Path Optimization

Which route should you take to walk to class when your campus is hilly?

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Problem. This is the proverbial question I ask daily: How can I minimize the time I take to walk to from my apartment to class/meeting/location X? What route should I take – the route that looks longer on the map, but has less uphill regions, or the route that looks shorter on the map, but has more uphill regions? This is a question I attempt to answer here.

Solution. We begin by noting that for any arbitrary path C in 3-dimensional space, the time T needed to get to my destination is given by the path integral

$$T = \int_C \frac{ds}{v}. \tag{1}$$

But v varies along C. Specifically, what I notice is that I walk slower when the *slope* of my path is steeply uphill. On the other hand, I walk faster when my path is slightly downhill. So for small slopes, I can make the educated "guess' that my speed is given by

$$v = v_0 \left(1 - \frac{dz}{d\rho}\right), \tag{2}$$

where v_0 is a constant, the speed at which I walk normally on a level curve, and ρ is the arc length along the *projection* of C onto the x-y plane, which we will call C'. Now let me explain my notation a bit. $dz/d\rho$ is just the slope of C with respect to the projection C'. We can think about this in terms of a directional derivative, that is:

$$\frac{dz}{d\rho} \equiv \nabla z(x, y) \cdot d\mathbf{s}',$$

where $d\mathbf{s}'$ is the line element along the path C'. This is just the height gradient in the direction along C'.

Assumptions. This is a good time to list our assumptions. One has already been listed above: *the angle/grade of the slope is small.* Mathematically, this means that

$$\frac{dz}{d\rho} \ll 1.$$

This is almost always a good approximation. According to this source[1] (see bottom of the link), the steepest grades in SF are around 40% (0.4 radians or 23°). That's the extreme extreme case, and even in that regime the assumption yields good approximations.

There is another assumption we have to make, and this is about walking. In general, you get tired if you've walked uphill for an hour. So v *ought* to depend on the history of your walking – if you've walked a long distance or if you've walked pretty steeply uphill, your v_0 will decrease, and thus $v_0 = v_0(dz/d\rho, t)$, and it is no longer a constant. For now we neglect those dependencies, which is OK because in my experience that as long as I am not sleep-deprived and I have not walked for more than a few minutes, v_0 is indeed constant. Equivalently, this means

$$T \ll 1 \text{ hour}$$

(like around 15 minutes or less, which is valid for walking within the confines of the UC Berkeley campus).

THE GENERAL RESULT. Our next step is to put ds in terms of $d\rho$. In general the 2-D path S' is easily plotted on a 2-D map, whereas the 3-D path S is not, so we prefer to use $d\rho$ if possible. We can split the horizontal and vertical components up, noting that

$$ds = \sqrt{d\rho^2 + dz^2} = d\rho\sqrt{1 + \left(\frac{dz}{d\rho}\right)^2}, \tag{3}$$

where $d\rho^2 = dx^2 + dy^2$ in the usual Cartesian basis. Plugging (2) and (3) into (1), we obtain the general result

$$\boxed{T = \int_{C'} \frac{1}{v_0} \frac{\sqrt{1 + (dz/d\rho)^2}}{1 - dz/d\rho}\, d\rho}. \tag{4}$$

Of course, this is *correct*, but it's not particularly illuminating. So let's see what we can do with this.

APPLYING APPROXIMATIONS. To make our next step easier, we define a new variable

$$x \equiv \frac{dz}{d\rho}.$$

We see that the integrand in (4) is a product of two factors, one of which can be Taylor expanded in a binomial series, the other in a geometric series, since we have made the assumption that $x \ll 1$:

$$\begin{aligned} \text{integrand} &= (1 + x^2)^{1/2}\left(\frac{1}{1-x}\right) \\ &\simeq \left(1 + \frac{x^2}{2} + \cdots\right)(1 + x + x^2 + \cdots) \\ &\simeq 1 + x + \frac{3}{2}x^2 + \cancel{O(x^3)}, \end{aligned} \tag{5}$$

[1] `http://www.datapointed.net/2009/11/the-steeps-of-san-francisco/`

where we can neglect terms of order x^3 and higher since $x \ll 1$.

The Final Answer. Replacing the integrand in (4) with the approximation in (5), and puttting in $dz/d\rho$ back for x, we get

$$T \simeq \frac{1}{v_0} \int_{C'} 1 + \frac{dz}{d\rho} + \frac{3}{2}\left(\frac{dz}{d\rho}\right)^2 \, d\rho.$$

Oh, but look!

- The zeroth-order term (i.e, 1), when integrated, gives just ρ/v_0, or the time it takes to walk to my destination if my elevation change was 0 the entire time. We can define this as T_0.
- The first-order term (i.e, $dz/d\rho$), when integrated, gives just $\Delta z/v_0$, or the time it takes to walk the distance equivalent to the elevation change between the endpoints of S'. We can define this as $T_{\Delta z}$.

In summary,

$$\boxed{T \simeq T_0 + T_{\Delta z} + \frac{3}{2v_0} \int_{C'} \left(\frac{dz}{d\rho}\right)^2 \, d\rho}, \tag{6}$$

with

$$T_0 \equiv \rho/v_0,$$
$$T_{\Delta z} \equiv \Delta z/v_0.$$

And this is the most useful result.

Upshot. Note that $T_{\Delta z}$ is a constant, since Δz is a constant (because the endpoints are fixed). Since the second-order term is fairly small, when your campus isn't *that* hilly, you should always take the route that minimizes the distance you travel. Don't take any wavy paths to avoid hills; just take the direct path to minimize ρ. However, when second-order contributions become significant, you may have to weigh that contribution relative to T_0. In general, the hills around my campus have <10% grade (around 0.1 radians or 6° slope). Second-order terms thus are pretty small.