Lecture 13: Fundamental Theorems of Asset Pricing

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1 How to Price a Derivative?

Let us recap the techniques used in pricing derivatives. For simplicity we restrict ourselves to European contracts.

Given an European option with payoff $V(T)$ and maturity T, the first attempt to price it is to use a portfolio of stocks and cash to replicate the payoff $V(T)$ along all possible path. Such a portfolio may not always be available. If there always exists such a replicating portfolio for any given derivative, then we say the market is **complete**.

Suppose the market is complete, and we are able to construct the replicating portfolio with $X(t)$ at time t, for $t \in [0, T]$. At time T, the replicating portfolio will always replicates the payoff $V(T)$. By no arbitrage argument, we know that the price of the option at time t must be the same as $X(t)$: $V(t) = X(t)$. Therefore, the price $V(t)$ is just the amount of money needed to setup the replicating portfolio at time t. Note that there is no probability measure involved in this discussion so far. The replicating strategy should work under any (possible) circumstances, regardless of the probabilities. The replicating portfolio should still work after any equivalent measure changes.

The next question is, how to obtain the price $X(t)$. A simple answer would be, first find the replicating/hedging strategy, then $X(t)$ is known. But this answer is not easy to implement[1]. An alternative solution is the risk-neutral pricing. Since the market is assumed to be complete, this replicating portfolio exists (but unknown). We know that $X(T) = V(T)$, and we want to know $X(t)$. We need to extract available information from $X(T)$ to compute $X(t)$. This backward mapping is the conditional expectation. Under the actual measure, $E(X(T)|\mathcal{F}_t)$ has little to do with $X(t)$. But under the risk-neutral measure, the discounted value process $\{D(t)X(t)\}_{t\in[0,T]}$ is a martingale, so we can compute the price:

$$X(t) \quad = \quad \frac{1}{D(t)}\tilde{E}(D(T)X(T)|\mathcal{F}_t) = \tilde{E}\left(e^{-\int_t^T R(u)du}V(T)\middle|\mathcal{F}_t\right). \tag{1}$$

We now talk about some fundamentals of the above discussion: when do we have a risk-neutral measure? What is the consequence of having it? Is it possible to have two different risk-neutral measure?

2 Existence of the Risk-neutral Measure

We consider a market consisting of m stocks driven by d independent Brownian motion:

$$dS_i(t) \quad = \quad \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), \ 1 \le i \le m. \tag{2}$$

In vector form we have

$$\begin{pmatrix} \frac{dS_1(t)}{S_1(t)} \\ \vdots \\ \frac{dS_i(t)}{S_i(t)} \\ \vdots \\ \frac{dS_m(t)}{S_m(t)} \end{pmatrix} = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_i(t) \\ \vdots \\ \alpha_m(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(t) & \dots & \sigma_{1j}(t) & \dots & \sigma_{1d}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{i1}(t) & \dots & \sigma_{ij}(t) & \dots & \sigma_{id}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1}(t) & \dots & \sigma_{mj}(t) & \dots & \sigma_{md}(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ \vdots \\ dW_j(t) \\ \vdots \\ dW_d(t) \end{pmatrix}. \tag{3}$$

[1] We will use this method next week.

To move to the risk-neutral measure, we check the discounted stock price process $\{D(t)S_i(t)\}$. Using Itô's product rule and the fact that $dD(t) = -R(t)D(t)dt$ we have

$$\begin{aligned} d(D(t)S_i(t)) &= D(t)[dS_i(t) - R(t)S_i(t)dt] \\ &= D(t)S_i(t)\big[(\alpha_i(t) - R(t))dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\big]. \end{aligned} \tag{4}$$

In order to make $\{D(t)S_i(t)\}$ be a martingale, we need to rewrite (4) as

$$d(D(t)S_i(t)) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)[\Theta_j(t)dt + dW_j(t)], \tag{5}$$

If we can find the market price of risk processes $\Theta_j(t)$ that makes (5) hold, then we can change the measure[2], so that under the new measure, $\tilde{W}_j(t) = \int_0^t \Theta_j(u)du + W_j(t)$ is a standard Brownian motion. In other words, we have a risk-neutral measure $\tilde{P}$ if we can solve the following system:

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t),\ 1 \le i \le m. \tag{6}$$

In vector form we have

$$\begin{pmatrix} \sigma_{11}(t) & \dots & \sigma_{1j}(t) & \dots & \sigma_{1d}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{i1}(t) & \dots & \sigma_{ij}(t) & \dots & \sigma_{id}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{m1}(t) & \dots & \sigma_{mj}(t) & \dots & \sigma_{md}(t) \end{pmatrix} \begin{pmatrix} \Theta_1(t) \\ \vdots \\ \Theta_j(t) \\ \vdots \\ \Theta_d(t) \end{pmatrix} = \begin{pmatrix} \alpha_1(t) - R(t) \\ \vdots \\ \alpha_i(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{pmatrix}. \tag{7}$$

If the above **market price of risk equations** are not solvable, then there is an arbitrage. See example below.

Example 1. Suppose there are two stocks driven by one Brownian motion. And further suppose that all coefficient processes are constant. Then the market price of risk equations are

$$\alpha_1 - r = \sigma_1\theta, \tag{8}$$
$$\alpha_2 - r = \sigma_2\theta. \tag{9}$$

If $\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma}$, then the above system has no solution. Let us define

$$\mu = \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma} > 0. \tag{10}$$

Consider the following trading strategy: hold $\Delta_1(t) = \frac{1}{S_1(t)\sigma_1}$ shares of stock one and $\Delta_2(t) = -\frac{1}{S_2(t)\sigma_2}$ shares of stock two, borrowing or investing money as necessary at the interest rate r. The intial capital required to take the stock position is $\frac{1}{\sigma_1} - \frac{1}{\sigma_2}$, we borrow or save money at time zero so that the total initial value of the portfolio is $X(0) = 0$. The increment of $X(t)$ is thus given by

$$\begin{aligned} dX(t) &= \Delta_1(t)dS_1(t) + \Delta_2(t)dS_2(t) + r(X(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t))dt \\ &= \frac{\alpha_1 - r}{\sigma_1}dt + dW(t) - \frac{\alpha_2 - r}{\sigma_2}dt - dW(t) + rX(t)dt = \mu dt + rX(t)dt. \end{aligned} \tag{11}$$

The discounted value follows

$$d(D(t)X(t)) = D(t)(dX(t) - rX(t)dt) = \mu D(t)dt, \tag{12}$$

which means the discounted portfolio value increases with time. So the trading strategy yields money for sure and the rate of growth is higher than the risk-free rate r. This is an arbitrage.

[2] We can use Girsanov theorem for multiple dimensions, see Theorem 5.4.1 in the textbook.

3 After Having a Risk-neutral Measure

Suppose market price of risk equations (6), (7) are solvable, so that the risk-neutral measure $\tilde{P}$ exists. Then a immediate consequence is that the market does not admit any arbitrage. Recall that an arbitrage is a trading strategy that needs no money, will not lose money and has a positive probability of making money. In mathematical symbols, that means, if $\{X(t)\}_{t\in[0,T]}$ is the wealth process of an arbitrage, then under the actual measure P, the discounted value process $\{D(t)X(t)\}$ satisfies

$$\begin{aligned} X(0) &= 0, & (13)\\ P(D(T)X(T)<0) &= 0, & (14)\\ E(D(T)X(T)) &> 0. & (15)\end{aligned}$$

Since the risk-neutral measure $\tilde{P}$ keeps null sets under measure P, we have under $\tilde{P}$ that

$$\begin{aligned} X(0) &= 0, & (16)\\ \tilde{P}(D(T)X(T)<0) &= 0, & (17)\\ \tilde{E}(D(T)X(T)) = E(Z(T)D(T)X(T)) &> 0, & (18)\end{aligned}$$

where $Z = \frac{d\tilde{P}}{dP} > 0$ is the Radon-Nikodým derivative.

However, under the risk-neutral measure, any discounted value process $\{D(t)X(t)\}$ is a martingale, so there must be

$$0 = X(0) = \tilde{E}(D(T)X(T)) > 0. \tag{19}$$

A contradiction! So if the risk-neutral measure exists, then there is no arbitrage. This is the **first fundamental theorem of asset pricing**.

4 Completeness of Market and Uniqueness of the Risk-neutral Measure

Before we talk about the uniqueness of the risk-neutral measure, let us first check the possibility of hedging derivatives with stocks and cash. In other words, the completeness of market. We assume (6), (7) are solvable so that we have a risk-neutral measure.

Given a derivative, we know that the discounted price process $\{D(t)V(t)\}$ is a martingale under the risk-neutral measure $\tilde{P}$. Using the martingale representation theorem, there are processes $\Gamma_1(u),\ldots,\Gamma_d(u)$ such that

$$D(t)V(t) = V(0) + \sum_{j=1}^{d}\int_0^t \Gamma_j(u)d\tilde{W}_j(u),\ 0\le t\le T. \tag{20}$$

On the other hand, consider a trading strategy of holding $\Delta_i(t)$ shares of stock i at time t, then the value process $\{X(t)\}$ follows

$$\begin{aligned} dX(t) &= \sum_{i=1}^{m}\Delta_i(t)dS_i(t) + R(t)\Big(X(t) - \sum_{i=1}^{m}\Delta_i(t)S_i(t)\Big)dt\\ &= R(t)X(t)dt + \sum_{i=1}^{m}\Delta_i(t)(dS_i(t) - R(t)S_i(t)dt)\\ &= R(t)X(t)dt + \sum_{i=1}^{m}\Delta_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)S_i(t)d\tilde{W}_j(t). \qquad (21)\end{aligned}$$

So the discounted value process $\{D(t)X(t)\}$ follows

$$d(D(t)X(t)) = D(t)(dX(t) - R(t)X(t)dt) = \sum_{i=1}^{m}\sum_{j=1}^{d}\Delta_i(t)D(t)\sigma_{ij}(t)S_i(t)d\tilde{W}_j(t). \tag{22}$$

In order to let this trading strategy hedge the option, i.e., $X(t) = V(t)$ at all times, we need

$$\frac{\Gamma_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij}(t). \tag{23}$$

In vector form, we are solving for $\{\Delta_i(t)\}$ in the following system:

$$\begin{pmatrix} \sigma_{11}(t) & \dots & \sigma_{1i}(t) & \dots & \sigma_{1m}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{j1}(t) & \dots & \sigma_{ji}(t) & \dots & \sigma_{jm}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{d1}(t) & \dots & \sigma_{di}(t) & \dots & \sigma_{dm}(t) \end{pmatrix} \begin{pmatrix} \Delta_1(t)S_1(t) \\ \vdots \\ \Delta_i(t)S_i(t) \\ \vdots \\ \Delta_m(t)S_m(t) \end{pmatrix} = \begin{pmatrix} \frac{\Gamma_1(t)}{D(t)} \\ \vdots \\ \frac{\Gamma_j(t)}{D(t)} \\ \vdots \\ \frac{\Gamma_d(t)}{D(t)} \end{pmatrix}. \tag{24}$$

If (23), (24) are solvable, then the market is complete.

Suppose the market is complete, then the market has a unique risk-neutral measure: if $\tilde{P}_1$ and $\tilde{P}_2$ are any two risk-neutral measures in this market, then for any fixed measurable set $A \in \mathcal{F}_T$, consider a derivative with payoff $V(T) = \mathbb{1}_A \frac{1}{D(T)}$. Because the market is complete, there exists a replicating portfolio whose value process is $\{X(t)\}$, so we have

$$\tilde{P}_1(A) = \tilde{E}_1(D(T)V(T)) = \tilde{E}_1(D(T)X(T)) = X(0) = \tilde{E}_2(D(T)X(T)) = \tilde{E}_2(D(T)V(T)) = \tilde{P}_2(A). \tag{25}$$

Since $\tilde{P}_1(A) = \tilde{P}_2(A)$ for any $A \in \mathcal{F}_T$, these two risk-neutral measure actually coincide.

Now suppose the market has a unique risk-neutral measure. Then the market uncertainty should only come from the d Brownian motion driving the market. Otherwise, we can assign arbitrage probabilities to those source of uncertainty without changing the distribution of the driving Brownian motion and hence without changing the distribution of assets. This would permit us to create multiple risk-neutral measures. Hence, the uniqueness of the risk-neutral measure implies that the market price of risk equation (6), (7) have only one solution. This implies that the system (24) is solvable[3]. This proves that, if the market has a unique risk-neutral measure, then the market is complete.

In conclusion we proved the **second fundamental theorem of asset pricing**: if a market has a risk-neutral measure, then the market is complete if and only if the risk-neutral measure is unique.

[3]Please refer to Appendix C for a proof. This is completely an algebraic result.