Characterizing algebraic curves using p-adic norms

Shuang-Yen Lee

1 Introduction

Let M be a compact Riemann surface of genus $g \geq 1$. There is a vast literature on characterizing M in terms of its holomorphic differential forms and studying their relations with the moduli space of complex structures. In this thesis we are interested in extending some of these fundamental considerations to the p-adic setting.

The classical Torelli theorem asserts that M is determined by its Jacobian variety $J(M)$ under the principal polarization. More precisely, $J(M)$ is defined by

$$J(M) = {H^0(M, \Omega^1)^\vee}\big/{H_1(M, \mathbb{Z})},$$

and its principal polarization $[\omega_M] \in H^2(J(M), \mathbb{Z})$ is the Poincaré paring:

$$\begin{aligned} [\omega_M] : H_2(J(M), \mathbb{Z}) = \textstyle\bigwedge^2 H_1(M, \mathbb{Z}) &\to \mathbb{Z} \\ \alpha \wedge \beta &\mapsto \langle \alpha, \beta \rangle. \end{aligned}$$

Then we have (cf. [1, p.359])

Theorem 1.1 (Torelli). Let M and M' be compact Riemann surfaces of genus $g \geq 1$. Suppose that there is an isomorphism

$$\big(J(M), [\omega_M]\big) \cong \big(J(M'), [\omega_{M'}]\big)$$

as principally polarized abelian varieties, i.e., there is an isomorphism $f : J(M) \to J(M')$ such that $[\omega_M] = f^*[\omega_{M'}]$. Then there is an isomorphism $\varphi : M \to M'$ such that f is induced by φ.

The Torelli theorem also holds for non-singular projective curves over any algebraically closed field under the purely algebraic definition of Jacobian varieties $J(M) = \mathrm{Pic}^0(M)$. Recently it was extended by Serre to curves over an arbitrary ground field in [2, Appendix].

The Torelli theorem concentrates on the first cohomology group $H^1(M,\mathbb{C})$ and the subspace of holomorphic forms $H^0(M,K_M) = H^0(M,\Omega^1)$ inside it. One might ask whether M is also determined by other cohomology groups, perhaps with some extra data. It is indeed the case for $H^0(M,2K_M)$, the space of quadratic differentials, as shown by Royden in [3]. To state the result in a better perspective we need to introduce the concept Teichmüller spaces (cf. [4, Chap. XV]).

Fix a compact oriented surface Σ of genus $g \geq 2$. A Teichmüller structure on a compact Riemann surface M is the isotopy class $[f]$ of an orientation-preserving homeomorphism

$$f : M \to \Sigma.$$

An isomorphism

$$\big(M,[f]\big) \cong \big(M',[f']\big)$$

between two Riemann surfaces with Teichmüller structure is an isomorphism of Riemann surfaces $\varphi : M \to M'$ such that $[f' \circ \varphi] = [f]$. The Teichmüller space $\mathbf{T}^g$ is the set of isomorphism classes of compact Riemann surfaces of genus g with Teichmüller structure.

There is a natural complex structure on $\mathbf{T}^g$: for $\big[M,[f]\big] \in \mathbf{T}^g$, let $\pi : \mathcal{C} \to (B,b_0)$ be a Kuranishi family for M (see [5]), i.e., M is the fiber of $b_0 \in B$ and there is a universal property for π:

> For any deformation $\mathcal{C}' \to (B',b_0')$ of M and sufficient small connected neighborhood U of b_0', there is a unique morphism of deformations of Riemann surfaces (i.e. a fiber square):

$$\begin{array}{ccc} \mathcal{C}'_U & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow \\ (U,b_0') & \longrightarrow & (B,b_0). \end{array}$$

By shrinking B if necessary, we may assume that there is a smooth trivialization $(F,\pi) : \mathcal{C} \to \Sigma \times B$ such that $F_{b_0} \circ \varphi = f$, which extends the Teichmüller structure on M uniquely to $\mathcal{C}$. By the definition of the isotopy class, we get a local chart U near $\big[M,[f]\big]$.

The Teichmüller metric on $\mathbf{T}^g$ is defined by

$$d\big(\big[M,[f]\big],\big[M',[f']\big]\big) = \frac{1}{2}\inf\{\,\log K(F) \mid F : M \to M',\ [F] = [\mathrm{id}]\,\},$$

where the infimum is taken over all quasiconformal maps F, i.e., $F_{\overline{z}} = \mu \cdot F_z$ for some function $\mu : M \to \mathbb{C}$ with $\sup |\mu| < 1$, and

$$K(F) := \sup_{x \in M} \frac{|F_z(x)| + |F_{\overline{z}}(x)|}{|F_z(x)| - |F_{\overline{z}}(x)|}$$

is the quasiconformal dilatation of F.

It is well-known that the space of quadratic differentials $H^0(M, 2K_M)$ is the cotangent space of $\mathbf{T}^g$ at $\big[M, [f]\big]$. Teichmüller then defined a norm on it by

$$\|\alpha\| := \int_M |\alpha|,$$

which induces a Finsler metric on $\mathbf{T}^g$. (There is also a Weil–Petersson metric on $\mathbf{T}^g$ which is indeed a Riemannian metric, but we will not discuss it here.) Using this norm, Teichmüller was lead to consider the following fundamental map

$$\Phi : B = \big\{\alpha \in H^0(M, 2K_M) \mid \|\alpha\| < 1\big\} \to \mathbf{T}^g,$$

which is defined as follows:

> For $\alpha \in B$, let Z be the set of zeroes of α. Define a new local coordinate system on $M \setminus Z$ by
>
> $$z = \frac{w + \|\alpha\|\overline{w}}{1 - \|\alpha\|}, \quad \text{where } w = \int \sqrt{\alpha}.$$
>
> These local coordinates give a new complex structure on $M \setminus Z$ and extend to one on M. Take $\Phi(\alpha)$ to be this complex structure.

It can be shown (highly non-trivially) that the map Φ is a homeomorphism (see [4, Chap. XV]), hence the Teichmüller space is homeomorphic to a unit ball in $\mathbb{C}^{3g-3}$.

With these backgrounds, we may now state Royden's results. In his study of isometries on Teichmüller spaces [3], Royden proved that the Finsler metric coincides with the Teichmüller metric [3, p. 369]. Furthermore, a compact Riemann surface is determined by the norm $\|\cdot\|$. Namely, the following Torelli type theorem for $H^0(M, 2K_M)$ holds:

Theorem 1.2 ([3, Theorem 1])**.** Let M and M' be compact Riemann surfaces of genus $g \geq 2$, and let

$$\Phi : H^0(M, 2K_M) \to H^0(M', 2K_{M'})$$

be a complex linear isometry between the spaces of quadratic differentials. Then there is a conformal map $\phi : M' \to M$ and a complex constant u with $|u| = 1$ such that $\Phi = u \cdot \phi^*$.

Royden proved his result by finding the “singular” part of the normed vector space $(H^0(M, 2K_M), \|\cdot\|)$ and then identifying the geometric data encoded in it. In fact, for α, $\beta \in H^0(M, 2K_M)$ and $t \in \mathbb{C}$, Royden computed the asymptotic order of

$$\|\alpha + t\beta\| - \|\alpha\|$$

in terms of the vanishing orders of α and β. Then he recovered the canonical curve using the asymptotic orders, which is invariant under isometries. (Using this, Royden also proved that the mapping class group, i.e., the group of all isotopy classes of orientation-preserving homeomorphism of Σ into itself, is the group of isometries on $\mathbf{T}^g$ by showing that the Teichmüller metric is invariant under biholomorphic map of $\mathbf{T}^g$ into itself [3, Theorem 3].)

The main purpose of this thesis is to study the p-adic analogue of Royden's theory.

Let K be a p-adic field, $\mathcal{O}$ its ring of integers, and $\mathbb{F}_q$ the residue field of $\mathcal{O}$. Let X be an n-dimensional projective scheme over $\mathcal{O}$. There are two parts of X, the generic fiber X_K and the special fiber $X_{\mathbb{F}_q}$. We assume that X_K is smooth over K in the following text. If we assume furthermore that X is smooth over $\mathcal{O}$, then the smoothness gives us a good reduction map

$$X(\mathcal{O}) \to X(\mathbb{F}_q).$$

If we view $X(K)$ as a K-analytic n-dimensional manifold (note that we used the smoothness of X_K here), then $X(K)$ is bianalytic to a finite disjoint union of copies of $\mathcal{O}^n$ (see [6, Sec. 7.5]). We define the quasinorm $\|\cdot\|_K$ on the space of r-differential forms $H^0(X, rK_X)$ as follows:

Definition 1.3. For any $\alpha \in H^0(X, rK_X)$, we can write it locally as $a(u)\,(du)^r$ on a chart $U \cong \mathcal{O}^n$ with coordinate $u = (u_1, \ldots, u_n)$. We define

$$\int_U |\alpha|^{1/r} = \int_{\mathcal{O}^n} |a(u)|^{1/r} d\mu,$$

where $|\cdot|$ is the normalized norm on K so that $|u| = q^{-v(u)}$, and μ is the normalized Haar measure on the locally compact abelian group K^n so that $\mu(\mathcal{O}^n) = 1$.

(1) We define the p-adic integral of α on X by

$$\int_{X(K)} |\alpha|^{1/r} := \sum_U \int_U |\alpha|^{1/r},$$

which is independent of the choice of atlas $\{U\}$ by the change of variables formula (see [6, Sec. 7.4]).

(2) We define the quasinorm of α to be

$$\|\alpha\|_K = \left(\int_{X(K)} |\alpha|^{1/r}\right)^r.$$

Notice that even for $r = 2$ the power in our definition is (necessarily) different form the case over complex numbers.

Under this analogous p-adic setting, Royden's method could be generalized to curves over K via p-adic integrals, and indeed the estimates are easier than the those in [3]. However, a p-adic field is not algebraically closed, so there will be some arguments on the rational points that are different from the case over the complex numbers.

If $X(K)$ is empty then the quasinorm $\|\cdot\|_K$ is trivial and we could not get any further information from it. So the condition that $X(K)$ being nonempty is needed. As in Royden's case, the quasinorm contains informations of the zero-divisors of α for each r-differential form α (see Proposition 2.1 and Proposition 2.3).

Here is the main result of this thesis:

Theorem 1.4. Let X and X' be smooth projective curves over $\mathcal{O}$ of genus $g \geq 2$, r a positive integer, and V, V' the spaces $H^0(X, rK_X)$, $H^0(X', rK_{X'})$ respectively. Let

$$\Phi : (V(K), \|\cdot\|_K) \to (V'(K), \|\cdot\|'_K)$$

be an isometry. Suppose that $\mathbb{F}_q = \mathcal{O}/\mathfrak{m}$ has at least $4g^2$ elements and the r-canonical maps

$$|rK_X| : X \to \mathbb{P}(V)^\vee \quad \text{and} \quad |rK_{X'}| : X' \to \mathbb{P}(V')^\vee$$

are embeddings. Then there exist an isomorphism

$$\varphi : X' \to X$$

and some $u \in \mathcal{O}^\times$ such that $\Phi = u \cdot \varphi_K^*$.

The condition $q \geq 4g^2$ is to guarantee that both $X(\mathcal{O})$ and $X'(\mathcal{O})$ are nonempty. Note that we can always do a finite extension on K so that this condition is satisfied. We will see that the theorem still holds with the assumption $q \geq 4g^2$ being replaced by that both $X(K)$ and $X'(K)$ contain some nonempty K-analytic discs. If X and X' are not necessarily smooth over $\mathcal{O}$. Then we can still get an isomorphism between the Néron models of them (cf. Definition 4.4 and Theorem 4.6).

It is natural to consider extensions of the above results to higher dimensions. In fact, Royden's theorem had been extended by Chi in [7] to complex projective manifolds M of general type. More precisely, M can be determined (up to a birational map) by the pseudonorm

$$\|\alpha\| = \int_M |\alpha|^{2/r}$$

on $H^0(M, rK_M)$ for sufficiently large and sufficiently divisible r.

Again we want to find a K-analytic analogue of the result. At this moment, we only achieve our goal partially, so we put the results in the Appendix. In the Appendix, we will provide estimates on the quasinormed spaces $H^0(X, rK_X)$, where X is a smooth projective scheme over $\mathcal{O}$. Surprisingly, the results on the estimates are similar to the complex case and all the estimates needed in the complex case are obtained also in the p-adic setting. A possible approach to the higher dimensional case is to recover the variety X_K using the data obtained from the estimates, and the remaining steps rely on the validity of the corresponding results in birational geometry (minimal model theory) over the non-algebraically closed field K.

Acknowledgement

This is an undergraduate thesis at National Taiwan University supervised by Professor Chin-Lung Wang. I would like to thank Professors Chin-Lung Wang and Jeng-Daw Yu for suggestions on my writing, and Professor Ming-Lun Hsieh for suggestions on Néron models. I also would like to mention that, after seeing the preliminary version of this article, Professor Chen-Yu Chi informed me that he was able to prove similar results for p-adic fields in higher dimensions, though in different methods.

2 p-adic quasinorms

For a p-adic field $(K, |\cdot|)$, let

- $\mathcal{O} = \{x \in K \mid |x| \leq 1\}$ be the ring of integers,
- $\mathfrak{m} = \{x \in K \mid |x| < 1\} = (\pi)$ the maximal ideal of $\mathcal{O}$,

- $v : K \to \mathbb{Z} \cup \{\infty\}$ the valuation on K, and

- $\mathbb{F}_q = {}^{\mathcal{O}}\!/_{\mathfrak{m}}$ the residue field.

Here we scale the norm so that $|u| = q^{-v(u)}$ for all $u \in K^\times$.

Fix a positive integer r. Let X be an n-dimensional projective scheme over $\mathcal{O}$ requiring X_K to be smooth over K and $X(K)$ to be nonempty. It follows from the valuative criterion for properness [8] that

$$X(\mathcal{O}) = X(K) \neq \varnothing.$$

We have defined the quasinorm $\|\cdot\|_K$ on the space of r-differentials $V = H^0(X, rK_X)$ via the p-adic integral:

$$\|\alpha\|_K = \left(\int_{X(K)} |\alpha|^{1/r}\right)^r.$$

Note that we have the identity $\|u\alpha\|_K = |u| \cdot \|\alpha\|_K$, so we may calculate the quasinorm $\|\alpha\|_K$ assuming that $\alpha \in V(\mathcal{O}) := H^0(X, rK_{X/\mathcal{O}})$.

For each $k \geq 1$, consider the mod-$\mathfrak{m}^k$ reductions:

$$\begin{array}{ccccccc} X(\mathcal{O}) & \xrightarrow{h_k} & X(\mathcal{O}/\mathfrak{m}^k) & \text{and} & V(\mathcal{O}) & \to & V(\mathcal{O}/\mathfrak{m}^k) \\ x & \mapsto & \overline{x}^{(k)} & & \alpha & \mapsto & \overline{\alpha}^{(k)}. \end{array}$$

We may calculate the quasinorm by the number of the zeros of $\alpha \in V(\mathcal{O})$ on each $X(\mathcal{O}/\mathfrak{m}^k)$ when X is smooth over $\mathcal{O}$:

Proposition 2.1. Suppose that X is smooth over $\mathcal{O}$. Let α be an element of $V(\mathcal{O})$. For each $k \geq 1$, let

$$N_k = \#\left\{\overline{x}^{(k)} \in X(\mathcal{O}/\mathfrak{m}^k) \;\middle|\; \overline{\alpha}^{(k)}\left(\overline{x}^{(k)}\right) = 0\right\}$$

be the cardinality of the zero set of $\alpha \in V(\mathcal{O})$ on $X(\mathcal{O}/\mathfrak{m}^k)$. Then

$$\|\alpha\|_K^{1/r} = \frac{\#X(\mathbb{F}_q)}{q^n} - (q^{1/r} - 1)\sum_{k=1}^{\infty} \frac{N_k}{q^{k(n+1/r)}}.$$

Proof. Since X is smooth over $\mathcal{O}$, by Hensel's lemma, each fiber of h_1 is K-bianalytic to $\pi\mathcal{O}^n$ with measure preserved. For $\overline{x}^{(1)} \in X(\mathbb{F}_q)$, let $u = (u_1, \ldots, u_n)$ be a local coordinate of $h_1^{-1}(\overline{x}^{(1)})$. Then

$$\int_{h_1^{-1}(\overline{x}^{(1)})} |\alpha|^{1/r} = \int_{\pi\mathcal{O}^n} |a(u)|^{1/r} du$$

for some analytic function $a(u) \in \mathcal{O}[[u]]$. Let

$$N_k\left(\overline{x}^{(1)}\right) = \#\left\{\overline{y}^{(k)} \in h_k(h_1^{-1}(\overline{x}^{(1)})) \;\middle|\; \overline{\alpha}^{(k)}\left(\overline{y}^{(k)}\right) = 0\right\}$$

be the cardinality of the zero set of $\overline{\alpha}^{(k)}$ on the fiber of $\overline{x}^{(1)}$. Then it follows from

$$\mu\left(|a|^{-1}\left([0, q^{-k}]\right)\right) = \mu\left(\left\{u \mid \overline{a}^{(k)}\left(\overline{u}^{(k)}\right) = 0\right\}\right) = \frac{N_k\left(\overline{x}^{(1)}\right)}{q^{kn}}$$

that

$$\begin{aligned}
\int_{\pi\mathcal{O}^n} |a(u)|^{1/r} du &= \sum_{k=0}^{\infty} q^{-k/r} \cdot \mu(|a|^{-1}(q^{-k})) \\
&= \frac{1}{q^n} + \sum_{k=1}^{\infty} (q^{-k/r} - q^{-(k-1)/r}) \cdot \mu(|a|^{-1}([0, q^{-k}])) \\
&= \frac{1}{q^n} - (q^{1/r} - 1) \sum_{k=1}^{\infty} q^{-k/r} \cdot \frac{N_k\left(\overline{x}^{(1)}\right)}{q^{kn}}.
\end{aligned}$$

Summing above equation over $\overline{x}^{(1)} \in X(\mathbb{F}_q)$, we get

$$\begin{aligned}
\|\alpha\|_K^{1/r} &= \sum_{\overline{x}^{(1)}} \left(\frac{1}{q^n} - (q^{1/r} - 1) \sum_{k=1}^{\infty} \frac{N_k\left(\overline{x}^{(1)}\right)}{q^{k(n+1/r)}}\right) \\
&= \frac{\#X(\mathbb{F}_q)}{q^n} - (q^{1/r} - 1) \sum_{k=1}^{\infty} \frac{N_k}{q^{k(n+1/r)}}.
\end{aligned} \qquad \blacksquare$$

Now we let $n = 1$, i.e., X is a smooth projective curve over $\mathcal{O}$.

Corollary 2.2. Suppose that the zero divisor of $\overline{\alpha}^{(1)}$ on $X(\mathbb{F}_q)$ is of the form $\overline{P}_1^{(1)} + \cdots + \overline{P}_\ell^{(1)}$ with $\overline{P}_i^{(1)}$ distinct. Then

$$\|\alpha\|_K^{1/r} = \frac{\#X(\mathbb{F}_q)}{q} - \ell \cdot \frac{q^{1/r} - 1}{q^{1+1/r} - 1}.$$

In particular, the quasinorm $\|\cdot\|_K$ is constant on the preimage $\alpha + \pi V(\mathcal{O})$ of $\overline{\alpha}^{(1)}$.

Proof. Since $\overline{\alpha}^{(1)}$ has multiplicity 1 at $\overline{P}_i^{(1)}$, by Hensel's lemma, we see that

$$N_k(\overline{P}_i^{(1)}) = \#\left\{\overline{x}^{(k)} \in h_k\left(h_1^{-1}(\overline{P}_i^{(1)})\right) \;\middle|\; \overline{\alpha}^{(k)}(\overline{x}^{(k)}) = 0\right\} = 1$$

for each $k \geq 1$. Hence, $N_k = \ell$ for each $k \geq 1$, which gives

$$\|\alpha\|_K^{1/r} = \frac{\#X(\mathbb{F}_q)}{q} - (q^{1/r} - 1) \sum_{k=1}^{\infty} \frac{\ell}{q^{k(1+1/r)}} = \frac{\#X(\mathbb{F}_q)}{q} - \ell \cdot \frac{q^{1/r} - 1}{q^{1+1/r} - 1}. \qquad \blacksquare$$

In order to recover the curve X from the quasinormed space $(V(K), \|\cdot\|_K)$, we need to detect the non-locally constant part of $V(K)$ with respect to the quasinorm. The following proposition gives some information about the "smoothness" of the quasinormed space $(V(K), \|\cdot\|_K)$.

Proposition 2.3. Let X be a projective curve over $\mathcal{O}$ such that X_K is smooth over K, and $V(K)$ the space of r-differential forms $H^0(X_K, rK_{X_K})$. Let α, $\beta \in V(K) \setminus \{0\}$, $(\alpha)_0 = n_1P_1 + \cdots + n_\ell P_\ell$ the zero divisor of α (in $X(K)$) and $N = \max\{n_1, \ldots, n_\ell\}$.

(i) If $N = 1$, we have

$$\|\alpha + t\beta\|_K = \|\alpha\|_K \qquad \text{as } |t| \to 0.$$

(ii) If $N > 1$, we have

$$\|\alpha + t\beta\|_K - \|\alpha\|_K = O\left(|t|^{1/N+1/r}\right) \qquad \text{as } |t| \to 0.$$

(iii) If $N > 1$, then

$$\|\alpha + t\beta\|_K - \|\alpha\|_K$$

is not $O(|t|^\rho)$ for any $\rho > \frac{1}{N} + \frac{1}{r}$ if and only if $\beta(P_i) \neq 0$ for some i with $n_i = N$.

Proof. If we define $\|\cdot\|$ by

$$\|\alpha\| = \int_X |\alpha|^{1/r},$$

which is a pseudonorm, then it is obvious that

$$\|\alpha + t\beta\| - \|\alpha\| = O(|t|^\rho) \iff \|\alpha + t\beta\|_K - \|\alpha\|_K = O(|t|^\rho)$$

for any $\rho > 0$ since $\|\alpha\| > 0$. So it suffices to estimate

$$\|\alpha + t\beta\| - \|\alpha\| = \int_X |\alpha + t\beta|^{1/r} - |\alpha|^{1/r}.$$

For a zero of α, say P, write

$$\alpha(u) = a(u)(du)^r = \left(a_n u^n + a_{n+1}u^{n+1} + \cdots\right)(du)^r, \ a_n \neq 0$$
$$\beta(u) = b(u)(du)^r = \left(b_m u^m + b_{m+1}u^{m+1} + \cdots\right)(du)^r, \ b_m \neq 0$$

locally in a neighborhood of P (with $P = 0$). Take $\varepsilon > 0$ small enough so that the expressions converge in the disc $B_P(\varepsilon)$ of radius ε. Then we take ε smaller so that

$$|a_n| > |a_{n+k}u^k|, \quad |b_m| > |b_{m+k}u^k| \tag{2.1}$$

for all $k \geq 1$ and $u \in B_P(\varepsilon)$ and that $B_P(\varepsilon)$ are pairwise disjoint for all zero P. Then we take t small enough so that $|\alpha + t\beta| = |\alpha|$ outside these P's neighborhoods.

For $|t| \ll 1$, we get

$$\|\alpha + t\beta\| - \|\alpha\| = \int_X |\alpha + t\beta|^{1/r} - |\alpha|^{1/r} = \sum_i \int_{B_{P_i}(\varepsilon)} |\alpha + t\beta|^{1/r} - |\alpha|^{1/r}. \tag{2.2}$$

Thus, for (i) and (ii) it suffices to show that for all zero P of α,

$$\int_{B_P(\varepsilon)} |\alpha + t\beta|^{1/r} - |\alpha|^{1/r} = \begin{cases} 0, & \text{if } n = 1, \\ O(|t|^{1/n+1/r}), & \text{if } n > 1, \end{cases}$$

where $n = \mathrm{mult}_P(\alpha)$.

When $n = 1$, we observe that

$$\frac{d}{du}\left(u + t \cdot \frac{b(u)}{a(u)/u}\right) \in \mathcal{O}^\times$$

for $|t| \ll 1$ and $u \in B_0(\varepsilon)$. Then there is a measure preserving K-bianalytic map

$$u \mapsto u + t \cdot \frac{b(u)}{a(u)/u}$$

from $B_P(\varepsilon)$ to

$$t \cdot \frac{b_m}{a_1} B_P(\varepsilon^m) + B_P(\varepsilon) = B_P(\varepsilon)$$

such that

$$\begin{aligned} \int_{B_P(\varepsilon)} |\alpha + t\beta|^{1/r} - |\alpha|^{1/r} &= \int_{B_P(\varepsilon)} |\alpha + t\beta|^{1/r} - \int_{B_P(\varepsilon)} |\alpha|^{1/r} \\ &= \int_{B_P(\varepsilon)} |a_1 u|^{1/r} du - \int_{B_P(\varepsilon)} |a_1 u|^{1/r} du = 0. \end{aligned}$$

This proves (i).

When $n > 1$, it follows from (2.1) that in the disc $B_P(\varepsilon)$,

$$|\alpha(u)| = |a_n||u|^n \, (du)^r, \quad |\beta(u)| = |b_m||u|^m \, (du)^r.$$

If $m \geq n$, we get

$$|\alpha(u) + t\beta(u)|^{1/r} - |\alpha(u)|^{1/r} = 0 \quad \forall u \in B_P(\varepsilon)$$

for $|t| < |a_n|/|b_m|$. If $m < n$, let

$$\delta = \left(\frac{|t||b_m|}{|a_n|}\right)^{1/(n-m)},$$

we get

$$\frac{|(\alpha+t\beta)(u)|^{1/r}-|\alpha(u)|^{1/r}}{du}=\begin{cases}0, & \text{if } |u|>\delta\\ (|t||b_m||u|^m)^{1/r}-(|a_n||u|^n)^{1/r}, & \text{if } |u|<\delta.\end{cases}$$

Thus,

$$\begin{aligned}\int_{B_P(\varepsilon)} & |\alpha+t\beta|^{1/r}-|\alpha|^{1/r}\\ &=\underbrace{\int_{|u|=\delta}|\alpha+t\beta|^{1/r}-|\alpha|^{1/r}}_{A}+\underbrace{\int_{|u|<\delta}(|t||b_m||u|^m)^{1/r}-(|a_n||u|^n)^{1/r}\,du}_{B}.\end{aligned}$$

Let $q^{-(s+1)}<\delta\leq q^{-s}$. We get the equation

$$\begin{aligned}B&=\int_{|u|<\delta}(|t||b_m||u|^m)^{1/r}-(|a_n||u|^n)^{1/r}\,du\\ &=|t|^{1/r}|b_m|^{1/r}\frac{(q-1)q^{-(s+1)(1+m/r)}}{q-q^{-m/r}}-|a_n|^{1/r}\frac{(q-1)q^{-(s+1)(1+n/r)}}{q-q^{-n/r}}.\end{aligned}$$

From the inequality

$$\left(\frac{|t||b_m|}{|a_n|}\right)^{1/(n-m)}\leq q^{-s}<q\cdot\left(\frac{|t||b_m|}{|a_n|}\right)^{1/(n-m)},$$

we see that

$$B=O(|t|^{(1+n/r)/(n-m)})\subseteq O(|t|^{1/n+1/r}).$$

It is much harder to compute the integral

$$A=\int_{|u|=\delta}|\alpha(u)+t\beta(u)|^{1/r}-|\alpha(u)|^{1/r},$$

but it is obvious from the inequality

$$0\geq A\geq-\int_{|u|=\delta}|\alpha(u)|^{1/r}=(q^{-1}-1)\left(\frac{(|t||b_m|)^{(1+n/r)/(n-m)}}{|a_n|^{(1+m/r)/(n-m)}}\right)$$

that $A=O(|t|^{(1+n/r)/(n-m)})$. Finally, we get

$$\int_{B_P(\varepsilon)}|\alpha+t\beta|^{1/2}-|\alpha|^{1/2}=A+B=O(|t|^{(1+n/r)/(n-m)})\subseteq O(|t|^{1/n+1/r}),$$

which implies (ii).

To get the lower bound, i.e.,

$$\|\alpha+t\beta\|-\|\alpha\|\neq O(|t|^{\rho})\quad\forall\,\rho>\frac{1}{n}+\frac{1}{r},$$

we make the integration "smoother". First, note that

$$\|\alpha+t\beta\|-\|\alpha\|=O(|t|^{\rho}) \implies \fint_{v(t)\geq k}\|\alpha+t\beta\|-\|\alpha\|\,dt=O(q^{-\rho k}). \tag{2.3}$$

So it suffices to show that the coefficient of the average integral

$$I(k):=q^{k(1/N+1/r)}\fint_{v(t)\geq k}\|\alpha+t\beta\|-\|\alpha\|\,dt$$

does not tend to 0 as $k\to\infty$.

According to (2.2), for $k\gg 1$, we know that

$$\fint_{v(t)\geq k}\|\alpha+t\beta\|-\|\alpha\|\,dt=\sum_i\fint_{v(t)\geq k}\int_{B_{P_i}(\varepsilon)}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}du\,dt.$$

For those i such that $n_i<N$ or $\beta(P_i)=0$, the difference n_i-m_i is strictly less than N. The estimates in (i) and (ii) show that

$$\int_{B_{P_i}(\varepsilon)}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}du=\begin{cases}O\left(|t|^{(1+n_i/r)/(n_i-m_i)}\right), & \text{if } m_i<n_i,\\ 0, & \text{if } m_i\geq n_i\end{cases}$$

as $|t|\to 0$.

So it follows from (2.3) that

$$q^{k(1/N+1/r)}\fint_{v(t)\geq k}\int_{B_{P_i}(\varepsilon)}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}dudt\to 0$$

as $k\to\infty$ since

$$\frac{1}{N}+\frac{1}{r}-\frac{1+n_i/r}{n_i-m_i}=-\left(\frac{1}{n_i-m_i}-\frac{1}{N}\right)-\frac{1}{r}\frac{m_i}{n_i-m_i}<0$$

when $m_i<n_i$. The inequality above also gives the "only if" part of (iii).

For those i such that $n_i=N$ and $\beta(P_i)\neq 0$, by Fubini's theorem,

$$\begin{aligned}&\fint_{v(t)\geq k}\int_{B_{P_i}(\varepsilon)}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}du\,dt\\ &=\int_{B_{P_i}(\varepsilon)}\left(\fint_{v(t)\geq k}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}dt\right)du.\end{aligned} \tag{2.4}$$

For each $u\in B_{P_i}(\varepsilon)$, we compute the integral as follows:

$$\begin{aligned}&\fint_{v(t)\geq k}|a_i(u)+tb_i(u)|^{1/r}-|a_i(u)|^{1/r}dt\\ &=\begin{cases}0, & \text{if } q^{-k}|b_i(u)|<|a_i(u)|^N\\ \lambda\cdot q^{-k/r}|b_i(u)|^{1/r}-|a_i(u)|^{1/r}, & \text{if } q^{-k}|b_i(u)|\geq|a_i(u)|^N,\end{cases}\end{aligned}$$

where

$$\lambda = \int_{\mathcal{O}} |t|^{1/r} dt = \frac{1-q^{-1}}{1-q^{-(1+1/r)}}.$$

Note that we used the measure preserving K-bianalytic map

$$t \mapsto t - \frac{a_i(u)}{b_i(u)}$$

for the second case $q^{-k}|b_i(u)| \geq |a_i(u)|^N$.

Since $|a_i(u)| = |a_{i,N}||u|^N$ and $|b_i(u)| = |b_{i,0}|$ on the disc $B_{P_i}(\varepsilon)$, the integral (2.4) is equal to

$$\int_{|u|\leq\delta_i(k)} \lambda \cdot q^{-k/r}|b_{i,0}|^{1/r} - |a_{i,N}|^{1/r}|u|^{N/r}du,$$

where $\delta_i(k) = \left(\frac{q^{-k}|b_{i,0}|}{|a_{i,N}|}\right)^{1/N}$. Let $y_i(k) = q^{\{(v(a_{i,N})-v(b_{i,0})-k)/N\}} \in [1,q)$ so that $\delta_i(k)/y_i(k)$ is an integer power of q. Then

$$\begin{aligned}
&\int_{|u|\leq\delta_i(k)} \lambda \cdot q^{-k/r}|b_{i,0}|^{1/r} - |a_{i,N}|^{1/r}|u|^{N/r}du \\
&= \left(\frac{\lambda}{y_i(k)} - \frac{1-q^{-1}}{y_i(k)^{1+N/r}(1-q^{-(1+N/r)})}\right)\left(\frac{(q^{-k}|b_{i,0}|)^{1/N+1/r}}{|a_{i,N}|^{1/N}}\right).
\end{aligned}$$

So we see that the coefficient is periodic with period N. The "average" of the coefficient is

$$\begin{aligned}
&\frac{1}{N}\sum_{j=0}^{N-1}\left(\frac{\lambda}{q^{j/N}} - \frac{1-q^{-1}}{q^{j(1/N+1/r)}(1-q^{-(1+N/r)})}\right)\cdot\frac{|b_{i,0}|^{1/N+1/r}}{|a_{i,N}|^{1/N}} \\
&= \frac{1}{N}\left(\frac{1-q^{-1}}{1-q^{-(1+1/r)}}\frac{1-q^{-1}}{1-q^{-1/N}} - \frac{1-q^{-1}}{1-q^{-(1/N+1/r)}}\right)\cdot\frac{|b_{i,0}|^{1/N+1/r}}{|a_{i,N}|^{1/N}} \\
&= \frac{1-q^{-1}}{N}\cdot\frac{(q^{-1/N}-q^{-1})(1-q^{-1/r})}{(1-q^{-(1+1/r)})(1-q^{-1/N})(1-q^{-(1/N+1/r)})}\cdot\frac{|b_{i,0}|^{1/N+1/r}}{|a_{i,N}|^{1/N}},
\end{aligned}$$

which is greater than 0 when $N > 1$. Summing over these i's we get

$$\lim_{k\to\infty}\sum_{j=k}^{k+N-1} I(j) > 0.$$

Hence, $I(k) \nrightarrow 0$ as $k \to \infty$. ■

Since rK_X is base-point-free when $g \geq 2$, this proposition shows that we can get the maximal zero order of α from the quasinorm $\|\cdot\|_K$. In the next section, we will give some ways to recover the curve from the maximal zero order of the r-differential forms.

Remark 2.4. We give an estimate of A when $v(tb_m) \equiv v(a_n) \pmod{(n-m)}$ requiring that $n - m < p$.

We know that, on $\{u \mid |u| = \delta\}$,

$$|\alpha(u) + t\beta(u)| \le |\alpha(u)|,$$

and the inequality is strict if and only if

$$0 \equiv \frac{\alpha(u) + t\beta(u)}{du^r} \equiv a_n u^n + tb_m u^m \pmod{\mathfrak{m}^{v(tb_m\delta^m)+1}},$$

which is equivalent to

$$\left(\frac{u}{\pi^s}\right)^{(n-m)} \equiv x := -\frac{tb_m}{a_n}\pi^{-s(n-m)} \pmod{\mathfrak{m}}.$$

Since $n - m < p$, by Hensel's lemma, for each $z \in \mathfrak{m}$ there exists a unique u with norm δ such that

$$\left(\frac{u}{\pi^s}\right)^{(n-m)} = x + z = z - \frac{tb_m}{a_n} \cdot \pi^{-s(n-m)} \iff a_n u^n + tb_m u^m = z a_n \pi^{s(n-m)}.$$

Let $U = \#\{u_0 \in \mathbb{F}_q \mid u_0^{n-m} = x\}$. We obtain

$$\begin{aligned}
A = U \cdot \frac{\delta}{q} \cdot |a_n|^{1/r} \delta^{n/r} \Bigg[&(1 - q^{-1})(q^{-1/r} - 1) + (q^{-1} - q^{-2})(q^{-2/r} - 1) + \cdots \\
&+ O\left(\max_{k \ge 1} \max \left\{ \left| \frac{a_{n+k}}{a_n} \right|, \left| \frac{b_{m+k}}{b_m} \right| \right\} \delta^k \right) \Bigg] \\
= U \cdot |a_n|^{1/2} \delta^{1+n/2} &\left(\frac{1 - q^{1/r}}{q^{1+1/r} - 1} \right) (1 + O(\delta)), \\
= U \cdot \left(\frac{1 - q^{1/r}}{q^{1+1/r} - 1} \right) &(1 + O(|t|^{1/(n-m)})) \left(\frac{(|t||b_m|)^{(1+n/r)/(n-m)}}{|a_n|^{(1+m/r)/(n-m)}} \right).
\end{aligned}$$

3 Dual r-canonical curve and dual of r-canonical curve

We want to cover the curve using the data given from the estimate Proposition 2.3, so we need to see how to use the vector space $V(K)$.

Fix an algebraic closure Ω of K (with a fixed embedding $K \hookrightarrow \Omega$). Let C be a curve over Ω of genus $g \ge 2$ and let $V = H^0(C, rK_C)$. We have the r-canonical map

$$\phi_r = |rK_C| : C \to \mathbb{P}(V)^\vee,$$

which is an embedding when

(i) $r \ge 3$,

(ii) $r = 2$ and $g \geq 3$, or

(iii) $r = 1$ and C is non-hyperelliptic.

Let C_r be the image of ϕ_r in $\mathbb{P}(V)^\vee$. We define the dual r-canonical map

$$\psi_r : C \to \mathbb{P}(V)$$

by sending $P \in C$ to the osculating hyperplane $H_P \in \mathbb{P}(V)$ of C_r at $\phi_r(P)$. Explicitly, write $V = \langle \alpha_1, \dots, \alpha_m \rangle_\Omega$, where

$$m = \dim V = \begin{cases} (2r-1)(g-1), & \text{if } r > 1, \\ g, & \text{if } r = 1, \end{cases}$$

we get

$$\phi_r(P) = [\alpha_1(P) : \dots : \alpha_m(P)] \in \mathbb{P}(V).$$

The osculating hyperplane H_P is

$$\left[\langle \phi_r(P), \phi_r'(P), \dots, \phi_r^{(k)}(P) \rangle_\Omega\right] = \ker \begin{pmatrix} \alpha_1(P) & \cdots & \alpha_m(P) \\ \vdots & \ddots & \vdots \\ \alpha_1^{(k)}(P) & \cdots & \alpha_m^{(k)}(P) \end{pmatrix} \in \mathbb{P}(V)$$

for some k. If we take α so that $\operatorname{mult}_P(\alpha)$ is maximal (which is unique up to a scalar, otherwise $h^0(rK_C - (\operatorname{mult}_P(\alpha) + 1)P) \geq 2 - 1 = 1$), then $\alpha(P) = \dots = \alpha^{(k)}(P) = 0$ and we get $H_P = [\alpha]$.

Theorem 3.1. Let C be an algebraic curve of genus g over a characteristic 0 field, and let Q be a g_d^r on C, i.e., a linear system on C of degree d, with $r = \dim Q$. Then

$$\sum_{P \in C} w_P(Q) = (r+1)(d + rg - r),$$

where

$$w_P(Q) = \sum_{i=1}^{r+1} (n_i - i),$$

and $n_1 < n_2 < \dots < n_{r+1}$ denote the gap numbers. In particular, there are only finitely many $P \in C$ such that $Q(-(r+1)P) \neq \varnothing$.

The proof of the theorem may be found in [9]. Applying this to the case $Q = |rK_C| = g_{r(2g-2)}^{m-1}$, we see that, for generic $P \in C$, the linear system $|rK_C - mP|$ is empty, and hence $\operatorname{mult}_P(\psi_r(P)) = m - 1$.

Definition 3.2. We say that $P \in C$ is an r-Weierstrass point if $\mathrm{mult}_P(\psi_r(P)) \geq m$.

Consider the set of r-Weierstrass points

$$W = \{P \in C \mid \mathrm{mult}_P(\psi_r(P)) \geq m\}.$$

For $P \in W$, let

$$L_P = \{[\alpha] \in \mathbb{P}(V) \mid \mathrm{mult}_P(\alpha) \geq m-1\}.$$

Then we see that $\psi_r(P) \in L_P$.

Lemma 3.3. The dual r-canonical map $\psi_r : C \to \mathbb{P}(V)$ is injective when

(i) $r \geq 3$, or

(ii) $r = 2$ and $g \geq 3$.

Proof. Since

$$h^0(rK_C - (m-1)P) \geq m - (m-1) = 1,$$

we have $\mathrm{mult}_P(\alpha) \geq m - 1 = 2rg - 2r - g$. It follows from

$$2(2rg - 2r - g) > r(2g-2) = \deg rK_C \iff r > 2 \text{ or } g > 2$$

that ψ_r is injective. ■

As in the original complex case for bi-canonical curves treated in [3, Lemma 4], we need the following slight extensions to the r-canonical case.

Lemma 3.4. The dual r-canonical curve $\psi_r(C)$ is not contained in any hyperplane of $\mathbb{P}(V)$.

Proof. Let H be a hyperplane of $\mathbb{P}(V)$. Then H is a $g^{m-2}_{r(2g-2)}$, so Theorem 3.1 shows that there are only finitely many points of C at which there is an $[\alpha] \in H$ with a zero of order at least $m-1$. Thus the dual r-canonical curve is not contained in H. ■

Now, we introduce the dual of the r-canonical curve $C_r^\vee$. For an irreducible projective variety $Y \subseteq \mathbb{P}^N$, we define its dual variety by

$$Y^\vee := \{H \in (\mathbb{P}^N)^\vee \mid H \text{ tangents to } Y\}.$$

Note that $Y^\vee$ is also irreducible since it is the image of the incidence correspondence

$$I = \{(P, H) \mid T_P Y \subseteq H\} \subseteq Y \times (\mathbb{P}^N)^\vee$$

under the projection $Y \times (\mathbb{P}^N)^\vee \to (\mathbb{P}^N)^\vee$. We have the following reflexivity theorem [10].

Theorem 3.5. For an irreducible variety $Y \subseteq \mathbb{P}^N$, we have $(Y^\vee)^\vee = Y$.

Applying this theorem to the case $Y = C_r$, we see that $(C_r^\vee)^\vee = C_r$. Thus, we can recover C by $C_r^\vee$ whenever ϕ_r is an embedding.

In the next section, we shall use these curves in the projective space $\mathbb{P}(V)$ to recover the original curve C.

4 Proof of the main result

As in section 3, we fix an algebraic closure Ω of K. For a smooth curve X over $\mathcal{O}$, we define X_Ω to be the geometric fiber of X over Ω, which is a smooth curve over Ω. Accordingly, we may apply the results in section 3.

For each $k \geq 2$, let

$$S_k = \{[\alpha] \in \mathbb{P}(V) \mid \exists P \in X_\Omega \text{ such that } \operatorname{mult}_P(\alpha) \geq k\} \subseteq \mathbb{P}(V).$$

From Proposition 2.3 we see that the K-rational set $S_k(K)$ contains the set $S_{k,K}$ of those $[\alpha]$ for which there is $\beta \in V(K)$ so that

$$\|\alpha + t\beta\|_K - \|\alpha\|_K$$

is not $O(|t|^\rho)$ for any $\rho > \frac{1}{k} + \frac{1}{r}$ since $|rK_X|$ is base-point-free for $g \geq 2$. This shows that $S_{k,K}$ is determined by the quasinorm $\|\cdot\|_K$.

Take $k = m - 1$. For $r \geq 2$, it is clear that

$$S_{m-1} = \psi_r(X_K) \cup \bigcup_{P \in W} L_P.$$

Define $\theta : S_{m-1} \dashrightarrow X_\Omega$ by sending $[\alpha] \in S_{m-1}$ to the point $P \in X_\Omega$ such that $\operatorname{mult}_P(\alpha) \geq m - 1$ (if it is unique). Then $\theta \circ \psi_r = \mathrm{id}_{X_K}$ and any fiber $\theta^{-1}(P)$ is always a linear space (of dimension ≥ 1 if and only if $P \in W$).

Take $k = 2$. We see from the definition that

$$S_2 = \{H \in \mathbb{P}(V) \mid H \text{ tangents to } C_r\} = C_r^\vee,$$

where C_r is the image of $|rK_X| : X_\Omega \to \mathbb{P}(V)^\vee$.

With these results, we are ready to prove the following proposition.

Proposition 4.1. Let X and X' be smooth projective curves over $\mathcal{O}$ of genus $g \geq 2$, r a positive integer, V, V' the spaces $H^0(X, rK_X)$, $H^0(X', rK_{X'})$, respectively, and let

$$\Phi : (V(K), \|\cdot\|_K) \to (V'(K), \|\cdot\|'_K)$$

be an isometry. Suppose that $q \geq 4g^2$ and the r-canonical maps

$$|rK_X| : X \to \mathbb{P}(V)^\vee \quad \text{and} \quad |rK_{X'}| : X' \to \mathbb{P}(V')^\vee$$

are embeddings. Then there is an isomorphism $\varphi_K : X'_K \to X_K$ and some $u \in \mathcal{O}^\times$ such that $\Phi = u \cdot \varphi_K^*$.

Proof. From $q \geq 4g^2$ we get $q + 1 > 2g\sqrt{q}$. It follows from the Riemann hypothesis for curves [11] that $X(\mathbb{F}_q)$ are nonempty. Consider the mod-$\mathfrak{m}$ reduction

$$h : X(\mathcal{O}) \to X(\mathbb{F}_q).$$

For any $\overline{x} \in X(\mathbb{F}_q)$, the preimage $h^{-1}(\overline{x})$ of $\overline{x}$ in $X(\mathcal{O})$ is isomorphic to $\mathfrak{m}$ in K-analytic sense. Hence, $X(K) = X(\mathcal{O})$ (by the valuative criterion for properness) contains infinitely many points. Similarly, $X'(K)$ contains infinitely many points.

Since Φ is an isometry, $S_{k,K}$ sends to $S'_{k,K}$ under the projective linear map

$$\overline{\Phi} : \mathbb{P}(V(K)) \xrightarrow{\sim} \mathbb{P}(V'(K))$$

for each $k \geq 2$. Extend $\overline{\Phi}$ to $\overline{\Phi}_\Omega : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V')$, we may assume that S_k, S'_k are contained in the same projective space $\mathbb{P}^{m-1}$.

Take $k = m - 1$, we have

$$\left(C(K) \cup \bigcup_{P \in W} L_P(K) \right) \cap \left(C'(K) \cup \bigcup_{P' \in W'} L_{P'}(K) \right) \supseteq S_{m-1,K} = S'_{m-1,K},$$

where $C = \psi_r(X_\Omega)$, $C' = \psi'_r(X'_\Omega)$. It follows from Lemma 3.4 that

$$|C'(K) \cap L_P(K)| \leq |C' \cap L_P| < \infty.$$

Then $|X'(K)| = \infty$ gives $|S'_{m-1,K}| = \infty$, and hence

$$|C \cap C'| \geq |C(K) \cap C'(K)| = \infty.$$

Thus $C = C'$ since they are both irreducible curves. Therefore,

$$\varphi_\Omega = \theta \circ \overline{\Phi}_\Omega^{-1} \circ \psi'_r : X'_\Omega \to X_\Omega$$

is an isomorphism. But this method only works when the dual r-canonical map is (generically) injective.

Take $k = 2$. By Theorem 3.5, it suffices to show that $S_2 = S'_2$. We have

$$S_2 \cap S'_2 \supseteq S_{2,K} = S'_{2,K}.$$

So it suffices to show that the Zariski closure of $S_{2,K}$ in $\mathbb{P}^{m-1}$ is S_2. Suppose not, the irreducibility of S_2 shows that there exists a codimension 1 reduced closed subscheme Y containing S_2.

Claim. The set $S_{2,K}$ has positive measure in $S_2(K)$.

Proof of Claim. First, we see that C_r contains only finitely many inflections. Suppose not, then $h^0(rK - 3P) \geq m - 2$ for infinitely many $P \in X$. So the linear system $|rK_X - mP|$ is nonempty for infinitely many $P \in X$. Together with Theorem 3.1, we see that

$$m < \dim |rK| + 1 = m,$$

a contradiction.

Let $U \cong \mathcal{O}$ be a local chart (with coordinate u) on $C_r(K)$ such that the U contains no inflection points (we need $X(\mathcal{O}) \neq \varnothing$ here). Then we get a map

$$\begin{array}{cccc} \eta : & \mathbb{P}\left(N^*_{C_r(K)/(\mathbb{P}^{m-1}(K))^\vee}\right)\Big|_U & \to & S_{2,K} \\ & (P, [\omega]) & \mapsto & [\omega^{-1}(0)], \end{array}$$

where $N^*_{C_r(K)/(\mathbb{P}^{m-1}(K))^\vee}$ is the conormal bundle of $C_r(K)$ in $(\mathbb{P}^{m-1})^\vee(K)$. Write the embedding $U \subseteq C_r(K) \to \mathbb{P}^{m-1}(K)$ locally by

$$u \mapsto \gamma(u) = (\gamma^1(u), \ldots, \gamma^{m-1}(u)) \in \mathbb{A}^{m-1} \subset \mathbb{P}^{m-1}, \quad \gamma(0) = 0.$$

Then the map η can be written explicitly in coordinate by

$$\left(u, \left[a_k dx^k\right]\right) \mapsto \left[-a_k \gamma^k(u) : a_1 : \cdots : a_{m-1}\right], \quad \sum a_k (\gamma^k)'(u) = 0.$$

Suppose that $(\gamma^1)'(u) \neq 0$ near $u = 0$. We take a trivialization of the projective conormal bundle with affine coordinate $(u, a_2, \ldots, a_{m-2}, a_{m-1} = 1)$. Then

$$\eta\left(u, a_k dx^k\right) = \left(-a_k \gamma^k(u), a_1, \cdots, a_{m-2}\right).$$

By direct computation, we see that

$$\begin{pmatrix} \eta_u \\ \eta_{a_2} \\ \vdots \\ \eta_{a_{m-2}} \end{pmatrix}_{(u,a_2,\ldots,a_{m-2})=0} = \begin{pmatrix} 0 & a_{1,u}(0) & 0 & \cdots & 0 \\ 0 & a_{1,2}(0) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{1,m-2}(0) & 0 & \cdots & 1 \end{pmatrix},$$

which is full rank if and only if

$$a_{1,u}(0) = \frac{d}{du}\bigg|_{u=0} \frac{(\gamma^{m-1})'(u)}{(\gamma^1)'(u)} \neq 0.$$

Note that we can always take i, j such that

$$(\gamma^i)'(0) \neq 0 \quad \text{and} \quad \frac{d}{du}\bigg|_{u=0} \frac{(\gamma^j)'(u)}{(\gamma^i)'(u)} \neq 0,$$

otherwise we see that $\gamma''(0) \in K\gamma'(0)$, which contradicts the condition that U contains no inflection points. Thus, the set $S_{2,K}$ contains a positive measure subset. □

Note that $Y(K)$ has measure zero, this leads to a contradiction. So we see again that there is an isomorphism

$$\varphi_\Omega : X'_\Omega \to X_\Omega.$$

Finally, we see that the maps ϕ_r, ϕ'_r and φ_Ω above are defined over K. So we get an isomorphism $\varphi_K : X'_K \to X_K$. Since $\overline{\Phi}$ is the projectivization of φ^*_K, we get $\Phi = u \cdot \varphi^*_K$ for some $u \in K$. Then $|u| = 1$ since both Φ and φ^*_K are isometries. ■

Using the following theorem, stated in [12], we can prove that the isomorphism $\varphi_K : X'_K \to X_K$ lifts to an isomorphism $\varphi : X' \to X$ and gives (1.4). This is also an arithmetic version of the theorem stated in [13].

Theorem 4.2. Let $(R, \mathfrak{m})$ be a discrete valuation ring with the quotient field K; let Y and Y' be smooth projective varieties, defined over K, and T the graph of an isomorphism, defined over K, between Y and Y'. Let D (resp. D') be an ample divisor on Y (resp. Y'), both rational over K, such that $D' = T(D)$. Let

$$(Y, Y', D, D', T) \to (\overline{Y}, \overline{Y}', \overline{D}, \overline{D}', \overline{T})$$

be the mod $\mathfrak{m}$-reduction and assume that $\overline{Y}$, $\overline{Y}'$ are smooth and that $\overline{D}$ (resp. $\overline{D}'$) is also ample on $\overline{Y}$ (resp. $\overline{Y}'$). Then $\overline{T}$ is the graph of an isomorphism between $\overline{Y}$ and $\overline{Y}'$, if one of the $\overline{Y}$, $\overline{Y}'$ is not ruled.

As we can see in the proof of Proposition 4.1, the assumption $q \geq 4g^2$ may be replaced by that $X(K)$ and $X'(K)$ containing some K-analytic discs. On the other hand, recall that:

Definition 4.3. A regular projective curve Y over $S = \operatorname{Spec} \mathcal{O}$ is minimal if it satisfies the following condition:

> Every birational map between regular projective curves $Z \dashrightarrow Y$ over S is a birational morphism.

Definition 4.4 ([14, Sec. 10.1])**.** Let Z be a separated smooth algebraic variety over K. A Néron model of Z over $S = \operatorname{Spec} \mathcal{O}$ is a smooth separated scheme X over S such that $X_K \cong Z$ and satisfing the following universal property:

> For any smooth scheme Y over S, the canonical map
>
> $$\operatorname{Hom}_S(Y, X) \to \operatorname{Hom}_K(Y_K, X_K)$$
>
> is a bijection.

If we only assume that X_K, X'_K are smooth over K, the isometry

$$\Phi : (V(K), \|\cdot\|_K) \to (V'(K), \|\cdot\|'_K)$$

still gives us an isomorphism $\varphi_K : X'_K \to X_K$. In this case we can extend the isomorphism to the Néron models of X_K and X'_K, where the existence of Néron model is due to the following theorem:

Theorem 4.5. Let C be a proper smooth connected curve of positive genus over K. Then the smooth locus $C_{S,\mathrm{sm}}$ of the minimal proper regular model of C over S is the Néron model of C.

Hence, we extend Theorem 1.4 to the following theorem:

Theorem 4.6. Let X and X' be projective curves over $\mathcal{O}$ such that X_K and X'_K are smooth curves over K of genus $g \geq 2$, r a positive integer, and V, V' the spaces $H^0(X, rK_X)$, $H^0(X', rK_{X'})$ respectively. Let

$$\Phi : (V(K), \|\cdot\|_K) \to (V'(K), \|\cdot\|'_K)$$

be an isometry. Suppose that $X(K)$ and $X'(K)$ contains some K-analytic discs and the r-canonical maps

$$|rK_X| : X \to \mathbb{P}(V)^\vee \quad \text{and} \quad |rK_{X'}| : X' \to \mathbb{P}(V')^\vee$$

are embeddings. Then there exist an isomorphism φ between the Néron models $X_{\mathcal{N}}$ and $X'_{\mathcal{N}}$ (of X_K and X'_K, respectively) and some $u \in \mathcal{O}^\times$ such that $\Phi = u \cdot \varphi_K^*$.

Remark 4.7. The proof for the curve case above only works when the r-canonical is an embedding. For $r = g = 2$, the dual r-canonical curve would be a $\mathbb{P}^1$ with 6 Weierstrass point on it (in the algebraically closed field Ω). Since K is not algebraically closed, the Weierstrass points may not be K-rational. A way to solve this is to take an extension $K \subset L$, so that the Weierstrass points are L-rational and try to compare the quasinorms $\|\cdot\|_K$ and $\|\cdot\|_L$. If we can compare the quasinorms $\|\cdot\|_K$ and $\|\cdot\|_L$, we can also take an extension so that the condition $q > 4g^2$ is satisfied.

Appendix: Higher dimensional case

Let X be a smooth projective scheme over $\mathcal{O}$ of relative dimension n. Recall that we have defined the quasinorm $\|\cdot\|_K$ on the vector space $V = H^0(X, rK_X)$ via the p-adic integral:

$$\|\alpha\|_K = \left(\int_X |\alpha|^{1/r}\right)^r.$$

We require that $X(K)$ to be nonempty, which is equivalent to $X(\mathbb{F}_q) \neq \varnothing$. This condition can be achieved by using the Weil conjectures [15]:

$$\#X(\mathbb{F}_q) \geq q^n + 1 - \sum_{i=1}^{2n-1} h^{2n-i} q^{(2n-i)/2},$$

with some bounds on q and the Betti numbers. For example,

$$q \geq \left(\sum_{i=1}^{2n-1} h^i\right)^2$$

will do.

Since $X(K)$ is nonempty, it contains a K-analytic open subset that is homeomorphic to a polydisc $\pi\mathcal{O}^n$, which is of positive measure. Hence, we see that the K-rational points $X(K)$ is Zariski dense in $X(\Omega)$ since X is an irreducible variety of dimension n. Note that this argument also works for any irreducible variety with nonempty smooth $\mathbb{F}_q$-rational points.

Suppose that X' is another smooth projective scheme over $\mathcal{O}$ of relative dimension n and there is a birational map $\varphi : X'_K \dashrightarrow X_K$. Then the smoothness of X' shows that there is an open set $U \subset X'_K$ on which $\varphi|_U : U \to X_K$ is a birational morphism with $\mathrm{codim}_{X'_K}(X'_K \setminus U) \geq 2$. We see that

$$\varphi^* : H^0(X, rK_X) \to H^0(U, rK_U) \cong H^0(X', rK_{X'})$$

is an isometry by change of variables. This means that we may always replace X by its birational model if we want.

As in the curve case, we are going to give an estimate on

$$\|\alpha + t\beta\|_K - \|\alpha\|_K.$$

For simplicity, we do the estimation on

$$\|\alpha + t\beta\| - \|\alpha\| = \int_X |\alpha + t\beta|^{1/r} - |\alpha|^{1/r},$$

where $\|\cdot\| = \|\cdot\|_K^{1/r}$.

In [7], which is a generalization of Royden's result for complex projective manifolds, there is a fundamental estimate:

> For multi-indices $A \in \mathbb{Z}^n_{\geq 0}$, $B \in \mathbb{R}^n_{\geq 0}$, holomorphic function h on the closed polydisc $\overline{\Delta}_0$, and smooth function χ on $\overline{\Delta}_0$, let
>
> $$\ell_j = \begin{cases} \frac{b_j+1}{a_j}, & \text{if } a_j > 0, \\ \infty, & \text{if } a_j = 0, \end{cases} \qquad \ell = \min_j \ell_j, \quad \mu = \#\{j \mid \ell_j = \ell\},$$
>
> $$\Psi(t) = \int_{\overline{\Delta}_0} \chi(z)|z^A + th(z)|^{2/r}|z|^{2B} dz.$$
>
> If $2\ell + \frac{2}{r} \geq 1$, then
>
> $$\Psi(t) - \Psi(0) = O\left(|t|(-\log|t|)^\mu\right).$$
>
> If $2\ell + \frac{2}{r} < 1$, then
>
> $$\Psi(t) - \Psi(0) = c|t|^{2\ell+\frac{2}{r}}(-\log|t|)^{\mu-1} + o\left(|t|^{2\ell+\frac{2}{r}}(-\log|t|)^{\mu-1}\right).$$
>
> Moreover, if $\chi(0) \neq 0$, then $c = 0$ if and only if $h(z) = 0$ for all z such that $z_j = 0$ for all $\ell_j = \ell$.

Using this fundamental estimate, Chi gets some estimation of the pseudonorm defined on $H^0(M, rK_M)$ and recover the projective manifold M by these data.

We want to find a similar estimate over the p-adic field K, say

$$\Psi(t) = \int_{\pi\mathcal{O}^n} |u^A + th(u)|^{1/r} |u|^B du,$$

where $h(u) \in \mathcal{O}[[u]]$. Same as above, we define

$$\ell_j = \begin{cases} \frac{b_j+1}{a_j}, & \text{if } a_j > 0, \\ \infty, & \text{if } a_j = 0, \end{cases} \qquad \ell = \min_j \ell_j, \qquad \mu = \#\{j \mid \ell_j = \ell\}.$$

In fact, we have the following upper bound.

Proposition A.1. We have

$$\Psi(t) - \Psi(0) = O(|t|^{\ell+1/r} v(t)^{\mu-1}) \qquad \text{as } |t| \to 0.$$

We start with the following lemma:

Lemma A.2. Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$ and $C = (c_1, \dots, c_n) \in \mathbb{R}^n_{>0}$ be multi-indices. Define

$$\ell_j = \begin{cases} \frac{c_j}{a_j}, & \text{if } a_j > 0 \\ \infty, & \text{if } a_j = 0 \end{cases}, \quad \ell = \min_j \ell_j \quad \text{and} \quad \mu = \#\{j \mid \ell_j = \ell\}.$$

Then

$$\sum_{\substack{A\cdot\vec{v} \geq k, \\ \vec{v} \in \mathbb{N}^n}} q^{-C\cdot\vec{v}} \sim q^{-k\ell} k^{\mu-1},$$

i.e., there exists some positive constants C_1, C_2 such that

$$C_1 \cdot q^{-k\ell} |k|^{\mu-1} \leq \sum_{\substack{A\cdot\vec{v} \geq k, \\ \vec{v} \in \mathbb{N}^n}} q^{-C\cdot\vec{v}} \leq C_2 \cdot q^{-k\ell} |k|^{\mu-1}$$

for $k \gg 1$.

Proof. Note that for each $\vec{v} \in \mathbb{N}^n$,

$$\int_{|w_j - v_j| \leq \frac{1}{2}} q^{-C\cdot\vec{w}} d\vec{w} \sim q^{-C\cdot\vec{v}}.$$

So we can estimate the sum by the integral

$$\sum_{A\cdot\vec{v}\geq k}\int_{|w_j-v_j|\leq\frac{1}{2}} q^{-C\cdot\vec{w}}d\vec{w}.$$

Let $a=\sum a_j$ and $c=\sum c_j$. For each κ, we define

$$\tilde{S}_\kappa=\left\{\vec{w}\in\mathbb{R}^n \;\middle|\; A\cdot\vec{w}\geq\kappa,\ w_j\geq\frac{1}{2}\ \forall j\right\} \quad \text{and}$$
$$S_\kappa=\left\{\vec{w}\in\mathbb{R}^n \mid A\cdot\vec{w}\geq\kappa,\ w_j\geq 0\ \forall j\right\}.$$

We see that

$$\int_{\tilde{S}_{k-1+a/2}} q^{-C\cdot\vec{w}}d\vec{w}\leq\sum_{A\cdot\vec{v}\geq k}\int_{|w_j-v_j|\leq\frac{1}{2}} q^{-C\cdot\vec{w}}d\vec{w}\leq\int_{\tilde{S}_{k-a/2}} q^{-C\cdot\vec{w}}d\vec{w}.$$

By change of variables,

$$\int_{\tilde{S}_\kappa} q^{-C\cdot\vec{w}}d\vec{w}=q^{-c/2}\cdot\int_{S_\kappa} q^{-C\cdot\vec{w}}d\vec{w}.$$

So it suffices to estimate

$$\int_{S_\kappa} q^{-C\cdot\vec{w}}$$

for $\kappa\gg 0$.

Without loss of generality, we may assume that $\ell_1=\cdots=\ell_\mu<\ell_{\mu+1}\leq\cdots\leq\ell_n$. Now we estimate the integration by induction on n that

$$\int_{S_\kappa} q^{-C\cdot\vec{w}}\sim q^{-\ell\kappa}\kappa^{\mu-1}.$$

If $\ell_n=\infty$, i.e., $a_n=0$, then

$$\int_{S_\kappa} q^{-C\cdot\vec{w}}=\int_{S_\kappa^{(n)}} q^{-C^{(n)}\cdot\vec{w}^{(n)}}\cdot\int_0^\infty q^{-c_nw_n}dw_n=\frac{1}{c_n\log q}\int_{S_\kappa^{(n)}} q^{-C^{(n)}\cdot\vec{w}^{(n)}},$$

where the superscript (n) means that we omit the last term of the vector. Then the estimate follows from the induction hypothesis. So we may assume that $a_j>0$ for all j. It follows that

$$\begin{aligned}\int_{S_\kappa} q^{-C\cdot\vec{w}}d\vec{w}&=\int_{S_\kappa^{(1)}} q^{-C^{(1)}\cdot\vec{w}^{(1)}}d\vec{w}^{(1)}\cdot\int_0^\infty q^{-c_1w_1}dw_1\\&\quad+\int_{A^{(1)}\cdot\vec{w}^{(1)}\leq\kappa}\int_{\frac{1}{a_1}(\kappa-A^{(1)}\cdot\vec{w}^{(1)})}^\infty q^{-C^{(1)}\cdot\vec{w}^{(1)}}q^{-c_1w_1}dw_1\\&=\frac{1}{c_1\log q}\cdot\int_{S_\kappa^{(1)}} q^{-C^{(1)}\cdot\vec{w}^{(1)}}d\vec{w}^{(1)}\\&\quad+\frac{1}{c_1\log q}\cdot q^{-\ell\kappa}\int_{A^{(1)}\cdot\vec{w}^{(1)}\leq\kappa} q^{-(C^{(1)}-\ell_1A^{(1)})\cdot\vec{w}^{(1)}}d\vec{w}^{(1)},\end{aligned}$$

where the superscript (1) means that we omit the first term of the vector. The first term of the equation can be estimated by the induction hypothesis.

For the second term, we see that after change of coordinate $z_j = a_j w_j \geq 0$,

$$\int_{A^{(1)}\cdot\vec{w}^{(1)}\leq\kappa} q^{-(C^{(1)}-c_1 A^{(1)})\cdot\vec{w}^{(1)}} d\vec{w}^{(1)} = \frac{1}{\prod a_j}\int_{\sum z_j\leq\kappa} q^{-L\cdot\vec{z}} d\vec{z},$$

where $L = (\ell_2-\ell_1,\ldots,\ell_n-\ell_1) = (0,\ldots,0,\ell_{\mu+1}-\ell_1,\ldots,\ell_n-\ell_1)$. We have the obvious upper bound

$$\int_{\sum z_j\leq\kappa} q^{-L\cdot\vec{z}} d\vec{z} \leq \int_{z_j\in[0,\kappa]} q^{-L\cdot\vec{z}} d\vec{z} \leq \left(\prod_{j=\mu+1}^{n}\frac{1}{(\ell_j-\ell_1)\log q}\right)\kappa^{\mu-1} \sim \kappa^{\mu-1}.$$

Fix any $\varepsilon > 0$ and let

$$I_\kappa = \left\{\vec{z}\in\mathbb{R}^{n-1}_{\geq 0}\;\middle|\; z_j\in\left[0,\tfrac{\kappa-\varepsilon}{\mu-1}\right]\ \forall j\leq\mu \ \text{ and }\ z_j\in\left[0,\tfrac{\varepsilon}{n-\mu}\right]\ \forall j>\mu\right\}.$$

We get the lower bound

$$\int_{\sum z_j\leq\kappa} q^{-L\cdot\vec{z}} d\vec{z} \geq \int_{I_\kappa} q^{-L\cdot\vec{z}} d\vec{z} = \left(\prod_{j=\mu+1}^{n}\frac{1-q^{(\ell_j-\ell_1)\varepsilon/(n-\mu)}}{(\ell_j-\ell_1)\log q}\right)\left(\frac{\kappa-\varepsilon}{\mu-1}\right)^{\mu-1} \sim \kappa^{\mu-1}$$

for $\kappa > 2\varepsilon$. Thus,

$$\begin{aligned}\int_{S_\kappa} q^{-C\cdot\vec{w}} d\vec{w} &\sim \int_{S_\kappa^{(1)}} q^{-C^{(1)}\cdot\vec{w}^{(1)}} d\vec{w}^{(1)} + q^{-\ell\kappa}\int_{A^{(1)}\cdot\vec{w}^{(1)}\leq\kappa} q^{-(C^{(1)}-\ell_1 A^{(1)})\cdot\vec{w}^{(1)}} d\vec{w}^{(1)}\\ &\sim o(q^{-\ell\kappa}\kappa^{\mu-1}) + q^{-\ell\kappa}\kappa^{\mu-1}\\ &\sim q^{-\ell\kappa}\kappa^{\mu-1}.\end{aligned}$$

■

Now we prove Proposition A.1 by using Lemma A.2. For any $u \in \pi\mathcal{O}^n$, let $v_i = v(u_i)$ and $\vec{v} = (v_1,\ldots,v_n)$. We see that

$$|u^A| \leq |th(u)| \implies A\cdot\vec{v} \geq v(t) =: v_0.$$

Hence,

$$\Psi(t)-\Psi(0) = \sum_{A\cdot\vec{v}\geq v_0}\int_{v(u_i)=v_i}\left(|u^A+th(u)|^{1/r}-|u^A|^{1/r}\right)|u|^B du.$$

From

$$|u^A+th(u)| \leq \max\{|u^A|,|th(u)|\} \leq \max\{|u^A|,|t|\},$$

we get

$$-\sum_{A\cdot\vec{v}\geq v_0}\int_{v(u_i)=v_i}|u|^{A/r+B}du \leq \Psi(t)-\Psi(0) \leq |t|^{1/r}\sum_{A\cdot\vec{v}>v_0}\int_{v(u_i)=v_i}|u|^B du.$$

Let $\mathbb{1} = (1, \ldots, 1)$ be the "all-ones" vector. Using the definition of the p-adic integral, we can easily see that

$$\int_{v(u_i)=v_i} |u|^{A/r+B} du = (1-q^{-1})^n q^{-(A/r+B+\mathbb{1})\cdot\vec{v}},$$
$$\int_{v(u_i)=v_i} |u|^{B} du = (1-q^{-1})^n q^{-(B+\mathbb{1})\cdot\vec{v}},$$

Applying Lemma A.2 to the cases $C = A/r + B + \mathbb{1}$ and $C = B + \mathbb{1}$, respectively, we get

$$\sum_{A\cdot\vec{v}\geq v_0} \int_{v(u_i)=v_i} |u|^{A/r+B} du \sim |t|^{\ell+1/r}|v(t)|^{\mu-1},$$
$$|t|^{1/r} \sum_{A\cdot\vec{v}>v_0} \int_{v(u_i)=v_i} |u|^{B} du \sim |t|^{1/r}\cdot|t|^{\ell}|v(t)|^{\mu-1}. \qquad \blacksquare$$

To get the lower bound, i.e.,

$$\Psi(t) - \Psi(0) \neq O\left(|t|^{\ell+1/r}|v(t)|^{\mu-1-\varepsilon}\right)$$

for all $\varepsilon > 0$, as in the curve case, we make the integration "smoother".

For each $m \in \mathbb{Z}$, we define

$$\begin{aligned} I(m) &= \frac{q^{m(\ell+1/r)}}{|m|^{\mu-1}} \fint_{v(t)\geq m} \Psi(t) - \Psi(0)\, dt \\ &= \frac{q^{m(\ell+1/r)}}{|m|^{\mu-1}} \int_{\pi\mathcal{O}^n} \left(\fint_{v(t)\geq m} |u^A + th(u)|^{1/r} - |u^A|^{1/r}\, dt \right) |u|^B du \\ &= \frac{q^{m(\ell+1/r)}}{|m|^{\mu-1}} \int_{|u^A|\leq q^{-m}|h(u)|} \left(\lambda q^{-m/r}|h(u)|^{1/r} - |u^A|^{1/r}\right) |u|^B du, \end{aligned}$$

where

$$\lambda = \frac{1-q^{-1}}{1-q^{-(1+1/r)}},$$

to be the coefficient of the average integration. (For simplicity on the notations, we let

$$\frac{q^{m(\ell+1/r)}}{|m|^{\mu-1}} = 1$$

when $m = 0$.)

Note that

$$\Psi(t) - \Psi(0) = O\left(|t|^{\ell+1/r}|v(t)|^{\mu-1-\varepsilon}\right) \quad \text{as } |t| \to 0 \implies \lim_{m\to\infty} I(m) = 0.$$

So we need to find a condition such that

$$\lim_{m\to\infty} I(m)$$

does not exist or not equal to zero.

For a positive integer a, we define the function

$$\eta_a(m) = \begin{cases} 1, & \text{if } m = -(a-1), -(a-2), \dots, 0 \\ 0, & \text{else.} \end{cases}$$

Proposition A.3. If $\ell < 1$ and h doesn't vanish on the intersection

$$Z = \bigcap_{\ell_j = \ell} (x_j = 0) \subseteq \pi\mathcal{O}^n,$$

then there exists a positive integer a such that

$$\limsup_{m\to\infty} (I * \eta_a)(m) > 0 \quad \text{and} \quad \liminf_{m\to\infty} (I * \eta_a)(m) \geq 0,$$

where

$$(I * \eta_a)(m) = \sum_{k=-\infty}^{\infty} I(k) \cdot \eta_a(m-k) = \sum_{i=0}^{a-1} I(m+i).$$

Note that if the statement is true for some positive integer a, then it is true for any positive multiple of a.

Proof. We deal with the special case $|h| \equiv 1$ first. Let $I_{A,B} = I$ in this case. We see that

$$I_{A,B}(m) = \frac{q^{m\ell}}{|m|^{\mu-1}} \sum_{m' \geq m} \left(\lambda - q^{-(m'-m)/r}\right) J(m'), \tag{A.1}$$

where J is defined by

$$J(m') = \sum_{A\cdot\vec{v}=m'} q^{-(\mathbb{1}+B)\cdot\vec{v}}.$$

Let $C = (c_1, \dots, c_n) = \mathbb{1} + B \geq \mathbb{1}$. Without loss of generality, we assume that $\ell_1 = \ell$. Then

$$\begin{aligned} J(m'+a_1) &= \sum_{A\cdot\vec{v}=m'+a_1} q^{-C\cdot\vec{v}} \\ &= \sum_{A\cdot\vec{v}=m'} q^{-c_1-C\cdot\vec{v}} + \sum_{A^{(1)}\cdot\vec{v}^{(1)}=m'+a_1} q^{-C^{(1)}\cdot\vec{v}^{(1)}} \\ &= q^{-c_1} J(m') + o\left(q^{-m'\ell}|m'|^{\mu-1}\right) \end{aligned} \tag{A.2}$$

as $m' \to \infty$ by Lemma A.2. Thus, for $m' \geq m > 0$ and $k \geq 1$,

$$
\begin{aligned}
&\frac{q^{m\ell}}{|m|^{\mu-1}} \left| J(m'+ka_1) - q^{-kc_1} J(m') \right| \\
&\leq \frac{q^{m\ell}}{|m|^{\mu-1}} \sum_{j=1}^{k} q^{-(k-j)c_1} \left| J(m'+ja_1) - q^{-c_1} J(m'+(j-1)a_1) \right| \\
&\leq k \left(\frac{|m'+ka_1|}{|m|} \right)^{\mu-1} q^{-kc_1-(m'-m)\ell} \cdot o(1) \subseteq o\left(q^{-kc_1/2}\right).
\end{aligned}
$$

Take $a = a_1$. We see from (A.1), $c_1 = a_1\ell$, and the above estimate that

$$
\begin{aligned}
I_{A,B}(m) &= \frac{q^{m\ell}}{|m|^{\mu-1}} \sum_{m' \geq m} \left(\lambda - q^{-(m'-m)s} \right) J(m') \\
&= \frac{q^{m\ell}}{|m|^{\mu-1}} \sum_{j=0}^{a-1} \left(\frac{\lambda}{1-q^{-a\ell}} - \frac{q^{-j/r}}{1-q^{-a(\ell+1/r)}} \right) J(m+j) + o(1).
\end{aligned}
$$

So

$$
\begin{aligned}
(I_{A,B} * \eta_a)(m) &= \sum_{i=0}^{a-1} I_{A,B}(m+i) \\
&= \sum_{i=0}^{a-1} \frac{q^{(m+i)\ell}}{|m+i|^{\mu-1}} \sum_{j=0}^{a-1} \left(\frac{\lambda}{1-q^{-a\ell}} - \frac{q^{-j/r}}{1-q^{-a(\ell+1/r)}} \right) J(m+i+j) + o(1) \\
&= \frac{q^{m\ell}}{|m|^{\mu-1}} \left(\sum_{k=0}^{a-1} \sum_{i=0}^{k} \left(\frac{q^{i\ell}\lambda}{1-q^{-a\ell}} - \frac{q^{i(\ell+1/r)-k/r}}{1-q^{-a(\ell+1/r)}} \right) J(m+k) \right. \\
&\qquad \left. + \sum_{k=a}^{2a-2} \sum_{i=k-a+1}^{a-1} \left(\frac{q^{i\ell}\lambda}{1-q^{-a\ell}} - \frac{q^{i(\ell+1/r)-k/r}}{1-q^{-a(\ell+1/r)}} \right) J(m+k) \right) \\
&\quad + o(1).
\end{aligned} \tag{A.3}
$$

For each $0 \leq k \leq a-1$, the estimate (A.2) shows that the coefficient of $\frac{q^{m\ell}}{|m|^{\mu-1}} J(m+k)$ in the above equation (with error term $o(1)$) is

$$
\begin{aligned}
&\sum_{i=0}^{k} \left(\frac{q^{i\ell}\lambda}{1-q^{-a\ell}} - \frac{q^{i(\ell+1/r)-k/r}}{1-q^{-a(\ell+1/r)}} \right) + q^{-a\ell} \cdot \sum_{i=k+1}^{a-1} \left(\frac{q^{i\ell}\lambda}{1-q^{-a\ell}} - \frac{q^{i(\ell+1/r)-k/r}}{1-q^{-a(\ell+1/r)}} \right) \\
&= \frac{q^{k\ell}\lambda}{1-q^{-\ell}} - \frac{q^{k\ell}}{1-q^{-(\ell+1/r)}} \\
&= \frac{q^{k\ell}}{1-q^{-\ell}} \left(\frac{1-q^{-1}}{1-q^{-(1+1/r)}} - \frac{1-q^{-\ell}}{1-q^{-(\ell+1/r)}} \right),
\end{aligned} \tag{A.4}
$$

which is in $(-q^{a\ell}, q^{a\ell})$ and greater than 0 if $\ell < 1$.

By Lemma A.2, we see that

$$
0 < \limsup_{m \to \infty} \frac{q^{\ell m}}{m^{\mu-1}} J(m) < \infty, \tag{A.5}
$$

otherwise for any $\varepsilon > 0$,

$$J(m') < \varepsilon q^{-\ell m}(m')^{\mu-1}, \quad \forall m \gg 0.$$

Using integration by parts we get

$$\begin{aligned}\sum_{m' \geq m} J(m') &< \varepsilon \cdot \sum_{m' \geq m} q^{-\ell m'}(m')^{\mu-1} \\ &< \varepsilon \int_{m-1}^{\infty} q^{-\ell t} t^{\mu-1}\, dt \\ &< M\varepsilon \cdot q^{-\ell m} m^{\mu-1}, \quad \forall m \gg 0\end{aligned}$$

for some $M > 0$ independent of ε. Thus, we see from (A.3), (A.4) and (A.5) that

$$0 < \limsup_{m\to\infty} (I_{A,B} * \eta_a)(m) < \infty \quad \text{ and } \quad \liminf_{m\to\infty} (I_{A,B} * \eta_a)(m) \geq 0.$$

For the general case ($|h|$ nonconstant). We partition $\pi\mathcal{O}^n$ into the disjoint union

$$h^{-1}(0) \sqcup \bigsqcup_{v=0}^{\infty} |h|^{-1}(q^{-v}).$$

Note that each $|h|^{-1}(q^{-v})$ is open and compact, so it can be written as a disjoint union

$$|h|^{-1}(q^{-v}) = \bigsqcup_{i=1}^{r_v} \left(x_{v,i} + \pi^{k_{v,i}+1}\mathcal{O}^n\right).$$

Here we take $x_{v,i} = (x_{v,i,1}, \ldots, x_{v,i,n})$ that minimizes $|x_{v,i,j}|$ for each j. For each pair (v, i), we define the following quantities

$$a_{v,i,j} = \begin{cases} a_j, & \text{if } x_{v,i,j} \equiv 0 \pmod{\mathfrak{m}^{k_{v,i}+1}} \\ 0, & \text{if } x_{v,i,j} \not\equiv 0 \pmod{\mathfrak{m}^{k_{v,i}+1}}, \end{cases} \qquad a_{v,i} = \sum_j a_{v,i,j}, \quad A_{v,i} = (a_{v,i,1}, \ldots, a_{v,i,n}),$$

$$b_{v,i,j} = \begin{cases} b_j, & \text{if } x_{v,i,j} \equiv 0 \pmod{\mathfrak{m}^{k_{v,i}+1}} \\ 0, & \text{if } x_{v,i,j} \not\equiv 0 \pmod{\mathfrak{m}^{k_{v,i}+1}}, \end{cases} \qquad b_{v,i} = \sum_j b_{v,i,j}, \quad B_{v,i} = (b_{v,i,1}, \ldots, b_{v,i,n}),$$

$$\ell_{v,i,j} = \begin{cases} \frac{b_{v,i,j}+1}{a_{v,i,j}}, & \text{if } a_{v,i,j} \neq 0, \\ \infty, & \text{if } a_{v,i,j} \neq 0, \end{cases}, \quad \ell_{v,i} = \min_j \ell_{v,i,j}, \quad \mu_{v,i} = \#\{j \mid \ell_{v,i,j} = \ell_{v,i}\}.$$

Then $\ell_{v,i} \geq \ell$ and if $\ell_{v,i} = \ell$, $\mu_{v,i} \leq \mu$. The equality $(\ell_{v,i}, \mu_{v,i}) = (\ell, \mu)$ occurs if and only if

$$x_{v,i,j} \equiv 0 \pmod{\mathfrak{m}^{k_{v,i}+1}} \quad \forall\, \ell_j = \ell.$$

We define the integrals and the coefficients locally on the polydisc $x_{v,i} + \pi^{k_{v,i}+1}\mathcal{O}^n$:

$$\Psi_{v,i}(t) = \int_{x+\pi^{k_{v,i}}\mathcal{O}^n} |u^A + th(u)|^{1/r}|u|^B du,$$
$$I_{v,i}(m) = \frac{q^{m\ell_{v,i}}}{|m|^{\mu_{v,i}-1}} \fint_{v(t)\geq m} \Psi_{v,i}(t) - \Psi_{v,i}(0)\, dt.$$

Then from the constant case we see that

$$\Psi_{v,i}(t) = q^{-k_{v,i}n} \int_{\pi\mathcal{O}^n} \left|(x+\pi^{k_{v,i}}u)^A + th(x+\pi^{k_{v,i}}u)\right|^{1/r} |x+\pi^{k_{v,i}}u|^B du,$$
$$I_{v,i}(m) = q^{-k_{v,i}(n+a_{v,i}+b_{v,i})} \cdot |x_{v,i,j}|^{A-A_{v,i}+B-B_{v,i}} \cdot I_{A_{v,i},B_{v,i}}(m+v-k_{v,i}a_{v,i}).$$

Let $C_{v,i} = (A - A_{v,i}) + (B - B_{v,i}) \geq 0$. Then we see that

$$\begin{aligned} I(m) &= \sum_{v,i} \frac{q^{m(\ell-\ell_{v,i})}}{|m|^{\mu-\mu_{v,i}}} \cdot I_{v,i}(m) \\ &= \underbrace{\sum_{(\ell_{v,i},\mu_{v,i})=(\ell,\mu)} q^{-k_{v,i}(n+a_{v,i}+b_{v,i})} \cdot |x_{v,i,j}|^{C_{v,i}} \cdot I_{A_{v,i},B_{v,i}}(m+v-k_{v,i}a_{v,i})}_{\text{(I)}} \\ &\quad + \underbrace{\sum_{(\ell_{v,i},\mu_{v,i})\neq(\ell,\mu)} q^{-k_{v,i}(n+a_{v,i}+b_{v,i})} \cdot |x_{v,i,j}|^{C_{v,i}} \cdot \frac{q^{m(\ell-\ell_{v,i})}}{|m|^{\mu-\mu_{v,i}}} \cdot I_{A_{v,i},B_{v,i}}(m+v-k_{v,i}a_{v,i})}_{\text{(II)}}. \end{aligned}$$

Let $a = \prod_{j=1}^n a_j$. For (II) in the equation above. Note that the pair of the multi-indices $(A_{v,i}, B_{v,i})$ has only finitely many choices and each

$$|I_{A_{v,i},B_{v,i}} * \eta_a|$$

is bounded because

$$\limsup_{m\to\infty}(I_{A_{v,i},B_{v,i}} * \eta_a)(m) < \infty, \quad \liminf_{m\to\infty}(I_{A_{v,i},B_{v,i}} * \eta_a)(m) \geq 0$$

and

$$0 \leq I_{A_{v,i},B_{v,i}}(-m) \leq \frac{q^{-m\ell}\lambda}{|m|^{\mu-1}} \sum_{m'\geq 0} J(m') \leq \frac{q^{-m\ell}\lambda}{|m|^{\mu-1}} \left(\frac{1}{q-1}\right)^n$$

for $m > 0$. Using the fact

$$\sum_{v,i} q^{-k_{v,i}n} = 1,$$

we have, from the estimate of the constant case that

$$\begin{aligned}
&\limsup_{m\to\infty} \sum_{(\ell_{v,i},\mu_{v,i})\neq(\ell,\mu)} \left| q^{-k_{v,i}(n+a_{v,i}+b_{v,i})} \cdot \frac{q^{m(\ell-\ell_{v,i})}}{|m|^{\mu-\mu_{v,i}}} \cdot (I_{A_{v,i},B_{v,i}} * \eta_a)(m+v-k_{v,i}a_{v,i}) \right| \\
&\le \limsup_{m\to\infty} \frac{1}{|m|} \sup_{\substack{(A_{v,i},B_{v,i}) \\ (\ell_{v,i},\mu_{v,i})\neq(\ell,\mu)}} \left|(I_{A_{v,i},B_{v,i}} * \eta_a)(m+v-k_{v,i}a_{v,i})\right| \\
&= 0
\end{aligned}$$

For (I), the condition $h|_Z \not\equiv 0$ shows that there exists a pair (v,i) such that $(\ell_{v,i},\mu_{v,i}) = (\ell,\mu)$. Again, the pair of the multi-indices $(A_{v,i},B_{v,i})$ has only finitely many choices, so we only need to estimate each choice $(A' = (a'_1,\ldots,a'_n), B' = (b'_1,\ldots,b'_n))$, i.e.,

$$\sum_{s=0}^{\infty} \sum_{\substack{k_{v,i}=s \\ (A_{v,i},B_{v,i})=(A',B')}} q^{-s(n+a'+b')}|x_{v,i}|^{C'} I_{A',B'}(m+v-sa'), \tag{A.6}$$

where $a' = \sum_j a'_j$, $b' = \sum_j b'_j$ and $C' = A - A' + B - B' \ge 0$. Take (v_0,i_0) such that $(A_{v_0,i_0},B_{v_0,i_0}) = (A',B')$ and which minimizes $s_0 := k_{v_0,i_0}$. From the constant case above, we know that for any $\delta > 0$, there exists m_0 such that, for all $m \ge m_0$,

$$(I_{A',B'} * \eta_a)(m) > -\delta.$$

We have seen that the function $I_{A',B'} * \eta_a$ has a lower bound $-M$ for some $M > 0$. Hence, by

$$\sum_{v,i} q^{-k_{v,i}n} = 1$$

and

$$m + v - sa' < m_0 \implies s > \frac{m-m_0}{a'},$$

we get the inequality

$$\begin{aligned}
&\sum_{s=0}^{\infty} \sum_{k_{v,i}=s} q^{-s(n+a'+b')}|x_{v,i}|^{C'}(I_{A',B'} * \eta_a)(m+v-sa') \\
&\ge q^{-s_0(n+a'+b')}|x_{v_0,i_0}|^{C'}(I_{A',B'} * \eta_a)(m+v-s_0a') \\
&\quad - q^{-s_0(a'+b')}\delta - M \cdot q^{-(a'+b')(m-m_0)/a'}.
\end{aligned}$$

So the limit superior and the limit inferior of (A.6) are at least

$$q^{-s_0(n+a'+b')}|x_{v_0,i_0}|^{C'} \limsup_{m\to\infty} (I_{A',B'} * \eta_a)(m+v-s_0a') - q^{-s_0(a'+b')}\delta$$

and

$$q^{-s_0(n+a'+b')}|x_{v_0,i_0}|^{C'} \liminf_{m\to\infty} (I_{A',B'} * \eta_a)(m+v-s_0a') - q^{-s_0(a'+b')}\delta,$$

respectively.

Since $\delta > 0$ is arbitrary, we see that the limit superior and the limit inferior of (A.6) are in fact greater than 0 and at least 0, respectively. We conclude that

$$\limsup_{m\to\infty} (I * \eta_a)(m) > 0 \quad \text{and} \quad \liminf_{m\to\infty} (I * \eta_a)(m) \geq 0. \qquad \blacksquare$$

Conversely, we have:

Proposition A.4. If h vanishes on the intersection

$$Z = \bigcap_{\ell_j = \ell} (x_j = 0) \subseteq \pi\mathcal{O}^n,$$

then

$$\Psi(t) - \Psi(0) = O\left(|t|^{\ell+1/r} v(t)^{\mu-1-\varepsilon}\right) \qquad \text{as } |t| \to 0$$

for some $\varepsilon > 0$.

Proof. Since h vanishes on Z, we can write

$$h(u) = \sum_{\ell_j \neq \ell} h_j(u) \cdot u_j$$

for some $h_j \in \mathcal{O}[[u]]$. Then

$$|u^A| \leq |th(u)| \leq |t| \cdot \max_{\ell_j = \ell} |u_j| \implies A \cdot \vec{v} - \min_{\ell_j = \ell} v_j \geq v_0.$$

Let

$$S = \left\{ \vec{v} \;\middle|\; A \cdot \vec{v} - \min_{\ell_j = \ell} v_j \geq v_0 \right\},$$

then

$$\begin{aligned} |\Psi(t) - \Psi(0)| &\leq \sum_{\vec{v} \in S} \int_{v_i(u) = v_i} |t|^{1/r} |u|^B + |u|^{A/r+B} du \\ &\sim \sum_{\vec{v} \in S} |t|^{1/r} q^{-(B+\mathbb{1})\cdot\vec{v}} + q^{-(A/r+B+\mathbb{1})\cdot\vec{v}} \end{aligned}$$

and

$$S = S_0 \cup \bigcup_{\ell_j = \ell} S_j,$$

where

$$S_0 = \left\{ \vec{v} \in S \;\middle|\; \min_{\ell_j = \ell} v_j \geq \delta \log_q v_0 \right\}, \quad S_j = \left\{ \vec{v} \in S \;\middle|\; v_j < \delta \log_q v_0 \right\},$$

and $\delta > 0$ sufficiently small. For a multi-index C, we get

$$\sum_{\vec{v} \in S} q^{-C \cdot \vec{v}} \leq \sum_{\vec{v} \in S_0} q^{-C \cdot \vec{v}} + \sum_{\ell_j = \ell} \sum_{\vec{v} \in S_j} q^{-C \cdot \vec{v}}.$$

Since

$$\sum_{\vec{v} \in S_0} q^{-C \cdot \vec{v}} \leq \sum_{A \cdot \vec{v} \geq v_0 + \delta \log_q v_0} q^{-C \cdot \vec{v}},$$

we see from Lemma A.2 that

$$\begin{aligned}
\sum_{\vec{v} \in S_0} |t|^{1/r} q^{-(\mathbb{1}+B) \cdot \vec{v}} + q^{-(\mathbb{1}+A/r+B) \cdot \vec{v}} &\lesssim q^{-(\ell+1/r)(v_0 + \delta \log_q v_0)} (v_0 + \log_q v_0)^{\mu-1} \\
&\lesssim q^{-(\ell+1/r) v_0} v_0^{\mu-1-(\ell+1/r)\delta+\varepsilon}
\end{aligned}$$

for all $\varepsilon > 0$.

For j such that $\ell_j = \ell$, denote

$$\begin{aligned}
A^{(j)} &= (a_1, \ldots, \widehat{a}_j, \ldots, a_n), \\
C^{(j)} &= (c_1, \ldots, \widehat{c}_j, \ldots, c_n), \\
\vec{v}^{(j)} &= (v_1, \ldots, \widehat{v}_j, \ldots, v_n).
\end{aligned}$$

Then $\vec{v} \in S_j$ implies

$$A^{(j)} \cdot \vec{v}^{(j)} \geq w_j := v_0 - a_j \cdot \delta \log_q v_0,$$

so

$$\sum_{\vec{v} \in S_j} q^{-C \cdot \vec{v}} \leq \sum_{\vec{v} \in S_j} q^{-C^{(j)} \cdot \vec{v}^{(j)}} \leq \sum_{A^{(j)} \cdot \vec{v}^{(j)} \geq w_j} q^{-C^{(j)} \cdot \vec{v}^{(j)}}.$$

- If $\mu > 1$, then

$$\begin{aligned}
\sum_{\vec{v} \in S_j} |t|^{1/r} q^{-(B+\mathbb{1}) \cdot \vec{v}} + q^{-(A/r+B+\mathbb{1}) \cdot \vec{v}} &\lesssim q^{-(\ell+1/r) w_j} |w_j|^{\mu-2} \\
&\leq q^{-(\ell+1/r) v_0} |v_0|^{\mu-2+(\ell+1/r) a_j \delta}.
\end{aligned}$$

- If $\mu = 1$, then

$$\ell^{(j)} := \min_{k \neq j} \ell_k > \ell,$$

so

$$\begin{aligned}
\sum_{\vec{v} \in S_j} |t|^{1/r} q^{-(B+\mathbb{1}) \cdot \vec{v}} + q^{-(A/r+B+\mathbb{1}) \cdot \vec{v}} &\lesssim q^{-(\ell^{(j)}+1/r) w_j} |w_j|^{n-1} \\
&\lesssim q^{-(\ell+1/r) v_0} |v_0|^{-1}.
\end{aligned}$$

So we see that

$$\Psi(t) - \Psi(0) = O\left(|t|^{\ell+1/r}|v(t)|^{\mu-1-\varepsilon}\right) \qquad \text{as } |t| \to 0$$

for some $\varepsilon > 0$. $\blacksquare$

We want to get an estimate of

$$\|\alpha + t\beta\| - \|\alpha\|$$

using these propositions.

Let D be the zero divisor of α on $X(K)$. Different from the curve case, the r-canonical linear system $|rK_X|$ might have nonempty base locus F. Consider a resolution of singularities $f : \widetilde{X} \to X$ such that there are normal crossing smooth divisors $\{E\}_{E\in\mathcal{E}}$ with

$$f^*D = \sum_E d_E E, \quad f^*F = \sum_E f_E E, \quad f^*K_X = K_Y + \sum_E j_E E.$$

We see from $D \geq F$ that $d_E \geq f_E$ for each $E \in \mathcal{E}$.

For any $\beta \in V$,

$$\|\alpha + t\beta\| = \int_X |\alpha + t\beta|^{1/r} = \int_{\widetilde{X}} |f^*\alpha + tf^*\beta|^{1/r}.$$

So we can do the estimate on $\widetilde{X}$.

In [6], there are some calculations on p-adic integrals that use the resolution of singularities. Here we do the same thing: Decompose $\widetilde{X}$ into disjoint compact charts U_i such that in coordinate we have

$$f^*\alpha = a_i(u) \cdot u^{D_i+rJ_i} du^r \quad \text{and} \quad f^*\beta = b_i(u) \cdot u^{F_i+rJ_i} du^r$$

where $a_i(u) \neq 0$ for all $u \in U_i$. We may further decompose U_i into finite pieces such that $|a_i(u)| \equiv a_i$ is constant on each piece. Then

$$\int_Y |f^*\alpha + tf^*\beta| = \sum_i a_i^{1/r} \int_{U_i} \left| u^{D_i - F_i} + t \cdot \frac{b_i(u)}{a_i(u)} \right|^{1/r} |u|^{F_i/r+J_i} du.$$

For each U_i, let

$$A_i = D_i - F_i, \quad B_i = F_i/r + J_i, \quad h_i(u) = \frac{b_i(u)}{a_i(u)}$$

and define $\ell_{i,j}$, ℓ_i and μ_i similarly. Applying Proposition A.1, A.3 and A.4 to these cases, we get the following proposition.

Proposition A.5. Let

$$\ell = \min_i \ell_i \quad \text{and} \quad \mu = \max_{\ell_i=\ell} \mu_i.$$

(i) We have

$$\|\alpha + t\beta\|_K - \|\alpha\|_K = O\left(|t|^{\ell+1/r} v(t)^{\mu-1}\right) \qquad \text{as } |t| \to 0.$$

(ii) If $\ell < 1$, we have

$$\|\alpha + t\beta\|_K - \|\alpha\|_K$$

is not $O\left(|t|^{\ell+1/r} v(t)^{\mu-1-\varepsilon}\right)$ for any $\varepsilon > 0$ if and only if $b_i(u)$ does not vanish on

$$\bigcap_{\ell_{i,j}=\ell_i} (u_j = 0) \subset U_i$$

for some i such that $(\ell_i, \mu_i) = (\ell, \mu)$.

Note that

$$\begin{aligned}
\ell(\alpha) &:= \ell = \min_i \ell_i = \min_{i,j} \frac{(f_{i,j}/r) + j_{i,j} + 1}{d_{i,j} - f_{i,j}} = \min_E \frac{(f_E/r) + j_E + 1}{d_E - f_E}, \\
\mu(\alpha) &:= \mu = \max_{\ell_i=\ell} \mu_i = \max_{\ell_i=\ell} \#\{\ell_{i,j} = \ell\} \\
&= \max\left\{ \#\mathcal{E}' \;\middle|\; \mathcal{E}' \subseteq \mathcal{E},\ \bigcap_{E\in\mathcal{E}'} E \neq \varnothing,\ \frac{(f_E/r) + j_E + 1}{d_E - f_E} = \ell,\ \forall E \in \mathcal{E}' \right\},
\end{aligned}$$

and the condition in (ii), $b_i(u)$ does not vanish on

$$\bigcap_{\ell_{i,j}=\ell_i} (u_i = 0) \subset U_i$$

for some i such that $(\ell_i, \mu_i) = (\ell, \mu)$, is equivalent to saying that $\beta \otimes s_F^{-1}$ does not vanish on the set

$$S(\alpha) := \{x \in X(K) \mid (\ell_x(\alpha), \mu_x(\alpha)) = (\ell(\alpha), \mu(\alpha))\},$$

where s_F is the image of 1 under the map

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(F))$$

and

$$\begin{aligned}
\ell_x(\alpha) &= \inf_{x\in f(E)} \frac{(f_E/r) + j_E + 1}{d_E - f_E}, \\
\mu_x(\alpha) &= \sup\left\{ \#\mathcal{E}' \;\middle|\; \mathcal{E}' \subseteq \mathcal{E},\ x \in \bigcap_{E\in\mathcal{E}'} E,\ \frac{(f_E/r) + j_E + 1}{d_E - f_E} = \ell_x(\alpha),\ \forall E \in \mathcal{E}' \right\}.
\end{aligned}$$

These quantities $\ell_x(\alpha)$ and $\mu_x(\alpha)$, and hence

$$\ell(\alpha) = \min_x \ell_x(\alpha), \quad \mu(\alpha) = \max_{\ell_x(\alpha)=\ell(\alpha)} \mu_x(\alpha) \quad \text{and} \quad S(\alpha),$$

are independent of the choice of the resolution $f : \widetilde{X} \to X$. Thus, we might want to recover the variety X from these data.

References

[1] Phillip Griffiths and Joseph Harris. Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[2] Kristin Lauter. Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields. J. Algebraic Geom., 10(1):19–36, 2001. With an appendix in French by J.-P. Serre.

[3] Halsey Royden. Automorphisms and isometries of Teichmüller space. In Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), pages 369–383. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[4] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. Geometry of algebraic curves. Volume II, volume 268 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[5] M. Kuranishi. On the locally complete families of complex analytic structures. Ann. of Math. (2), 75:536–577, 1962.

[6] Jun-ichi Igusa. An introduction to the theory of local zeta functions, volume 14 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000.

[7] Chen-Yu Chi. Pseudonorms and theorems of Torelli type. J. Differential Geom., 104(2):239–273, 2016.

[8] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[9] Rick Miranda. Algebraic curves and Riemann surfaces, volume 5 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1995.

[10] Evgueni Tevelev. Projectively dual varieties of homogeneous spaces. In Surveys in geometry and number theory: reports on contemporary Russian mathematics, volume 338 of London Math. Soc. Lecture Note Ser., pages 183–225. Cambridge Univ. Press, Cambridge, 2007.

[11] André Weil. Sur les courbes algébriques et les variétés qui s'en déduisent. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1041. Hermann et Cie., Paris, 1948. Publ. Inst. Math. Univ. Strasbourg 7 (1945).

[12] T. Matsusaka and D. Mumford. Correction to: "Two fundamental theorems on deformations of polarized varieties". Amer. J. Math., 91:851, 1969.

[13] Chin-Lung Wang. Cohomology theory in birational geometry. J. Differential Geom., 60(2):345–354, 2002.

[14] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[15] Pierre Deligne. La conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math., (43):273–307, 1974.