MNM Study Guide Coursework 2

Notation:

Scalars – a or *a*
Vectors – $\mathbf{v}$ or $\vec{\mathbf{v}}$ or $\bar{\mathbf{v}}$, vector length $|\mathbf{v}|$
Tensors – $\mathbf{T}$ or (if rank 2) $\underline{\mathbf{T}}$
Unit vector along direction of $\mathbf{v}$: $\hat{\mathbf{e}}_v = \frac{\mathbf{v}}{|\mathbf{v}|}$
Unit outward normal for a plane: $\hat{\mathbf{n}}$

Equations/concepts you are expected to know and be able to apply:

Examples given here all for 3-D, orthonormal Cartesian reference frame

- Index notation: vector or tensor components written as v_i or T_{ij} with i,j=1,2,3 or i,j =x,y,z

- Einstein convention – implied summation of the same index repeated twice within a single term, e.g. $v_i w_i = \sum_{i=1}^{3} v_i w_i$

- Vector and tensor products:
  - dot product: $\mathbf{v}\cdot\mathbf{w}=v_i w_i$ or $\mathbf{T}\cdot\mathbf{v}=T_{ij}v_j$
  - multiple contraction, e.g. $\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\varepsilon} = C_{ijkl}\,\varepsilon_{kl}$
  - cross product: $\mathbf{v} \times \mathbf{w} = \varepsilon_{ijk} v_i w_j \hat{\mathbf{e}}_k$
  - tensor product: $\mathbf{vw}=v_i w_j$

- Transpose: $T_{ji}=T^T_{ij}$

- Tensor symmetry:
  - Symmetric in i,j: $T_{ji}=T_{ij}$,
  - Antisymmetric in i,j: $T_{ji}=-T_{ij}$

- Tensor trace: for rank 2 tensor $\mathrm{tr}(\mathbf{T})=T_{11}+T_{22}+T_{33}=T_{ii}$.

- Kronecker delta $\delta_{ij}=1$ if i=j, =0 if i≠j

- Levi-Civita tensor $\varepsilon_{ijk}=1$ for even permutations of 1,2,3, $\varepsilon_{ijk}=-1$ for odd permutations of 1,2,3, $\varepsilon_{ijk}=0$ if any i,j,k are equal

- Lagrangian or material – description of motion following a 'particle', all fields described as a function of position $\boldsymbol{\xi}$ at a reference time t_0 and time *t*.

- Eulerian or spatial – description of motion from a fixed observation point. All fields described as a function of position $\mathbf{x}$ and time *t*.

- Material Derivative
  - in spatial description, the full time derivative of a field P($\mathbf{x}$,*t*) becomes: $\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v}\cdot\nabla P$, i.e., contains a time and advective term
  - in material description, the time derivative of the field P($\boldsymbol{\xi}$,*t*) is $\frac{DP}{Dt} = \frac{\partial P}{\partial t}$

- Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$
  - represents source/sink of a **v** field
  - can also be applied to tensors, e.g. $(\nabla \cdot \mathbf{T})_i = \frac{\partial T_{1i}}{\partial x_1} + \frac{\partial T_{2i}}{\partial x_2} + \frac{\partial T_{3i}}{\partial x_3}$

- Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$
  - represents vorticity of a **v** field

- Gradient:
  - of a scalar $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$
  - or of a vector $(\nabla \mathbf{v})_{ij} = \frac{\partial v_j}{\partial x_i}$

- Laplacian: $\nabla \cdot \nabla f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x_i \partial x_i}$

- Cauchy Stress tensor:
  - stress tensor component σ_{ij} represents a force in $\hat{\mathbf{e}}_j$ direction on a plane with normal in $\hat{\mathbf{e}}_i$ direction. Positive normal stress corresponds to extension.
  - stress tensor is symmetric: $\sigma_{ij} = \sigma_{ji}$ (conservation of angular momentum)
  - traction **t** on a plane with normal $\hat{\mathbf{n}}$ is $\mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$
  - the stress tensor can be diagonalised, with principal components σ_1, σ_2, σ_3 which include maximum and minimum normal stress
  - Can be decomposed into isotropic stress (pressure $p = -\sigma_{kk}/3$) and deviatoric stress $\boldsymbol{\sigma}'$ such that $\sigma_{ij}=-p\delta_{ij}+ \sigma'_{ij}$

- Conservation of linear momentum (per unit volume): $\rho \frac{D^2\mathbf{u}}{Dt^2} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$, where ρ is density, **u** is displacement and **f** is body force.

- Infinitesimal strain tensor:
  - Infinitesimal strain tensor component $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, where **u** is the displacement field. Applicable if $\nabla\mathbf{u} \ll 1$.
  - An original line segment described by vector **x** deforms to a new line segment $\mathbf{x}'$ as: $\mathbf{x}' = \boldsymbol{\varepsilon} \cdot \mathbf{x}$
  - Diagonal components of ε_{ij} represent fractional length changes, i.e., if **x** is a vector in $\hat{\mathbf{e}}_1$ direction than $\boldsymbol{\varepsilon}_{11} = \frac{|\mathbf{x}'|-|\mathbf{x}|}{|\mathbf{x}|}$. Similarly, for a given vector **s**, the product $\mathbf{s}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{s}$ corresponds to the fractional change in |**s**| by the strain **ε**.
  - Off-diagonal components represent changes in angle (i.e., shape), such that $2\varepsilon_{12}$ equals the change in angle between a line segment originally in $\hat{\mathbf{e}}_1$ direction and one originally in $\hat{\mathbf{e}}_2$ direction. Given two originally perpendicular vectors **s** and **p**, 2 x the product $\mathbf{p}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{s}$ corresponds to the change in the angle between **s** and **p** by the strain **ε**.
  - $tr(\boldsymbol{\varepsilon}) = \nabla \cdot \mathbf{u}$ and represents the fractional change in volume.
  - ε_{ij} is symmetric and can be diagonalised, such that principal strain components ε_1, ε_2, ε_3 include the maximum and minimum fractional length changes in the strain field described by **ε**.

- Can be decomposed into isotropic and deviatoric strain, like the stress tensor

- Strain rate tensor
  - Strain rate tensor $\mathbf{D}$=D$\boldsymbol{\varepsilon}$/Dt has same kind of properties as the infinitesimal strain tensor, but depends on the velocity field $\mathbf{v}$: $D_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$.
  - $tr(\mathbf{D}) = \nabla \cdot \mathbf{v} = 0$ means no change in volume and is the conservation of mass equation for an incompressible material.

- Energy equation – if given the equation, understand the different terms (conduction, advection, heat production, power dissipated by deformation) and be able to use (e.g., to solve for temperature for simple case)

- Rheology – know difference between elastic and viscous rheology. Be able to apply. If more complex equations are necessary, they will be given.

  - Elasticity – $\boldsymbol{\sigma}$=$\mathbf{C}$:$\boldsymbol{\varepsilon}$, linear relationship between stress and infinitesimal strain. For an isotropic medium, only two independent parameters, e.g. Lamé parameters (λ,μ), bulk and shear moduli (K,μ), or Young's modulus and Poisson's ratio (E,ν). In terms of Lamé parameters: $\boldsymbol{\sigma} = \lambda\theta \mathbf{I} + 2\mu\boldsymbol{\varepsilon}$, where θ = tr($\boldsymbol{\varepsilon}$). Bulk modulus:$-p=K\theta$, $K = \lambda + \frac{2}{3}\mu$; In uniaxial stress: Young's modulus $E = \sigma_{11}/\varepsilon_{11}$, and Poisson's ratio $\nu=-\varepsilon_{33}/\varepsilon_{11}$

  - Newtonian Viscosity – linear relationship between deviatoric stress $\boldsymbol{\sigma}$' and strain rate $\mathbf{D}$. If isotropic => bulk viscosity ζ and shear viscosity η as the two material parameters: $\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$

- Equations of motion – wave equation for elastic media and Navier Stokes for fluids, understand terms and derive simple solutions if equations given.

- Dimensional analysis
  - For non-dimensionalising equations, understand how to produce dimensionless versions of the dependent and independent variables and to use those to form the dimensionless the dimensionless version of the equations.
  - Understand how to use Buckingham Pi theory to determine the number of dimensionless groups required to describe a system and to produce an appropriate set of dimensionless groups