Outline

Part 1: Stress and tensors

- Cauchy stress tensor
- (Stress) tensor symmetry
- Coordinate transformation (stress) tensors
- Shear and normal stresses
- Tensor invariants

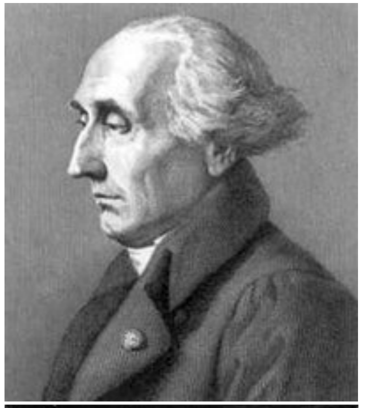
Part 2: Kinematics *chapter3.ipynb*

- Material and spatial description of variables

Learning Objectives
Kinematics

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor

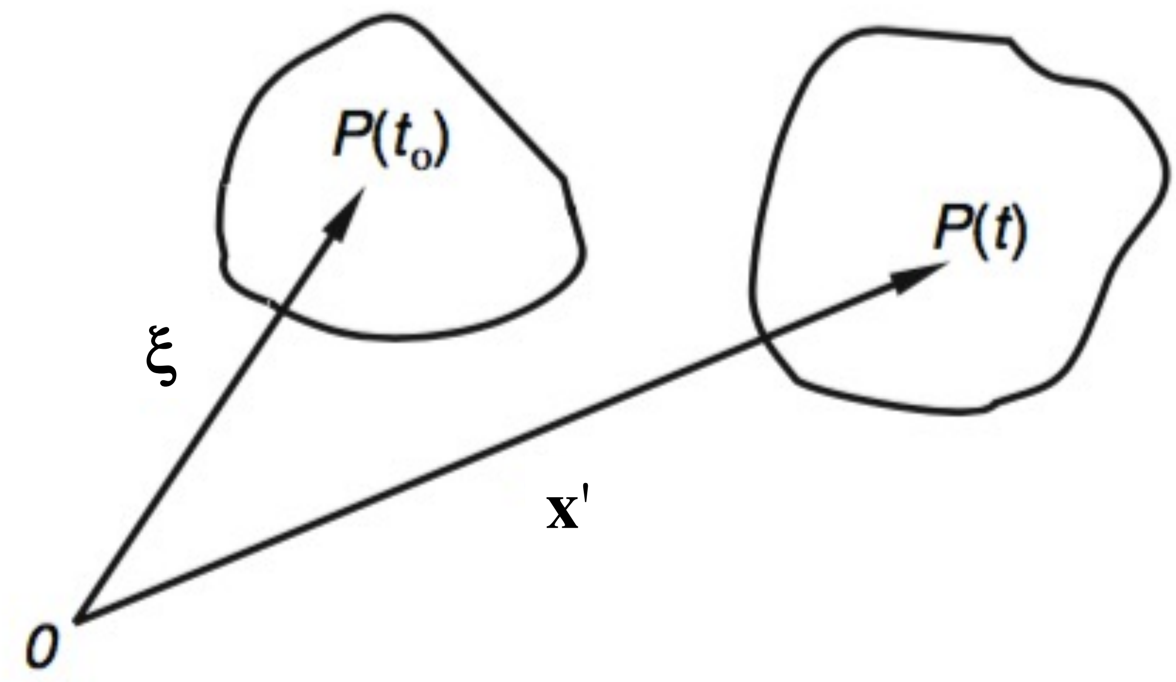
Two ways to describe motion



- Material (Lagrangian)
  - following a “particle”
- Spatial (Eulerian)
  - from a fixed observation point

Preferred description depends on application

Material description

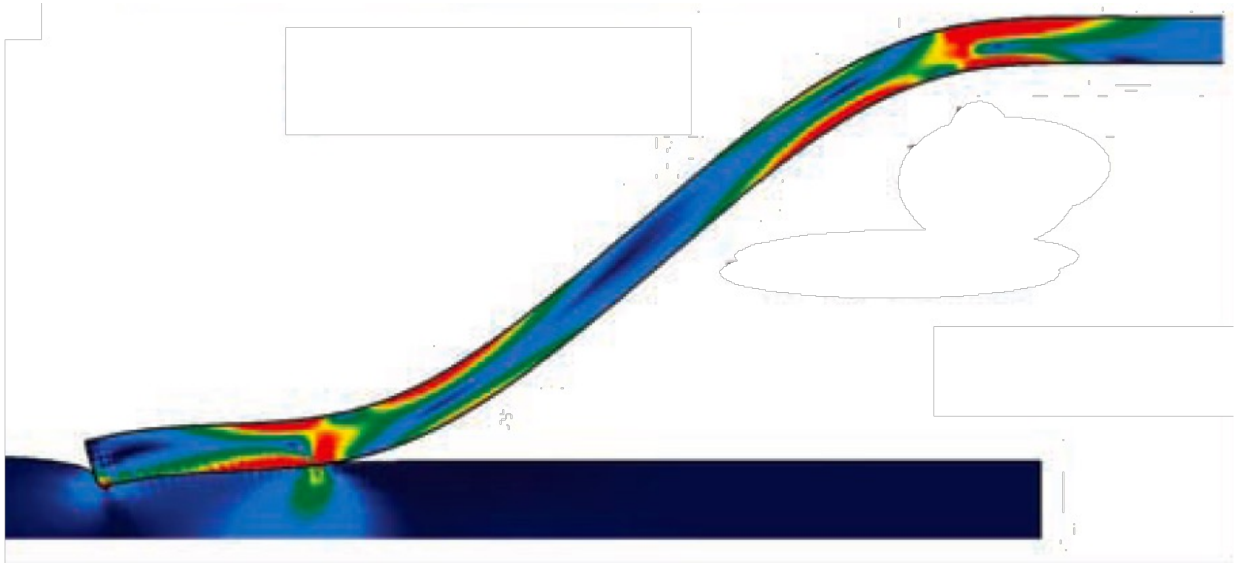


Position vector
$\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$

"Particle" at point $\boldsymbol{\xi}$ at a reference time t_0,
moves to point $\mathbf{x}'$ at a later time t
Field P described as function of $\boldsymbol{\xi}$ and t

Often the preferred description for solids

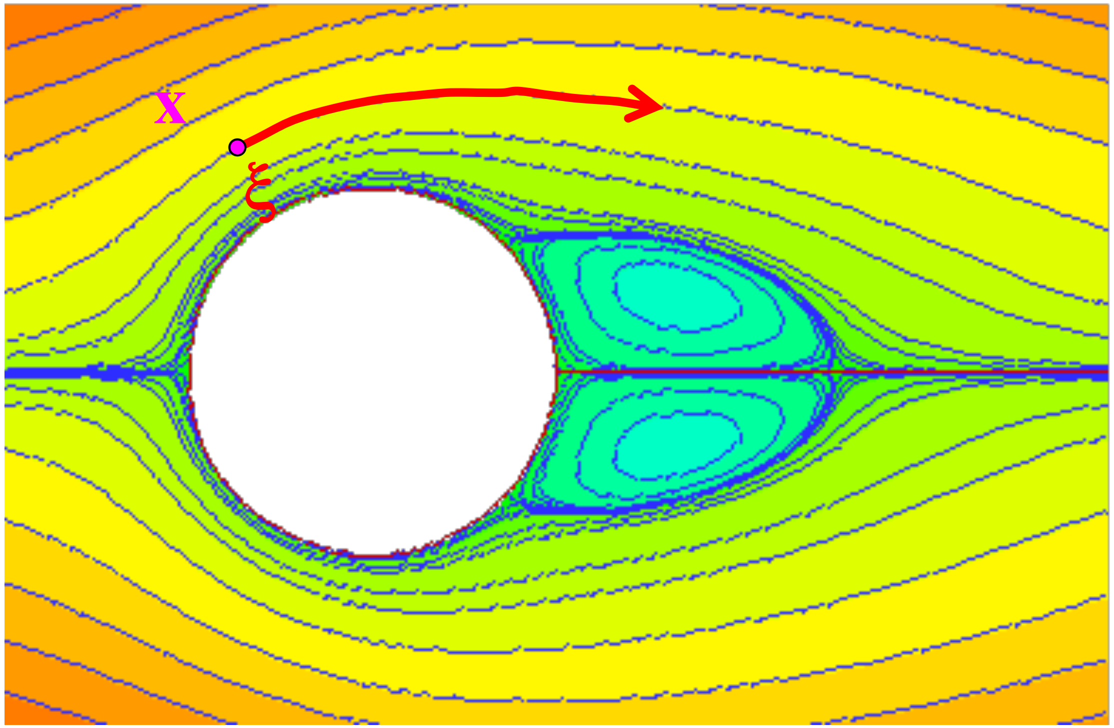
Material description



"Particle" at point ξ at a reference time t_0,
moves to point $\mathbf{x}'$ at a later time t
Field P described as function of ξ and t

Often the preferred description for solids

Spatial description



Field *P* described as function of a given position **x** and *t*

In the example flow, velocity in point **x** *does not change with time, but velocity that a particle originally in same position* ξ *experiences with time does change*

Often the preferred description for fluids

Material Time Derivative

- Rate of change (with time) of a quantity (e.g., T,$\mathbf{v}$,$\boldsymbol{\sigma}$) for a material particle
- In <u>material description</u>, time derivative of P: $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi}$$

  Note: here $P(\boldsymbol{\xi},t)$

- In <u>spatial description</u>, $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi} = \left(\frac{\partial P}{\partial t}\right)_{\mathbf{x}} + \frac{\partial P}{\partial x_i}\left(\frac{\partial x'_i}{\partial t}\right)_{\xi}$$

  Note: here $P(\mathbf{x},t)$

  where $\underbrace{\left(\frac{\partial \mathbf{x}'}{\partial t}\right)_{\xi}}_{\textit{material}} = \underbrace{\frac{D\mathbf{x}}{Dt}}_{\textit{spatial}}$ velocity of particle $\boldsymbol{\xi}$

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P$$

This definition works in any coordinate frame

Example: Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \boldsymbol{v}}{\partial t} + \mathbf{v} \cdot \nabla\, \mathbf{v}$

Let's determine the acceleration of a particle in a spatial velocity field:

$$v_i = \frac{kx_i}{(1+kt)}$$

For a_1:

$$\frac{\partial v_1}{\partial t} = -\frac{kx_1(k)}{(1+kt)^2} = -\frac{k^2 x_1}{(1+kt)^2}$$

$$v_1 \frac{\partial v_1}{\partial x_1} = \frac{kx_1}{(1+kt)} \frac{k}{(1+kt)} = \frac{k^2 x_1}{(1+kt)^2} \qquad v_2 \frac{\partial v_1}{\partial x_2} = 0$$

Hence:

$$a_1 = \frac{Dv_1}{Dt} = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} = -\frac{k^2 x_1}{(1+kt)^2} + \frac{k^2 x_1}{(1+kt)^2} = 0$$

Acceleration

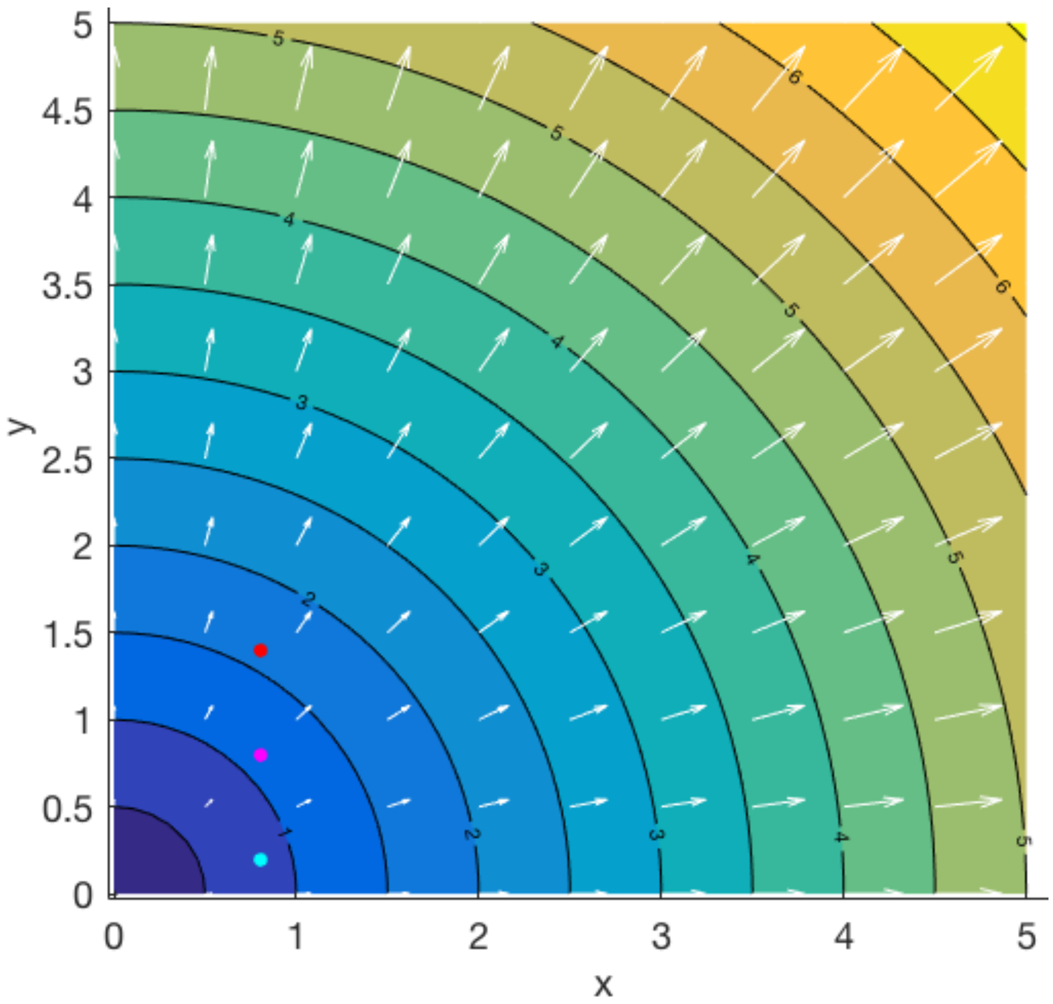
Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

velocity field at t=0 (k=1)



contours for magnitude, arrows direction and size

Acceleration

velocity field at t=2 (k=1)

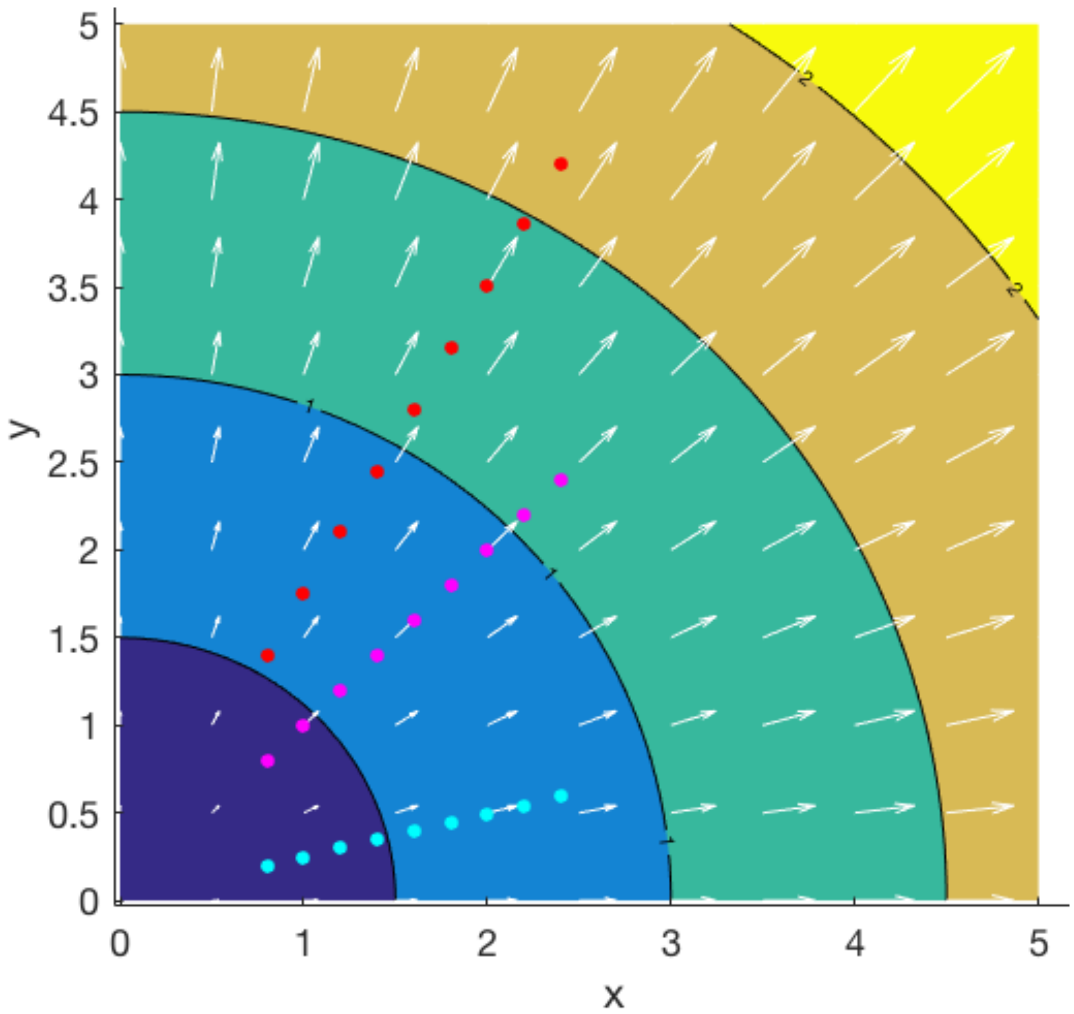
Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

How can you see that ***a*** *= 0?*



marker positions at constant time intervals between [0:2]

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

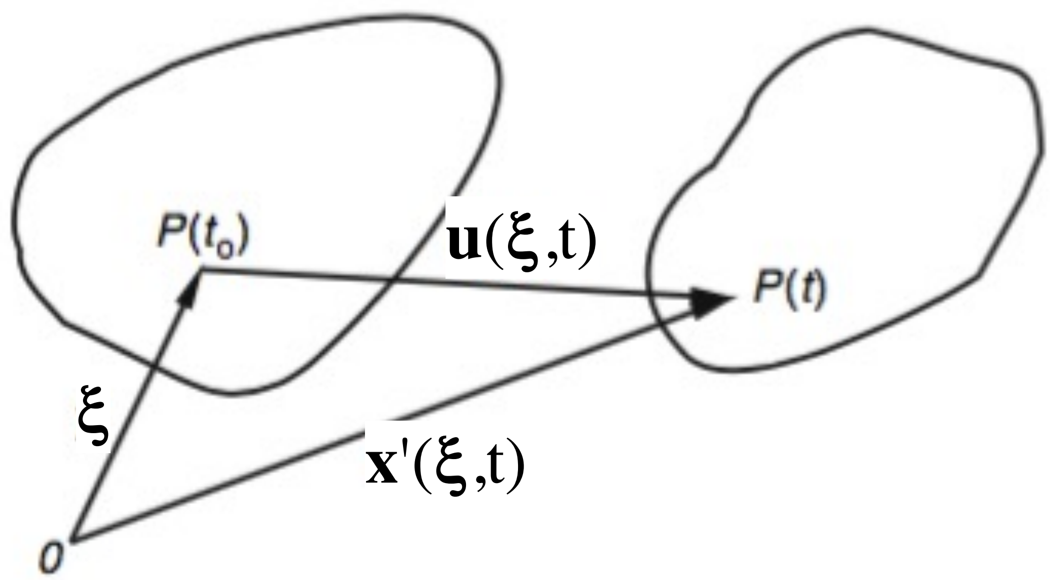
Force balance:
F=m**a** or per unit volume **f**=ρ**a**
becomes:

$$\mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right)$$

Displacement

Motion of a continuum can be described by:

- path lines $\mathbf{x}'=\mathbf{x}'(\xi,t)$
- displacement field $\mathbf{u}(\xi,t)=\mathbf{x}'(\xi,t)-\xi$



Pathlines

Let's determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

Realise that:

$$v_i = \frac{\partial x'_i}{\partial t} = \frac{kx_i}{1+kt}$$

$$\int_{\xi_i}^{x'_i} \frac{dx_i}{kx_i} = \int_0^t \frac{dt}{1+kt}$$

$$\frac{1}{k}\left[\ln x'_i - \ln \xi_i\right] = \frac{1}{k}\left[\ln(1+kt) - \ln(1)\right]$$

$$x'_i(\xi, t) = (1+kt)\xi_i$$

Pathlines

Determine the pathline for the x'_i component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

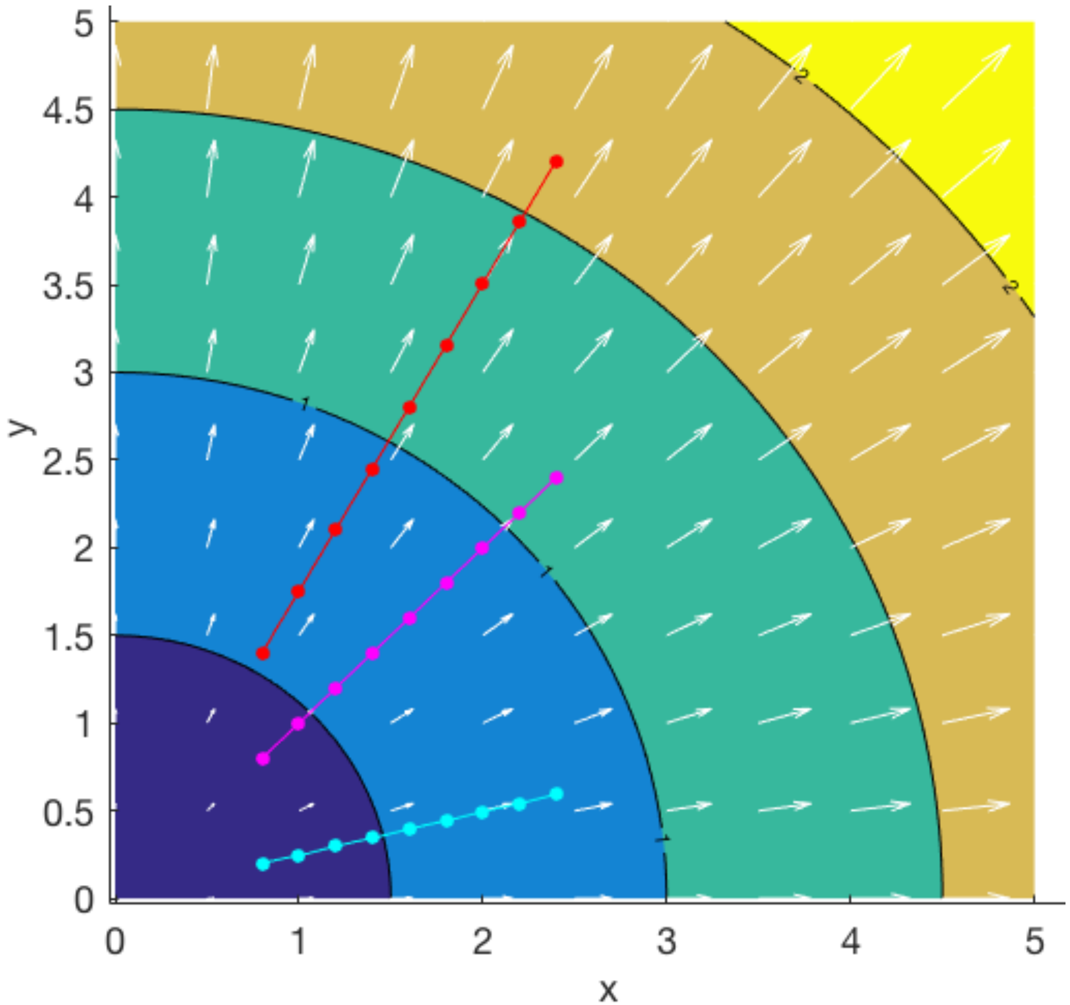
$$x'_i(\xi,t) = (1+kt)\xi_i$$

Material displacement field:

$$u'_i = kt\xi_i$$

Material velocity field:

$$v'_i = v_i = k\xi_i$$



Kinematics of Continua
description of deformation, motion of a continuum

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Kinematics

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Try yourself

For this part of the lecture, first try **Exercise 1** or **3** in *chapter3.ipynb*

In the afternoon workshop, after completing the *chapter2.ipynb* exercises, work on *chapter3.ipynb*:

Exercise 1, 2, 5

- Additional practise: **Exercise 3, 6**
- Advanced practise: **Exercise 4**