Simulating Synchrotron Radiation

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ABSTRACT
Synchrotron radiation is an astrophysical phenomenon that results from relativistic electrons moving helically in a magnetic field. By measuring synchrotron spectra, we can measure the strength of the magnetic field of many astronomical objects. For my computational project, I will simulate the electric/magnetic (EM) fields emitted from a synchrotron, and also simulate a power spectrum observation at an arbitrary point. In this paper I will give brief derivations for relevant concepts and explain how I will use them in my simulation.

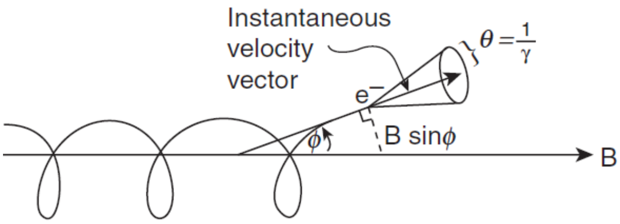

Figure 1. Diagram of the problem setup. Figure adapted from Irwin (2007).

1 DERIVATIONS

1.1 Synchrotron Motion

The following derivation references Irwin (2007). For a non-relativistic electron moving at an angle ϕ to a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, we have the Lorentz force

$$F_e = \frac{ev}{c} B \sin\phi, \tag{1}$$

where e is the charge of the electron, v is the velocity of the electron (as fraction of speed of light), and c is the speed of light. Equating this to the centripetal force we have

$$r_0 = \frac{m_e v_\perp c}{eB} = \frac{v_\perp}{2\omega_0}, \tag{2}$$

where r_0 is the gyroradius, m_e is the mass of the electron, ν_0 is the gyrofrequency, and $v_\perp = v \sin\phi$ is the component of velocity perpendicular to the magnetic field. For a relativistic electron (see Figure 1), the energy is increased by a factor of γ, so we have gyroradius and gyrofrequency:

$$r = \gamma r_0 = \frac{\gamma m_e v_\perp c}{eB}, \tag{3}$$

$$\omega = \gamma \omega_0 = \frac{eB}{\gamma m_e c}. \tag{4}$$

This allows us to write the form of the position of the synchrotron as a function of the time in the electron's frame τ:

$$\mathbf{r}(\tau) = r\cos(\omega\tau)\hat{\boldsymbol{\rho}} + v_\parallel \tau \hat{\mathbf{z}}, \tag{5}$$

where τ is the time in the frame of the moving charge, and $v_\parallel = v\cos\phi$ is the parallel component of the velocity to **B**. This equation, as well as its time derivatives, will be coded into my simulation and used to calculate the EM fields from the synchrotron, the formula for which is given below.

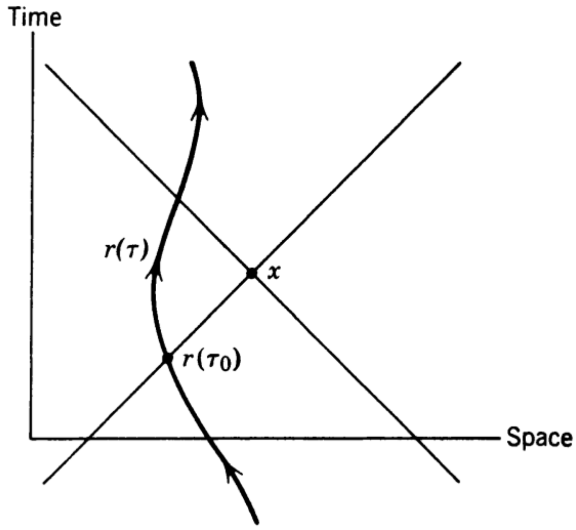

Figure 2. The charge is moving along the curved bold line with position $\mathbf{r}(\tau)$, and the potential/field is being measured at $\mathbf{x}$. Adapted from Jackson (1999).

1.2 EM Fields of an Accelerating Charge

Deriving the EM fields of an accelerating charge in general is a long task and outside the scope of this project, so I will only give a very condensed version of it, referencing Jackson's Classical Electrodynamics (Jackson 1999).

For an arbitrarily moving charge, the relativistically correct potential, called the Lieńard-Wiechert potential, is given by

$$\Phi(\mathbf{x}, t) = \left[\frac{e}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret},$$
$$\mathbf{A}(\mathbf{x}, t) = \left[\frac{e\boldsymbol{\beta}}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \tag{6}$$

where Φ is the scalar potential, $\mathbf{A}$ is the vector potential,

$\boldsymbol{\beta}$ is the velocity of the particle $\mathbf{v}(\tau)$ over c, $\mathbf{n}$ is a unit vector in the direction of $\mathbf{x}-\mathbf{r}(\tau)$, and $R \equiv |\mathbf{x}-\mathbf{r}(\tau_0)|$ (see Figure 2). The "*ret*" subscript means that the quantity in the square brackets is evaluated at the retarded time τ_0, given by $r_0(\tau_0) = x_0 - R$. Using the formulas

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{7}$$

we get

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret} \\ &+ \frac{e}{c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}, \\ \mathbf{B} &= [\nabla\times\mathbf{E}]_{ret}. \end{aligned} \tag{8}$$

This are the field equations that I will use for my simulation. Since the observation point will be placed very far away from the charge, the location of the synchrotron will stay approximately at the origin, and $R \approx |\mathbf{x}|$. However, $\mathbf{r}(\tau)$ will still be needed to calculate $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$.

1.3 Synchrotron Radiation Spectrum

Now I will brief the derivation of the power from synchrotron radiation, in order to be able to simulate observations of it from a faraway point, again referencing Jackson (1999). From Equation 8 we calculate the radial component of the Poynting vector to be

$$[\mathbf{S}\cdot\mathbf{n}]_{ret} = \frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right|^2\right\}_{ret}, \tag{9}$$

which is the power per unit area detected at an observation point at time t of radiation emitted by the charge at $t' = t - R(t')/c$. We can then derive the power output per unit solid angle to be

$$\frac{dP(t')}{d\Omega} = R^2(\mathbf{S}\cdot\mathbf{n})\frac{dt}{dt'} = R^2\mathbf{S}\cdot\mathbf{n}(1-\boldsymbol{\beta}\cdot\mathbf{n}). \tag{10}$$

Then, plugging in Equation 9, we get

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}. \tag{11}$$

In our case, $\boldsymbol{\beta}$ is perpendicular to $\dot{\boldsymbol{\beta}}$. Using the coordinate system defined in Figure 3, Equation 11 reduces to

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]. \tag{12}$$

While not immediately obvious, this equation corresponds to a narrow beam of light of width $\Delta\theta \sim 1/\gamma$ pointed in the direction of the instantaneous velocity of the charge.

To simulate the observed signal, I will approximate the light beam to be rigid. That is, there will be no measured signal until the edge of the light beam at $\Delta\theta$ hits the observation point, at which the measured signal can be approximated as

$$P(t') \approx 4\pi\sin^2(\Delta\theta)\frac{dP(t')}{d\Omega}, \tag{13}$$

which is just the solid angle of a spherical cap with angle $\Delta\theta$ times Equation 12 evaluated at (θ, ϕ) of the observation point. The observed signal in time-domain should look somewhat like Figure 4. I will run the simulation for many pulses, and then do a fast fourier transform to obtain the frequency-domain data. This is what is then used by astronomers to calculate the magnetic field of the astronomical object that contains the synchrotrons.

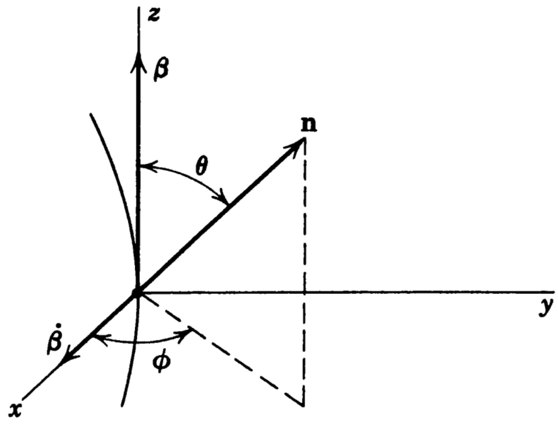

Figure 3. Coordinate system used to calculate power angular distribution.

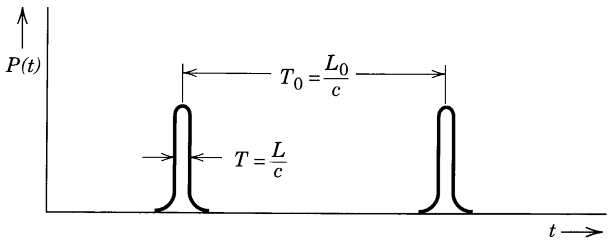

Figure 4. The observed signal power over time.

2 CONCLUSION

I will be simulating synchrotron radiation, which is a very useful probe in astronomy. There are two major components to my simulation: the EM field emitted from the rest frame of the synchrotron, and the mock observation from a point far away from the synchrotron. In this paper I have derived (hopefully) all the equations that I will need to successfully simulate this process. While I did not go into detail on many of these derivations, they can mostly be found in Jackson (1999) or probably any other grad-level E&M textbook.

REFERENCES

Irwin J., 2007, Astrophysics: Decoding the Cosmos. Wiley, https://books.google.com/books?id=srDOZUpAT_cC

Jackson J. D., 1999, Classical electrodynamics, 3rd ed. edn. Wiley, New York, NY, http://cdsweb.cern.ch/record/490457