Modelling and Numerical Methods
Lecture 2

Stress and Tensors
Kinematics

Outline

Part 1: Stress and tensors   *chapter2.ipynb*

- Cauchy stress tensor

- (Stress) tensor symmetry

- Coordinate transformation (stress) tensors

- Shear and normal stresses

- Tensor invariants

Part 2: Kinematics   *chapter3.ipynb*

- Material and spatial description of variables

Learning Objectives
Stress and tensors

- Understand meaning of different components of 3D Cauchy stress tensor

- Know how to determine state of stress on given plane

- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components

- Be able to transform rank 2 tensor to a new basis.

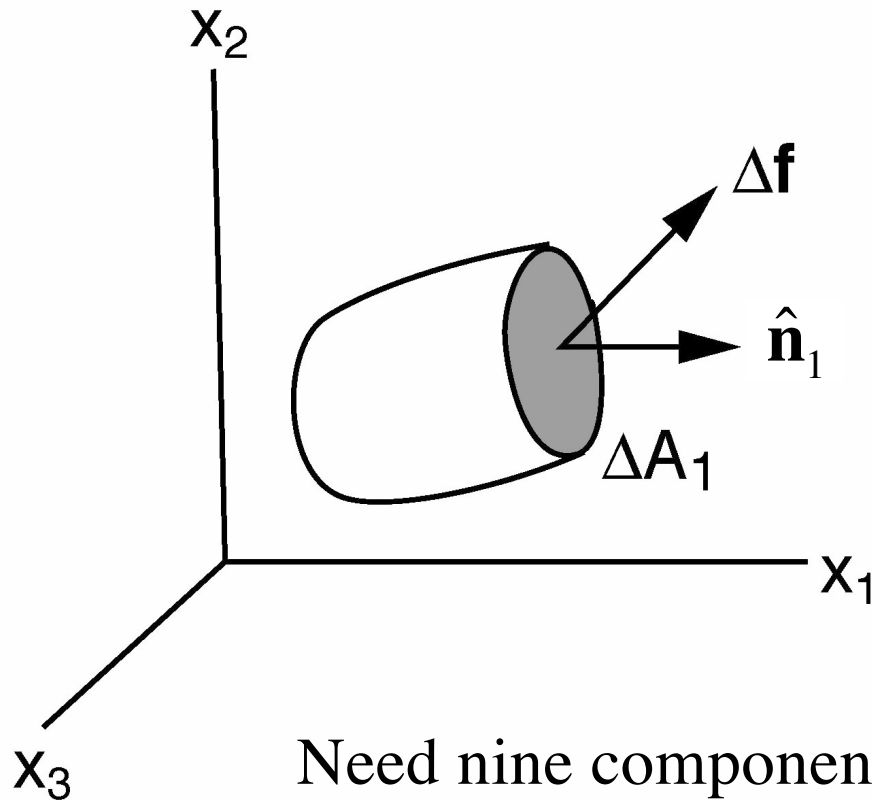- Be able to determine shear and normal stresses on a plane

Cauchy Stress

Stress in a point, measured in medium as deformed
by the stress experienced.
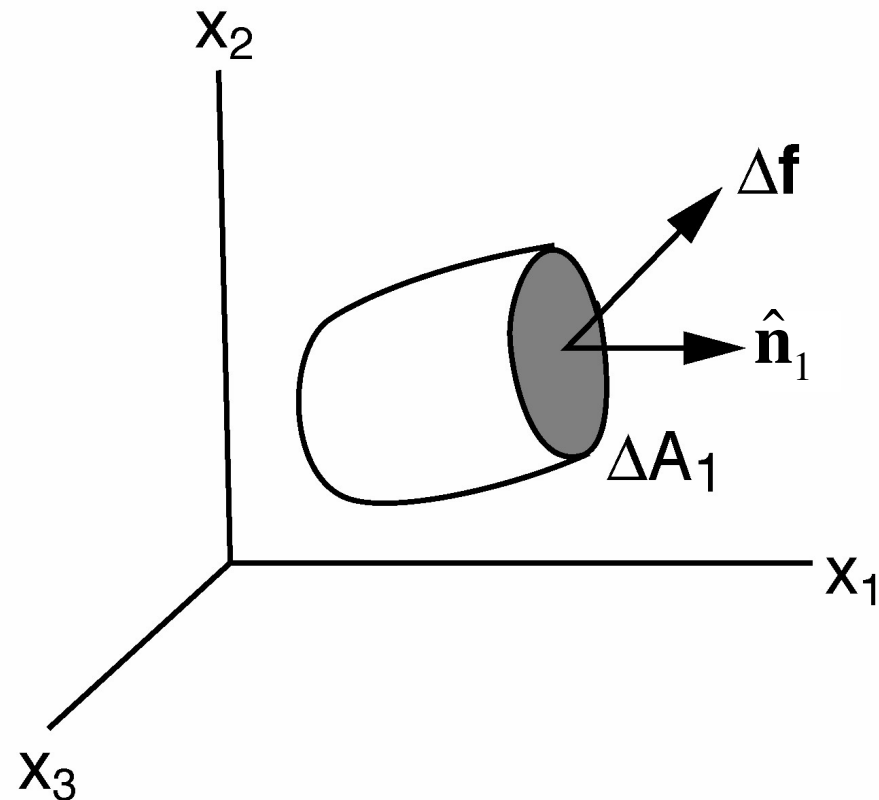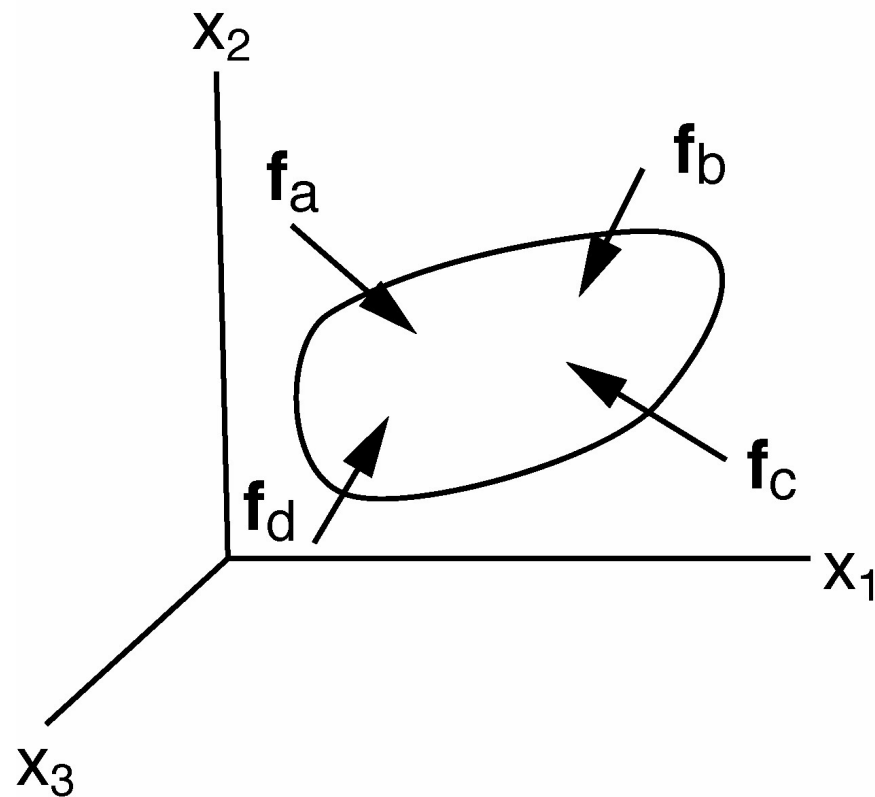
forces introduce a state of stress in a body

(Other stress measures, e.g., Piola-Kirchhoff tensor,
used in Lagrangian formulations)

traction, stress vector

$$t_1 = t(\hat{n}_1) = \lim_{\Delta A \to 0} \Delta f/\Delta A_1$$

$$t_1 = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1

$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2

$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

first index = orientation of plane
second index = orientation of force

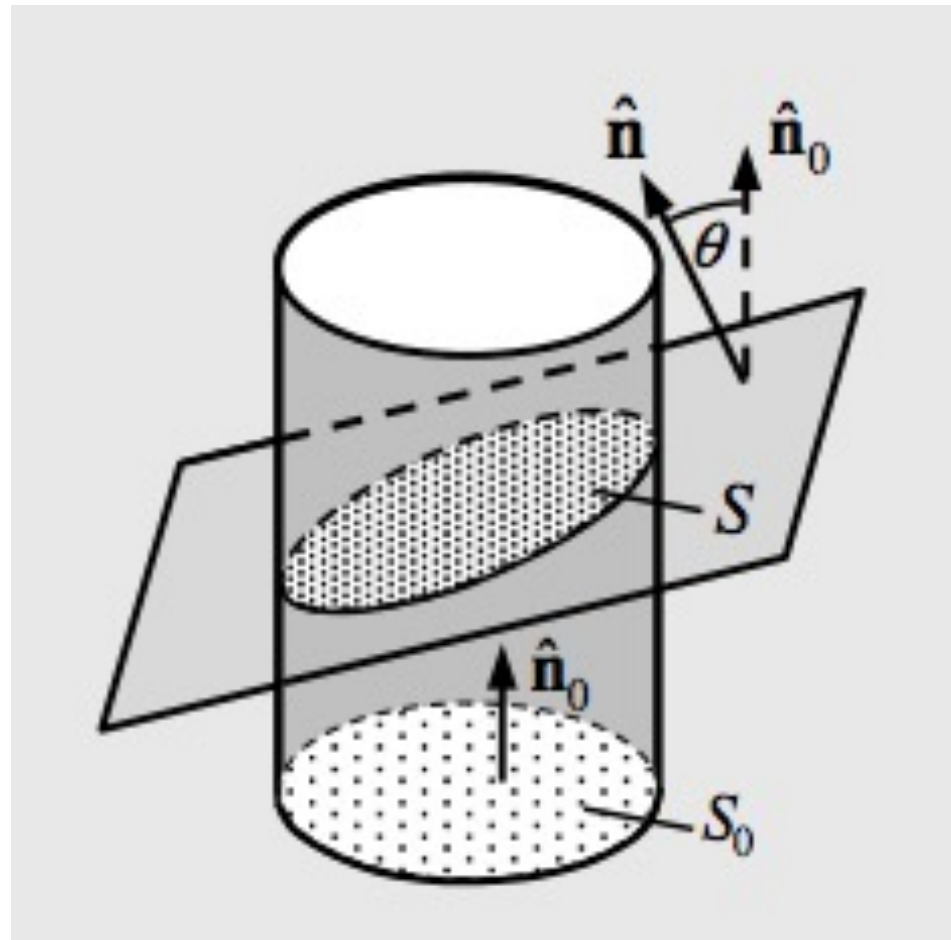*Are nine components sufficient?*

Plane area as a vector

The area of plane S can be defined in terms of vectors assuming S_0 and θ are known.

$$\mathbf{S} = S\hat{\mathbf{n}}$$

$$S_0 = \mathbf{S} \cdot \hat{\mathbf{n}}_0 =$$

$$S\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 = S\cos\theta$$
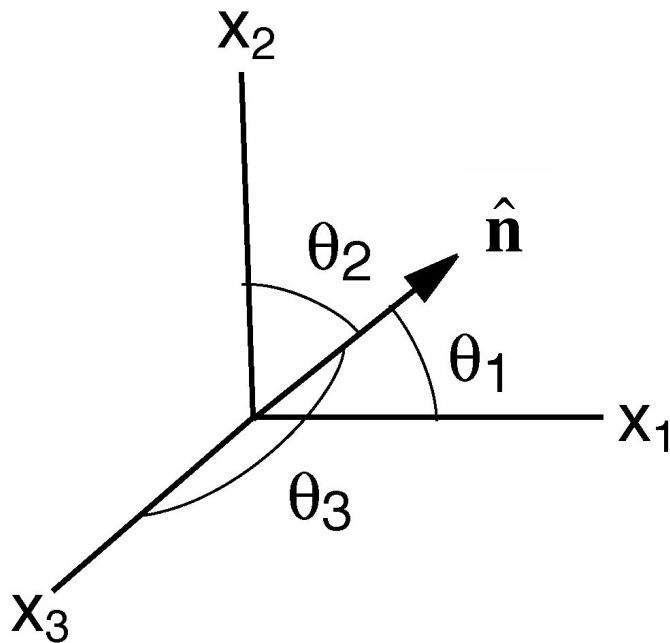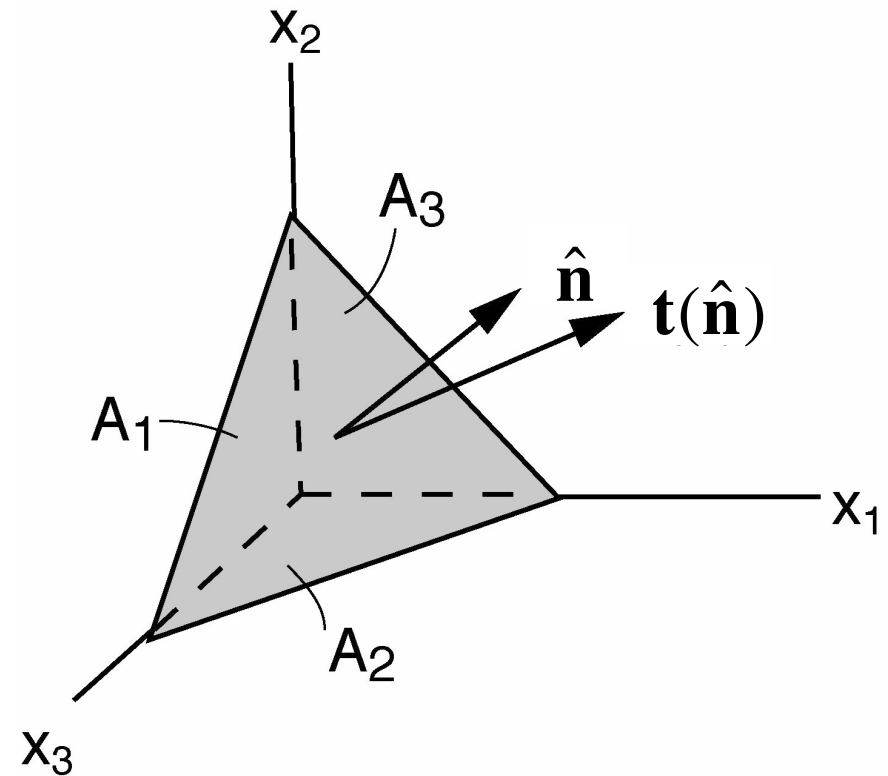
$$\Rightarrow S = S_0 / \cos\theta$$

Are nine components sufficient?
Demonstrate with equilibrium for a
tetrahedron

Given: stress on A_1, A_2, A_3
Find: $\mathbf{t}(\hat{\mathbf{n}})$

1: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_1$, $\Delta A_1 = \Delta A \cos\theta_1$
2: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_2$, $\Delta A_2 = \Delta A \cos\theta_2$
3: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_3$, $\Delta A_3 = \Delta A \cos\theta_3$
4: $\hat{\mathbf{n}} = \left(n_1, n_2, n_3\right)$, $n_i = \cos\theta_i$, $\Delta A_4 = \Delta A$

$$\Sigma f_1 = t_1 \Delta A - \sigma_{11} \Delta A \cos\theta_1 - \sigma_{21} \Delta A \cos\theta_2 - \sigma_{31} \Delta A \cos\theta_3 = 0$$

$$\Sigma f_1 = t_1 \Delta A - \sigma_{11} \Delta A \cos\theta_1 - \sigma_{21} \Delta A \cos\theta_2 - \sigma_{31} \Delta A \cos\theta_3 = 0$$
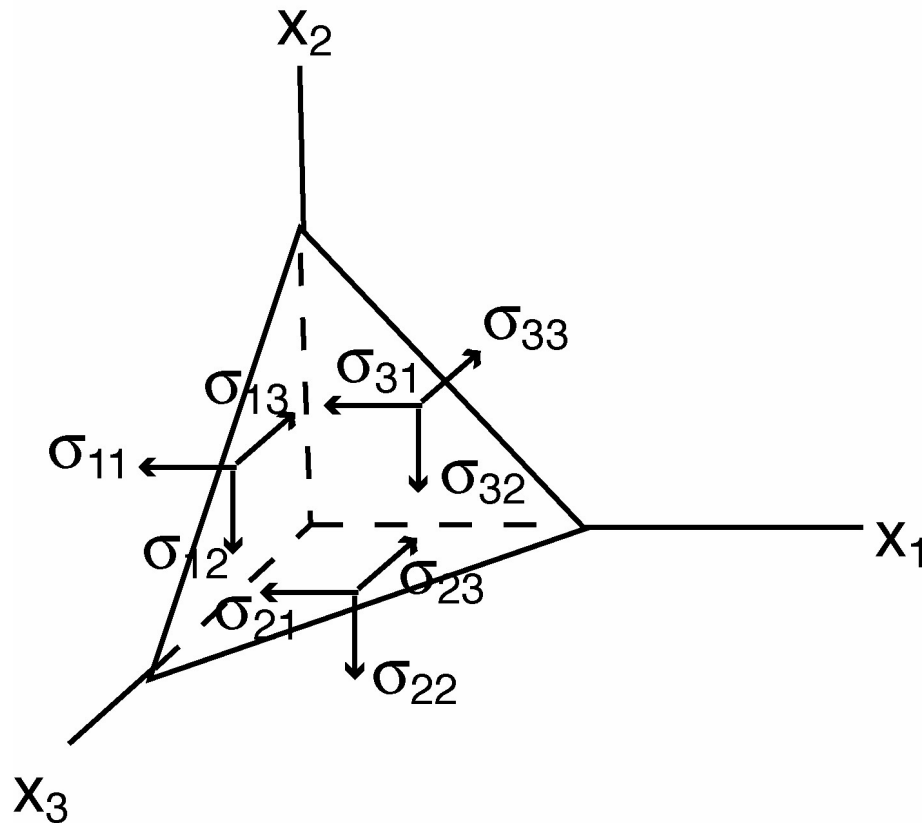
this gives:

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

similarly:
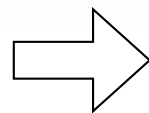
$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
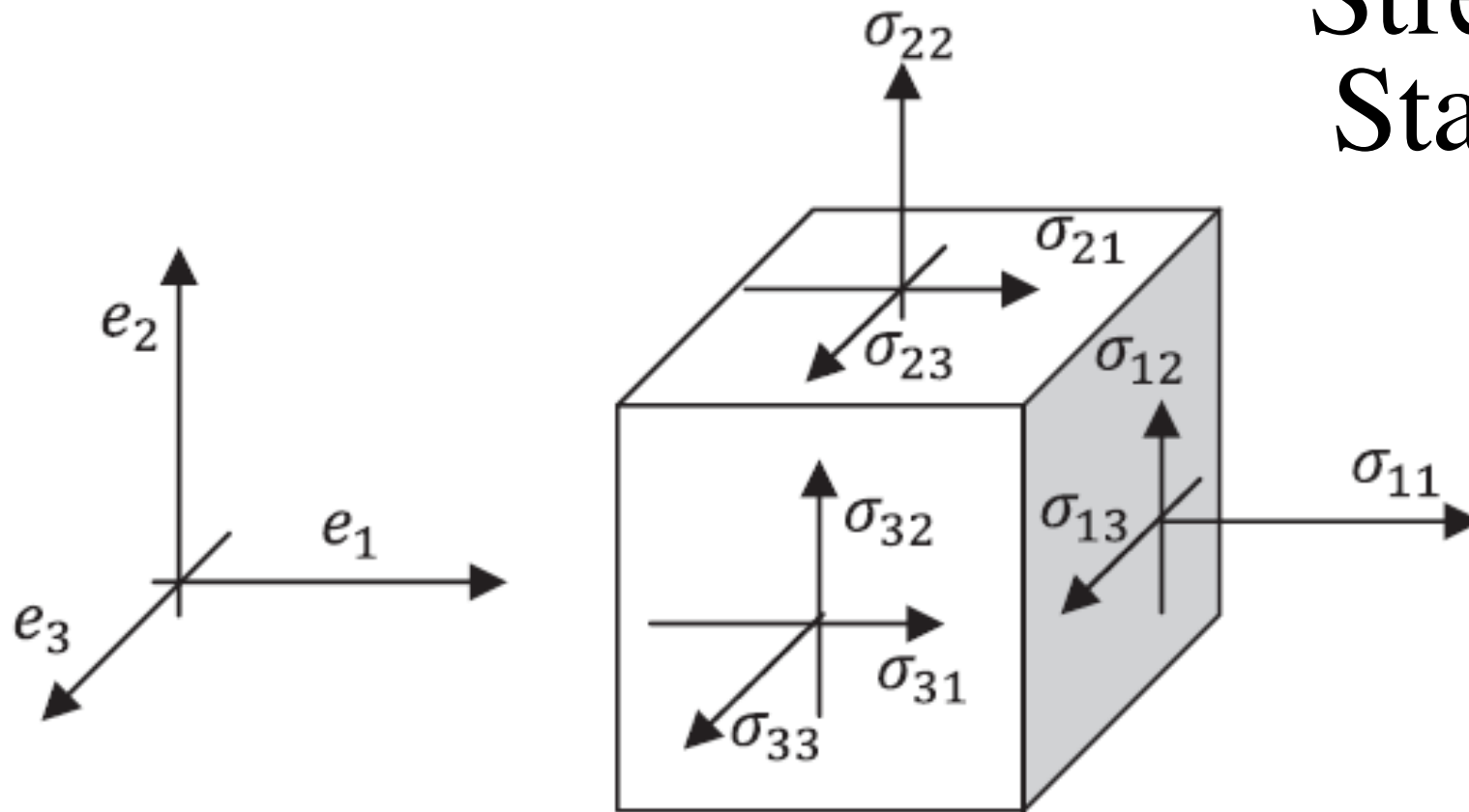$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

$\Rightarrow$ $\boxed{t_i = \sigma_{ji} n_j}$ (Einstein convention)

How many stress components required in 2D?

3-D Stress State

first index = orientation of plane
second index = orientation of force

Positive if force in direction of normal (as shown)

$t_i = \sigma_{ji} n_j$ $\qquad\qquad$ $\mathbf{t} = \mathbf{\sigma}^T \cdot \hat{\mathbf{n}}$ $\qquad\qquad$ Transpose: $\sigma_{ji} = \sigma^T_{ij}$

Note: unusual index order

in matrix notation: $\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$

$\mathbf{t}$ and $\hat{\mathbf{n}}$ - tensors of rank 1 (vectors) in 3-D

$\underline{\underline{\sigma}}$ - tensor of rank 2 in 3-D

compression - negative $\qquad\qquad$ σ_{ji} where i=j - normal stresses

tension - positive $\qquad\qquad\qquad$ σ_{ji} where i≠j - shear stresses

rank 2 tensors can be written as **square matrices** *and have algebraic properties similar to some of those of matrices.*

Stress components
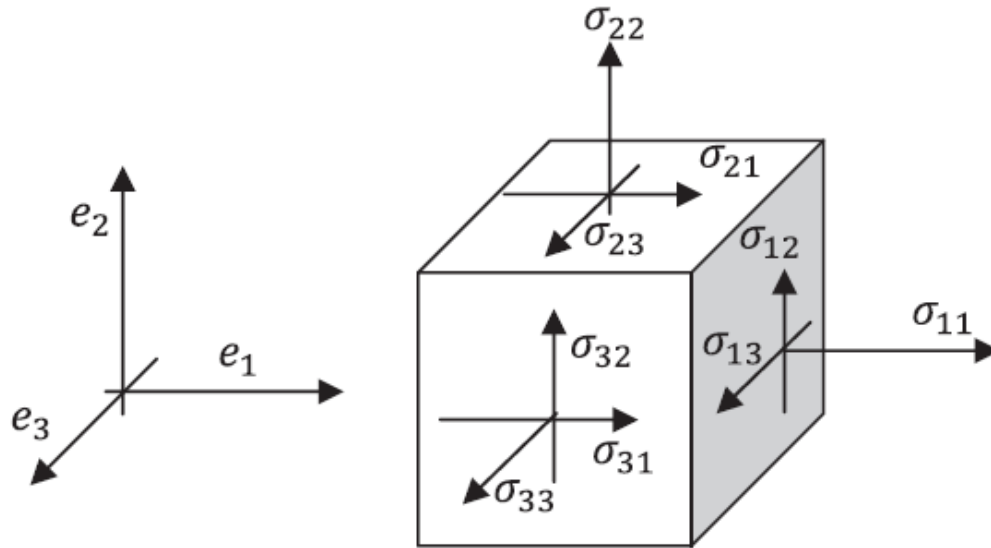
traction on a plane

$$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

what is (1) $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \sigma^T \cdot \hat{\mathbf{n}}$?

what is (2) $\hat{\mathbf{e}}_1 \cdot \sigma^T \cdot \hat{\mathbf{e}}_1$?   what is (3) $\hat{\mathbf{e}}_1 \cdot \sigma^T \cdot \hat{\mathbf{e}}_2$?

Tensor symmetry

A tensor can be symmetric in 1 or more indices

For rank 2:

$S_{ij} = S_{ji} \Rightarrow \mathbf{S} = \mathbf{S}^T$ symmetric

$S_{ij} = -S_{ji} \Rightarrow \mathbf{S} = -\mathbf{S}^T$ antisymmetric

Higher rank:

e.g., $S_{ijk} = S_{jik}$ for all i,j,k $\Rightarrow$ symmetric in i,j

antisymmetric $\mathbf{T}$ of rank 2

Write out general antisymmetric $\mathbf{T}$ rank 2, $n=3 \Rightarrow$ how many independent components?

symmetric $\mathbf{T}$ of rank 2
      has $n(n+1)/2$ independent components

Any $\mathbf{T}$ of rank 2 can be decomposed in symm. and antisymm. part:
      $T_{ij} = (T_{ij} + T_{ji})/2 + (T_{ij} - T_{ji})/2$

Tensor symmetry

A tensor can be symmetric in one or more indices

For rank 2:

$$S_{ij} = S_{ji} \Rightarrow \mathbf{S} = \mathbf{S}^T \qquad \text{symmetric}$$

$$S_{ij} = -S_{ji} \Rightarrow \mathbf{S} = -\mathbf{S}^T \qquad \text{antisymmetric}$$

Higher rank:

e.g., $S_{ijk} = S_{jik}$ for all i,j,k => symmetric in i,j

antisymmetric $\mathbf{T}$ of rank 2

$\Rightarrow T_{ij} = 0$ for i=j, trace($\mathbf{T}$)=0

has n(n-1)/2 independent components

symmetric $\mathbf{T}$ of rank 2

has n(n+1)/2 independent components

Any $\mathbf{T}$ of rank 2 can be decomposed in symm. and antisymm. part:

$$T_{ij} = (T_{ij} + T_{ji})/2 + (T_{ij} - T_{ji})/2$$

Symmetry of the stress tensor

$$\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} \Delta x_2$$

$$\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} \Delta x_1$$

$$\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} \Delta x_1$$

stress couple 1

$\Delta x_1 / 2$

σ_{12}

stress couple 2

$$\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} \Delta x_2$$

$\Delta x_2 / 2$

σ_{21}
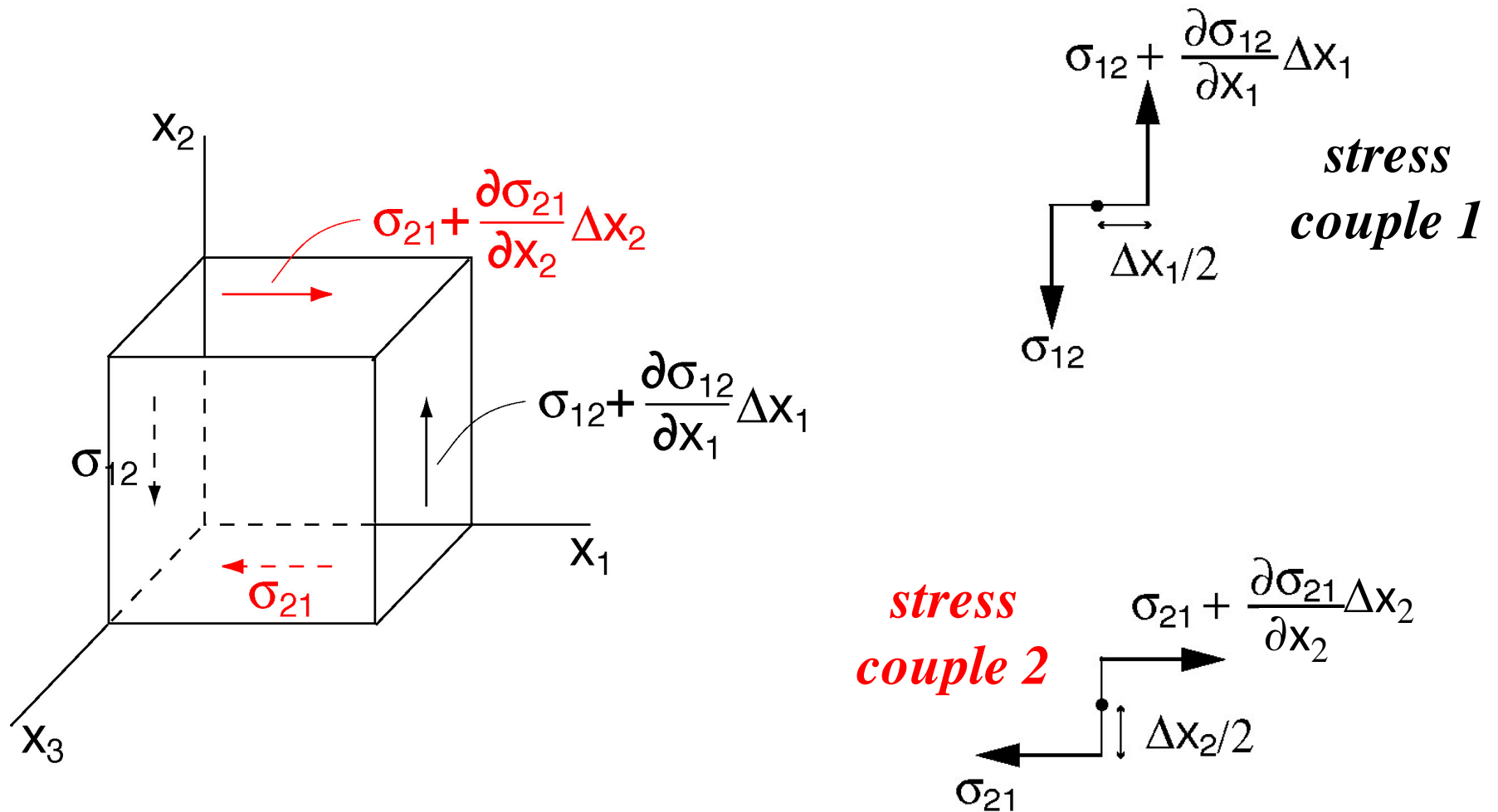
assuming static equilibrium

balance of moments in x₃ direction

A balance of moments in x₃ direction:

$$m_3 = \overbrace{[\sigma_{12} + (\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})]}^{\text{\color{blue}stress couple 1}} \overbrace{\Delta x_2 \Delta x_3}^{\text{\color{blue}area 1}} \cdot \overbrace{\Delta x_1 / 2}^{\text{\color{blue}arm 1}}$$

$$-\underbrace{[\sigma_{21} + (\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})]}_{\text{\color{red}stress couple 2}} \underbrace{\Delta x_1 \Delta x_3}_{\text{\color{red}area 2}} \cdot \underbrace{\Delta x_2 / 2}_{\text{\color{red}arm 2}} = 0$$

$\sigma_{12} + \dfrac{\partial \sigma_{12}}{\partial x_1} \Delta x_1$

$\Delta x_1/2$

σ_{12}

**stress couple 1**

**stress couple 2**

$\sigma_{21} + \dfrac{\partial \sigma_{21}}{\partial x_1} \Delta x_1$

$\Delta x_2/2$

σ_{21}

A balance of moments in x_3 direction:

stress couple 1   area 1   arm 1

$$m_3 = [\sigma_{12} + (\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})]\Delta x_2 \Delta x_3 \cdot \Delta x_1 / 2$$

stress couple 2   area 2   arm 2

$$-[\sigma_{21} + (\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})]\Delta x_1 \Delta x_3 \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \rightarrow 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

Balancing m_1 and m_2: $\boxed{\sigma_{23} = \sigma_{32}}$ *and* $\boxed{\sigma_{13} = \sigma_{31}}$

*thus, the stress tensor is **symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

Take a break

Then try Exercises 1 & 2 in the notebook (*chapter2.ipynb*)

Learning Objectives
Stress and tensors

- Understand meaning of different components of 3D Cauchy stress tensor

- Know how to determine state of stress on given plane

- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components

- Be able to transform rank 2 tensor to a new basis.

- Be able to determine shear and normal stresses on a plane

physical parameters should not depend on coordinate frame
$\Rightarrow$ **tensors follow linear transformation laws**

for vectors on orthonormal basis:
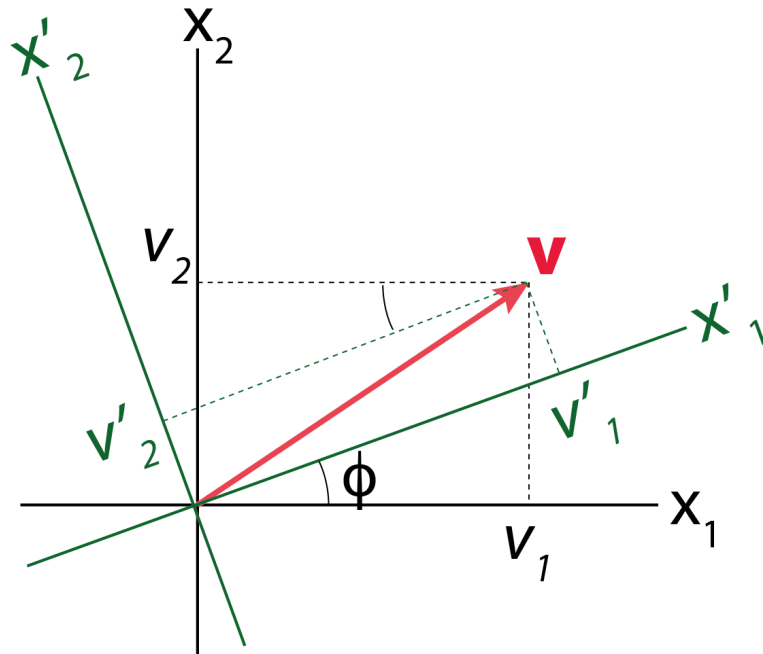
$$\mathbf{v'} = \mathbf{A}\mathbf{v}$$

$$\Rightarrow \mathbf{v'} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v'} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix} \mathbf{v} \qquad \boxed{\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j}$$

Inverse transform: $v_j = \alpha_{ji}\, v'_i$ $\qquad \alpha_{ji} = \hat{e}_j \cdot \hat{e}'_i$

$$\mathbf{t} = \boldsymbol{\sigma}^T \mathbf{n}$$

$$\mathbf{t}' = \boldsymbol{\sigma}'^T \mathbf{n}'$$

Relation $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$?

Vectors in a new coordinate system:
Traction $\mathbf{t}' = \mathbf{A}\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{A}^T \mathbf{t}'$
normal $\mathbf{n}' = \mathbf{A}\mathbf{n} \Rightarrow \mathbf{n} = \mathbf{A}^T \mathbf{n}'$

$\Rightarrow$ *transformation for stress tensor*

$$\mathbf{t}' = \color{blue}{\mathbf{A}}\boldsymbol{\sigma}^T \mathbf{n}$$

$$\mathbf{t}' = \color{blue}{\mathbf{A}}\boldsymbol{\sigma}^T \color{red}{\mathbf{A}^T} \mathbf{n}'$$

$$\mathbf{t}' = \boldsymbol{\sigma}'^T \mathbf{n}'$$

$$\Rightarrow \boldsymbol{\sigma}'^T = \color{blue}{\mathbf{A}}\boldsymbol{\sigma}^T \color{red}{\mathbf{A}^T}$$

- transformation matrices are orthogonal
  $$\alpha_{ij}^{-1} = \alpha_{ji} \quad (\mathbf{A}^{-1} = \mathbf{A}^T)$$
- *remember* $\quad \alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$
  $$\alpha_{ij}^{-1} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j = \alpha_{ji} = \alpha_{ij}^T$$

$\Rightarrow \sigma'^T_{ij} = \color{blue}{\alpha_{ik}}\sigma^T_{kl}\color{red}{\alpha_{jl}} = \color{blue}{\alpha_{ik}}\color{red}{\alpha_{jl}}\sigma^T_{kl}$ index notation

$\Rightarrow$ each dependence on direction transforms as a vector, requiring two transformations

In general:

An *n-dimensional* tensor of *rank r* consists of n^r components

This tensor $T_{i1,i2,\ldots,in}$ is defined relative to a basis of the real, linear n-dimensional space S_n

and under a coordinate transformation **T** transforms as:

$$T'_{ij\ldots n} = \alpha_{ip}\alpha_{jq}\ldots\alpha_{nt}\, T_{pq\ldots t} \quad \text{(i.e., one transformation per rank)}$$

For *orthonormal* bases the matrices α_{ik} are *orthogonal* transformations, i.e. $\alpha_{ik}^{-1} = \alpha_{ki}$. (i.e., $\mathbf{A} = \mathbf{A}^T$; columns and rows are orthogonal and have length = 1, i.e., perpendicular unit vectors are transformed to perpendicular unit vectors)

If the basis is *Cartesian*, α_{ik} are *real*.

Transforming the 2-D stress tensor
(determining normal and shear stress on a plane)
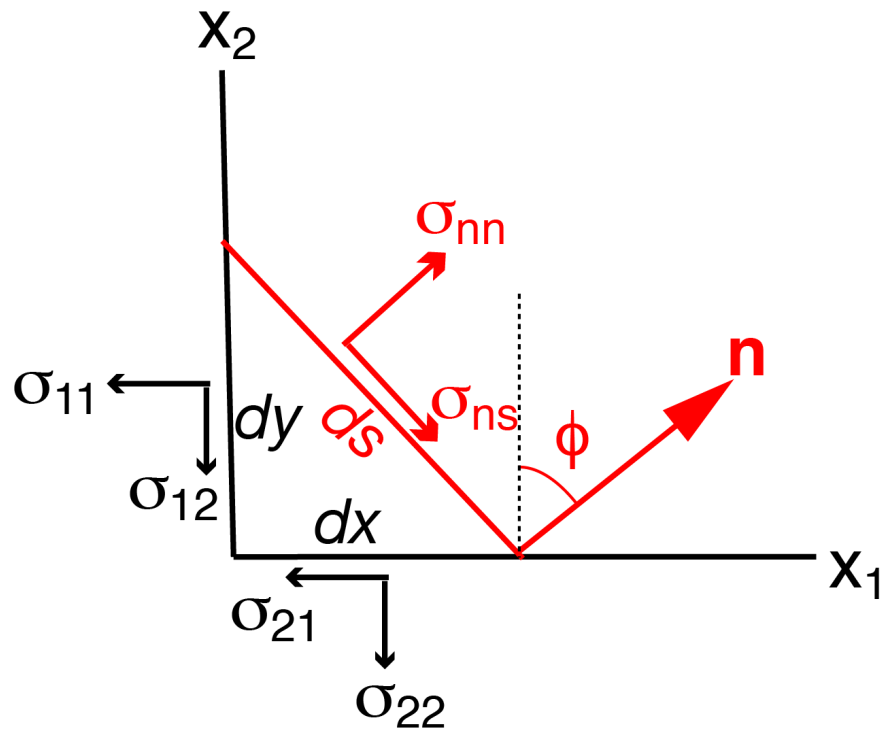
Normal to the plane:

$$\hat{\mathbf{n}} = (\sin \phi, \cos \phi)$$

Normal stress on plane:

$$\sigma_{nn} = \hat{\mathbf{n}} \cdot \underline{\underline{\sigma}}^T \cdot \hat{\mathbf{n}}$$

This gives:

$$\sigma_{nn} = (\sin \phi, \cos \phi) \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{pmatrix} \sin \phi \\ \cos \phi \end{pmatrix}$$

Similarly,

$$\sigma_{ns} = \hat{\mathbf{s}} \cdot \underline{\underline{\sigma}}^T \cdot \hat{\mathbf{n}} \qquad \hat{\mathbf{s}} = (\cos \phi, -\sin \phi)$$

Multiplying out:

$$\sigma_{nn} = \sigma_{11} \sin^2 \phi + \sigma_{21} \cos \phi \sin \phi + \sigma_{12} \cos \phi \sin \phi + \sigma_{22} \cos^2 \phi$$

$$\sigma_{ns} = \sigma_{11} \cos \phi \sin \phi + \sigma_{21} \cos^2 \phi - \sigma_{12} \sin^2 \phi - \sigma_{22} \cos \phi \sin \phi$$

This is equivalent to the **tensor transformation**

$$\boldsymbol{\sigma}'^T = \mathbf{A} \boldsymbol{\sigma}^T \mathbf{A}^T$$

$$\sigma'_{qp} = \alpha_{pi} \alpha_{qj} \sigma_{ji}$$
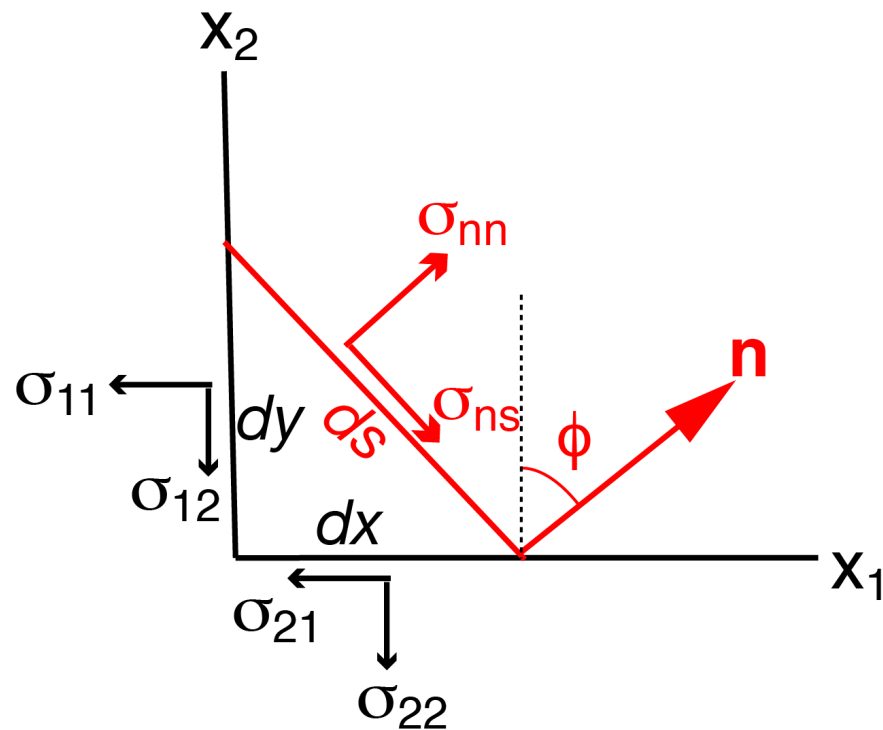
$$\sigma'_{nn} = \alpha_{ni} \alpha_{nj} \sigma_{ji}$$
$$\sigma'_{ns} = \alpha_{si} \alpha_{nj} \sigma_{ji}$$

With $\alpha_{n1} = \sin \phi, \ \alpha_{n2} = \cos \phi, \ \alpha_{s1} = \cos \phi, \ \alpha_{s2} = -\sin \phi$
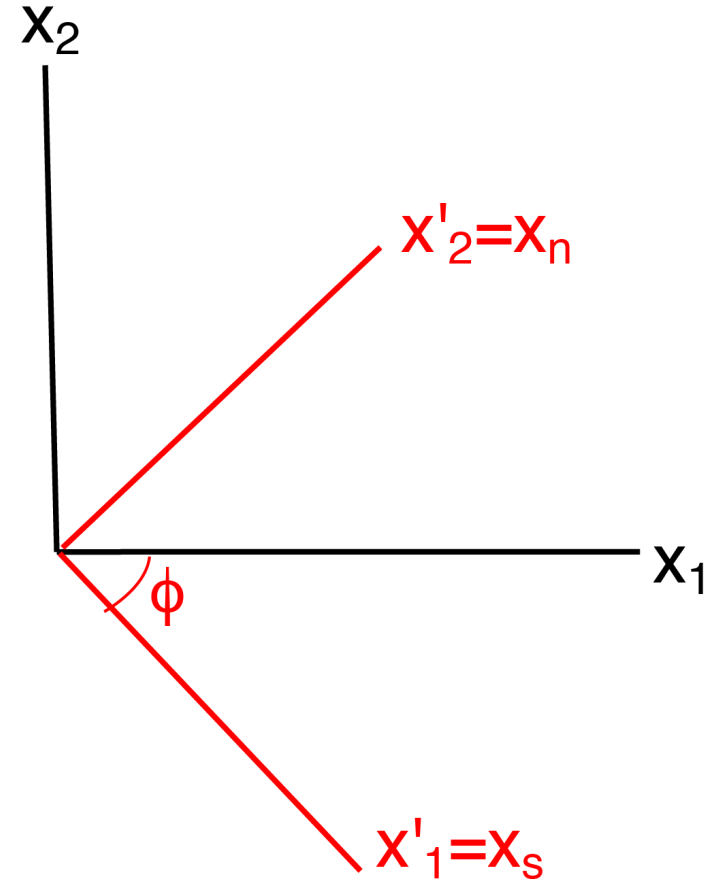
Transforming the 2-D stress tensor
(determining normal and shear stress on a plane)

$dx = \cos\phi \, ds$

$dy = \sin\phi \, ds$

Write out transformation

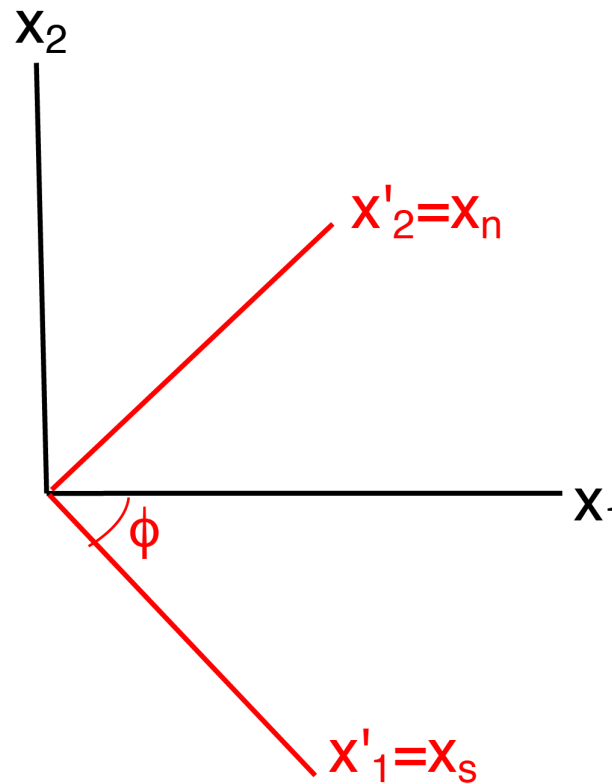$$\sigma'_{qp} = \alpha_{pi}\alpha_{qj}\sigma_{ji}$$

In tensor notation:

$$\sigma'^{T} = A \cdot \sigma^{T} \cdot A^{T}$$

$$\alpha_{ij} = \hat{\mathbf{e}}'_{i} \cdot \hat{\mathbf{e}}_{j}$$

$$\alpha_{s1} = \hat{\mathbf{e}}_{s} \cdot \hat{\mathbf{e}}_{1} = \cos\phi$$

$$\alpha_{s2} = \hat{\mathbf{e}}_{s} \cdot \hat{\mathbf{e}}_{2} = -\sin\phi$$

$$\alpha_{n2} = \hat{\mathbf{e}}_{n} \cdot \hat{\mathbf{e}}_{2} = \cos\phi$$

$$\alpha_{n1} = \hat{\mathbf{e}}_{n} \cdot \hat{\mathbf{e}}_{1} = \sin\phi$$

In matrix notation:

$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}$$

Write out matrices A and A^{T}

Check that the expressions for σ_{nn}, σ_{ns} of previous slide obtained

Write out transformation

$$\sigma'_{qp} = \alpha_{pi}\alpha_{qj}\sigma_{ji}$$

In tensor notation:

$$\sigma'^{T} = A \cdot \sigma^{T} \cdot A^{T}$$

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$

$$\alpha_{s1} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_1 = \cos\phi$$

$$\alpha_{s2} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_2 = -\sin\phi$$

$$\alpha_{n2} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \cos\phi$$

$$\alpha_{n1} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1 = \sin\phi$$

In matrix notation:

$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \alpha_{s1} & \alpha_{s2} \\ \alpha_{n1} & \alpha_{n2} \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} \alpha_{s1} & \alpha_{n1} \\ \alpha_{s2} & \alpha_{n2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$
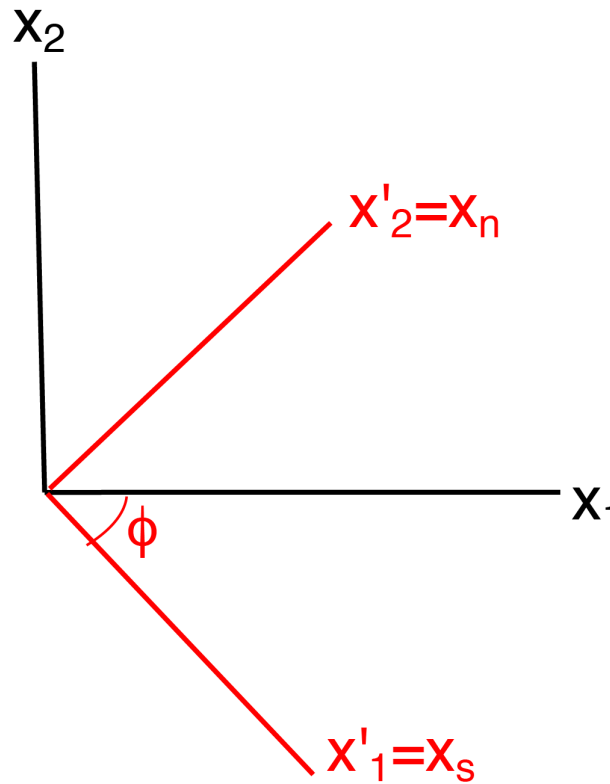
For $\hat{\mathbf{x}}_1=(1,0)$, $\hat{\mathbf{x}}_2=(0,1)$, $\hat{\mathbf{x}}'_1 = (\cos\phi, -\sin\phi)$ $\hat{\mathbf{x}}'_2 = (\sin\phi, \cos\phi)$
first **row** of $\mathbf{A}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$
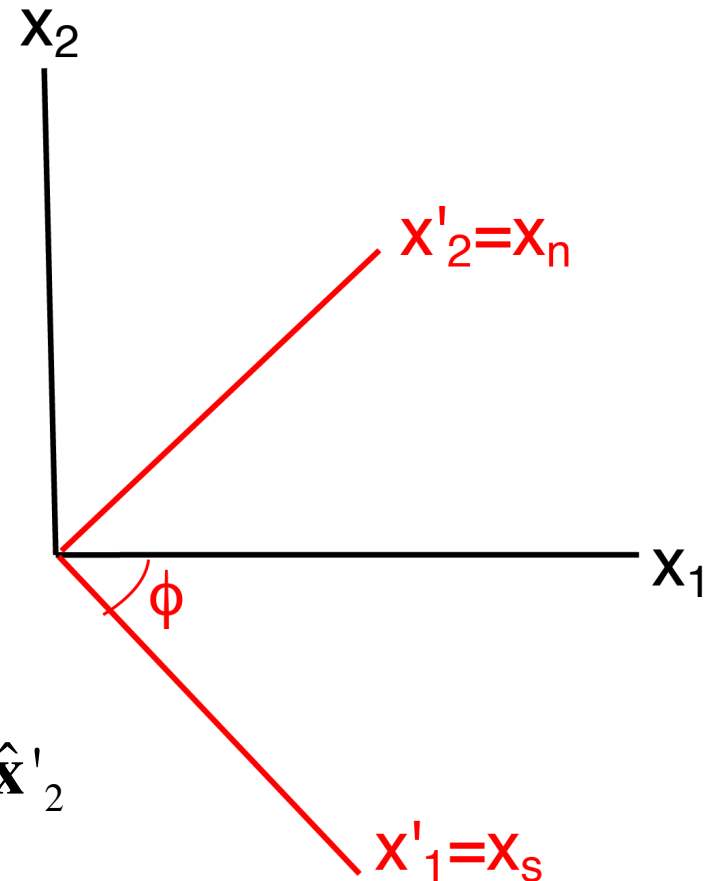
$$\mathbf{A} = \begin{bmatrix} \mathbf{x}'_1 \cdot \mathbf{x}_1 & \mathbf{x}'_1 \cdot \mathbf{x}_2 \\ \mathbf{x}'_2 \cdot \mathbf{x}_1 & \mathbf{x}'_2 \cdot \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

*You may recognise $\mathbf{A}$ as a matrix that describes an **anticlockwise** rigid-body rotation over an angle ϕ*

*A^T describes **clockwise** rotation over angle ϕ*

First **column** of $\mathbf{A}^T$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}'_1 & \mathbf{x}_1 \cdot \mathbf{x}'_2 \\ \mathbf{x}_2 \cdot \mathbf{x}'_1 & \mathbf{x}_2 \cdot \mathbf{x}'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

x_2

$x'_2 = x_n$

ϕ

x_1

$x'_1 = x_s$

Diagonalizing/Principal Stresses

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, $\sigma_1, \sigma_2, \sigma_3$ are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T} \cdot \mathbf{x} = \lambda \mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors
and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow (\mathbf{T} - \lambda \mathbf{I}) \cdot \mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T} - \lambda \mathbf{I}) = 0$

Stress/Tensor Invariants

Invariants are properties of a tensor that do not change if the coordinate system is changed.

A rank-2 tensor has three invariants:

$$I_1 = \text{tr}(\mathbf{T}) \qquad = T_{11} + T_{22} + T_{33}$$

$$I_2 = \text{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$= T_{11}T_{22} + T_{22}T_{33} + T_{11}T_{33} - T_{21}{}^2 - T_{32}{}^2 - T_{31}{}^2$$

$$I_3 = \det(\mathbf{T}) \quad = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} \begin{aligned} &= T_{11}T_{22}T_{33} + 2T_{21}T_{32}T_{31} \\ &\quad - T_{11}T_{32}{}^2 - T_{22}T_{31}{}^2 - T_{33}T_{21}{}^2 \end{aligned}$$

These invariants are important in e.g. diagonalising

Hydrostatic and Deviatoric stress

$$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$\text{tr}(\sigma)$ = trace of stress tensor

= sum of normal stresses = $\sigma_{11} + \sigma_{22} + \sigma_{33}$

is an invariant of the stress tensor,

i.e. has same value in any coordinate system,

$\text{tr}(\sigma)/3$ = - pressure p = average normal stress = **hydrostatic stress**

$\Rightarrow$ **leads to volume change**

σ'_{ij} is **deviatoric stress** = $\sigma_{ij} + p\delta_{ij}$

$\Rightarrow$ **leads to shape change**

$\text{tr}(\sigma')$ = ?

Note that in geologic applications $\sigma'_{ij} \neq \sigma_{ij} + \rho g z \delta_{ij}$

Learning Objectives
Stress and tensors

- Understand meaning of different components of 3D Cauchy stress tensor

- Know how to determine state of stress on given plane

- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components

- Be able to transform rank 2 tensor to a new basis.

- Be able to determine shear and normal stresses on a plane

Summary
Stress Tensors

- Cauchy stress tensor

- (Stress) tensor symmetry

- Coordinate transformation (stress) tensors

- Shear and normal stresses

- Tensor invariants

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 2.18 through 2.25, 4.4 through 4.7

Take a break

Then try Exercise 3 and 5 in the *chapter2.ipynb* notebook

Try to finish in the afternoon workshop in *chapter2.ipynb*:

Exercise 1, 2, 3, 4, 6

Advanced practise: **Exercise 5**