Exploring Density Profiles of Dark Matter Halos

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ABSTRACT
One of the most well-known indicators for the existence of dark matter is in the rotational velocity curves of galaxies. While with only baryonic matter we expect the gravitational acceleration to be proportional to r^{-2} at large distances r, implying the rotational velocity to follow a $r^{-1/2}$ relationship, our observations show that the rotational velocity profile in fact flattens out at large radii. This indicates the existence of a non-baryonic type of mass contributing to the structure of galaxies. In this work, I experiment with two different galaxy density profile models, integrating them using a fourth-order Runge-Kutta algorithm (RK4) to acquire the rotational velocity curves from these models. I find that RK4 is decently effective in accurately computing the rotational curve from these simple density profiles, and is a great tool for simulating and studying halo models.

1 THEORY

According to Newtonian gravity, the gravitational field can be described by the Poisson equation:

$$\nabla^2 \phi = -4\pi G \rho, \tag{1}$$

where *phi* is the gravitational field, G is the gravitational constant, and ρ is the density, which in our case will be a function of radius r. With spherical symmetry, this becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d\phi}{dr}) = -4\pi G \rho. \tag{2}$$

We can turn this into a system of two first-order differential equations:

$$\begin{cases} \frac{d\phi}{dr} = -g \\ \frac{dg}{dr} = 4\pi G \rho - \frac{2g}{r}. \end{cases} \tag{3}$$

In order to numerically solve this system of ODEs, we need a boundary condition. Since our ρ will be finite at $r = 0$, $\frac{d\phi}{dr}$ will equal 0 there, and $\phi(0)$ is a constant.

Since $\frac{dg}{dr}$ has a term proportional to r^{-1}, if we integrate from $r = 0$ we will get an invalid value, so we will need a non-diverging solution near $r = 0$. By Taylor expanding ϕ up to the second-order term we can get the following approximate formulas for r near 0:

$$\begin{cases} \phi = 1 - \frac{2\pi G \rho_c}{3} r^2 \\ \frac{d\phi}{dr} = -\frac{4\pi G \rho_c}{3} r. \end{cases} \tag{4}$$

We will apply the above formula to evaluate our first non-zero t, and use the ODEs to evaluate the rest.

The most basic ODE solving method involves evaluating the right hand side (RHS) of the ODE at a small step h away from the original point and adding that value to the original value. The RK4 method, however, also evaluates the ODE at 3 test points between each step to give us a more accurate result. Let $\frac{dy}{dt} = f(y, t)$. The formula for RK4 is:

$$\begin{aligned} k_1 &= hf(y_n, t_n) \\ k_2 &= hf(y_n + \frac{k_1}{2}, t_n + \frac{h}{2}) \\ k_3 &= hf(y_n + \frac{k_2}{2}, t_n + \frac{h}{2}) \\ k_4 &= hf(y_n + k_3, t_n + h) \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \end{aligned} \tag{5}$$

where the subscript n denotes the index of the array of t we want to evaluate over. To apply this method to our system of equations above, I only need to input a 1x2 array of the previous ϕ and g values. After getting my array of g through this algorithm, I can calculate the rotational velocity v with

$$v = \sqrt{gr}. \tag{6}$$

2 DENSITY PROFILES

2.1 Analytic Profile

The first density profile I will be working with is one that has an analytic solution for v, described by

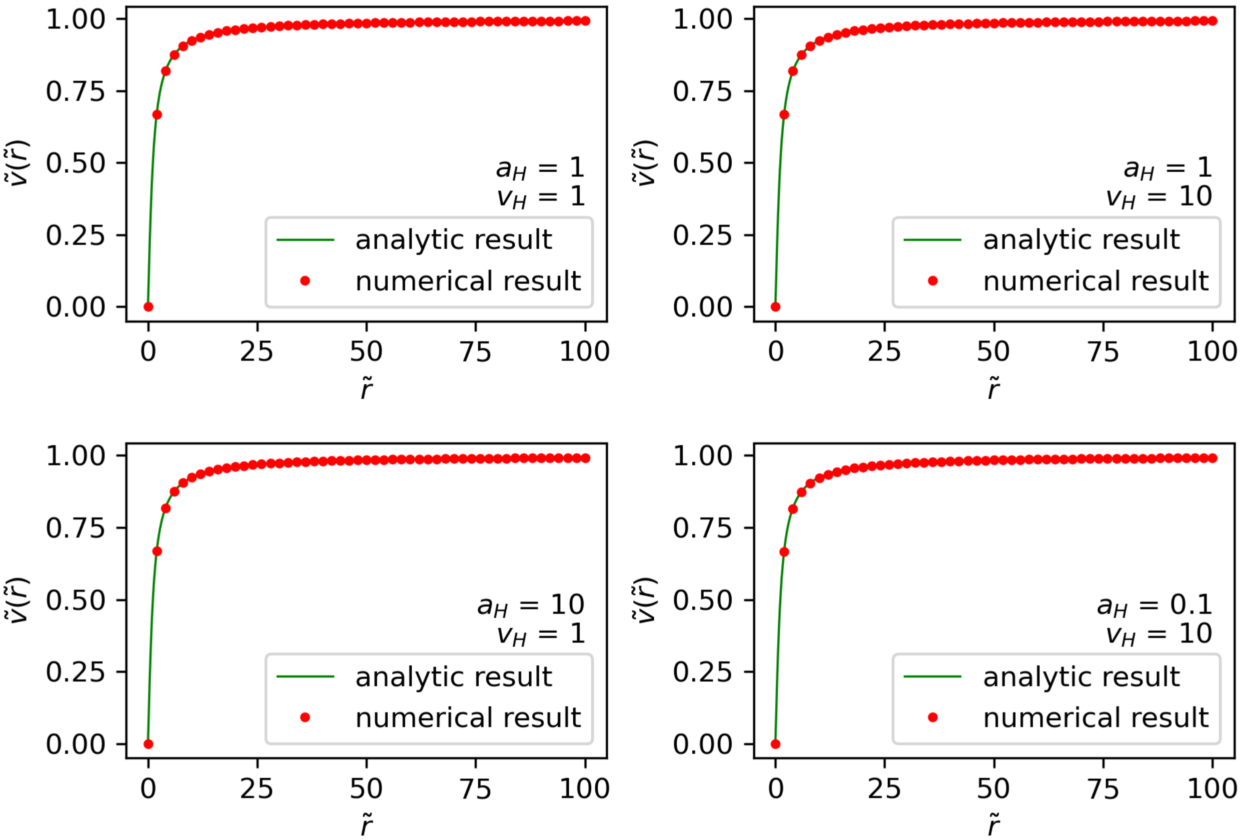

Figure 1. Velocity profile plots for the analytic density profile, varying a_H and v_H. For the numerical result plots, only every 200 points were plotted to prevent clumping of data points. As can be seen, the shape of the curve is independent of my choice of parameters, and the RK4 algorithm works well to match our analytic solution almost exactly.

$$4\pi G\rho = \frac{v_H^2}{r^2 + a_H^2}, \tag{7}$$

where v_H is a scale velocity related to the halo, and a_H is a scale length. The analytic solution for the velocity profile is given by

$$\frac{v(r)}{v_H} = \sqrt{1 - \frac{a_H}{r}\arctan(\frac{r}{a_H})}. \tag{8}$$

In order to compare my numerical solution to the above solution and preserve the shape of my curves independent of adjustable parameters v_H and a_H, I defined dimensionless parameters $\tilde{r}$, $\tilde{g}$, and $\tilde{v}$:

$$\begin{aligned} \tilde{r} &= \frac{r}{a_H} \\ \tilde{g} &= \frac{g}{r\rho G} \\ \tilde{v} &= \frac{v}{v_H}. \end{aligned} \tag{9}$$

Picking a step size h of 0.01, I ran the RK4 algorithm for different a_H and v_H values. Then for each pair of a_H and v_H I calculated the analytic velocity profile using Equation 8. My results are shown in Figure 1. As expected, the shape of the profile did not change with my changes in a_H and v_H, showing that I've successfully implemented the transform into dimensionless parameters. My numerical result overlaps with my analytic result almost perfectly, meaning I was successful in implementing my RK4 algorithm.

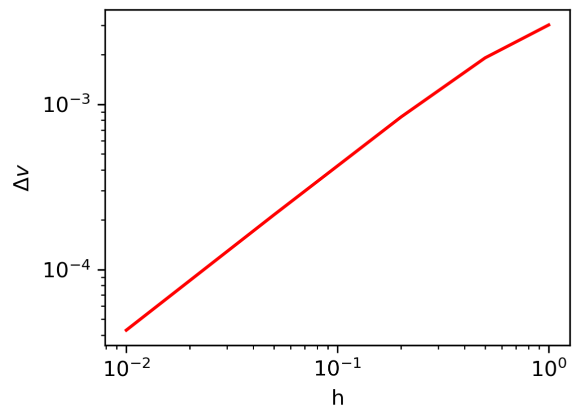

Figure 2. Convergence plot for the analytic density profile, made by comparing the middle point of the numerical result for each step size with the middle point of the analytic result. Judging by eye the order of convergence is h.

I then ran the procedure for numerous step sizes to analyze the convergence of my algorithm. I did this by taking the difference between the middle point of each numerical result and my analytic result. As shown in Figure 2, the convergence is roughly on the order of h. While one may expect the order of convergence to be h^4 due to us using a fourth-order ODE solving algorithm, that order could have dropped down due to the complex dependence of v on r. My choice of transforming g to the dimensionless $\tilde{g}$ using our density profile ρ, which itself also depends on r, could have also complicated things.

Despite my algorithm working as expected, though, the velocity profile given by this density profile does not seem to match what is observed. In our Milky Way the rotational velocity has an early peak, and decays before coming back up and flattening out (see Sofue (2020)). My velocity profile here simply asymptotes to a value as $\tilde{r}$ increases, and so is too simple in shape to describe a physical halo.

2.2 NFW Profile

Navarro et al. (1997) showed that dark matter halos have a universal profile irrespective of the halo mass which is given by

$$\frac{\rho(r)}{\rho_c} = \frac{\delta_c}{\frac{r}{r_s}(1+\frac{r}{r_s})^2}, \tag{10}$$

where ρ_c is a constant critical density, δ_c is a dimensionless constant, and r_s is a scale radius. I will be using a slightly modified version of this profile which assumes a small core, described by

$$\frac{\rho(r)}{\rho_c} = \frac{\delta_c}{(\frac{r}{r_s}+0.01)(1+\frac{r}{r_s})^2}. \tag{11}$$

Just like the previous profile, I define dimensionless parameters $\tilde{r}$, $\tilde{g}$, and $\tilde{v}$, where only the definition of $\tilde{r}$ changed to

$$\tilde{r} = \frac{r}{r_s}. \tag{12}$$

Then, similarly, I implemented RK4 to get a numerical result for the velocity profile, which is shown in Figure 3. Although we don't have an analytic result to compare with this time, in my previous subsection I have already shown the accuracy and robustness of RK4 using the analytic density profile, and so I expect my numerical solution here to also match the true profile. The shape of the curve have also been tested to be robust to changes in r_s, ρ_c, and δ_c.

As there is no analytic solution to compare with, I ran the algorithm for a list of decreasing h by factors of 2, and calculated the self-convergence using the result from the smallest step size. The self-convergence plot is shown in Figure 4, which shows an order of convergence of roughly h^2. This is higher than the previous density profile I used, and may be due to the r^{-3} dependence of the NFW profile compared to the r^{-2} dependence in the analytic profile.

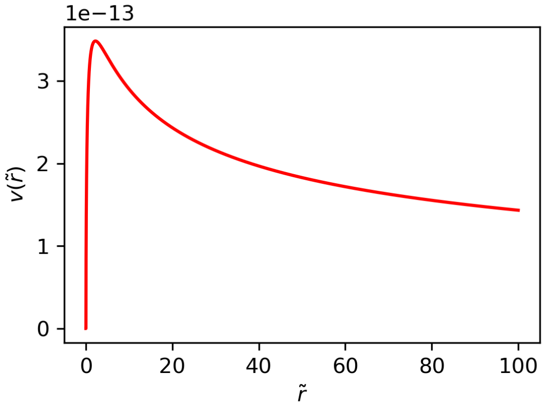

Figure 3. Velocity profile plot for the NFW profile. Constant parameters r_s, ρ_c, and δ_c in Equation 11 are arbitrarily defined, but the shape of the curve is robust to changes in those parameters.

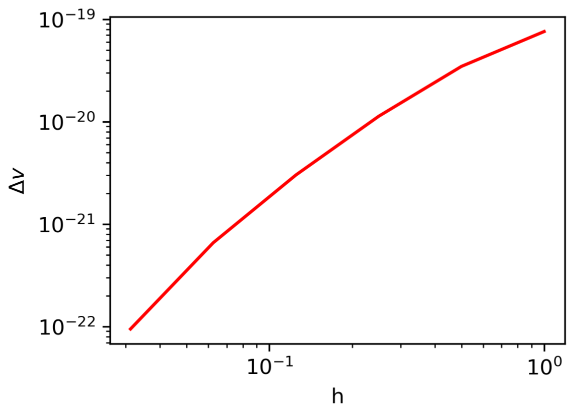

Figure 4. Self convergence plot for the NFW velocity profile, created by comparing the middle point of each result from decreasing step sizes with the middle point from the smallest step size.

Knowing the order of convergence, we can further improve our error estimations using Richardson Extrapolation. Since we have an $O(h^2)$ convergence, we can write our approximations as

$$v_0 \approx v_{0,n} + K_1(h_n^2) + K_2(h_n^4) + ... \tag{13}$$

where n denotes the index of the step size we are using, where in our case $h_{n+1} = h_n/2$. Using Richardson Extrapolation, we can get a better estimate of the velocity by combining the results of two of our adjacent step sizes using the following formula

$$v_{1,n} \approx \frac{4v_{0,n} - v_{0,n-1}}{3} \tag{14}$$

which would cancel out the $O(h^2)$ error term in Equation 13. Doing this for the two smallest step sizes gives us a new, more accurate reference point to calculate errors. Thus I repeated the self-convergence plot using the new

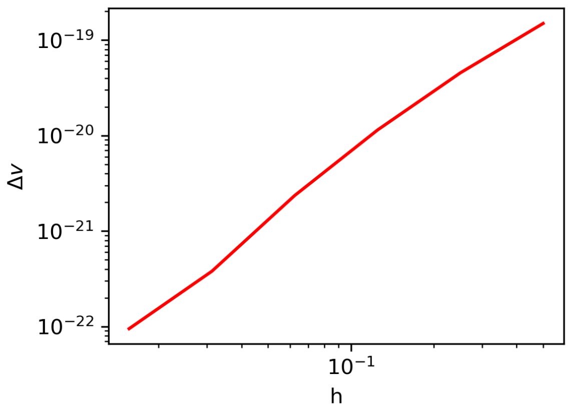

Figure 5. Self convergence plot for the NFW velocity profile after applying Richardson Extrapolation. New reference point is calculated from Equation 14 using the two smallest step sizes.

reference point, which is shown in Figure 5. As expected, we still see an $O(h^2)$ convergence, but this time the plot is visibly straighter.

Comparing with my results from the previous density profile, as well as the observed profile in Milky Way, the NFW profile does seem to more accurately describe a physical halo. The velocity profile decays after peaking but slowly flattens out, which matches with the rotational velocity profile of the Milky Way at small radii. Of course, modelling the real velocity profile accurately requires more parameters than the ones used in this paper, but as a simplified test of concept, I believe the results that I got were successful in demonstrating the utility of the RK4 algorithm as a great tool for galaxy halo modelling.

REFERENCES

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