Bayesian inference in NDLM: Known variances

Consider a NDLM given by

$$
\begin{aligned}
y_t &= \boldsymbol{F}_t'\boldsymbol{\theta}_t + \nu_t, \quad \nu_t \sim N(0, v_t), && (1)\\
\boldsymbol{\theta}_t &= \boldsymbol{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \sim N(0, \boldsymbol{W}_t), && (2)
\end{aligned}
$$

with $\boldsymbol{F}_t, \boldsymbol{G}_t, v_t$ and $\boldsymbol{W}_t$ known. We also assume a prior distribution of the form $(\boldsymbol{\theta}_0|\mathcal{D}_0) \sim N(\boldsymbol{m}_0, \boldsymbol{C}_0)$, with $\boldsymbol{m}_0, \boldsymbol{C}_0$ known.

Filtering

We are interested in finding $p(\boldsymbol{\theta}_t|\mathcal{D}_t)$ for all t. Assume that the posterior at $t-1$ is such that

$$(\boldsymbol{\theta}_{t-1}|\mathcal{D}_{t-1}) \sim N(\boldsymbol{m}_{t-1}, \boldsymbol{C}_{t-1}).$$

Then, we can obtain the following:

1. *Prior at time t:* $(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}) \sim N(\boldsymbol{a}_t, \boldsymbol{R}_t)$, with

   $$\boldsymbol{a}_t = \boldsymbol{G}_t\boldsymbol{m}_{t-1}$$

   and

   $$\boldsymbol{R}_t = \boldsymbol{G}_t\boldsymbol{C}_{t-1}\boldsymbol{G}_t' + \boldsymbol{W}_t.$$

2. *One-step forecast:* $(y_t|\mathcal{D}_{t-1}) \sim N(f_t, q_t)$, with

$$f_t = \boldsymbol{F}'_t \boldsymbol{a}_t, \quad q_t = \boldsymbol{F}'_t \boldsymbol{R}_t \boldsymbol{F}_t + v_t$$

3. *Posterior at time t:* $(\boldsymbol{\theta}_t|\mathcal{D}_t) \sim N(\boldsymbol{m}_t, \boldsymbol{C}_t)$, with

$$\begin{aligned} \boldsymbol{m}_t &= \boldsymbol{a}_t + \boldsymbol{R}_t \boldsymbol{F}_t q_t^{-1}(y_t - f_t), \\ \boldsymbol{C}_t &= \boldsymbol{R}_t - \boldsymbol{R}_t \boldsymbol{F}_t q_t^{-1} \boldsymbol{F}'_t \boldsymbol{R}_t. \end{aligned}$$

Now, denoting $e_t = (y_t - f_t)$ and $\boldsymbol{A}_t = \boldsymbol{R}_t \boldsymbol{F}_t q_t^{-1}$, we can rewrite the equations above as

$$\begin{aligned} \boldsymbol{m}_t &= \boldsymbol{a}_t + \boldsymbol{A}_t e_t, \\ \boldsymbol{C}_t &= \boldsymbol{R}_t - \boldsymbol{A}_t q_t \boldsymbol{A}'_t. \end{aligned}$$