Voting Paradoxes in the Real World

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1 Introduction

The mathematical theory of voting is filled with instances of paradoxes and counterintuitive outcomes that can occur in elections involving three or more candidates when the voters are asked to rank their candidates in order of preference rather than just vote for a single candidate. Unfortunately, while it is relatively easy to construct hypothetical situations that lead to such paradoxes (see Section 3.2 for examples), finding voting paradoxes in real world elections is much more challenging, if not impossible, as the necessary ballot data is usually not publicly available[1]. Thus, the question whether, and to what extent, real world elections are susceptible to voting paradoxes, remains largely open.

In this project, we examine the occurrence of voting paradoxes in the real world, focusing specifically on two kinds of "elections" in sports contexts: the AP (Associated Press) Top 25 Polls for college football, and the MVP (Most Valuable Player) voting in Major League Baseball (MLB). Both of these "elections" are of a type that lends itself well to the occurrence of paradoxes in that they involve a large number of "candidates" (i.e., college football teams or baseball players) and voters submit a ranked list of these candidates rather than simply choosing a single preferred candidate. Moreover, in both cases complete ballots of all voters are publicly available. It is these two features that make the AP Top 25 Polls and the MLB MVP voting unique real world data sources that can be mined for voting paradoxes.

2 Description of Data

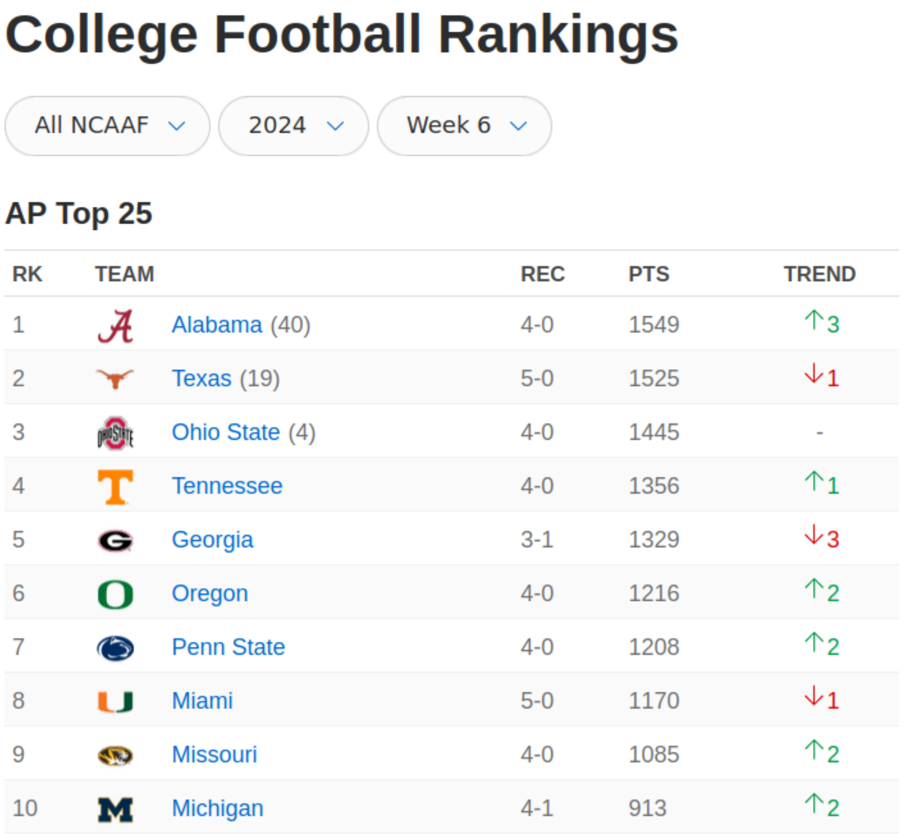
2.1 AP Top 25 College Football Polls

The AP Top 25 College Football Polls are polls conducted by the Associated Press (AP) each week during the college football season. Approximately 60 sports journalists are selected by the AP as voters in these polls. Every week during the season, each of these voters submits a ballot consisting of a ranked list of the voter's top 25 college football teams. Each team is assigned points according to where they were ranked by the voters: 25 points for first place, 24 points for second place, and so on, up to 1 point for 25th (last) place[2]. The total number of points earned by each team determines the official AP Top 25 Ranking.

[1] See [2] for a compilation of studies on voting paradoxes in real world elections. Most of these studies are based on small, and rather obscure, local elections, and some rely on questionable data such as voter preferences extrapolated from opinion polls. The work closest to our project is a 1992 paper by J.P. Benoit [1], who investigated paradoxes in MLB MVP voting from 1943 to 1984.

[2] This is the standard Borda Count Voting Method with 25 "candidates" (teams); see Section 3.1 for a complete description of this method.

Figure 1 shows a sample AP Poll, taken in Week 6 of the 2024 college football season. (Only the top 10 teams of the 25 teams ranked in the poll are shown in the figure.) The column labeled "PTS" shows the total number of points accumulated by the team and determines the team's ranking. The numbers in parentheses after a team denote the number of first place votes this team has received.



College Football Rankings

All NCAAF | 2024 | Week 6

AP Top 25

RK	TEAM	REC	PTS	TREND
1	Alabama (40)	4-0	1549	↑3
2	Texas (19)	5-0	1525	↓1
3	Ohio State (4)	4-0	1445	-
4	Tennessee	4-0	1356	↑1
5	Georgia	3-1	1329	↓3
6	Oregon	4-0	1216	↑2
7	Penn State	4-0	1208	↑2
8	Miami	5-0	1170	↓1
9	Missouri	4-0	1085	↑2
10	Michigan	4-1	913	↑2

Figure 1: 2024 Week 6 AP College Football Poll (Source: `espn.com`)

The complete ballots of all voters in the AP Poll are made public by the Associated Press. The current version is available at `https://apnews.com/hub/ap-top-25-college-football-poll`, and archived versions of past polls are available at `https://collegepolltracker.com`. We used the latter site to collect voter ballot data from all weekly college football polls conducted since 2014. Altogether, we collected complete ballot data from **167 weekly polls (i.e., 167 "elections")**.

2.2 MLB MVP Voting

The MVP voting in Major League Baseball is conducted by the Baseball Writers' Association of America (BBWAA), a professional organization. At the end of each season, two MVP awards are given out, one for a player from the National League (NL) and one for a player from the American League (AL). The awards are based on separate votes—one for the NL MVP and another for the AL MVP—conducted by the BBWAA. For each of the two leagues, the BBWAA selects approximately 30 of its members as voters in that league's MVP voting. The voters are asked to submit a ballot ranking their top 10 players in their assigned league. The ballots are aggregated using a point system that assigns 14 points for a first place vote, 9 points for a second place vote, and so on, up to 1 point for a tenth place vote. The player receiving the most points is declared the MVP for the given league.

Since 2012 individual voter ballots have been made available by the Baseball Writers' Association of

America and published at its website, `https://bbwaa.com`. We collected this data for the 12 seasons from 2012–2023. As there are two MVP votes each season (NL MVP and AL MVP), this represents a total of **24 MVP votes (i.e., 24 "elections")**.

3 Voting Theory Basics

In this section we introduce the basic concepts in the mathematical theory of voting and we present examples that illustrate some of the paradoxes that can arise in this theory; more details can be found in [3].

3.1 Voting Methods

We consider elections in which each voter is asked to *rank* all candidates (or the top n candidates) in order of preference, rather than simply voting for a single candidate. This situation is relatively common in elections in sports contexts, but much less so in political elections. A voting method is a method to determine a winner given these preferential ballots, or to determine a single overall ranking that reflects the individual voter rankings in as fair a manner as possible. The most common approaches are the following:

- **Plurality method:** Only first place votes are counted. The candidate with the most first place votes is the **plurality winner**.
- **Condorcet method:** A **Condorcet winner** is a candidate that beats each of the other candidates in pairwise comparisons, in the sense that the candidate is ranked higher than each of the other candidates by a majority of voters.
- **Borda count method:** Assume each voter ranks their top n candidates in order of preference. In its standard form, the Borda count method assigns n points for each first place vote, $n-1$ points for each second place vote, and so on. The points earned in this manner by each candidate are added up, and the candidates are ranked according to their accumulated points. The candidate who has earned the most points is the **Borda count winner**. The AP Top 25 polls are conducted using this method with $n = 25$.
- **Generalized Borda count method:** This method generalizes the standard Borda count method, with weights (points) $w_1, w_2, \dots, w_n$ in place of the standard Borda count weights $n, n-1, \dots, 1$ for 1st place through nth place. The weights w_i are assumed to satisfy $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$, but can otherwise be arbitrary. The MVP voting in baseball is an example of a generalized Borda count method, corresponding to the weight vector $(w_1, \dots, w_{10}) = (14, 9, 8, \dots, 1)$. The plurality method can also be regarded as a special case of the generalized Borda count method, corresponding to the weight vector $(1, 0, \dots, 0)$.

To illustrate these methods, consider an election with three candidates, A, B, and C, and 21 voters whose preferences among the candidates are given in Table 1.

# of Voters	1st	2nd	3rd
1	A	B	C
7	A	C	B
7	B	C	A
6	C	B	A

Table 1: **An "election" involving 3 candidates and 21 voters.**

- **Plurality method:** Candidate A received $1+7=8$ first place votes, B received 7 first place votes, and C received 6 first place votes. Since A received the most first place votes, **A is the Plurality winner**.
- **Condorcet method:** We compute, for each of the pairs (A,B), (B,C), and (A,C), the pairwise winner by comparing the number of voters that rank the first candidate in the pair higher than the second with the number of voters that rank the second candidate higher than the first:
  - **A versus B:** 8 votes to 13 votes, so **B wins over A**.
  - **B versus C:** 8 votes to 13 votes, so **C wins over B**.
  - **A versus C:** 8 votes to 13 votes, so **C wins over A**.

  Since C wins both of its pairwise contests, **C is the Condorcet winner.**
- **Borda count method:** We compute the number of Borda count points accumulated by each of the three candidates using the standard Borda count weights $3, 2, 1$ for first, second, and third place:
  - **Points for A:** $3 \cdot 8 + 2 \cdot 0 + 1 \cdot 13 = 37$
  - **Points for B:** $3 \cdot 7 + 2 \cdot 7 + 1 \cdot 7 = 42$
  - **Points for C:** $3 \cdot 6 + 2 \cdot 14 + 1 \cdot 1 = 47$

  Since C has the highest Borda points total, **C is the Borda count winner**.

3.2 Voting Paradoxes

When aggregating votes to determine a winner of an election, a variety of paradoxes and counterintuitive outcomes can arise, no matter what voting method is used. Some of the most famous examples of such paradoxes are the following:

Paradox I: There exists a nontransitive cycle: If a majority of voters rank A over B, a majority rank B over C, while at the same time a majority rank C over A, the candidates A, B, and C form a nontransitive cycle of length 3: $A > B > C > A$. Examples of elections that have nontransitive cycles are easy to construct. Table 2 shows a particularly simple example, one that involves just three voters:

# Voters	1st	2nd	3rd
1	A	B	C
1	B	C	A
1	C	A	B

Table 2: **Illustration of the nontransitive cycle paradox.**

In this example, two out of the three voters (i.e., a 2/3 majority) rank A above B, two voters rank B above C, and two voters rank C above A, resulting in the nontransitive cycle A > B > C > A.

Paradox II: The Condorcet winner is different from the Borda or Plurality winner. A Condorcet winner is a candidate that is preferred by a majority of voters over *each* of the other candidates. Put differently, a Condorcet winner is a candidate that wins *every* single pairwise contest. This is a very strong requirement—winning the *majority* of pairwise contests is not enough to be declared the Condorcet winner. A Condorcet winner need not exist, but if it exists, it would seem a matter of basic fairness that this candidate should also be the winner of other "fair" voting methods.

This is not the case: Both the Borda Count Method and the Plurality Method can violate this seemingly natural fairness requirement. The example in Table 1 illustrates this for the Plurality Method: Candidate C wins the pairwise contests against both of the other candidates, while Candidate A is the winner under the Plurality Method[3].

Paradox III: The Independence of Irrelevant Alternatives (IIA) Paradox: A reasonable requirement on any "fair" election method should be that removing some candidates from all ballots (e.g., as a result of disqualification of these candidates) should not affect the relative ranking of the remaining candidates. This requirement is called the **Independence of Irrelevant Alternatives (IIA) Axiom**. The Borda Count Method does not satisfy this requirement and thus is susceptible to the **Independence of Irrelevant Alternatives Paradox**.

As an illustration of this paradox, consider the example shown in Table 3, which involves four candidates, A, B, C, D, and 7 voters.

# Voters	1st	2nd	3rd	4th
3	A	B	C	$\boxed{\text{D}}$
2	B	C	$\boxed{\text{D}}$	A
2	C	$\boxed{\text{D}}$	A	B

# Voters	1st	2nd	3rd
3	A	B	C
2	B	C	A
2	C	A	B

Table 3: **Illustration of the IIA paradox.**

The table on the left shows the original preference rankings among the voters when all 4 candidates are on the ballot. The table on the right shows the preference rankings resulting after candidate D is deleted from each of the ballots, without changing the order of the remaining candidates. The Borda count point totals and the resulting Borda rankings for these tables are as follows:

[3] In fact, more is true: The plurality winner, A, is the *Condorcet loser* (in the sense that A is the loser in *all* pairwise contests), while the Condorcet winner, C, is the *Plurality loser* (since it received the fewest first place votes). Thus we have the paradoxical situation that one perfectly reasonable and seemingly fair voting method produces the *opposite* outcome as another—equally reasonable and fair—method.

- **Left table (with D included):** A: 18 points; B: 19 points; C: 20 points; D: 13 points.
  Borda count ranking: C > B > A > D.
- **Right table (with D excluded):** A: 15 points; B: 14 points; C: 13 points.
  Borda count ranking: A > B > C.

Thus, we have the surprising result that removing candidate D from each ballot changes the relative order of the other three candidates in the Borda ranking from C > B > A to A > B > C, i.e., completely reverses the ranking of these three candidates. This is all the more remarkable since the removed candidate, D, is the lowest ranked candidate in the original Borda count ranking.

4 Results

We systematically searched our data sets for occurrences of the three paradoxes mentioned above. The results are as follows:

4.1 Occurrences of Paradox I (nontransitive cycles)

Nontransitive cycles in the AP Top 25 Poll Data: Altogether, we found **146 nontransitive cycles** among the 167 college football polls we analyzed, representing an average of 0.87 cycles per "election" (college poll). Of the 146 cycles found, 137 were of length 3, 8 were of length 4, and 1 was of length 5. The most "extreme" example (in the sense of involving the highest ranked teams) occurred in the 2019 Week 10 Poll, where the three top ranked teams formed a nontransitive cycle:
LSU (#1) > Ohio State (#3) > Alabama (#2) > LSU (#1)

The distribution of cycles per week in the season is shown in Figure 2. The figure suggests that polls conducted in the first half of a season are more likely to contain nontransitive cycles than polls conducted later in the season. This may be explained by the fact that near the beginning of a season there is less known—and hence more uncertainty—about the true strengths of the teams than towards the end of a season.

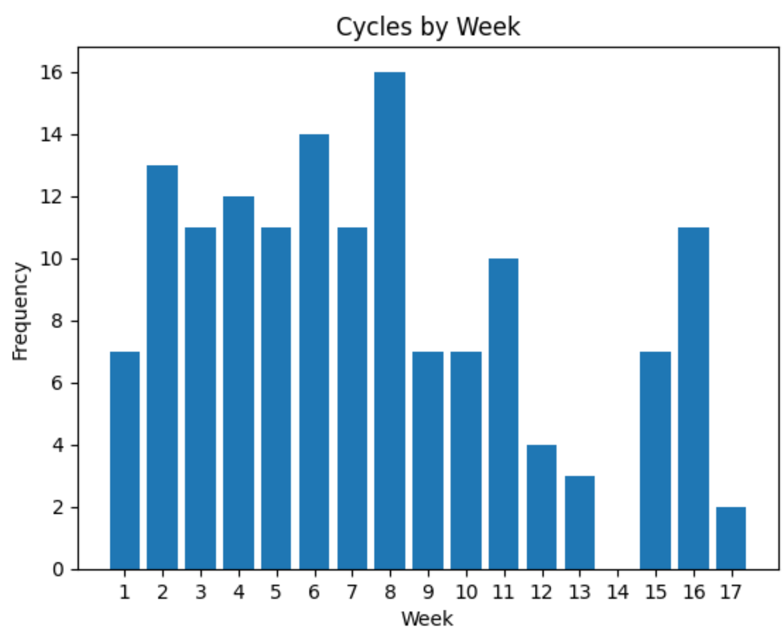


Figure 2: Number of nontransitive cycles of length 3 per week in the season.

Nontransitive cycles in the MLB MVP Voting Data: We found **13 nontransitive cycles** among the 24 MVP votes in our data sets, representing an average of 0.54 cycles per "election" (MVP vote). Of these cycles, 9 were of length 3, 3 of length 4, and 1 of length 5. None of the cycles found involved a top three team; the cycle with the highest ranked players we found occurred in the 2013 NL MVP Vote, where the players ranked 5th (Freeman), 6th (Votto), and 7th (Kershaw) formed a nontransitive cycle: Freeman (#5) > Kershaw (#7) > Votto (#6) > Freeman (#5).

4.2 Occurrences of Paradox II (Condorcet winner different from Borda winner)

We found no instance of this paradox in the MLB MVP Voting Data, and only two instances in the AP Top 25 Poll Data:

- In the 2014 Week 15 college football poll, Alabama was the Borda winner, while Florida State was the Condorcet winner.
- In the 2022 Week 6 college football poll, Alabama was the Borda winner, while Georgia was the Condorcet winner.

4.3 Occurrences of Paradox III (Independence of Irrelevant Alternatives Paradox)

In the AP Top 25 Poll Data, there were 32 cases of the IIA paradox if 1 team is removed, and 146 cases if 2 teams are removed. The most extreme example occurred in the 2022 Week 10 College football poll, where removing the 4th ranked team (Michigan) causes the ranks of the second-ranked team (Ohio State) and the third-ranked team (Tennessee) to be reversed.

In the MLB MVP Voting data, there was 1 case of the IIA paradox if 1 player is removed, and 9 cases if 2 players are removed. The most extreme example occurred in the 2017 NL MVP Vote and is illustrated in Figure 3: Removing the 5th ranked player (Blackmon) from all ballots causes the ranks of the two highest-ranked players (Stanton and Votto) to be reversed, and thus would result in a change in the MVP winner.

Figure 3: Real world example of the IIA Paradox (from the 2017 NL MVP Vote): If the 5th ranked player (Blackmon) is removed from all ballots, the winner of the MVP Vote changes from Stanton to Votto.

References

[1] Benoit, J. P. (1992), *Scoring reversals: a major league dilemma*, Social Choice and Welfare 9(2), 89–97.

[2] Lagerspetz, E. (2016), *Social Choice in the Real World*, in: Social Choice and Democratic Values, Springer-Verlag, 383–430.

[3] Pacuit, E. (2011), *Voting methods,* Stanford Encyclopedia of Philosophy, `http://plato.stanford.edu/entries/voting-methods/`.