Introduction Tensors

- Tensors, generalisation of vectors to more dimensions
- Use when properties depend on direction in more than one way.
- A physical quantity that is independent of coordinate system used
- Derives from the word tension (= stress)
- Stress tensor as example
- *Not* just a multidimensional array

Tensors

Used in Stress, strain, moment tensors

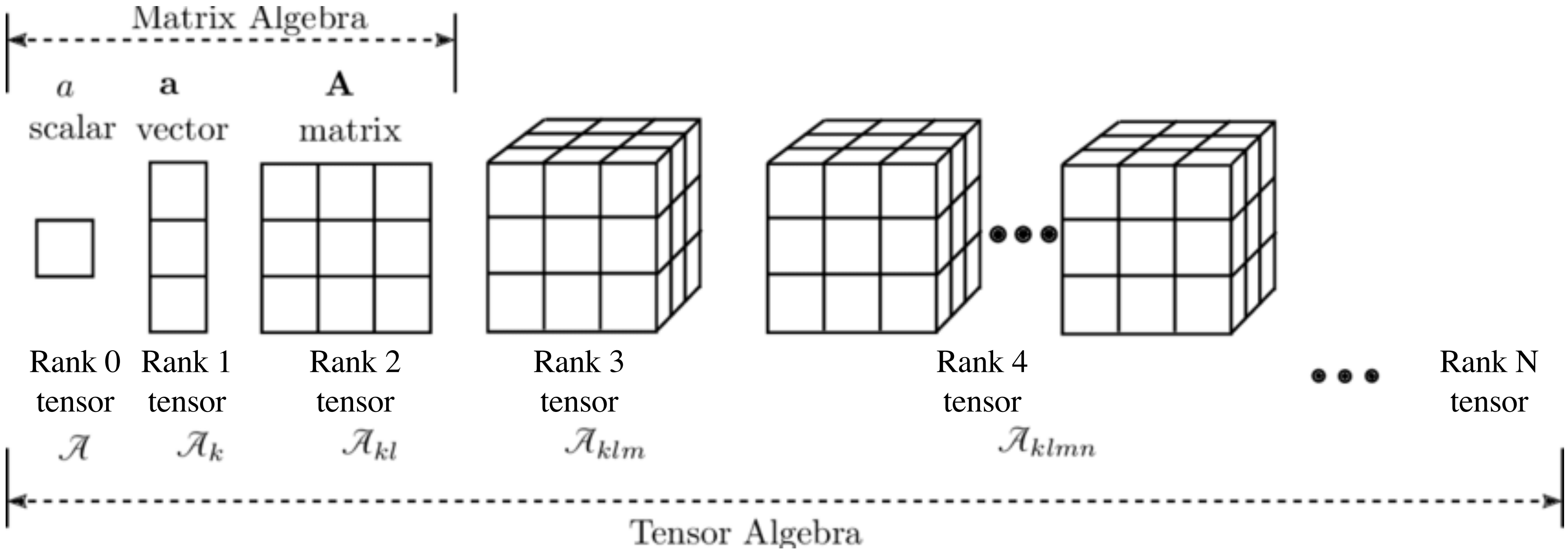
Electrostatics, electrodynamics, rotation, crystal properties

Tensors describe properties that depend on direction

Tensor rank 0 - scalar - independent of direction

Tensor rank 1 - vector - depends on direction in 1 way

Tensor rank 2 - tensor - depends on direction in 2 ways



Notation

- Tensors as **T**
- for second order: $\overline{\overline{T}}$ or $\underline{\underline{T}}$
- Index notation T_{ij}, $i,j=x,y,z$ or $i,j=1,2,3$
- For higher order T_{ijkl}

An example tensor

Gradient of velocity depends on direction in two ways

Component of velocity

$$\nabla v = \frac{\partial v_j}{\partial x_i} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Spatial variation in this direction

This tensor gradient definition common in fluid dynamics

An example tensor

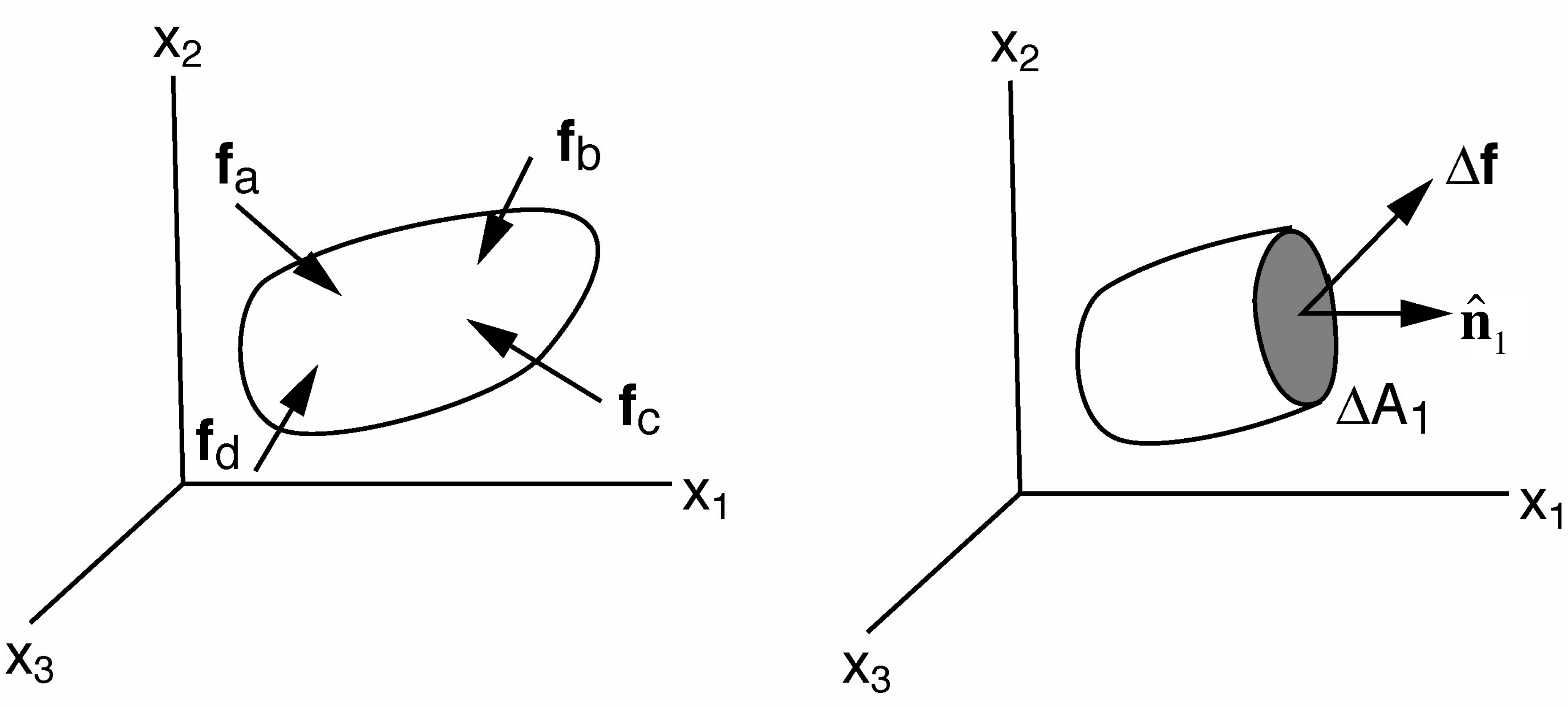
Gradient of velocity depends on direction in two ways

$$\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

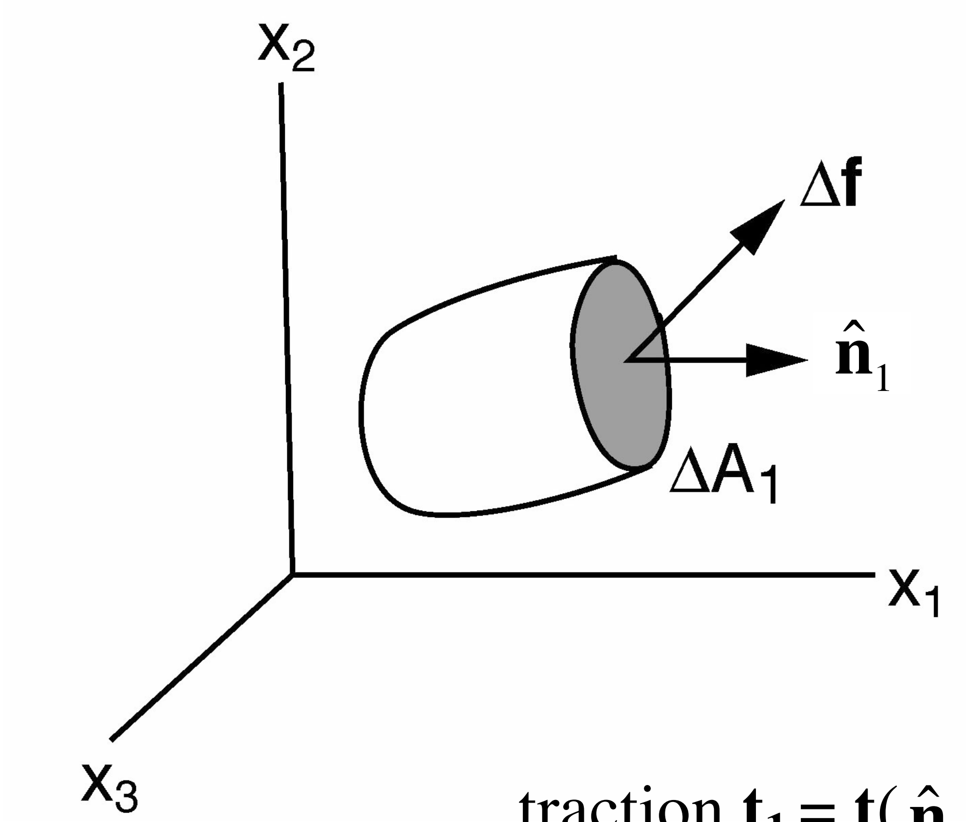
NOTE: some texts (including Lai et al., Reddy) use this ***transposed*** definition

Another example: Stress

- *Body forces* - depend on volume, e.g., gravity
- *Surface forces* - depend on surface area, e.g., friction



forces introduce a state of stress in a body

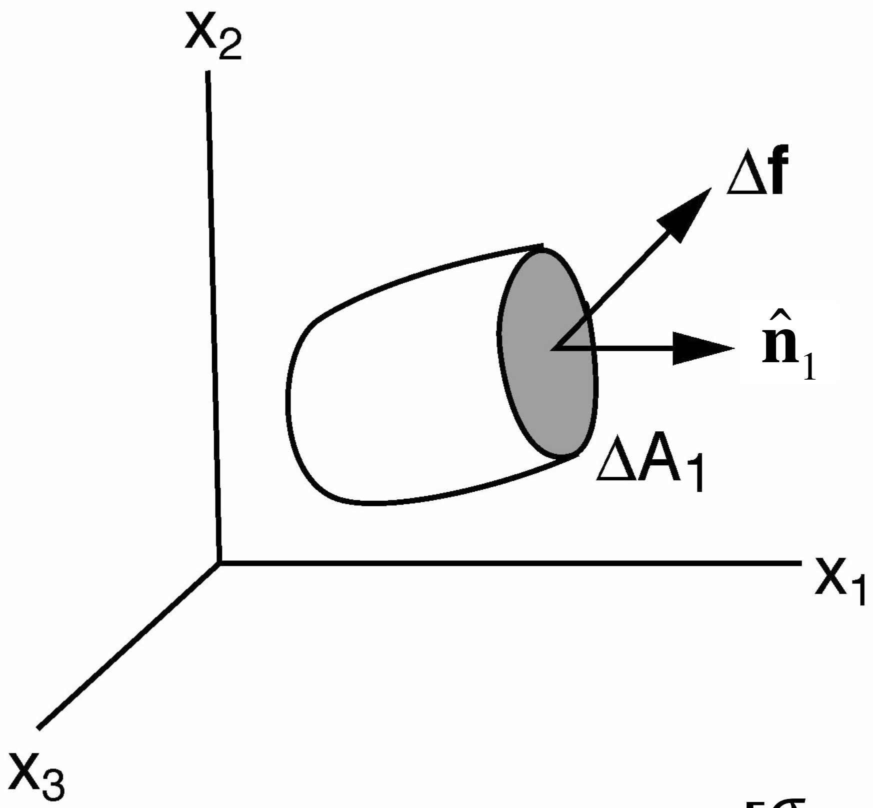


- $\Delta \mathbf{f}$ necessary to maintain equilibrium depends on orientation of the plane, $\hat{\mathbf{n}}_1$

traction $\mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta \mathbf{f}/\Delta A_1$

$$\mathbf{t_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

$$\sigma_{11} = \lim_{\Delta A1 \to 0} \Delta \mathbf{f}_1/\Delta A_1$$
$$\sigma_{12} = \lim_{\Delta A1 \to 0} \Delta \mathbf{f}_2/\Delta A_1$$
$$\sigma_{13} = \lim_{\Delta A1 \to 0} \Delta \mathbf{f}_3/\Delta A_1$$



Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

first index = orientation of plane
second index = orientation of force

Difference between a tensor and its matrix

Tensor – physical quantity that is independent of coordinate system used

Matrix of a tensor – contains components of that tensor in a particular coordinate frame

Could test that indeed tensor addition and multiplication satisfy transformation laws

Summation (Einstein) convention

When an index in a single term is a duplicate, dummy index, summation implied without writing summation symbol

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = a_i v_i$$

$$\begin{aligned}\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i y_j = a_{ij} x_i y_j &= a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{13} x_1 y_3 \\ &+ a_{21} x_2 y_1 + a_{22} x_2 y_2 + a_{23} x_2 y_3 \\ &+ a_{31} x_3 y_1 + a_{32} x_3 y_2 + a_{33} x_3 y_3\end{aligned}$$

Invalid, indices repeated more than twice

$$\sum_{i=1}^{3} a_i b_i v_i \neq a_i b_i v_i$$

Notation conventions

index notation

$$\alpha_{ij}x_iy_j=$$

matrix-vector notation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y} =$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

other versions index notation

$$\alpha_{ij}x_iy_j= x_i\alpha_{ij}y_j=$$
$$\alpha_{ij}y_jx_i$$

Dummy vs free index

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = \sum_{k=1}^{3} a_k v_k$$

- i,k – dummy index – appears in duplicates and can be substituted without changing equation

$$F_j = A_j \sum_{i=1}^{3} B_i C_i \quad \Rightarrow$$

$$F_1 = A_1 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right)$$
$$F_2 = A_2 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right)$$
$$F_3 = A_3 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right)$$

- j – free index, appears once in each term of the equation

Exercise 7

1. $g_k = h_k(2-3a_ib_i) - p_jq_jf_k$ - *Which dummy, which free indices, how many equations, how many terms in each?*
2. *Are these valid expressions?*
   a) $a_m b_s = c_m(d_r - f_r)$
   b) $x_i x_i = r^2$
   c) $a_i b_j c_j = 3$

Addition and subtraction of tensors

$\mathbf{W} = a\mathbf{T}+b\mathbf{S}$

add each component: $W_{ijkl} = aT_{ijkl} + bS_{ijkl}$

T and **S** must have same rank, dimension and units

W has same rank, dimension and units as **T** and **S**

T and **S** are tensors => **W** is a tensor

commutative, associative

This is the same as how vectors and matrices are added.

Multiplication of tensors

Inner product = dot product

$\mathbf{W} = \mathbf{T}\cdot\mathbf{S}$
involves contraction over one index: $W_{ik} = T_{ij}S_{jk}$
As normal matrix and matrix-vector multiplication

T and **S** can have different rank, but same dimension
rank$\mathbf{W}$ = rank$\mathbf{T}$+rank$\mathbf{S}$-2, dimension as **T** and **S**,
units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$
$\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\varepsilon}$ or $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ (Hooke's law)

Multiplication of tensors

Tensor product=outer product = dyadic product
≠ cross product

$\mathbf{W} = \mathbf{TS}$ often written as $\mathbf{W} = \mathbf{T} \otimes \mathbf{S}$
no contraction: $W_{ijkl} = T_{ij}S_{kl}$

T and **S** can have different rank, but same dimension
rank**W** = rank**T**+rank**S**, dimension as **T** and **S**,
units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $\nabla\mathbf{v}$ (gradient of a vector) ≠ $\nabla\cdot\mathbf{v}$ (divergence)

remember gradient is a vector $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

Multiplication of tensors

For both multiplications

Distributive: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

Associative: $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

Not commutative: $\mathbf{TS} \neq \mathbf{ST}$, $\mathbf{T{\cdot}S} \neq \mathbf{S{\cdot}T}$
but: $\mathbf{T{\cdot}S}=\mathbf{S}^{T}\cdot\mathbf{T}^{T}$
and: $\mathbf{ab}=(\mathbf{ba})^{\mathbf{T}}$ but only for rank 2

Remember **transpose**: $\mathbf{a{\cdot}T{\cdot}b}=\mathbf{b{\cdot}T}^{T}\mathbf{\cdot a}$ => $T_{ji}=T^{T}{}_{ij}$

Special tensor:
Kronecker delta δ_{ij}

Isotropic tensors, invariant upon coordinate transformation
- scalars
- **0** vector
- δ_{ij}

$$\delta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$$

$\delta_{ij} = 1$ for $i=j$, $\delta_{ij} = 0$ for $i \neq j$

In 3-D:

$$\delta = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{T \cdot \delta = T \cdot I = T}$ or $T_{ij}\delta_{jk} = T_{ik}$

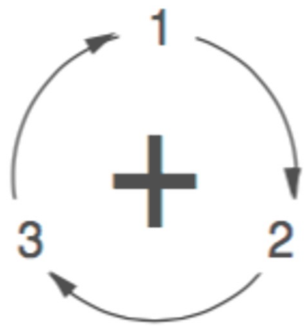
δ is isotropic: $\delta_{ij} = \delta'_{ij}$ upon coordinate transformation
can be used to write dot product: $T_{ij}S_{jl} = T_{ij}S_{kl}\delta_{jk}$
can be used to write trace: $A_{ii} = A_{ij}\delta_{ij}$
orthonormal transformation: $\alpha_{ij}\alpha^T{}_{jk} = \delta_{ik}$

Exercise 8

Special tensor: Permutation symbol ε_{ijk}

$$\varepsilon_{ijk} = \left(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j\right) \cdot \hat{\mathbf{e}}_k$$

ε_{ijk} = 1 if i,j,k an even permutation of 1,2,3
ε_{ijk} = -1 if i,j,k an odd permutation of 1,2,3
ε_{ijk} = 0 for all other i,j,k



$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$
$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{222} = \ldots = 0$

Note that $\varepsilon_{ijk} a_i b_j \hat{e}_k$ where $\hat{e}_k$ is the unit vector in k direction is index notation for cross product **a** x **b**

Exercise: useful identity $\varepsilon_{ijm}\,\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

Exercise 8

Vector derivatives - curl

Curl of a vector: $$\nabla \times \mathbf{v} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} v_j \hat{\mathbf{e}}_k = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\[2ex] \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\[2ex] \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

In index notation, using special tensor

Some tensor calculus

Gradient of a vector is a tensor: $\boldsymbol{\nabla}\mathbf{v} = \dfrac{\partial v_j}{\partial x_i} = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_2}{\partial x_2} & \dfrac{\partial v_3}{\partial x_2} \\ \dfrac{\partial v_1}{\partial x_3} & \dfrac{\partial v_j}{\partial x_3} & \dfrac{\partial v_3}{\partial x_3} \end{bmatrix}$

Such that the change **dv** in field **v** in direction **dx** is: $\mathbf{dv} = \mathbf{dx} \cdot \boldsymbol{\nabla}\mathbf{v}$

Divergence of a vector: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_i}{\partial x_i} = \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3}$

$$\nabla \cdot \mathbf{v} = tr\left(\nabla \mathbf{v}\right)$$

Trace of a tensor is the sum of diagonal elements

Some tensor calculus

Divergence of a tensor:

$$\nabla \cdot \mathbf{T} = \frac{\partial T_{ij}}{\partial x_i} = \begin{bmatrix} \frac{\partial T_{i1}}{\partial x_i} \\ \frac{\partial T_{i2}}{\partial x_i} \\ \frac{\partial T_{i3}}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} \\ \frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} \\ \frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{bmatrix}$$

vector

Laplacian = div(grad *f*), where *f* is a scalar function

$$\nabla \cdot \nabla f = \nabla^2 f = \Delta f = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Learning Objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention

Summary

- **Vectors**
  - Addition, linear independence
  - Orthonormal Cartesian bases, transformation
  - Multiplication
  - Derivatives, del, div, curl
- **Tensors**
  - Tensors, rank, stress tensor
  - Index notation, summation convention
  - Addition, multiplication
  - Special tensors, δ_{ij} and ε_{ijk}
  - Tensor calculus: gradient, divergence, curl, ..

Further reading/studying e.g: ***Reddy*** *(2013) 2.2.1-2.2.3, 2.2.5, 2.2.6, 2.4.1, 2.4.4, 2.4.5, 2.4.6, 2.4.8 (not co/contravariant),* ***Lai, Rubin, Kremple*** *(2010): 2.1-2.13, 2.16, 2.17, 2.27-2.32, 4.1-4.3,* ***Khan Academy*** *– linear algebra, multivariate calculus*

Try yourself

- For this part of the lecture, try **Exercise 7** and *optional advanced* **Exercise 8**

- Try to finish in the afternoon workshop: **Exercise 2, 3, 5, 6, 7, 9**

- Additional practise: **Exercise 1, 4**
- Advanced practise: **Exercise 8, 10**