Prova 2

Probabilidade — Primeiro Semestre

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Questão 1

Sejam $\boldsymbol{X}$ e $\boldsymbol{Y}$ variáveis aleatórias independentes com função de probabilidade geométrica de parâmetros $\alpha > 0$ e $\beta > 0$ respectivamente.

$$\mathbb{P}(\mathbf{X} = k) = \alpha(1-\alpha)^{k-1}, \qquad \mathbb{P}(\mathbf{Y} = k) = \beta(1-\beta)^{k-1}, \qquad k = 1, 2, \ldots$$

Definimos:

$$\mathbf{U} = \min\{\mathbf{X}, \mathbf{Y}\}, \qquad \mathbf{V} = \max\{\mathbf{X}, \mathbf{Y}\}, \qquad \mathbf{W} = \mathbf{V} - \mathbf{U}$$

1. Calcular a probabilidade conjunta de $(\mathbf{U}, \mathbf{V})$

Solução:

Probabilidade conjunta de $(\mathbf{U}, \mathbf{V})$:

$$\begin{aligned}
\mathbb{P}(\mathbf{U} = u, \mathbf{V} = v) &= \mathbb{P}(\min\{\mathbf{X}, \mathbf{Y}\} = u, \max\{\mathbf{X}, \mathbf{Y}\} = v, \mathbf{X} \geq \mathbf{Y}) + \\
&\quad + \mathbb{P}(\min\{\mathbf{X}, \mathbf{Y}\} = u, \max\{\mathbf{X}, \mathbf{Y}\} = v, \mathbf{X} < \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{Y} = u, \mathbf{X} = v, v \geq u) + \mathbb{P}(\mathbf{X} = u, \mathbf{Y} = v, u < v) \\
&= \mathbb{P}(\mathbf{Y} = u)\mathbb{P}(\mathbf{X} = v)\mathbb{1}(v \geq u) + \mathbb{P}(\mathbf{X} = u)\mathbb{P}(\mathbf{Y} = v)\mathbb{1}(u < v) \\
&= \alpha(1-\alpha)^{v-1}\beta(1-\beta)^{u-1}\mathbb{1}(v \geq u) + \alpha(1-\alpha)^{u-1}\beta(1-\beta)^{v-1}\mathbb{1}(u < v) \\
&= \alpha\beta\left[(1-\alpha)^{v-1}(1-\beta)^{u-1}\mathbb{1}(v \geq u) + (1-\alpha)^{u-1}(1-\beta)^{v-1}\mathbb{1}(u < v)\right]
\end{aligned}$$

- resultados adicionais:

  função de probabilidade do $\min\{\mathbf{X}, \mathbf{Y}\} = \mathbf{U}$

$$
\begin{aligned}
\mathbb{P}(\mathbf{U}=u) &= \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=k) \\
&= \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=k,\mathbf{X}\leq\mathbf{Y})) + \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=k,\mathbf{X}>\mathbf{Y})) \\
&= \mathbb{P}(\mathbf{X}=k,\mathbf{Y}\geq k) + \mathbb{P}(\mathbf{X}>k,\mathbf{Y}=k)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbf{X}>k) &= \mathbb{P}(\mathbf{X}\geq k) - \mathbb{P}(\mathbf{X}=k) \\
&= (1-\alpha)^{k-1} - \alpha(1-\alpha)^{k-1} \\
&= (1-\alpha)^{k-1}(1-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbf{X}\geq k) &= \sum_{k=u}^{\infty}\alpha(1-\alpha)^{k-1} \\
&= \alpha(1-\alpha)^{u-1}\left[1+(1-\alpha)^1+(1-\alpha)^2+\ldots\right] \\
&= \alpha(1-\alpha)^{u-1}\sum_{i=0}^{\infty}(1-\alpha)^i \\
&= \alpha(1-\alpha)^{u-1}\frac{1}{1-(1-\alpha)} \;=\; (1-\alpha)^{u-1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbf{U}=u) &= \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=k) \\
&= \left[\alpha(1-\alpha)^{k-1}(1-\beta)^{k-1} + (1-\alpha)^{k-1}(1-\alpha)\beta(1-\beta)^{k-1}\right] \\
&= \left[(1-\alpha)(1-\beta)\right]^{k-1}\left[\alpha+\beta(1-\alpha)\right] \\
&= \left[(1-\alpha)(1-\beta)\right]^{k-1}\left[\alpha+\beta-\alpha\beta\right]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\mathbb{P}(\mathbf{U}=k) &= \sum_{k=1}^{\infty}\left[(1-\alpha)(1-\beta)\right]^{k-1}\left[\alpha+\beta(1-\alpha)\right] \\
&= \left[\alpha+\beta-\alpha\beta\right]\cdot\frac{1}{1-(1-\alpha)(1-\beta)} \;=\; 1
\end{aligned}
$$

Logo,

$$
\mathbb{P}(\mathbf{U}=u) \;=\; \left[(1-\alpha)(1-\beta)\right]^{k-1}\left[\alpha+\beta-\alpha\beta\right]
$$

Função de probabilidade do $\max\{\mathbf{X},\mathbf{Y}\}=\mathbf{V}$

$$
\begin{aligned}
\mathbb{P}(\max\{\mathbf{X},\mathbf{Y}\}) &= \mathbb{P}(\max\{\mathbf{X},\mathbf{Y}\}=k,\mathbf{X}\le\mathbf{Y}) \ + \ \mathbb{P}(\max\{\mathbf{X},\mathbf{Y}\}=k,\mathbf{X}>\mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X}\le k,\mathbf{Y}=k)+\mathbb{P}(\mathbf{X}=k,\mathbf{Y}<k) \\
&= \mathbb{P}(\mathbf{X}\le k)\mathbb{P}(\mathbf{Y}=k)+\mathbb{P}(\mathbf{X}=k)\mathbb{P}(\mathbf{Y}<k)
\end{aligned}
$$

$$
\mathbb{P}(\mathbf{X}\le k) = 1-\mathbb{P}(\mathbf{X}>k) \ = \ 1-\left[(1-\alpha)^{k-1}(1-\alpha)\right]
$$

$$
\mathbb{P}(\mathbf{Y}<k) = 1-\mathbb{P}(\mathbf{X}\ge k) \ = \ 1-\left[(1-\beta)^{k-1}\right]
$$

$$
\begin{aligned}
\mathbb{P}(\max\{\mathbf{X},\mathbf{Y}\}) &= \left\{1-\left[(1-\alpha)^{k-1}(1-\alpha)\right]\right\}\beta(1-\beta)^{k-1}+ \\
&\quad + \ \left\{1-\left[(1-\beta)^{k-1}\right]\right\}\alpha(1-\alpha)^{k-1} \\
&= \beta(1-\beta)^{k-1}-\beta\beta(1-\beta)^{k-1}(1-\alpha)^{k-1}(1-\alpha)+ \\
&\quad +\alpha(1-\alpha)^{k-1}-\alpha(1-\alpha)^{k-1}(1-\beta)^{k-1} \\
&= (1-\beta)^{k-1}\left[\beta-\beta(1-\alpha)^{k-1}(1-\alpha)-\alpha(1-\alpha)^{k-1}\right] \ + \ \alpha(1-\alpha)^{k-1} \\
&= \beta(1-\beta)^{k-1} \ + \ \alpha(1-\alpha)^{k-1} \ - \ [(1-\alpha)(1-\beta)]^{k-1}\,[\beta(1-\alpha)+\alpha]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\mathbb{P}(\mathbf{V}=k) &= \sum_{k=1}^{\infty}\left\{\beta(1-\beta)^{k-1}+\alpha(1-\alpha)^{k-1}-[\beta(1-\alpha)+\alpha]\,[(1-\alpha)(1-\beta)]^{k-1}\right\} \\
&= \beta\sum_{k=1}^{\infty}(1-\beta)^{k-1}+\alpha\sum_{k=1}^{\infty}(1-\alpha)^{k-1}-[\beta(1-\alpha)+\alpha]\cdot \\
&\quad \cdot\sum_{k=1}^{\infty}[(1-\alpha)(1-\beta)]^{k-1} \\
&= \beta\cdot\frac{1}{1-(1-\beta)}+\alpha\cdot\frac{1}{1-(1-\alpha)}-[\beta(1-\alpha)+\alpha]\cdot\frac{1}{1-(1-\alpha)(1-\beta)} \\
&= 1+1-1 \ = \ 1
\end{aligned}
$$

Logo,

$$
\mathbb{P}(\mathbf{V}=k) = \beta(1-\beta)^{k-1}+\alpha(1-\alpha)^{k-1}-[(1-\alpha)(1-\beta)]^{k-1}\,[\beta(1-\alpha)+\alpha]
$$

2.Provar que **U** e **V** são independentes

Solução:

$$
\mathbf{U}=\min\{\mathbf{X},\mathbf{Y}\}, \qquad \mathbf{W}=\max\{\mathbf{X},\mathbf{Y}\}-\mathbf{U}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbf{U}=u,\mathbf{W}=w) &= \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=u,\max\{\mathbf{X},\mathbf{Y}\}-\min\{\mathbf{X},\mathbf{Y}\}=w)\\
&= \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=u,\max\{\mathbf{X},\mathbf{Y}\}-\min\{\mathbf{X},\mathbf{Y}\}=w,\mathbf{X}\le\mathbf{Y}))\\
&\quad + \mathbb{P}(\min\{\mathbf{X},\mathbf{Y}\}=u,\max\{\mathbf{X},\mathbf{Y}\}-\min\{\mathbf{X},\mathbf{Y}\}=w,\mathbf{X}>\mathbf{Y}))\\
&= \mathbb{P}(\mathbf{X}=u,\mathbf{Y}-\mathbf{X}=w,\mathbf{Y}\ge\mathbf{X})+\mathbb{P}(\mathbf{Y}=u,\mathbf{X}-\mathbf{Y}=w,\mathbf{X}>\mathbf{Y})\\
&= \mathbb{P}(\mathbf{X}=u,\mathbf{Y}=w+u,w+u\ge u)+\mathbb{P}(\mathbf{Y}=u,\mathbf{X}=w+u,w+u>u)\\
&= \mathbb{P}(\mathbf{X}=u,\mathbf{Y}=w+u,w\ge 0)+\mathbb{P}(\mathbf{Y}=u,\mathbf{X}=w+u,w>0)\\
&= \alpha(1-\alpha)^{u-1}\beta(1-\beta)^{w+u-1}\mathbb{1}_{\{0,1,2,\ldots\}}(w)+\\
&\quad + \alpha(1-\alpha)^{w+u-1}\beta(1-\beta)^{u-1}\mathbb{1}_{\{1,2,\ldots\}}(w)\\
&= \alpha\beta(1-\alpha)^{u-1}\beta(1-\beta)^{u-1}\cdot\\
&\quad\cdot\left[(1-\beta)^{w}\mathbb{1}_{\{0,1,2,\ldots\}}(w)+(1-\alpha)^{w}\mathbb{1}_{\{1,2,\ldots\}}(w)\right]
\end{aligned}
$$

Agora, iremos encontrar as marginais,

$$
\begin{aligned}
\mathbb{P}(\mathbf{U}=u) &= \sum_{w=0}^{\infty}\alpha\beta(1-\alpha)^{u-1}(1-\beta)^{u-1}(1-\beta)^{w}+\\
&\quad+\sum_{w=1}^{\infty}\alpha\beta(1-\alpha)^{u-1}(1-\beta)^{u-1}(1-\alpha)^{w}\\
&= \alpha\beta(1-\alpha)^{u-1}(1-\beta)^{u-1}\cdot\\
&\quad\cdot\left[\sum_{w=0}^{\infty}(1-\beta)^{w}+\sum_{w=1}^{\infty}(1-\alpha)^{w}\right]\\
&= \alpha\beta\left[(1-\alpha)(1-\beta)\right]^{u-1}\cdot\\
&\quad\cdot\left[\frac{1}{1-(1-\beta)}+\frac{(1-\alpha)}{1-(1-\alpha)}\right]\\
&= \left[(1-\alpha)(1-\beta)\right]^{u-1}(\alpha+\beta-\alpha\beta)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\mathbf{W}=w) &= \sum_{w=0}^{\infty}\alpha\beta(1-\alpha)^{u-1}(1-\beta)^{u-1}\cdot\\
&\quad\cdot\left[(1-\beta)^{w}\mathbb{1}_{\{0,1,2,\ldots\}}(w)+(1-\alpha)^{w}\mathbb{1}_{\{1,2,\ldots\}}(w)\right]\\
&= \alpha\beta\left[(1-\beta)^{w}\mathbb{1}_{\{0,1,2,\ldots\}}(w)+(1-\alpha)^{w}\mathbb{1}_{\{1,2,\ldots\}}(w)\right]\cdot\\
&\quad\cdot\sum_{w=0}^{\infty}\left[(1-\alpha)(1-\beta)\right]^{u-1}\\
&= \alpha\beta\left[(1-\beta)^{w}\mathbb{1}_{\{0,1,2,\ldots\}}(w)+(1-\alpha)^{w}\mathbb{1}_{\{1,2,\ldots\}}(w)\right]\cdot\\
&\quad\cdot\frac{1}{1-(1-\alpha)(1-\beta)}
\end{aligned}
$$

$$= \quad \alpha\beta\frac{1}{(\alpha+\beta-\alpha\beta)} \cdot \left[(1-\beta)^w \mathbb{1}_{\{0,1,2,\dots\}}(w) + (1-\alpha)^w \mathbb{1}_{\{1,2,\dots\}}(w)\right]$$

Logo, como $\mathbb{P}(\mathbf{U} = u, \mathbf{W} = w) = \mathbb{P}(\mathbf{U} = u) \cdot \mathbb{P}(\mathbf{W} = w)$ então, $\mathbf{U}$ e $\mathbf{W}$ são independentes, assim está provado.

Questão 2

Seja o espaço amostral $\Omega = \{a, b, c\}$ e considere a sigma álgebra como o conjunto das partes. Definimos as probabilidades sobre este espaço por

$$\mathbb{P}(\{\mathbf{a}\}) = \frac{1}{2} \quad \mathbb{P}(\{\mathbf{b}\}) = \frac{1}{4} \quad \mathbb{P}(\{\mathbf{c}\}) = \frac{1}{4}.$$

Sejam as variáveis aleatórias $\mathbf{X}$ e $\mathbf{Y}$ definidas por

$$\mathbf{X}(\omega) = \mathbf{I}_{\{a\}}(\omega) - \mathbf{I}_{\{b,c\}}(\omega) \quad \text{e} \quad \mathbf{Y}(\omega) = \mathbf{I}_{\{b\}}(\omega) - \mathbf{I}_{\{c\}}(\omega).$$

Onde o indicador é definido por

$$\mathbf{I}_{\{A\}}(\omega) \quad = \quad \begin{cases} 1 & \text{se} \quad \omega \in A \\ 0 & \text{se} \quad \text{caso contrário.} \end{cases}$$

1. Calcular as distribuições de probabilidade de $\mathbf{X}$ e $\mathbf{Y}$.

Solução:

$$\mathbf{X}(\{\mathbf{a}\}) \quad = \quad 1, \quad \mathbf{X}(\{\mathbf{b}\}) \quad = \quad -1, \quad \mathbf{X}(\{\mathbf{c}\}) \quad = \quad -1$$

$$\mathbb{P}(\mathbf{X} = 1) = \frac{1}{2}, \qquad \mathbb{P}(\mathbf{X} = -1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

assim a distribuição de probabilidade de $\mathbf{X}$ é:

Table 1: Distribuição de probabilidade de $\mathbf{X}$

$\mathbf{X}$	-1	1
$\mathbb{P}(\mathbf{X} = k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\mathbf{Y}(\{\mathbf{a}\}) \quad = \quad 0, \quad \mathbf{Y}(\{\mathbf{b}\}) \quad = \quad 1, \quad \mathbf{Y}(\{\mathbf{c}\}) \quad = \quad -1$$

$$\mathbb{P}(\mathbf{Y} = 0) = \frac{1}{2}, \qquad \mathbb{P}(\mathbf{Y} = 1) = \frac{1}{4}, \qquad \mathbb{P}(\mathbf{Y} = -1) = \frac{1}{4}.$$

assim a distribuição de probabilidade de $\mathbf{Y}$ é:

Table 2: Distribuição de probabilidade de $\mathbf{Y}$

$\mathbf{Y}$	-1	0	1
$\mathbb{P}(\mathbf{Y} = k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

2. Calcular $\mathbb{E}[\mathbf{XY}]$. $\mathbf{X}$ e $\mathbf{Y}$ são independentes?

Solução:

Como,

$$\begin{aligned}
\mathbb{E}(\mathbf{X}) &= -1 \cdot \mathbb{P}(\mathbf{X} = -1) + 1 \cdot \mathbb{P}(\mathbf{X} = 1) \\
&= -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \;=\; 0 \\
\mathbb{E}(\mathbf{Y}) &= -1 \cdot \mathbb{P}(\mathbf{Y} = -1) + 0 \cdot \mathbb{P}(\mathbf{Y} = 0) + 1 \cdot \mathbb{P}(\mathbf{Y} = 1) \\
&= -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \;=\; 0
\end{aligned}$$

Logo, $\mathbf{X}$ e $\mathbf{Y}$ são independentes, dado que a probabilidade de um evento em $\mathbf{X}$ ocorrer não depende de nenhum evento ocorrer em logo, $\mathbf{X}$ e $\mathbf{Y}$ são independentes, dado que a probabilidade de um evento em $\mathbf{Y}$.

Portanto, pela propriedade da esperança,

$$\mathbb{E}(\mathbf{XY}) \;=\; \mathbb{E}(\mathbf{X}) \cdot \mathbb{E}(\mathbf{Y}) \;=\; 0$$

Questão 3

Sejam $\mathbf{X}$ e $\mathbf{Y}$ variáveis aleatórias independentes com função densidade de probabilidade uniforme no intervalo $[0, a]$. Definimos $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$

1. Fazer um desenho da função de distribuição de $\mathbf{Z}$, para $a > 0$.

Solução:

2. Calcular a função densidade de $\mathbf{Z}$, para $a > 0$.

Solução:

$$\begin{aligned}
f_x(x) &= \frac{1}{a} = f_y(y);\; a > 0 \\
f_z(z) &= \int_{\mathbf{X}} f_x(x) \cdot f_y(z - x) dx
\end{aligned}$$

Logo,

se $0 < z < a$

$$f_z(z) \;=\; \int_0^z \frac{1}{a} \cdot \frac{1}{a} dx = \frac{1}{a^2} \cdot x \Big|_0^z \;=\; z \cdot \frac{1}{a^2} \cdot \mathbb{1}_{(0,a)}(z)$$

se $a < z < 2a$

$$f_z(z) = \int_{z-a}^{a} \frac{1}{a}\cdot\frac{1}{a}dx = \frac{1}{a^2}\cdot x\Big|_{z-a}^{a} = (a-z+a)\cdot\frac{1}{a^2} = (2a-z)\cdot\frac{1}{a^2}\cdot \mathbb{1}_{(a,2a)}(z)$$

Questão 4

Seja o vetor $(\mathbf{X},\mathbf{Y})$ aleatório com função densidade conjunta densidade conjunta dada por

$$f_{\mathbf{XY}}(x,y) = \begin{cases} 2e^{-x-y} & \text{se} \quad 0 < x < y < \infty \\ 0 & \text{se} \quad \text{caso contrário.} \end{cases}$$

1. Calcular $\mathbb{E}[\mathbf{X}]$ e $\mathbb{E}[\mathbf{Y}]$.

Solução:

$$\begin{aligned} f_x(x) &= \int_x^{\infty} 2e^{-x-y}dy = 2e^{-x}\int_x^{\infty} e^{-y}dy \\ &= 2e^{-x}\left(-e^{-y}\big|_x^{\infty}\right) = 2e^{-x}e^{-x} = 2e^{-2x}\mathbb{1}_{(0,\infty)}(x) \end{aligned}$$

$$\begin{aligned} f_y(y) &= \int_0^{y} 2e^{-x-y}dx = 2e^{-y}\int_0^{y} e^{-x}dx \\ &= 2e^{-y}\left(-e^{-x}\big|_0^{y}\right) = 2e^{-y}(-e^{-y}+1) = 2e^{-y}(1-e^{-y})\mathbb{1}_{(0,\infty)}(y) \end{aligned}$$

Assim,

$$\begin{aligned} \mathbb{E}(\mathbf{X}) &= \int_0^{\infty} x\cdot 2e^{-2x}dx = 2\int_0^{\infty} xe^{-2x}dx \\ &= 2\left[x\cdot -\frac{1}{2}e^{-2x}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{2}e^{-2x}dx\right] = \\ &= 2\left[\frac{1}{2}\int_0^{\infty} e^{-2x}dx\right] = \left(-\frac{1}{2}e^{-2x}\Big|_0^{\infty}\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbb{E}(\mathbf{Y}) &= \int_0^{\infty} y\cdot 2e^{-2y}(1-e^{-2y})dy = 2\int_0^{\infty}\left(ye^{-2y} - ye^{-2y}\right)dy \\ &= 2\left\{\left[\left(ye^{-y}\big|_0^{\infty}\right) - \int_0^{\infty} -e^{-2y}dy\right] - \left[\left(y\cdot -\frac{1}{2}e^{-2y}\Big|_0^{\infty}\right) - \int_0^{\infty} -\frac{1}{2}e^{-2y}dy\right]\right\} \\ &= 2\left[\int_0^{\infty} e^{-y}dy - \left(\frac{1}{2}\int_0^{\infty} -e^{-2y}dy\right)\right] \\ &= 2\left[\left(-e^{-y}\big|_0^{\infty}\right) - \frac{1}{2}\cdot\left(-\frac{1}{2} - e^{-2y}\big|_0^{\infty}\right)\right] \\ &= 2\left[1 - \left(\frac{1}{2}\cdot\frac{1}{2}\right)\right] = 2\left(1-\frac{1}{4}\right) = 2\left(\frac{3}{4}\right) = \frac{3}{2} \end{aligned}$$

2. Calcular a esperança condicional de $\mathbb{E}[\mathbf{Y}|\mathbf{X}]$.

Solução:

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}] = \int y \cdot f_{\mathbf{Y}|\mathbf{X}}(y|x)dy$$

$$f_{\mathbf{Y}|\mathbf{X}}(y|x) = \frac{2e^{-x-y}}{2e^{-2x}} = e^{-x-y+2x} = e^{x-y}, \qquad \text{logo,}$$

$$\begin{aligned}
\mathbb{E}[\mathbf{Y}|\mathbf{X}=x] &= \int_x^\infty y \cdot e^{x-y}dy = e^x \cdot \int_x^\infty y \cdot e^{-y}dy \\
&= e^x \left[y \cdot -e^{-y}|_x^\infty - \int_x^\infty -e^{-y}dy \right] \\
&= e^x \left[x \cdot e^{-x} + \int_x^\infty e^{-y}dy \right] = e^x \left[x \cdot e^{-x} + (-e^{-y}|_x^\infty) \right] \\
&= e^x \left[x \cdot e^{-x} + e^{-x} \right] = x + 1
\end{aligned}$$

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}] = \mathbf{X} + 1$$