

Outline Lecture 3

Part 1: Kinematics

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Strain Tensor

chapter3.ipynb

Part 2: Conservation Equations

- Conservation of Mass
- Conservation of Momentum

chapter4.ipynb

Learning Objectives

Conservation Equations

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.

Today:

- Use the conservation of mass equation
- Use the conservation of linear momentum equation, i.e. balance body forces and stresses

Continuum Mechanics Equations

General:

1. Kinematics – describing deformation and velocity without considering forces
2. Dynamics – equations that describe force balance, conservation of linear and angular momentum
3. Thermodynamics – relations temperature, heatflux, stress, entropy

Material-specific

4. Constitutive equations – relations describing how material properties vary as a function of T,P, stress,.... Such material properties govern dynamics (e.g., *density*), response to stress (*viscosity, elastic parameters*), heat transport (*thermal conductivity, heat capacity*)

Kinematics:

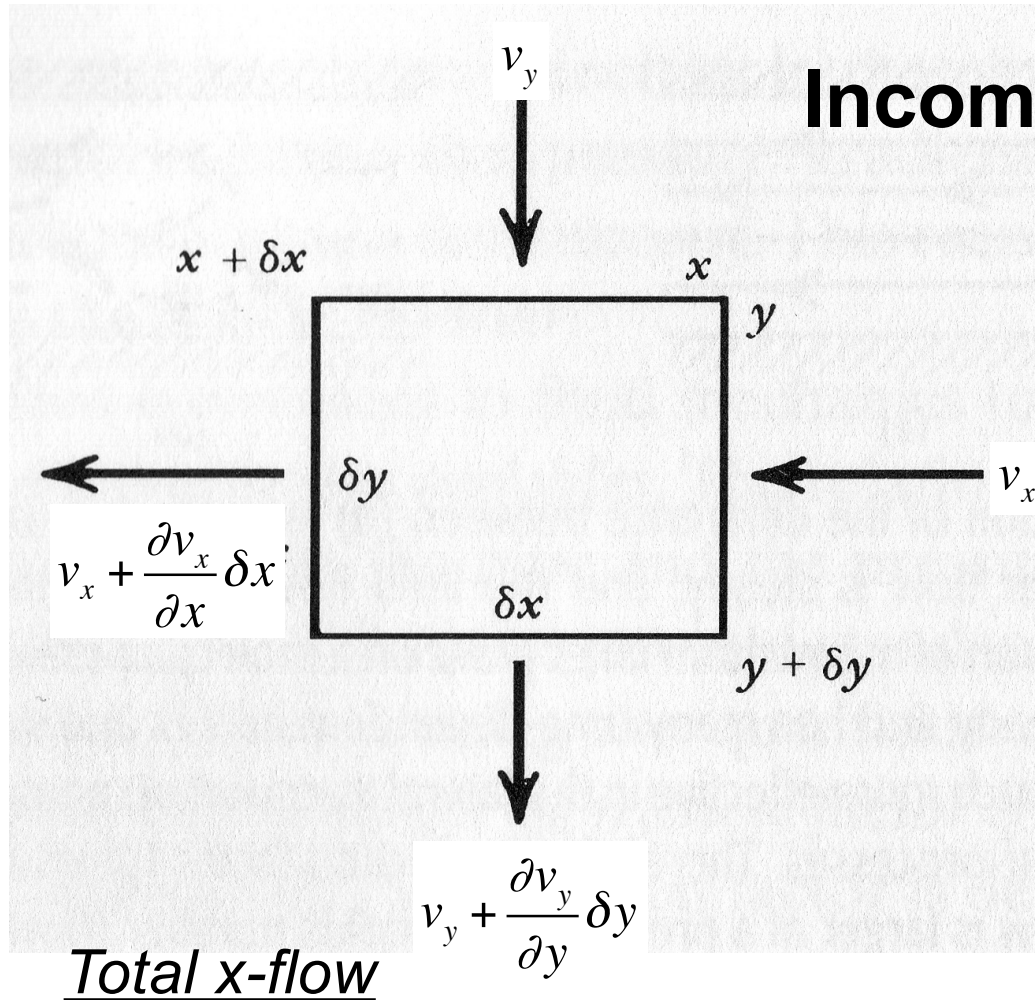
Conservation of Mass

- Describes that no material lost during flow or deformation
- Material-in balances material-out
- Need to take into account any potential changes in density (e.g. due to changes in temperature, pressure, phase)

2-D Conservation of Mass

Incompressible (ρ constant)

Continuity Equation

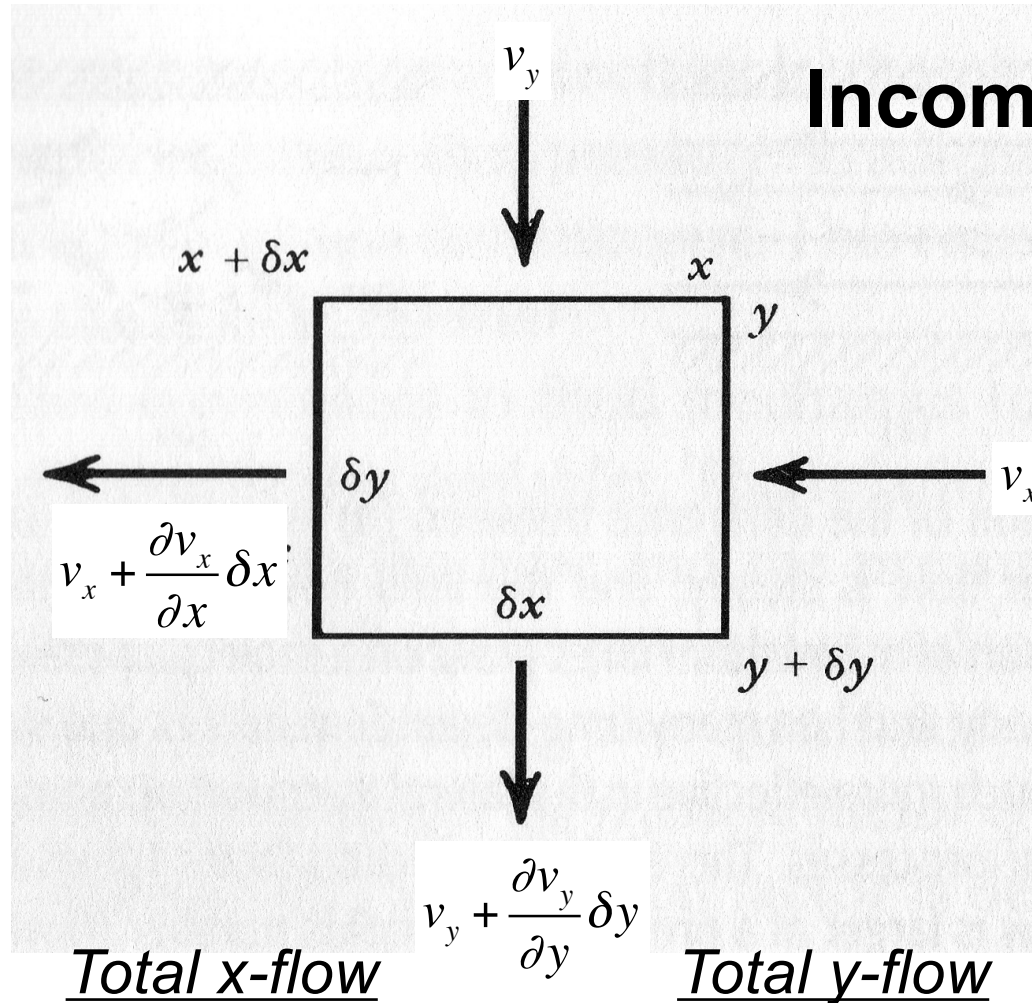


x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x} \delta x) \delta y$

$$\left\{ (v_x + \frac{\partial v_x}{\partial x} \delta x) - v_x \right\} \delta y$$

2-D Conservation of Mass



x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x} \delta x) \delta y$

Per unit area

$$\left\{ (v_x + \frac{\partial v_x}{\partial x} \delta x) - v_x \right\} \delta y + \left\{ (v_y + \frac{\partial v_y}{\partial y} \delta y) - v_y \right\} \delta x = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

i.e.,

$$\nabla \cdot \mathbf{v} = 0$$

which also applies in 3-D

No volume
changes!

Conservation of Mass

Full expression: compressible

$$\frac{D\rho dV}{Dt} = 0$$

ρ – density
 dV – infinitesimal volume

density changes \rightarrow

$$\frac{D\rho}{Dt} dV + \rho \frac{DdV}{Dt} = 0$$

\nearrow volume changes

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

In spatial description: $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$, so $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

$\rho(\text{time})$ advected

Dynamics:

Conservation of Momentum

- Linear force balance = Newton's second law, $\mathbf{F} = m\mathbf{a}$
- Relates force \mathbf{F} to motion, acceleration \mathbf{a} – hence also called “equation of motion”
- Conservation angular momentum assumed in symmetry of stress tensor

Equation of motion

Force balance:

$$\mathbf{F}_{\text{body}} + \mathbf{F}_{\text{stress}} = m\mathbf{a}$$

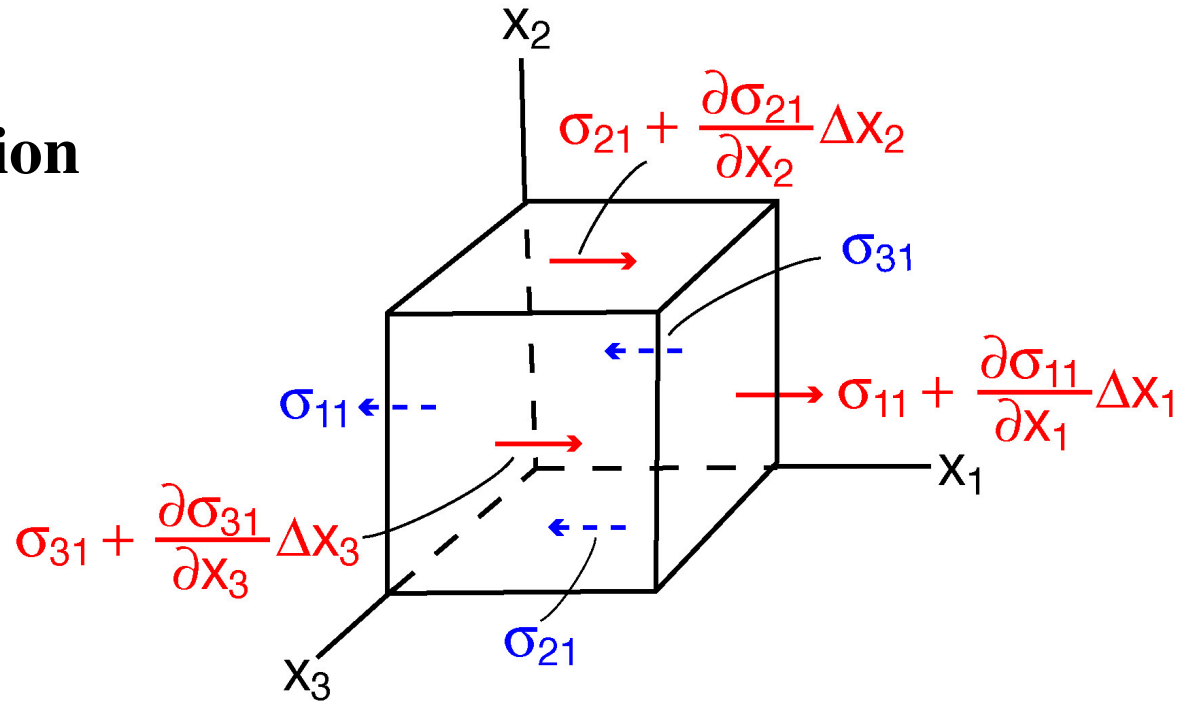
In x_1 - direction:

$$\begin{aligned} & f_1 \Delta x_1 \Delta x_2 \Delta x_3 + \\ & (\sigma_{11} + \Delta x_1 \partial \sigma_{11} / \partial x_1 - \sigma_{11}) \Delta x_2 \Delta x_3 + \\ & (\sigma_{21} + \Delta x_2 \partial \sigma_{21} / \partial x_2 - \sigma_{21}) \Delta x_1 \Delta x_3 + \\ & (\sigma_{31} + \Delta x_3 \partial \sigma_{31} / \partial x_3 - \sigma_{31}) \Delta x_1 \Delta x_2 \\ & = \rho \Delta x_1 \Delta x_2 \Delta x_3 \partial^2 u_1 / \partial t^2 \end{aligned}$$

$$\Rightarrow f_1 + \partial \sigma_{11} / \partial x_1 + \partial \sigma_{21} / \partial x_2 + \partial \sigma_{31} / \partial x_3 = \rho \partial^2 u_1 / \partial t^2$$

$$\Rightarrow f_i + \partial \sigma_{ji} / \partial x_j = \rho \partial^2 u_i / \partial t^2$$

$$\Rightarrow \mathbf{f} + \nabla \cdot \underline{\underline{\sigma}} = \rho \partial^2 \mathbf{u} / \partial t^2$$



Try yourself

chapter4.ipynb

Exercises 1 and 2

now/tomorrow

Learning Objectives

Kinematics & Conservation

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Use the conservation of mass equation
- Use the conservation of linear momentum equation, i.e. balance body forces and stresses