Voting Paradoxes in the Real World
Theory Materials
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Voting Methods

- **Plurality**
- **Borda Count**
- **Condorcet**

Voting method 1: Plurality Method

- The candidate(s) with the most first place votes wins.
- **Example:**
  - **Candidates**: A, B, C
  - **Ballots**:

Number of Voters	1st	2nd	3rd
1	A	B	C
7	A	C	B
7	B	C	A
6	C	B	A

  - **Plurality Calculation**:
    - 1st place votes for A: 8
    - 1st place votes for B: 7
    - 1st place votes for C: 6
  - **Plurality Winner: A**

Voting method 2: Borda Count Method

- Assume there are n candidates.
- Each voter ranks the candidates in the order of preference.
- Points are assigned based on the rank:
  - **Standard Borda Count:** n points for 1st place, $n-1$ for 2nd place, etc, and 1 point for the last place.
  - **Generalized Borda Count:** w_1 points for 1st place, w_2 for 2nd place, etc, and w_n point for the last place, where $w_1 \geq w_2 \geq \cdots \geq w_n$.
- The points for each candidate are added up and the candidate with the highest total points wins.

Voting method 2: Borda count example

- **Candidates**: A, B, C
- **Ballots**:

Number of Voters	1st	2nd	3rd
1	A	B	C
7	A	C	B
7	B	C	A
6	C	B	A

- **Borda Calculation**:
  - **Points for A:** $3 \cdot 8 + 2 \cdot 0 + 1 \cdot 13 = 37$
  - **Points for B:** $3 \cdot 7 + 2 \cdot 7 + 1 \cdot 7 = 42$
  - **Points for C:** $3 \cdot 6 + 2 \cdot 14 + 1 \cdot 1 = 47$
- **Borda Winner: C**

Voting method 3: Condorcet Method

- Each voter ranks the candidates in the order of preference.
- For each pair of candidates A and B, the number of voters that rank A over B is compared to the number of voters that rank B over A.
- A Condorcet winner is a candidate who can beat every other candidate in a head-to-head comparison.

Voting method 3: Condorcet example

- **Candidates**: A, B, C
- **Ballots**:

Number of Voters	1st	2nd	3rd
1	A	B	C
7	A	C	B
7	B	C	A
6	C	B	A

- **Condorcet Calculation**:
  - A vs B: 8 votes to 13 votes: B wins over A
  - B vs C: 8 votes to 13 votes: C wins over B
  - A vs C: 8 votes to 13 votes: C wins over A
- **Condorcet Winner: C**

Some Voting Paradoxes

- **The Borda count winner is the Plurality loser**
- **The Borda count winner is different from the Cordorcet winner**
- **Condorcet Paradox:** There exists a cycle of candidates A, B, and C such that
  - a majority of voters rank A over B
  - a majority of voters rank B over C
  - a majority of voters rank C over A.

Independence of Irrelevant Alternatives (IIA)

- **Independence of Irrelevant Alternatives Axiom**: The addition or removal of a candidate (team, player) should not change the relative ranking of the remaining candidates.
- **Example**: Suppose there are four candidates A, B, C, and D, ranked $A > D > C > B$.
  Suppose D is disqualified. Then under the IIA axiom, the ranking between A, B, and C should be $A > C > B$.
- **Paradox**: Adding or removing a candidate changes the relative ranking of some other candidates (ie, IIA axiom is violated).

IIA Paradox Example

Number of Voters	1st	2nd	3rd	4th
3	A	B	C	D
2	B	C	D	A
2	C	D	A	B

- **Borda Calculation by considering A, B, C, and D**:
  - **Points for A:** $4 \cdot 3 + 3 \cdot 0 + 2 \cdot 2 + 1 \cdot 2 = 18$
  - **Points for B:** $4 \cdot 2 + 3 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 19$
  - **Points for C:** $4 \cdot 2 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 0 = 20$
  - **Points for D:** $4 \cdot 0 + 3 \cdot 2 + 2 \cdot 2 + 1 \cdot 3 = 13$
  - **Borda Ranking:** **C** $>$ **B** $>$ **A** $>$ **D**
- **Borda Calculation by considering A, B, and C**:
  - **Points for A:** $3 \cdot 3 + 2 \cdot 2 + 1 \cdot 2 = 15$
  - **Points for B:** $3 \cdot 2 + 2 \cdot 3 + 1 \cdot 2 = 14$
  - **Points for C:** $3 \cdot 2 + 2 \cdot 2 + 1 \cdot 3 = 13$
  - **Borda Ranking:** **A** $>$ **B** $>$ **C**

Arrow's Impossibility Theorem

- **Arrow's Impossibility Theorem**: If there are three or more candidates, there does not exist a voting method that satisfies every reasonable fairness criteria.