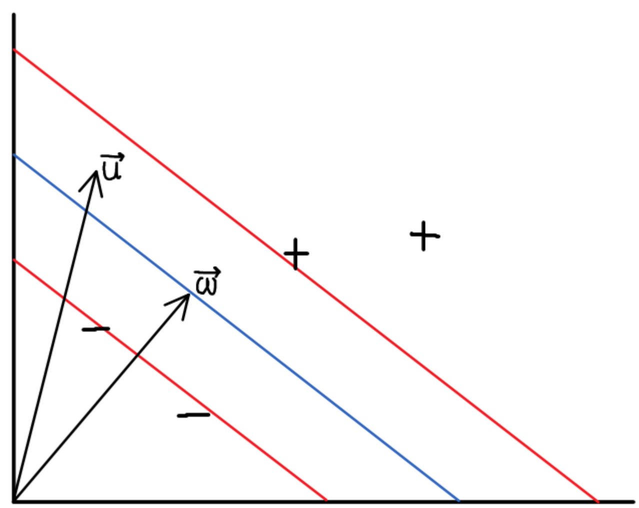
SVM

Steven Wang

2017

1 Intuition

Assume there are 2 classes: - and + in a 2-dimensional space.
We want to classify unknown samples into these 2 classes with SVM, by finding a hyperplane that best separates the 2 classes.
In this case:
$\vec{u}$ is an unknown sample.
$\vec{w}$ is a vector perpendicular to the separation hyperplane.



Decision rule:

$$if \quad \vec{\omega} \cdot \vec{u} \geq c, \ then \quad \vec{u} \in +$$
$$(The\ projection\ of\ \vec{u}\ on\ \vec{w})$$

More generally($c = -b$):

$$if \quad \vec{\omega} \cdot \vec{u} + b \geq 0, \quad then \Rightarrow \vec{u} \in + \tag{1}$$

2 Question: What is $\vec{w}$ and b?

Assume for known samples:

$$\vec{\omega} \cdot \vec{x}_+ + b \geq 1$$

$$\vec{\omega} \cdot \vec{x}_- + b \leq -1$$

Now let's introduce a variable y_i for mathematical convenience:

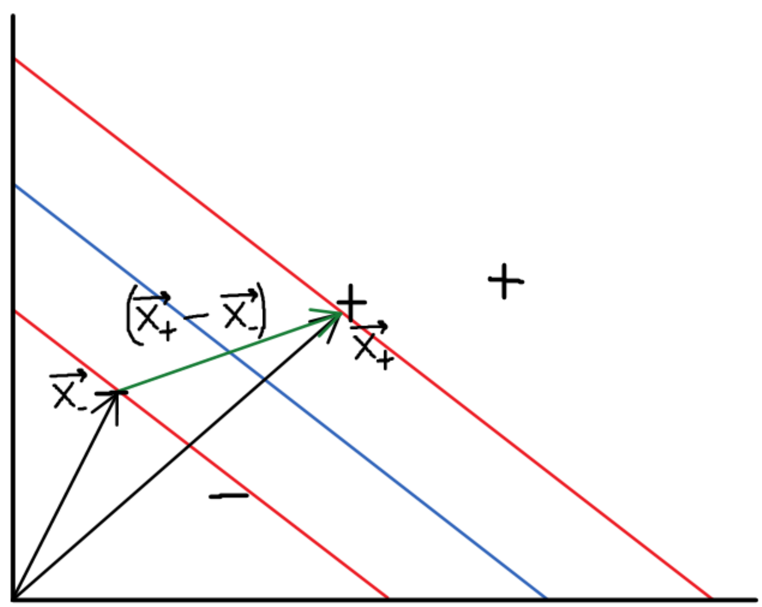
$$y_i = \begin{cases} +1 & for + samples \\ -1 & for - samples \end{cases}$$

$$\Rightarrow \begin{cases} y_i(\vec{\omega} \cdot \vec{x}_+ + b) \geq 1 \\ y_i(\vec{\omega} \cdot \vec{x}_- + b) \geq 1 \end{cases}$$

$$\Rightarrow y_i(\vec{\omega} \cdot \vec{x}_i + b) - 1 \geq 0$$

Assume for $\vec{x}_i$ between(&on) the 2 solid lines:

$$\Rightarrow y_i(\vec{\omega} \cdot \vec{x}_i + b) - 1 = 0 \tag{2}$$



$$\textbf{width}\ (between\ the\ red\ lines) = (\vec{x}_+ - \vec{x}_-) \cdot \underbrace{\frac{\vec{\omega}}{||\vec{\omega}||}}_{\text{unit vector}}$$

$\frac{\vec{\omega}}{||\vec{\omega}||}$: This makes a unit vector that gives the direction perpendicular to separation hyperplane(a straight line here in this case).

$$(2) \Rightarrow \begin{cases} \vec{x}_+\vec{\omega} = 1 - b \\ -\vec{x}_-\vec{\omega} = 1 + b \end{cases}$$

$$\Rightarrow \textbf{width} = (1 - b + 1 + b)\frac{1}{||\vec{\omega}||} = \frac{2}{||\vec{\omega}||} \tag{3}$$

$\Rightarrow$ We want to max $\frac{1}{||\vec{\omega}||}$(maximize the margin of the separation hyperplane between the 2 classes).

$$\Rightarrow \quad min\{||\vec{\omega}||\} \ \Rightarrow \quad min\{\frac{1}{2}||\vec{\omega}||^2\} \quad (for\ mathematical\ convenience)$$

(How find extremum with constraints? $\Rightarrow$ *Lagrange*)

$$L = \frac{1}{2}||\vec{\omega}||^2 - \sum_i \alpha_i[y_i(\vec{\omega} \cdot \vec{x}_i + b) - 1]$$

(How to take derivative of vectors? It has the same form of scalars.)

$$\frac{\partial L}{\partial \vec{\omega}} = \vec{\omega} - \sum_i \alpha_i y_i \vec{x}_i = 0$$

$$\Rightarrow \quad \vec{\omega} = \sum_i \underbrace{\alpha_i}_{\text{(scalar)}} \cdot \underbrace{y_i}_{\text{(+1,-1)}} \cdot \vec{x}_i \tag{4}$$

It is proved to be a convex space, so no local extremum.

2.1 *Important observation 1

$\vec{\omega}$ is a linear sum of the sample vectors.

$$\frac{\partial L}{\partial b} = -\sum_i \alpha_i y_i = 0 \Rightarrow \sum_i \alpha_i y_i = 0$$

Plug (4) into L:

$$L = \frac{1}{2}(\sum_i \alpha_i y_i \vec{x}_i)(\sum_j \alpha_j y_i \vec{x}_j) - \sum_i \alpha_i y_i \vec{x}_i (\sum_j \alpha_j y_i \vec{x}_j) - \underbrace{\sum_i \alpha_i y_i}_{=0} \cdot b + \sum_i \alpha_i$$

$$L = \sum \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j \underline{\vec{x}_i \vec{x}_j} \tag{5}$$

2.2 *Important observation 2

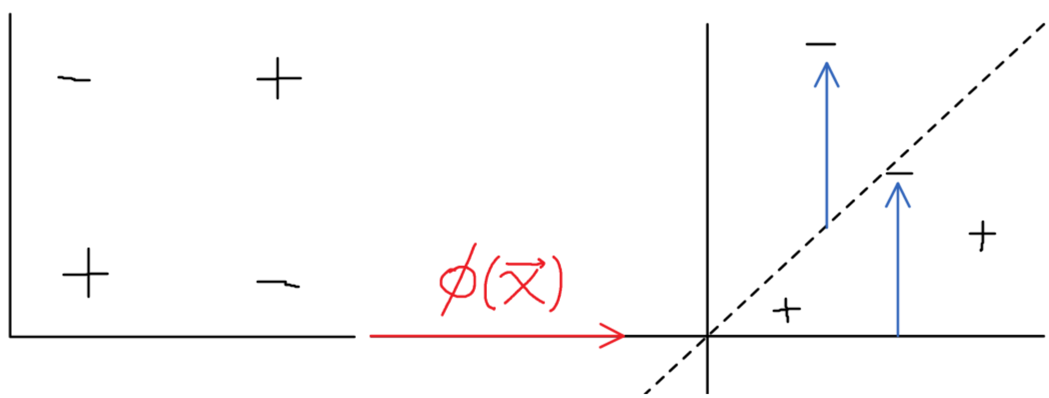
Maximization depends only on the dot products of sample vectors.

Decision rule:

$$if \quad \sum_i \alpha_i y_i \vec{x}_i \cdot \vec{u} + b \geq 0, \quad then\ +$$

3 Linearly inseparable instances

Apply transformation to sample vectors: $\phi(\vec{x})$
(In this case, convert the samples from 2d to 3d space)



Now the goal is:

$$max\{\phi(\vec{x}_i) \cdot \phi(\vec{x}_j)\}$$

And the decision rule contains:

$$\phi(\vec{x}_i) \cdot \phi(\vec{u}_j)$$

If we can find kernel function:

$$K(\vec{x}_i \cdot \vec{x}_j) = \phi(\vec{x}_i) \cdot \phi(\vec{x}_j)$$

Then we do not need the transformation ϕ. Because K provides the product of those 2 vectors in another space, I don't have to know the transformation ϕ.

4 Popular kernels

- linear kernel:

$$(\vec{u}\vec{v} + 1)^n$$

- radio basis function(rbf):

$$e^{-\frac{||x_i - x_j||}{\sigma}}$$