discretely monitored Asian options under the DEJD model

n	K	Fusai and Meucci	CTMC	Abs. err.	Rel. err. (%)
12	90	12.71236	12.70857	−0.00379	0.03
	100	5.01712	5.01254	−0.00458	0.09
	110	1.04142	1.03988	−0.00154	0.15
50	90	12.74369	12.74016	−0.00353	0.03
	100	5.05809	5.05358	−0.00451	0.09
	110	1.06878	1.06725	−0.00153	0.14
250	90	12.75241	12.74875	−0.00366	0.03
	100	5.06949	5.06491	−0.00458	0.09
	110	1.07646	1.07489	−0.00157	0.15

(II) Continuously monitored Asian options under the DEJD model

K	Cai and Kou	CTMC	Abs. err.	Rel. err. (%)	Cai and Kou	CTMC	Abs. err.	Rel. err. (%)
		$\sigma = 0.05$				$\sigma = 0.1$		
90	13.47952	13.46823	−0.01129	0.08	13.55964	13.56418	0.00454	0.03
95	9.16588	9.18472	0.01884	0.21	9.41962	9.42931	0.00969	0.10
100	5.38761	5.37399	−0.01362	0.25	5.91537	5.91365	−0.00172	0.03
105	2.72681	2.71628	−0.01053	0.39	3.35071	3.34830	−0.00241	0.07
110	1.28264	1.30224	0.01960	1.53	1.74896	1.75431	0.00535	0.31
		$\sigma = 0.2$				$\sigma = 0.3$		
90	14.17380	14.17589	0.00209	0.01	15.33688	15.33545	−0.00143	0.01
95	10.53795	10.53824	0.00029	0.00	12.10723	12.10414	−0.00309	0.03
100	7.48805	7.48621	−0.00184	0.02	9.35336	9.34883	−0.00453	0.05
105	5.09001	5.08708	−0.00293	0.06	7.08059	7.07520	−0.00539	0.08
110	3.32061	3.31802	−0.00259	0.08	5.26109	5.25561	−0.00548	0.10
		$\sigma = 0.4$				$\sigma = 0.5$		
90	16.81490	16.81130	−0.00360	0.02	18.46259	18.45288	−0.00971	0.05
95	13.87995	13.87460	−0.00535	0.04	15.75006	15.73859	−0.01147	0.07
100	11.33257	11.32581	−0.00676	0.06	13.36027	13.34737	−0.01290	0.10
105	9.16131	9.15366	−0.00765	0.08	11.27716	11.26330	−0.01386	0.12
110	7.34063	7.33266	−0.00797	0.11	9.47826	9.46389	−0.01437	0.15

Notes. Part (I) compares our numerical prices via CTMC approximation ($N = 50$) with those obtained via the recursive algorithm in Fusai and Meucci (2008) for discretely monitored Asian options under the DEJD model. The unvarying parameters are the same as in Fusai and Meucci (2008), i.e., $r = 0.0367$, $d = 0$, $S_0 = 100$, $T = 1$, $\sigma = 0.120381$, $p = 0.2071$, $\eta = 9.65997$, $\theta = 3.13868$, and $\lambda = 0.330966$. It takes about 0.2, 0.7, and 3.6 seconds to generate one price via our method for $n = 12$, 50, and 250, respectively. Part (II) compares our numerical prices via CTMC approximation ($N = 100$) with the double-Laplace inversion results in Cai and Kou (2012) for continuously monitored Asian options under the DEJD model. The unvarying parameters are $r = 0.09$, $d = 0$, $S_0 = 100$, $T = 1$, $p = 0.6$, $\eta = \theta = 25$, and $\lambda = 5$. It takes about 1.1 seconds to obtain one price via our method.

absolute error is around 0.00736 (respectively, 0.01960) and the average (respectively, maximum) relative error is around 0.14% (respectively, 1.53%). Besides, it takes about 1.1 seconds to obtain one price via our method.

5.4. Merton's Jump Diffusion Model

In Table 6, we compare in part (I) the numerical results of discretely monitored Asian option prices obtained via our method with those obtained via the recursive method in Fusai and Meucci (2008), and we compare in part (II) our numerical results for continuously monitored Asian option prices with Monte Carlo simulation estimates under the MJD model (see Merton 1976). It can be seen that our method is quite accurate in both cases. Indeed, in the discrete monitoring case, the average (respectively, maximum) absolute error is around 0.00417 (respectively, 0.00592) and the average (respectively, maximum) relative error is around 0.12% (respectively, 0.21%). In the continuous monitoring case, the average (respectively, maximum) absolute error is around 0.00235 (respectively, 0.00822) and the average (respectively, maximum) relative error is around 0.27% (respectively, 0.76%). In addition, the CPU times for producing one numerical result via our method are 0.2 seconds for $n = 12$, 0.7 seconds for $n = 50$, 3.6 seconds for $n = 250$, and 0.3 seconds for continuously monitored Asian options.

Table 6. Pricing Asian options under the MJD model.

(I) Discretely monitored Asian options under the MJD model

n	K	Fusai and Meucci	CTMC	Abs. err.	Rel. err. (%)
12	90	12.71066	12.70620	−0.00446	0.04
	100	5.01127	5.00539	−0.00588	0.12
	110	1.05162	1.04941	−0.00221	0.21
50	90	12.74093	12.73659	−0.00434	0.03
	100	5.05246	5.04654	−0.00592	0.12
	110	1.07959	1.07736	−0.00223	0.21
250	90	12.74917	12.74485	−0.00432	0.03
	100	5.06381	5.05790	−0.00591	0.12
	110	1.08740	1.08515	−0.00225	0.21

(II) Continuously monitored Asian options under the MJD model

K	MC value	Std. err.	CTMC	Abs. err.	Rel. err. (%)
90	12.74587	0.00371	12.74705	0.00117	0.01
100	5.05974	0.00399	5.05740	−0.00235	0.05
110	1.08413	0.00280	1.09235	0.00822	0.76

Notes. Part (I) compares our numerical prices via CTMC approximation ($N = 50$) with those obtained via the recursive algorithm in Fusai and Meucci (2008) for discretely monitored Asian options under the MJD model. It takes about 0.2, 0.7, and 3.6 seconds to generate one price via our method for $n = 12$, 50, and 250, respectively. Part (II) compares our numerical prices via CTMC approximation ($N = 50$) with Monte Carlo simulation estimates (denoted by "MC value." The sample size is 1,000,000 and the number of time steps is 10,000) for continuously monitored Asian options under the MJD model. It takes about 0.3 seconds to obtain one price via our method. For both parts, the unvarying parameters are the same as in Fusai and Meucci (2008) and obtained from calibration, i.e., $r = 0.0367$, $d = 0$, $S_0 = 100$, $T = 1$, $\sigma = 0.126349$, $\alpha = -0.390078$, $\delta = 0.338796$, and $\lambda = 0.174814$.

Table 7. Test cases of parameter settings for Asian options under the VG model.

Case no.	r	d	δ	ν	θ
1	0.0533	0.011	0.17875	0.13317	−0.30649
2	0.0536	0.012	0.18500	0.22460	−0.28837
3	0.0549	0.011	0.19071	0.49083	−0.28113
4	0.0541	0.012	0.20722	0.50215	−0.22898

Note. These parameter settings are obtained from calibration for S&P 500 on June 30, 1999 by Hirsa and Madan (2004).

5.5. The Variance Gamma Model

In this subsection, we consider the VG model, a pure jump model, under the four cases of parameter settings calibrated for S&P 500 on June 30, 1999 by Hirsa and Madan (2004) except $T = 1$ (see Table 7). Table 8 provides the numerical results for both discretely and continuously monitored Asian options obtained via our method as well as the benchmark prices computed by Monte Carlo simulation. We can see that the average (respectively, maximum) absolute error is around 0.0099 (respectively, 0.0187) and the average (respectively, maximum) relative error is around 0.17% (respectively, 0.30%). Besides, the CPU times for producing one numerical result via our method are 0.2 seconds for

Table 8. Pricing Asian options under the VG model.

n	Case no.	MC values	Std. err.	CTMC	Abs. err.	Rel. err. (%)
12	1	5.5193	0.0034	5.5068	−0.0125	0.23
	2	5.7773	0.0036	5.7692	−0.0082	0.14
	3	6.3873	0.0039	6.3802	−0.0071	0.11
	4	6.1591	0.0040	6.1404	−0.0187	0.30
50	1	5.5740	0.0034	5.5655	−0.0085	0.15
	2	5.8455	0.0037	5.8301	−0.0154	0.26
	3	6.4541	0.0040	6.4436	−0.0105	0.16
	4	6.2146	0.0040	6.2011	−0.0135	0.22
250	1	5.5975	0.0034	5.5815	−0.0160	0.29
	2	5.8606	0.0037	5.8467	−0.0139	0.24
	3	6.4752	0.0040	6.4610	−0.0143	0.22
	4	6.2246	0.0041	6.2177	−0.0069	0.11
∞	1	5.5932	0.0034	5.5869	−0.0063	0.11
	2	5.8585	0.0037	5.8533	−0.0052	0.09
	3	6.4687	0.0040	6.4673	−0.0014	0.02
	4	6.2289	0.0041	6.2295	0.0007	0.01

Notes. Pricing Asian options under the VG model via CTMC approximation ($N = 50$). The columns "MC value" and "Std. err." denote the Monte Carlo simulation estimates and associated standard errors obtained by simulating 1,000,000 sample paths and using 10,000 time steps for continuous monitoring cases. The four cases of parameter settings are taken from Table 1 of Hirsa and Madan (2004) except $T = 1$ (see Table 7 in this paper) and obtained from calibration of S&P 500 on June 30, 1999. The CPU times for producing one numerical result via our method are 0.2 seconds for $n = 12$, 0.8 seconds for $n = 50$, 3.8 seconds for $n = 250$, and 0.3 seconds for continuously monitored Asian options (i.e., $n = +\infty$).

Table 9. Pricing Asian options under the CGMY model.

(I) Discretely monitored Asian options under the CGMY model

n	K	Fusai and Meucci	CTMC	Abs. err.	Rel. err. (%)
12	90	12.70625	12.70406	−0.00219	0.02
	100	5.03492	5.02551	−0.00941	0.19
	110	1.02115	1.01464	−0.00651	0.64
50	90	12.73854	12.73745	−0.00109	0.01
	100	5.07570	5.06651	−0.00919	0.18
	110	1.04674	1.04012	−0.00662	0.63
250	90	12.74737	12.74653	−0.00084	0.01
	100	5.08694	5.07783	−0.00911	0.18
	110	1.05389	1.04725	−0.00664	0.63

(II) Continuously monitored Asian options under the CGMY model

K	MC values	Std. err.	CTMC	Abs. err.	Rel. err. (%)
90	12.74788	0.00396	12.74689	−0.00099	0.01
100	5.08865	0.00405	5.08019	−0.00846	0.17
110	1.05810	0.00280	1.06028	0.00218	0.21

Notes. Part (I) compares our numerical prices via CTMC approximation ($N = 50$) with those obtained via the recursive algorithm in Fusai and Meucci (2008) for discretely monitored Asian options under the CGMY model. It takes about 0.2, 0.8, and 3.6 seconds to generate one price via our method for $n = 12$, 50, and 250, respectively. Part (II) compares our numerical prices via CTMC approximation ($N = 50$) with Monte Carlo simulation estimates (denoted by "MC value." The sample size is 1,000,000 and the number of time steps is 10,000) for continuously monitored Asian options under the CGMY model. It takes about 0.3 seconds to obtain one price via our method. For both parts, the unvarying parameters are the same as in Fusai and Meucci (2008) and obtained from calibration, i.e., $r = 0.0367$, $d = 0$, $S_0 = 100$, $T = 1$, $C = 0.0244$, $G = 0.0765$, $M = 7.5515$, and $Y = 1.2945$.

$n = 12$, 0.8 seconds for $n = 50$, 3.8 seconds for $n = 250$, and 0.3 seconds for continuously monitored Asian options.

5.6. The CGMY Model

In this subsection, we consider another pure jump model, the CGMY model (see Carr et al. 2002). In Table 9, we compare in part (I) the numerical results of discretely monitored Asian option prices obtained via our method with those obtained via the recursive method in Fusai and Meucci (2008), and we compare in part (II) our numerical results for continuously monitored Asian option prices with those generated by the Monte Carlo simulation method of Madan and Yor (2008). It can be seen that our method is quite accurate in both cases. In the discrete monitoring case, the average (respectively, maximum) absolute error is around 0.00573 (respectively, 0.00941) and the average (respectively, maximum) relative error is around 0.28% (respectively, 0.64%). In the continuous monitoring case, the average (respectively, maximum) absolute error is around 0.00388 (respectively, 0.00846) and the average (respectively, maximum) relative error is around 0.13% (respectively, 0.21%). In addition, the CPU times for producing one numerical result via our method are 0.2 seconds for $n = 12$, 0.8 seconds for $n = 50$, 3.6 seconds for $n = 250$, and 0.3 seconds for continuously monitored Asian options.

5.7. Computing the Greeks

Our method can also be applied to compute the sensitivities of the option prices approximately. Below we take the delta as an example. To approximate the delta $\Delta(S_0)$ of either continuously or discretely monitored Asian option price for an initial stock price S_0, we first use our CTMC approximation method to obtain the Asian option prices $V(S_0+\epsilon)$ and $V(S_0-\epsilon)$ with the initial stock price equal to $S_0+\epsilon$ and $S_0-\epsilon$, respectively (here ϵ is a small number). Then $\Delta(S_0)$ can be approximated by $\Delta(S_0) \approx (1/(2\epsilon))(V(S_0+\epsilon) - V(S_0-\epsilon))$.

Table 10 gives the numerical results for the delta (denoted by "CTMC") under the CEV model obtained by our method. The parameter settings are the same as in Table 4, and the benchmarks are constructed using the finite difference approximation of Monte Carlo simulation estimates. It can be seen that our results are very accurate. For the daily monitored ($n = 250$) Asian options, part (I) in Table 10 shows that our numerical results of delta have average (respectively, maximum) absolute error 0.00092 (respectively, 0.00170) and average (respectively, maximum) relative error 0.33% (respectively, 1.15%). Besides, it takes around 7.1 seconds to generate one numerical result of delta via our method. For continuously monitored Asian options, as shown in part (II) of Table 10, the average (respectively, maximum) absolute error is 0.00086 (respectively, 0.00149), and the average (respectively, maximum) relative error is 0.27% (respectively, 0.96%). It takes around 0.2 seconds to generate one numerical result of delta via our method. For a detailed study for computing greeks of Asian options under exponential Lévy models, we refer to the paper by Ballotta et al. (2014).

5.8. Pricing of Floating Strike Asian Options

In the previous sections, we focus our discussions on the Asian options with fixed strike prices. There also exist floating strike Asian options. For instance, the price of a continuously monitored floating strike Asian put option at time 0 is given by

$$\tilde{V}_c(T, \xi; x) = e^{-rT}\mathbb{E}^x\left[\left(\frac{1}{T}A_T - \xi S_T\right)^+\right], \quad \text{with } A_T := \int_0^T S_t\,dt,$$

and the price of a discretely monitored floating strike Asian put option at time 0 is given by

$$\tilde{V}_d(n, \xi; x) = e^{-rT}\mathbb{E}^x\left[\left(\frac{1}{n+1}B_n - \xi S_T\right)^+\right], \quad \text{with } B_n := \sum_{i=0}^{n} S_{i\Delta},$$

where ξ is a positive constant.

Table 10. Computing deltas for discretely and continuously monitored Asian options under the CEV model.

	(I) Discretely monitored Asian options under the CEV model ($n=250$)				(II) Continuously monitored Asian options under the CEV model			
K	MC value	CTMC	Abs. err.	Rel. err. (%)	MC value	CTMC	Abs. err.	Rel. err. (%)
		$\beta=0.25$				$\beta=0.25$		
80	0.94395	0.94429	0.00033	0.04	0.94395	0.94415	0.00020	0.02
90	0.81128	0.81269	0.00141	0.17	0.81126	0.81250	0.00123	0.15
100	0.56919	0.56920	0.00002	0.00	0.56844	0.56923	0.00080	0.14
110	0.32310	0.32193	−0.00118	0.36	0.32315	0.32212	−0.00102	0.32
120	0.15470	0.15312	−0.00158	1.02	0.15487	0.15338	−0.00149	0.96
		$\beta=-0.25$				$\beta=-0.25$		
80	0.93839	0.93884	0.00045	0.05	0.93815	0.93873	0.00058	0.06
90	0.81468	0.81601	0.00133	0.16	0.81458	0.81582	0.00124	0.15
100	0.58549	0.58547	−0.00002	0.00	0.58509	0.58552	0.00043	0.07
110	0.33351	0.33234	−0.00116	0.35	0.33375	0.33252	−0.00122	0.37
120	0.15113	0.14950	−0.00163	1.08	0.15080	0.14977	−0.00102	0.68
		$\beta=-0.5$				$\beta=-0.5$		
80	0.93583	0.93634	0.00052	0.06	0.93586	0.93624	0.00038	0.04
90	0.81682	0.81811	0.00129	0.16	0.81732	0.81791	0.00059	0.07
100	0.59358	0.59359	0.00001	0.00	0.59266	0.59364	0.00097	0.16
110	0.33836	0.33718	−0.00118	0.35	0.33802	0.33735	−0.00067	0.20
120	0.14866	0.14696	−0.00170	1.15	0.14829	0.14725	−0.00104	0.70

Notes. Part (I) and (II) compare the numerical results of delta via our CTMC approximation method ($N=50$) with Monte Carlo simulation estimates (denoted by "MC value") under the CEV model. For the simulation method, the sample size is 1,000,000, and the number of time steps for approximating continuously monitored Asian options is 10,000; the deltas are calculated by the central difference scheme with step size 0.50. It takes around 7.1 seconds and 0.2 seconds to generate one numerical result via our method for the discretely and continuously monitored Asian options, respectively. The unvarying parameters are $r=0.05$, $d=0$, $S_0=100$, $T=1$, and $\sigma S_0^{\beta}=0.25$.

When the underlying asset follows an exponential Lévy model, Eberlein and Papapantoleon (2005) prove that under some regularity conditions, a floating strike Asian option is equivalent to a fixed strike Asian option under another closely related exponential Lévy model. More precisely, suppose $\{S_t, t\geqslant 0\}$ follows an exponential Lévy model with the infinitesimal generator

$$(r-d)xf'(x)+\frac{1}{2}\sigma^2x^2f''(x) + \int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu(dy),$$

and assume there exists a constant $M>1$ such that

$$\mathbb{E}^x[S_t^b]<\infty, \quad \text{for any } |b|<M.$$

Furthermore, consider another process $\{S_t^*, t\geqslant 0\}$ with the infinitesimal generator

$$(d-r)xf'(x)+\frac{1}{2}\sigma^2x^2f''(x) + \int_{\mathbb{R}}[f(xe^y)-f(x)-xf'(x)(e^y-1)]\nu^*(dy),$$

where $\nu^*(dy)=e^{-y}\nu(-dy)$. In other words, $\{S_t^*, t\geqslant 0\}$ follows another exponential Lévy model with risk free rate d, dividend rate r, and Lévy measure ν^*. Then the price of an (either discretely or continuously monitored) Asian put (respectively, call) option with floating strike and multiplier ξ under $\{S_t, t\geqslant 0\}$ is equal to the price of an Asian call (respectively, put) option with fixed strike ξS_0 under $\{S_t^*, t\geqslant 0\}$.

As a consequence, this correspondence enables us to price the floating strike Asian options under an exponential Lévy model $\{S_t, t\geqslant 0\}$ by pricing the corresponding fixed strike Asian options under the closely related exponential Lévy model $\{S_t^*, t\geqslant 0\}$ via our CTMC approximation method. Table 11 provides some numerical results for pricing Asian options with floating strike under Merton's jump diffusion model. We can see that the average (respectively, maximum) absolute error is around 0.0062 (respectively, 0.0114), and the average (respectively, maximum) relative error is around 0.19% (respectively, 0.35%). Besides, the CPU times for producing one numerical result

Table 11. Pricing floating strike Asian options under the MJD model.

n	MC value	Std. err.	CTMC	Abs. err.	Rel. err. (%)
12	3.3002	0.0041	3.3116	0.0114	0.35
50	3.3463	0.0041	3.3565	0.0103	0.31
250	3.3676	0.0042	3.3688	0.0012	0.04
∞	3.3658	0.0042	3.3678	0.0020	0.06

Notes. Pricing Asian put options with floating strikes ($\xi=1$) under Merton's jump diffusion model via our CTMC approximation with $N=50$. The parameters are the same as in Table 6. The columns "MC value" are generated by Monte Carlo simulation, for which the sample size is 1,000,000 and the number of time steps for the continuously monitored Asian option is 10,000. The CPU times to generate one numerical price via our method are about 0.2, 0.7, and 3.8 seconds for $n=12$, 50, and 250 respectively, and about 0.3 seconds for $n=+\infty$, i.e., for continuously monitored Asian options.

via our method are 0.2, 0.7, and 3.8 seconds for $n=12$, 50 and 250, respectively, and 0.3 seconds for continuously monitored Asian options. Under general Markov processes, there may exist no such simple relationship between floating strike and fixed strike Asian option prices, and the problem of using CTMC approximation to price floating strike Asian options remains an interesting research topic.

6. Error Analysis for Our Pricing Method

This section is devoted to the analysis on the errors associated with our pricing method, including the CTMC approximation error in Step 1 and the transform inversion error in Step 2.

6.1. The Transform Inversion Error in Step 2

According to Choudhury et al. (1994), the numerical inversion of double transforms in Step 2 may incur two types of errors: the discretization error and the truncation error. For simplicity, we assume $r\neq d$ in the following analysis. For the discretely monitored Asian options, the discretization error for the double inversion (B2) is given by

$$\mathrm{Error}_d=\sum_{\substack{j=0\\ \text{not } j=l=0}}^{\infty}\sum_{l=0}^{\infty}e^{-Aj}\rho^{2ln}v_d((1+2l)n,(1+2j)k;x).$$

Note that $|v_d(n,k;x)|=|\mathbb{E}^x[(B_n-k)^+]|\leqslant\mathbb{E}^x[B_n]=x((e^{(r-d)\Delta(n+1)}-1)/(e^{(r-d)\Delta}-1))$. Then for any $\rho\in(0,\min\{e^{-(r-d)\Delta},1\})$ and $A>0$, we can obtain the following computable error bound for Error_d:

$$\begin{aligned}|\mathrm{Error}_d| &\leqslant\frac{x}{e^{(r-d)\Delta}-1}\sum_{\substack{j=0\\ \text{not } j=l=0}}^{\infty}\sum_{l=0}^{\infty}e^{-Aj}\rho^{2ln}\big[e^{(r-d)\Delta[(1+2l)n+1]}-1\big]\\ &=\frac{x}{e^{(r-d)\Delta}-1}\\ &\quad\cdot\left[e^{(r-d)\Delta(n+1)}\frac{e^{-A}+(\rho e^{(r-d)\Delta})^{2n}-e^{-A}(\rho e^{(r-d)\Delta})^{2n}}{(1-e^{-A})[1-(\rho e^{(r-d)\Delta})^{2n}]}\right.\\ &\qquad\left.-\frac{e^{-A}+\rho^{2n}-e^{-A}\rho^{2n}}{(1-e^{-A})(1-\rho^{2n})}\right],\end{aligned}\tag{8}$$

which implies that $|\mathrm{Error}_d|=O(e^{-A})+O(\rho^{2n})$ as $A\to+\infty$ and $\rho\to0$. Therefore the discretization error can be controlled by properly specifying inversion parameters A and ρ. For the numerical examples in this paper, we choose $A=20$ and ρ such that $e^{-A}=\rho^{2n}$. It follows from (8) that this setting is able to bound the discretization errors of the option prices by 10^{-6}.

Similarly, for the continuously monitored Asian options, the discretization error for the double inversion (B1) is given by

$$\mathrm{Error}_c=\sum_{\substack{j=0\\ \text{not } j=l=0}}^{\infty}\sum_{l=0}^{\infty}e^{-(A_1j+A_2l)}v_c((1+2j)t,(1+2l)k;x).$$

Since $|v_c(t,k;x)|=|\mathbb{E}^x[(A_t-k)^+]|\leqslant\mathbb{E}^x[A_t]=(x/(r-d))(e^{(r-d)t}-1)$, for any $A_1>\max\{2(r-d)t,0\}$ and $A_2>0$, we can also derive a computable error bound for Error_c:

$$\begin{aligned}|\mathrm{Error}_c| &\leqslant\frac{x}{r-d}\sum_{\substack{j=0\\ \text{not } j=l=0}}^{\infty}\sum_{l=0}^{\infty}e^{-(A_1j+A_2l)}\big[e^{(r-d)(1+2j)t}-1\big]\\ &=\frac{x}{r-d}\left[e^{(r-d)t}\frac{e^{-A_1+2(r-d)t}+e^{-A_2}-e^{-A_1-A_2+2(r-d)t}}{(1-e^{-A_1+2(r-d)t})(1-e^{-A_2})}\right.\\ &\qquad\left.-\frac{e^{-A_1}+e^{-A_2}-e^{-A_1-A_2}}{(1-e^{-A_1})(1-e^{-A_2})}\right].\end{aligned}\tag{9}$$

It follows that $|\mathrm{Error}_c|=O(e^{-A_1})+O(e^{-A_2})$ as $A_1\to+\infty$ and $A_2\to+\infty$. Namely, the discretization error has an exponential decay as A_1 and A_2 go to $+\infty$. In our implementation, we set $A_1=A_2=20$ based on (9) such that the discretization error is no greater than 10^{-6}.

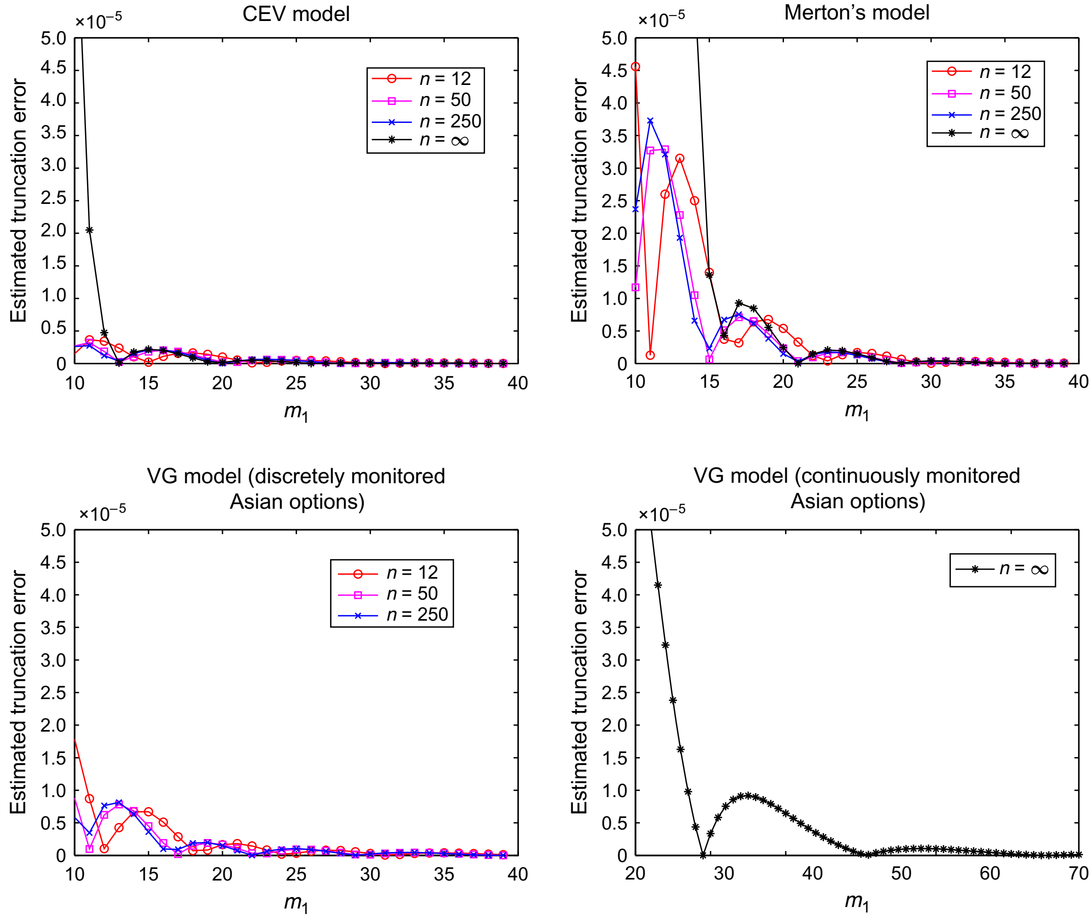
The truncation errors are related to the parameters m_1 and m_2 used in the Euler transformation $E(m_1,m_2)$ in (B3), which is applied to accelerate the convergence of the partial sums to the corresponding infinite series in (B1) and (B2). As suggested by Abate and Whitt (1992), the truncation errors can be estimated by $|E(m_1,m_2)-E(m_1-1,m_2)|$ and $|E(m_1,m_2+1)-E(m_1,m_2)|$.

Figure 1 plots the estimated truncation errors with respect to different choices of m_1 (with the fixed $m_2=15$) under the CEV model, Merton's jump diffusion model, and the VG model and for different monitoring frequencies, $n=12$, 50, 250, and $+\infty$ (continuously monitored). As we can see, the estimated truncation errors decay very fast as m_1 increases. For instance, when $m_1\geqslant30$, all the errors are less than 10^{-6} except the continuously monitored Asian option under the VG model. In the latter case, the error is less than 10^{-5} that is still sufficiently small in practice. This might be due to the power decay of the moment generating function of the VG process. Nonetheless, this effect seems not significant in the discretely monitored cases under the VG model for commonly used discretely monitoring frequencies such as $n=12$, 50, and 250.

6.2. The CTMC Approximation Error in Step 1

In Step 1, we approximate the targeted Markov process model with a CTMC. With the inversion errors in Step 2 controlled at a low level, the approximation error in Step 1 decreases as the number of states for the approximate CTMC increases. Figure 2 plots the absolute pricing errors for both discretely and continuously monitored Asian options obtained via our method with respect to an increasing number of states in the approximate CTMC, under the CIR model, Merton's jump diffusion model and the VG model, respectively. The errors are estimated by $\mathrm{Error}(N)=|\mathrm{Price}(N)-\mathrm{Price}(N^*)|$, where $\mathrm{Price}(N)$

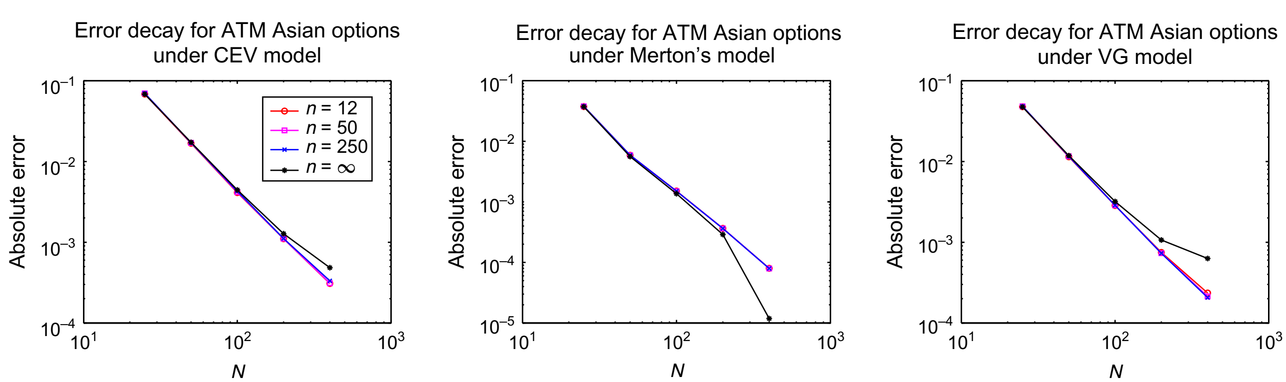
Figure 1. (Color online) The decay of the truncation errors of the double inversions.



Notes. The estimated truncation errors of the double inversions with respect to different choice of m_1 and fixed $m_2 = 15$. All the options computed in these figures are at the money. Other parameters are set as follows: the parameters for the CEV model are the same as in Table 4 ($\beta = -0.25$); the parameters for Merton's jump diffusion model are the same as in Table 6; the parameters for the VG model are the same as in Table 8 (case no. 2).

represents the price computed by CTMC approximation with N states and N^* is a fixed large natural number. Here we set $N^* = 1{,}000$. It can be seen from Figure 2 that for the discretely monitored Asian options, the logarithms of absolute pricing errors are approximately linear in the logarithm of N and the slopes of the lines are about -2, which suggests that the absolute pricing errors appear to be proportional to $1/N^2$. For the continuously monitored Asian

Figure 2. (Color online) The decay of the CTMC approximation errors for pricing Asian options.



Notes. The absolute errors of discretely and continuously monitored Asian option prices obtained via our CTMC approximation with the number of states $N = 25$, 50, 100, 200, and 400. Left: The CEV model with the same parameters as in Table 4 ($\beta = -0.25$). Middle: Merton's jump diffusion model with the same parameters as in Table 6. Right: The VG model with the same parameters as in Table 8 (case no. 2).

options, the slopes of the decay lines in Figure 2 vary from -1.6 to -2.8, which suggests that the decay of the absolute pricing errors of our method appears to be at the speed of $1/N^{1.6}$ to $1/N^{2.8}$. This is similar to the observation by Mijatović and Pistorius (2013) in the CTMC pricing of continuously monitored double barrier options, where they find that the absolute pricing error is approximately proportional to $1/N^{1.2}$ to $1/N^2$ under different models.

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Appendix A. Proof of Theorem 1

Proof. (i) Suppose that f is any bounded solution to the functional Equation (4). Define

$$M_n := f(S_{t_n})e^{\theta S_{t_n}} \cdot z^n e^{-\theta B_n} + \sum_{k=0}^{n-1} z^k e^{-\theta B_k} \quad \text{for } n = 0, 1, \ldots.$$

Then we claim that $\{M_n\colon n \geqslant 0\}$ is a martingale with respect to $\{\mathcal{F}^{(n)}, n \geqslant 0\}$, where $\mathcal{F}^{(n)} := \mathcal{F}_{n\Delta}$. Indeed, on the one hand,

$$\begin{aligned}|M_n| &\leqslant |f(S_{t_n})e^{\theta S_{t_n}}||z^n e^{-\theta B_n}| + \sum_{k=0}^{n-1}|z^k e^{-\theta B_k}| \\ &= |1 + zP_\Delta f(S_{t_n})||z^n e^{-\theta B_n}| + \sum_{k=0}^{n-1}|z^k e^{-\theta B_k}| \\ &\leqslant 1 + C + \frac{1}{1-|z|} < +\infty, \end{aligned} \tag{A1}$$

where $P_\Delta f(x) := \mathbb{E}^x[f(S_\Delta)]$. On the other hand, for $n \geqslant 1$, since $e^{\theta S_{t_n}} e^{-\theta B_n} = e^{-\theta B_{n-1}}$,

$$\begin{aligned}&\mathbb{E}^x[M_n \mid \mathcal{F}^{(n-1)}] - M_{n-1} \\ &\quad = z^n e^{-\theta B_{n-1}} P_\Delta f(S_{t_{n-1}}) + z^{n-1} e^{-\theta B_{n-1}} \\ &\qquad - f(S_{t_{n-1}})e^{\theta S_{t_{n-1}}} \cdot z^{n-1} e^{-\theta B_{n-1}} \\ &\quad = z^{n-1}e^{-\theta B_{n-1}}[zP_\Delta f(S_{t_{n-1}}) + 1 - f(S_{t_{n-1}})e^{\theta S_{t_{n-1}}}] = 0.\end{aligned}$$

Therefore, $\{M_n, n \geqslant 0\}$ is a martingale with respect to $\{\mathcal{F}^{(n)}, n \geqslant 0\}$, and hence $\mathbb{E}^x[M_n] = M_0 = f(x)$. Letting $n \to +\infty$ and noticing that $\lim_{n\to+\infty} M_n = \sum_{k=0}^{+\infty} z^k e^{-\theta B_k}$ almost surely, we obtain

$$\begin{aligned}f(x) &= \lim_{n\to+\infty} E^x[M_n] = \mathbb{E}^x\Big[\lim_{n\to+\infty} M_n\Big] \\ &= \mathbb{E}^x\left[\sum_{k=0}^{+\infty} z^k e^{-\theta B_k}\right] = l(z, \theta; x),\end{aligned}$$

where the second equality holds because of (A1) and the dominated convergence theorem. This implies that $l(z, \theta; x)$ is the unique bounded solution to (4).

(ii) Suppose that f is any bounded solution to the functional Equation (5). Define

$$M_t := f(S_t)e^{-\mu t - \theta A_t} + \int_0^t e^{-\mu u - \theta A_u}\, du \quad \text{for } t \geqslant 0.$$

We claim that

$$\frac{d}{dt}\mathbb{E}^x[f(S_t)e^{-\mu t - \theta A_t}] = -\mathbb{E}^x[e^{-\mu t - \theta A_t}]. \tag{A2}$$

If (A2) holds, we have

$$\begin{aligned}&\frac{d}{d\tau}\mathbb{E}^x[f(S_{t+\tau})e^{-\mu(t+\tau)-\theta A_{t+\tau}} \mid \mathcal{F}_t] \\ &= \frac{d}{d\tau}\mathbb{E}^y[f(S_\tau)e^{-\mu\tau - \theta A_\tau}]|_{y=S_t} \cdot e^{-\mu t - \theta A_t} \\ &= -\mathbb{E}^y[e^{-\mu\tau-\theta A_\tau}]|_{y=S_t} \cdot e^{-\mu t - \theta A_t} = -\mathbb{E}^x[e^{-\mu(t+\tau)-\theta A_{t+\tau}} \mid \mathcal{F}_t],\end{aligned}$$

where the second equality holds because of (A2). Then it follows that

$$\begin{aligned}&\mathbb{E}^x[f(S_{t+u})e^{-\mu(t+u)-\theta A_{t+u}} \mid \mathcal{F}_t] - \mathbb{E}^x[f(S_t)e^{-\mu t - \theta A_t}] \\ &\quad = -\int_0^u \mathbb{E}^x[e^{-\mu(t+\tau)-\theta A_{t+\tau}} \mid \mathcal{F}_t]\, d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}^x[M_{t+u} \mid \mathcal{F}_t] - M_t &= -\int_0^u \mathbb{E}^x[e^{-\mu(t+\tau)-\theta A_{t+\tau}} \mid \mathcal{F}_t]\, d\tau \\ &\quad + \mathbb{E}^x\left[\int_t^{t+u} e^{-\mu s - \theta A_s}\, ds \mid \mathcal{F}_t\right] = 0.\end{aligned}$$

Also, noticing that $|M_t| \leqslant C + \mathrm{Re}(\mu)^{-1} < +\infty$, we conclude that $\{M_t, t \geqslant 0\}$ is a martingale with respect to $\{\mathcal{F}_t, t \geqslant 0\}$. Then by the dominated convergence theorem, we obtain

$$\begin{aligned}f(x) = M_0 &= \lim_{t\to\infty}\mathbb{E}^x[M_t] = \mathbb{E}^x\Big[\lim_{t\to\infty} M_t\Big] \\ &= \mathbb{E}^x\left[\int_0^\infty e^{-\mu u - \theta A_u}\, du\right] = m(\mu, \theta; x).\end{aligned}$$

To prove (A2), first note that for any fixed $t \geqslant 0$,

$$\begin{aligned}&\frac{\mathbb{E}^x[f(S_{t+u})e^{-\mu(t+u)-\theta A_{t+u}}] - \mathbb{E}^x[f(S_t)e^{-\mu t - \theta A_t}]}{u} \\ &= \frac{1}{u}\cdot(\mathbb{E}^x[f(S_{t+u})e^{-\mu(t+u)-\theta A_{t+u}} - f(S_{t+u})e^{-\mu t - \theta A_t}] \\ &\qquad + \mathbb{E}^x[f(S_{t+u})e^{-\mu t - \theta A_t} - f(S_t)e^{-\mu t - \theta A_t}]) \\ &= \mathbb{E}^x\left[\frac{f(S_{t+u})(e^{-\mu(t+u)-\theta A_{t+u}} - e^{-\mu t - \theta A_t})}{u}\right] \\ &\quad + \mathbb{E}^x\left[\frac{(f(S_{t+u}) - f(S_t))e^{-\mu t - \theta A_t}}{u}\right].\end{aligned} \tag{A3}$$

We intend to let $u \downarrow 0$ on both sides of (A3). For the first term on the right-hand side (RHS) of (A3), when $u \in (0, u_0)$ with any fixed $u_0 > 0$,

$$\begin{aligned}&\left|\frac{f(S_{t+u})(e^{-\mu(t+u)-\theta A_{t+u}} - e^{-\mu t - \theta A_t})}{u}\right| \\ &= \left|\frac{f(S_{t+u})\int_0^u(-\mu - \theta S_{t+s})e^{-\mu(t+s)-\theta A_{t+s}}\, ds}{u}\right| \\ &\leqslant \frac{C}{u}\int_0^u |\mu + \theta S_{t+s}| \cdot |e^{-\mu(t+s)-\theta A_{t+s}}|\, ds \\ &\leqslant C\cdot\Big(|\mu| + |\theta|\max_{t\leqslant s\leqslant t+u} S_s\Big) \leqslant C\cdot\Big(|\mu| + |\theta|\max_{0\leqslant s\leqslant t+u_0} S_s\Big).\end{aligned}$$

By Doob's inequality,

$$\mathbb{E}^x\Big[\max_{0\leqslant s\leqslant t+u_0} S_s\Big] \leqslant (1-e^{-1})^{-1}(1+\mathbb{E}^x[|S_{t+u_0}\log S_{t+u_0}|]) \\ \leqslant C_1 + C_2\cdot \mathbb{E}^x[S_{t+u_0}^{1+\epsilon}] < +\infty,$$

for some $C_1, C_2 > 0$, where the second inequality holds because $|x\log x| \leqslant C_3 + |x|^{1+\epsilon}$ for some $C_3 > 0$. Therefore, applying the dominated convergence theorem yields

$$\lim_{u\downarrow 0}\mathbb{E}^x\left[\frac{f(S_{t+u})(e^{-\mu(t+u)-\theta A_{t+u}} - e^{-\mu t-\theta A_t})}{u}\right] \\ = \mathbb{E}^x[f(S_t)(-\mu-\theta S_t)e^{-\mu t-\theta A_t}]. \tag{A4}$$

For the second term on the RHS of (A3), by Proposition 1.1.5 in Ethier and Kurtz (2005) we obtain

$$\mathbb{E}^x\left[\frac{(f(S_{t+u})-f(S_t))e^{-\mu t-\theta A_t}}{u}\right] \\ = \mathbb{E}^x\left[(\mathbb{E}^x[f(S_{t+u})\mid \mathcal{F}_t] - f(S_t))\frac{e^{-\mu t-\theta A_t}}{u}\right] \\ = \mathbb{E}^x\left[\int_0^u \mathbb{E}^x[Gf(S_{t+s})\mid \mathcal{F}_t]\,ds\frac{e^{-\mu t-\theta A_t}}{u}\right] \\ = \mathbb{E}^x\left[\int_0^u Gf(S_{t+s})\,ds\frac{e^{-\mu t-\theta A_t}}{u}\right].$$

Since $|Gf(x)| = |(\mu+\theta x)f(x)-1| \leqslant (|\mu|+|\theta x|)C+1$, we have

$$\left|\int_0^u Gf(S_{t+s})\,ds\frac{e^{-\mu t-\theta A_t}}{u}\right| \leqslant C\Big(|\mu|+|\theta|\max_{t\leqslant s\leqslant t+u} S_s\Big)+1.$$

Then it follows from the dominated convergence theorem that

$$\lim_{u\downarrow 0}\mathbb{E}^x\left[\frac{(f(S_{t+u})-f(S_t))e^{-\mu t-\theta A_t}}{u}\right] \\ = \mathbb{E}^x\left[\lim_{u\downarrow 0}\frac{\int_0^u Gf(S_{t+s})\,ds}{u}e^{-\mu t-\theta A_t}\right] \\ = \mathbb{E}^x[Gf(S_t)\cdot e^{-\mu t-\theta A_t}]. \tag{A5}$$

Now letting $u\downarrow 0$ on both sides of (A3) and using (A4) and (A5) yields

$$\frac{d}{dt}\mathbb{E}^x[f(S_t)e^{-\mu t-\theta A_t}] \\ \equiv \lim_{u\downarrow 0}\frac{\mathbb{E}^x[f(S_{t+u})e^{-\mu(t+u)-\theta A_{t+u}}] - \mathbb{E}^x[f(S_t)e^{-\mu t-\theta A_t}]}{u} \\ = \mathbb{E}^x[f(S_t)(-\mu-\theta S_t)e^{-\mu t-\theta A_t}] + \mathbb{E}^x[Gf(S_t)\cdot e^{-\mu t-\theta A_t}] \\ = -\mathbb{E}^x[e^{-\mu t-\theta A_t}],$$

where the last equality follows from the functional Equation (5). □

Appendix B. Double Transform Inversion Formulas

To numerically invert the double transforms involved in our pricing algorithms for both discretely and continuously monitored Asian options, we shall use the transform inversion algorithms proposed by Choudhury et al. (1994). Specifically, to evaluate $f(t_1,t_2)$ from its double-Laplace transform $\tilde{f}(t_1,t_2)$, we have

$$f(t_1,t_2) = \frac{e^{(A_1+A_2)/2}}{4t_1t_2}\sum_{j=-\infty}^{\infty}(-1)^{j+1}\sum_{k=-\infty}^{\infty}(-1)^{k+1} \\ \cdot\tilde{f}\left(\frac{A_1}{2t_1}-\frac{\mathrm{i}\pi}{t_1}-\frac{\mathrm{i}j\pi}{t_1}, \frac{A_2}{2t_2}-\frac{\mathrm{i}\pi}{t_2}-\frac{\mathrm{i}k\pi}{t_2}\right) - \mathrm{e}_f, \tag{B1}$$

where

$$\mathrm{e}_f = \sum_{\substack{j=0\\ \text{not } j=k=0}}^{\infty}\sum_{k=0}^{\infty} e^{-(A_1 j+A_2 k)}f((1+2j)t_1,(1+2k)t_2).$$

Similarly, we use the formula below to evaluate $g(t,n)$ from its Z-Laplace transform $\tilde{g}(t,n)$

$$g(t,n) = \frac{e^{A/2}}{4n\rho^n t}\sum_{j=-\infty}^{\infty}(-1)^{j+1}\sum_{k=-n}^{n-1}(-1)^k \\ \cdot\tilde{g}\left(\frac{A}{2t}-\frac{\mathrm{i}\pi}{t}-\frac{\mathrm{i}j\pi}{t}, \rho e^{(\mathrm{i}k\pi)/n}\right) - \mathrm{e}_g, \tag{B2}$$

where

$$\mathrm{e}_g = \sum_{\substack{j=0\\ \text{not } j=k=0}}^{\infty}\sum_{k=0}^{\infty} e^{-Aj}\rho^{2kn}g((1+2j)t,(1+2k)n).$$

The involved parameters, i.e., $A_1 > 0$ and $A_2 > 0$ in (B1) and $A > 0$ and $\rho\in(0,1)$ in (B2), are chosen to control the errors of the corresponding inversion formulas.

Besides, when evaluating series of the form $\sum_{j=0}^{+\infty}(-1)^j a_j$ in the formulas (B1) and (B2), we can accelerate the convergence via Euler transformation. Namely, we approximate $\sum_{j=0}^{+\infty}(-1)^j a_j$ by

$$E(m_1,m_2) := \sum_{j=0}^{m_1}\frac{m_1!}{j!(m_1-j)!}2^{-m_1}S_{m_2+j}, \tag{B3}$$

where $S_j := \sum_{k=0}^{j}(-1)^k a_k$. We set two parameters $m_1 = 10$ and $m_2 = 15$ in our implementation. See Abate and Whitt (1992) for more details about Euler transformation.

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