Bayesian inference in the AR(1): Example with the full likelihood

We consider a prior distribution that assumes that ϕ and v are independent and

$$
\begin{aligned}
p(v) &\propto \frac{1}{v}, \\
p(\phi) &= \frac{1}{2}, \quad for \quad \phi \in (-1, 1),
\end{aligned}
$$

i.e., we assume a Uniform prior for $\phi \in (-1, 1)$. Combining this prior with the full likelihood in the AR(1) case we obtain the following posterior density

$$p(\phi, v | y_{1:T}) \propto \frac{(1-\phi^2)^{1/2}}{v^{T/2+1}} \exp\left(-\frac{Q^*(\phi)}{2v}\right), \quad -1 < \phi < 1,$$

with

$$Q^*(\phi) = y_1^2(1-\phi^2) + \sum_{t=2}^{T} (y_t - \phi y_{t-1})^2.$$

It is not possible to get a closed form expression for this posterior or to perform direct simulation. Therefore, we use simulation-based Markov Chain Monte Carlo methods to obtain samples from the posterior distribution.

We first consider the following transformation on ϕ :

$$\eta = \log\left(\frac{1-\phi}{\phi+1}\right),$$

so that $\eta \in (-\infty, \infty)$. The inverse transformation on η is:

$$\phi = \frac{1-\exp(\eta)}{1+\exp(\eta)}.$$

Writing down the posterior density for η and v we obtain

$$p(\eta, v|y_{1:T}) \propto \frac{(1-\phi^2)^{1/2}}{v^{T/2+1}} \exp\left(-\frac{Q^*(\phi)}{2v}\right) \cdot \frac{2\exp(\eta)}{(1+\exp(\eta))^2},$$

with ϕ written as a function of η, and proceed to obtain samples from this posterior distribution using the MCMC algorithm outlined below. Once we have obtained M samples from η and v after convergence, we can use the inverse transformation above to obtain posterior samples for ϕ.

Algorithm MCMC: Bayesian inference for AR(1), full likelihood

Initialize $\eta^{(0)}$ and $\beta^{(0)}$

for m in 1:M **do**

Sample $v^{(m)} \sim \mathcal{IG}\left(\frac{T}{2}, \frac{Q^*(\phi^{(m-1)})}{2}\right)$

Sample $\eta^{(m)}$ using Metropolis-Hastings:

(a) Sample $\eta^* \sim \mathcal{N}\left(\eta^{(m-1)}, c\right)$, where c is a tuning parameter

(b) Compute the importance ratio:

$$r = \frac{p\left(\eta^*, v^{(m)}|y_{1:T}\right)}{p\left(\eta^{(m-1)}, v^{(m)}|y_{1:T}\right)}$$

(c) Set:

$$\eta^{(m)} = \begin{cases} \eta^* & \text{with probability = min(r, 1),} \\ \eta^{(m-1)} & \text{otherwise.} \end{cases}$$