How often should you fill up on gas?

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Over the past two weeks, I have observed an enormous increase in gas prices. Unfortunately, having not filled up on gas in awhile, I missed it when the prices bottomed out. Perhaps, if I had filled up on gas more often, I may have been able to get it when it was at its lowest. This inspired me to ask the question: in general, is it better to fill up more or less frequently, in smaller or bigger chunks?

Gas prices are known to fluctuate back-and-forth. Mathematically, we can model this with a sinusoidal function. (Recall that any oscillatory signal can be expressed us a sum of sinusoids, so this is a good starting point.) Let $G(t)$ be the gas price as a function of time. With this model,

$$G(t) = G_0 + G_a \sin\left(\frac{2\pi t}{f}\right), \tag{1}$$

where G_0 is the average gas price, G_a is the maximum oscillation from the average, and f is the frequency of oscillation. The values of G_a/G_0 and f determine the volatility of prices.

Let's say we fill up on gas at constant intervals of time τ, and we do this for a total time T, such that during the total time we fill up N times, where

$$N = \frac{T}{\tau}.$$

Every time, we want to fill up our vehicle until it is full, which will require $b\tau$ amount of gas, where b is some proportionality constant with dimensions of [amount of gas] / [time]. The cost of filling up gas the kth time C_k is the amount of gas multiplied by the cost of gas at time $t = k\tau$, or

$$C_k = b\tau\, G(t = k\tau) = b\tau\left(G_0 + G_a \sin\left(\frac{2\pi k\tau}{f}\right)\right). \tag{2}$$

The total cost of gas through time T, which we will denote as C, is

$$\begin{aligned} C = \sum_{k=1}^{N} C_k &= \sum_{k=1}^{N} b\tau\left(G_0 + G_a \sin\left(\frac{2\pi k\tau}{f}\right)\right) \\ &= \underbrace{\sum_{k=1}^{N} b\tau G_0}_{=Nb\tau G_0} + b\tau G_a \sum_{k=1}^{N} \sin\left(\frac{2\pi k\tau}{f}\right). \end{aligned} \tag{3}$$

Define

$$C_0 \equiv Nb\tau G_0 = TbG_0$$

as the base cost, the cost of gas if you hit the average cost every time you filled up gas. Moving all terms without τ to the LHS yields

$$\frac{C - C_0}{bG_a} = \tau \sum_{k=1}^{N} \sin\left(\frac{2\pi k\tau}{f}\right). \tag{4}$$

Nondimensionalizing the equation, we define a dimensionless time q

$$q \equiv \frac{\tau}{f}.$$

In terms of q, we can rewrite (4) as

$$\frac{C - C_0}{bG_a f} = q \sum_{k=1}^{N} \sin\left(2\pi k q\right). \tag{5}$$

For simplicity's sake, we will hereafter write the LHS as $L \equiv (C - C_0)/bG_a f$.

Recall now that we defined N in terms of τ (and by extension, q). So we are not yet done. We have to evaluate the sum to extract N, which, after an involved calculation, results in

$$L = q\,\frac{\sin(N\pi q)\sin((N+1)\pi q)}{\sin(\pi q)}. \tag{6}$$

Substituting

$$N = \frac{T}{\tau} = \frac{T}{fq},$$

we get

$$L = q\,\frac{\sin\left(\frac{T\pi}{f}\right)\sin\left(\frac{T\pi}{f} + \pi q\right)}{\sin(\pi q)}. \tag{7}$$

To simplify (7), define a new dimensionless time p as

$$p \equiv \frac{T}{f}.$$

Then, we get our final result,

$$\boxed{L = \frac{C - C_0}{bG_a f} = q\,\frac{\sin(\pi p)\sin(\pi(p+q))}{\sin(\pi q)}}. \tag{8}$$

Interpretation – L is a dimensionless measure of how much we actually pay for gas through time T, from our "base" cost. We want to see how that varies with our variables, in this case the dimensionless times p and q.

We recall $p = T/f$, $q = \tau/f$. Think of this as a "sampling" problem, where we are sampling a signal (the gas prices) at periodic intervals of τ, for a length of time T. The dimensionless forms of those time metrics are just scaled against a reference time which is the frequency at which the gas prices fluctuate. In summary, p is a measure of for how long you are doing the analysis. q is a measure of how often you want to fill up gas. q is the critical variable to answer the question in the title of this document.

The sinusoids in (8) make it difficult to predict the behavior, compounded by the presence of three of them in products/ratios. In addition, as $-1 \leq \sin u \leq 1$, the ratio in (8) could range from $-\infty$ to $+\infty$. Instead, to analyze the result, and to answer the question posed in the title, we must look at some limiting cases.

Limiting Cases

1) $q \to 0$, p finite: In this case,

$$\begin{aligned} \sin(\pi q) &\to \pi q, \\ \sin(\pi(p+q)) &\to \sin(\pi p), \text{ so} \\ L = q\,\frac{\sin(\pi p)\sin(\pi(p+q))}{\sin(\pi q)} &\to \frac{\sin^2(\pi p)}{\pi}. \end{aligned} \tag{9}$$

A few remarks:

- The result in (9) does not depend on q.
- Instead of the range from $-\infty$ to $+\infty$, it only goes from 0 to $1/\pi$.

But *mathematically* speaking, what is $q \to 0$? It is when our "sampling" becomes continuous. In other words, it is when the discrete sum that we started setting up in (3) approaches an *integral.* With that said, the result in (9) can be checked. Instead of evaluating the sum in (5), we will convert it into an integral and evaluate that. The results should match.