Stochastic Basics

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1 Measure Theory Concepts

1.1 Banach Space

Banach Space: A complete normed vector space.

$$\lim_{n\to\infty} x_n = x$$

$$\lim_{n\to\infty} ||x_n - x||_x = 0$$

A normed space X is a Banach space if and only if each absolutely convergent series in X converges:

$$\sum_{n=1}^{\infty} ||v_n||_X < \infty \Rightarrow \sum_{n=1}^{\infty} v_n \; converges \; in \; X$$

1.2 L^p Space

L^p Space: uncountable infinite dimension

$$(S, \Sigma, \mu) \quad L^p(S, \mu) = \{f; ||f||_p = (\int_S |f|^p d\mu)^{\frac{1}{p}} < \infty\}$$

$$||(x_n)_{n\in N}||_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p + ...) = (\sum_{n\in N} |x_n|^p)^{\frac{1}{p}}$$

The series on the right of the above euqation converges.

L^p space is the subset of all the sequences whose elements make the series on the right of above equation converge.

$$if \; p < q, then \; L^p < L^q, \quad this \; is \; a \; proper \; subset.$$

- All L^p spaces are Banach space.

- If and only if $p = 2$, L^2 is Hilbert space.(???)

Norm: A function that assigns a strictly positive length or size to each vector in a vector space, except 0 for zero vector:

1. zero vector: $l(v) = 0 \Leftrightarrow v = 0$
2. linear: $\forall \lambda \in R, l(\lambda v) = \lambda l(v)$
3. triangle inequality: $l(u) + l(v) \geq l(u + v)$

length function: $[\sum (X_i)^p]^{\frac{1}{p}}$

1.3 Hilbert Space

Hilbert Space:

- It extends the methods of vector algebra and calculus from 2-dimension or 3-dimension to finite or infinite number of dimensions.
- A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured.
- As a complete normed space, Hilbert spaces are by definition also Banach spaces.

1.4 L^2 Space

L^2 Space: Countable infinite dimension.

1.5 Some Notations

- $lim\ sup$: limit superior
- $lim\ inf$: limit inferior
- inf: infimum, greatest lower bound(GLB)
- sup: supremum, least upper bound(LUB)

$$\begin{aligned}
\lim_{n\to\infty} sup\ x_n = b \quad &\Rightarrow \quad There\ exists\ N, \epsilon \in R^+,\ x_n < b + \epsilon, \forall n > N \\
\lim_{n\to\infty} inf\ x_n = b \quad &\Rightarrow \quad There\ exists\ N, \epsilon \in R^+,\ x_n > b - \epsilon, \forall n > N
\end{aligned}$$

2 Ito Calculus

Brownian Process: B_t
Quadratic Variation: $(dB_t)^2 = dt$

2.1 Ito's Lemma $f(B_t)$

$$df = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

$$df(t, X_t) = f_t dt + f_x dX_t + \frac{1}{2}f_{xx}d[X]_t$$

For normal calculus, we only have the first term.
The second term here is because of Quadratic Variation.

For $f(t, x)$:

$$f(t+\Delta t, x+\Delta x) = f(t,x) + \frac{\partial f(t,x)}{\partial t}\Delta t + \frac{\partial f(t,x)}{\partial x}\Delta x + \frac{1}{2}(\frac{\partial^2 f(t,x)}{\partial t^2}(\Delta t)^2 + 2\frac{\partial^2 f(t,x)}{\partial t \partial x}\Delta t \Delta x + \frac{\partial^2 f(t,x)}{\partial x^2}$$

$$= f + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial t \partial x}dtdx + \frac{\partial^2 f}{\partial x^2}(dx)^2) + O^3$$

(Use Taylor series)

$$df(t, B_t) = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2})dt + \frac{\partial f}{\partial B_t}dB_t$$

$$(df(t, X) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}d[X]_t)$$

If we have $dX_t = \mu dt + \sigma dW_t$:

$$\Rightarrow df(t, X_t) = (\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2})dt + \sigma\frac{\partial f}{\partial X}dW_t$$

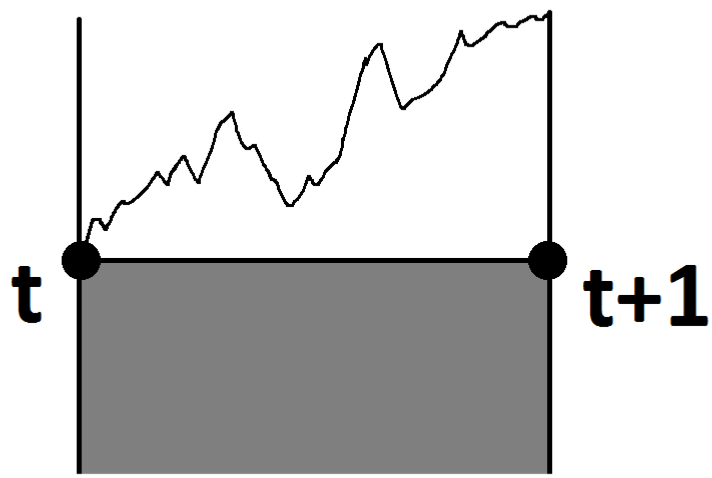
(Plugin dX_t, $d[X]_t = \sigma^2 dt$)

2.2 Integral

Definition:

$$F(t, B_t) = \int f(t, B_t)dB_t + \int g(t, B_t)dt, \quad if \quad dF = fdB_t + gdt$$

Does there exist a Riemannian sum type integration here?
In Riemannian integration, no matter which point you take, the sum is always the same.
If you do this in stochastic integration, the result is different. Eg. : If taking left points all the time, the result is different from taking right points all the time.



This is due to the quadratic variation. Because that much variance can accumulate overtime.

Remarks:
→ Ito integral is a limit of Riemannian sums when always taking left point of each interval.
→ Taking left points means for each interval, you cannot see the future.
→ 'equivalent' of Ito's Calculus: $(dB_t) = -dt$

2.3 Adapted Process

Definition:
$\Delta(t)$ is adapted to X_t if $\forall t >= 0$, $\Delta(t)$ depends only on $X_0 \sim X_t$.(only dependent on the past)
Examples:

1. X_t is dependent on X_t
2. $Y_t = X_{t+1}$ is not dependent on X_t.
3. $\Delta(t) = minX_t$ is dependent on X_t.
4. $T > 0, \Delta(t) = macX_s, (t < s < T)$ is not adaptable.

2.4 Theorem 1

If $\Delta(t)$ is a process depends only on t, then:

$X(t) = \int \Delta(t)dB_t$ has Normal Distribution.

Intuition: Sum of Normal Distributions(Random Variables) is Normal Distribution(Random Variable).

2.5 Theorem 2: Ito Isometry

If X_t is adapted to W_t,

$$E[(\int_0^T X_t dW_t)^2] = E[\int_0^T X_t^2 dt]$$

Eg.(Quadratic Variation)
If $\Delta(t) = 1$, then $E[B(t)^2] = t$.

2.6 Theorem 3

Question: When is Ito Integral a martingale?

If $g(t, B_t)$ is adapted to B_t, then

$$\int g(t, B_t) dB_t$$

is a martingale as long as $\int \int g(t, B_t)^2 < \infty$.

- $E[X_s|F_t] = X_t \; \forall t < s$. (where $F_t = X_0 \sim X_t$)
- Eg.: If $dX_t = \mu(t, B_t)dt + \sigma(t, B_t)dBt$, then X_t is a martingale if and only if $\mu = 0$.
- For SDE: No Drift $\Leftrightarrow$ martingale.

3 Lebesgue Integral

$$\{E, X, \mu\}$$

E: set
X: E's σ-algebra
μ: measure

$$\int_X f d\mu$$

- X: measurable space
- μ: measure

4 Log-Normal

$$Y = ln(X) \sim N(\mu, \sigma^2)$$
$$X = exp(Y) \sim LN(\mu, \sigma^2)(Log - Normal)$$
$$E[e^x] = e^{E[x]+Var(x)}$$

5 Black Scholes

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

By Ito's Lemma

$$(ln(S))' = \frac{1}{S}, (ln(S))'' = -\frac{1}{S^2}$$

$$dln(S_t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

we get

$$\int_0^t dlnS_u = \int_0^t (\mu - \frac{1}{2}\sigma^2)du + \sigma \int_0^t dW_u$$

$$ln(S_t) - ln(S_0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2) + \sigma W_t}$$

5.1 Black Scholes Equation

5.2 Black Scholes Formula

European call option price:

$$C(S_t, t) = Pr(S_T > K)(E[S_T | S_T > K] - K)$$

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}[ln(\frac{S_t}{k}) + (r + \frac{\sigma^2}{2})(T-t)]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

6 Doleans Dade exponential(ε)

(In the change of measure, we need to define a density process, during which we will need ε.)
Let X_t be a measurable process adaptd to the filtration.
Doleans Dade exponential is the solution to:

$$dY_t = Y_t dX_t$$

which is the form of density process $dQ = \eta_t dP$.
Where η_t is Radon-Nikodym derivative.

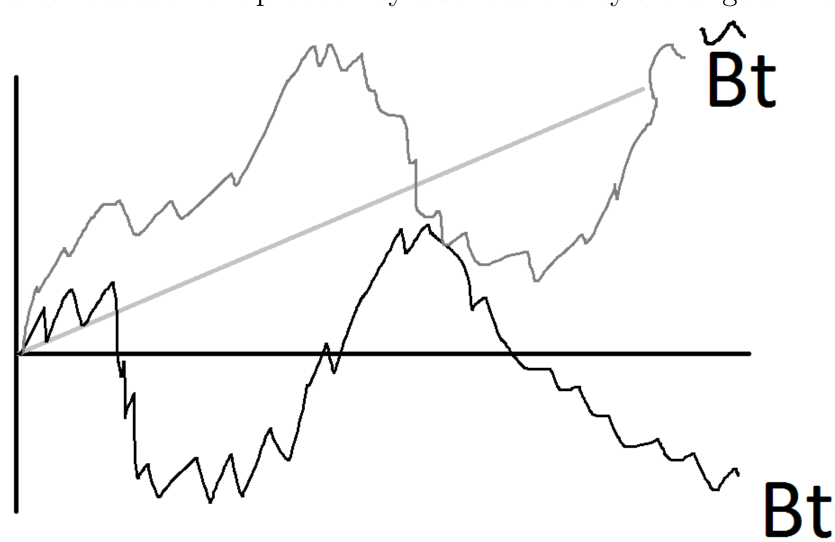
$$\varepsilon(X)_t = exp[X_t - \frac{1}{2}[X,X]_t]$$

L is the inverse of ε.

$$W_t^* = W_t - [W,X]_t$$

7 Change of Measure

Question:
B_t: Brownian Motion without drift.
$\tilde{B}_t$: Brownian Motion with drift.
Can we switch between the probability distributions by a change of measure?



$p(\omega)$: pdf by B_t.
$\tilde{p}(\omega)$: pdf by $\tilde{B}_t$.
$\exists Z = Z(\omega)\ s.t.\ p(\omega) = Z(\omega)\tilde{p}(\omega)$?

(Ω, P): Probaility distribution without drift.
$(\Omega, \tilde{P})$: Probability distribution with drift.

Definition: P and $\tilde{P}$ are equivalent if $P(A) > 0 \Leftrightarrow \tilde{P}(A) > 0$ $(\forall A \in \Omega)$.

7.1 Radon-Nikodym derivative

$$\exists Z \ s.t. \ p(\omega) = Z(\omega)\tilde{p}(\omega)$$

if and only if P and $\tilde{P}$ are equivalent.
Z: Radon-Nikodym derivative.

7.2 Girsanov Theorem

P: Probability distribution over $[0, T]^\infty$ defined by a BM with drift μ.
(where $[0, T]^\infty$ means paths from 0 to T)

$\tilde{P}$: Probability distribution over $[0, T]^\infty$ defined by a BM w/o drift μ.

The P and $\tilde{P}$ are equivalent.

$$\Rightarrow Z(\omega) = \frac{d\tilde{P}}{dP}(\omega) = e^{-\mu W_T - \frac{1}{2}\mu^2 T}$$

$$\tilde{E}[V_t] = E[Z_t V_t]$$

(where $\tilde{E}$ in Probability space $(\Omega, \tilde{P})$, E in Probability space (Ω, P))

8 Notation

$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$
V_t : Value process
$\pi_{it} = \phi_{it} S_{it}$
ϕ : # of shares π : dollar amount

9 Mallivian Calculus

$$M_v = E_v[F]$$

$$D_t M_v = E_v[D_t F]$$