Inference in the NDLM with unknown but constant observational variance

Let $v_t = v$ for all t, with v unknown and consider a DLM with the following structure:

$$
\begin{aligned}
y_t &= \boldsymbol{F}_t'\boldsymbol{\theta}_t + \nu_t, \quad \nu_t \sim N(0, v), \\
\boldsymbol{\theta}_t &= \boldsymbol{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{\omega}_t, \quad \boldsymbol{\omega}_t \sim N(0, v\boldsymbol{W}_t^*),
\end{aligned}
$$

with conjugate prior distributions:

$$
(\boldsymbol{\theta}_0|\mathcal{D}_0, v) \sim N(\boldsymbol{m}_0, v\boldsymbol{C}_0^*), \quad (v|\mathcal{D}_0) \sim IG(n_0/2, d_0/2),
$$

and $d_0 = n_0 s_0$.

Filtering

Assuming $(\boldsymbol{\theta}_{t-1}|\mathcal{D}_{t-1}, v) \sim N(\boldsymbol{m}_{t-1}, v\boldsymbol{C}_{t-1}^*)$, we have the following results:

- $(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}, v) \sim N(\boldsymbol{a}_t, v\boldsymbol{R}_t^*)$, with $\boldsymbol{a}_t = \boldsymbol{G}_t\boldsymbol{m}_{t-1}$ and $\boldsymbol{R}_t^* = \boldsymbol{G}_t\boldsymbol{C}_{t-1}^*\boldsymbol{G}_t' + \boldsymbol{W}_t^*$, and unconditional on v, $(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}) \sim T_{n_{t-1}}(\boldsymbol{a}_t, \boldsymbol{R}_t)$, with $\boldsymbol{R}_t = s_{t-1}\boldsymbol{R}_t^*$. The expression for s_t for all t is given below.

- $(y_t|\mathcal{D}_{t-1}, v) \sim N(f_t, vq_t^*)$, with $f_t = \boldsymbol{F}_t'\boldsymbol{a}_t$, and $q_t^* = (1 + \boldsymbol{F}_t'\boldsymbol{R}_t^*\boldsymbol{F}_t)$ and unconditional on v we have $(y_t|\mathcal{D}_{t-1}) \sim T_{n_{t-1}}(f_t, q_t)$, with $q_t = s_{t-1}q_t^*$.

- $(v|\mathcal{D}_t) \sim IG(n_t/2, s_t/2)$, with $n_t = n_{t-1} + 1$ and

$$s_t = s_{t-1} + \frac{s_{t-1}}{n_t}\left(\frac{e_t^2}{q_t} - 1\right).$$

  Here $e_t = y_t - f_t$.

- $(\boldsymbol{\theta}_t|\mathcal{D}_t, v) \sim N(\boldsymbol{m}_t, v\boldsymbol{C}_t^*)$, with $\boldsymbol{m}_t = \boldsymbol{a}_t + \boldsymbol{A}_t e_t$, and $\boldsymbol{C}_t^* = \boldsymbol{R}_t^* - \boldsymbol{A}_t\boldsymbol{A}_t' q_t^*$. Similarly, unconditional on v we have

$$(\boldsymbol{\theta}_t|\mathcal{D}_t) \sim T_{n_t}(\boldsymbol{m}_t, \boldsymbol{C}_t),$$

  with $\boldsymbol{C}_t = s_t\boldsymbol{C}_t^*$.

Forecasting

Similarly, we have the forecasting distributions:

$$\begin{aligned}
(\boldsymbol{\theta}_{t+h}|\mathcal{D}_t) &\sim T_{n_t}(\boldsymbol{a}_t(h), \boldsymbol{R}_t(h)), \\
(y_{t+h}|\mathcal{D}_t) &\sim T_{n_t}(f_t(h), q_t(h)),
\end{aligned}$$

with $\boldsymbol{a}_t(h) = \boldsymbol{G}_{t+h}\boldsymbol{a}_t(h-1)$, $\boldsymbol{a}_t(0) = \boldsymbol{m}_t$, and

$$\boldsymbol{R}_t(h) = \boldsymbol{G}_{t+h}\boldsymbol{R}_t(h-1)\boldsymbol{G}'_{t+h} + \boldsymbol{W}_{t+h}, \quad \boldsymbol{R}_t(0) = \boldsymbol{C}_t,$$

$f_t(h) = \boldsymbol{F}'_{t+h}\boldsymbol{a}_t(h)$, and

$$q_t(h) = \boldsymbol{F}'_{t+h}\boldsymbol{R}_t(h)\boldsymbol{F}_{t+h} + s_t.$$

Smoothing

Finally, the smoothing distributions have the form:

$$(\boldsymbol{\theta}_t|\mathcal{D}_T) \;\sim\; T_{n_T}(\boldsymbol{a}_T(t-T), \boldsymbol{R}_T(t-T)s_T/s_t),$$

with

$$\begin{aligned}
\boldsymbol{a}_T(t-T) &= \boldsymbol{m}_t - \boldsymbol{B}_t[\boldsymbol{a}_{t+1} - \boldsymbol{a}_T(t-T+1)], \\
\boldsymbol{R}_T(t-T) &= \boldsymbol{C}_t - \boldsymbol{B}_t[\boldsymbol{R}_{t+1} - \boldsymbol{R}_T(t-T+1)]\boldsymbol{B}'_t,
\end{aligned}$$

with $\boldsymbol{B}_t = \boldsymbol{C}_t\boldsymbol{G}'_{t+1}\boldsymbol{R}^{-1}_{t+1}$, and $\boldsymbol{a}_T(0) = \boldsymbol{m}_T$, $\boldsymbol{R}_T(0) = \boldsymbol{C}_T$.