Polynomial trend models

First-order polynomial

$$
\begin{aligned}
y_t &= \mu_t + \nu_t, \quad \nu_t \sim N(0, v_t) \\
\mu_t &= \mu_{t-1} + \omega_t, \quad \omega_t \sim N(0, w_t).
\end{aligned}
$$

In this case we have $\theta_t = \mu_t$, $F_t = 1, G_t = 1$ for all t, resulting in $\{1, 1, v_t, w_t\}$. Forecast function $f_t(h) = E(\mu_t|\mathcal{D}_t) = k_t$, for all $h > 0$.

Second-order polynomial

$$
\begin{aligned}
y_t &= \theta_{t,1} + \nu_t, \quad \nu_t \sim N(0, v_t) \\
\theta_{t,1} &= \theta_{t-1,1} + \theta_{t-1,2} + \omega_{t,1}, \quad \omega_{t,1} \sim N(0, w_{t,11}) \\
\theta_{t,2} &= \theta_{t-1,2} + \omega_{t,2}, \quad \omega_{t,2} \sim N(0, w_{t,22}),
\end{aligned}
$$

we can also have $Cov(\theta_{t,1}, \theta_{t,2}) = w_{t,12} = w_{t,21}$. This can be written as a DLM with state-space vector $\boldsymbol{\theta}_t = (\theta_{t,1}, \theta_{t,2})'$, and $\{\boldsymbol{F}, \boldsymbol{G}, v_t, \boldsymbol{W}_t\}$ with $\boldsymbol{F} = (1, 0)'$,

$$
\boldsymbol{G} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{W}_t = \begin{pmatrix} w_{t,11} & w_{t,12} \\ w_{t,21} & w_{t,22} \end{pmatrix}.
$$

Note that

$$\boldsymbol{G}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

and so $f_t(h) = (1, h)E(\boldsymbol{\theta}_t|\mathcal{D}_t) = (1, h)(k_{t,0}, k_{t,1})' = (k_{t,0} + hk_{t,1})$. Here $\boldsymbol{G} = \boldsymbol{J}_2(1)$ (see below). Also, we denote $\boldsymbol{E}_2 = (1, 0)'$ and so, the short notation for this model is $\{\boldsymbol{E}_2, \boldsymbol{J}_2(1), \cdot, \cdot\}$.

General p-th order polynomial model

We can consider a so called p-th order polynomial model. This model will have a state-space vector of dimension p and a polynomial of order $p - 1$ forecast function on h. The model can be written as $\{\boldsymbol{E}_p, \boldsymbol{J}_p(1), v_t, \boldsymbol{W}_t\}$ with $\boldsymbol{F}_t = \boldsymbol{E}_p = (1, 0, \ldots, 0)'$ and $\boldsymbol{G}_t = \boldsymbol{J}_p(1)$, with

$$\boldsymbol{J}_p(1) = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ & & & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The forecast function is given by

$$f_t(h) = (k_{t,0} + k_{t,1}h + \dots k_{t,p-1}h^{p-1}).$$

There is also an alternative parameterization of this model that leads to the same algebraic form of the forecast function given by $\{\boldsymbol{E}_p, \boldsymbol{L}_p, v_t, \boldsymbol{W}_t\}$, with

$$\boldsymbol{L}_p = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Dynamic regression models

Simple dynamic regression

$$\begin{aligned} y_t &= \beta_{t,0} + \beta_{t,1}x_t + \nu_t \\ \beta_{t,0} &= \beta_{t-1,0} + \omega_{t,0} \\ \beta_{t,1} &= \beta_{t-1,1} + \omega_{t,1}, \end{aligned}$$

and so $\boldsymbol{\theta}_t = (\beta_{t,0}, \beta_{t,1})'$, $\boldsymbol{F}_t = (1, x_t)'$ and $\boldsymbol{G} = \boldsymbol{I}_2$. This results in a forecast function of the form $f_t(h) = k_{t,0} + k_{t,1}x_{t+h}$.

General dynamic regression

$$\begin{aligned} y_t &= \beta_{t,0} + \beta_{t,1}x_{t,1} + \ldots \beta_{t,M}x_{t,M} + \nu_t \\ \beta_{t,m} &= \beta_{t-1,m} + \omega_{t,m}, \quad m = 0 : M. \end{aligned}$$

Then, $\boldsymbol{\theta} = (\beta_{t,0}, \ldots, \beta_{t,M})'$, $\boldsymbol{F}_t = (1, x_{t,1}, \ldots, x_{t,M})'$ and $\boldsymbol{G} = \boldsymbol{I}_M$. The forecast function is given by

$$f_t(h) = k_{t,0} + k_{t,1}x_{t+h,1} + \ldots + k_{t+h,M}x_{t+h,M}.$$

A particular case is of dynamic regressions is the case of **time-varying autoregressions (TVAR)** with

$$\begin{aligned} y_t &= \phi_{t,1}y_{t-1} + \phi_{t,2}y_{t-2} + \ldots + \phi_{t,p}y_{t-p} + \nu_t, \\ \phi_{t,m} &= \phi_{t-1,m} + \omega_{t,m}, \quad m = 1 : p. \end{aligned}$$