Probability Notes
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The proof for the **Binomial Theorem** in Page 34 of DeGoot (2012).

Theorem 1 (Binomial Theorem)**.** *For all real numbers x and y and each positive integer n.*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We are going to prove using induction.

1. First lets prove for $n = 1$.

$$\begin{aligned}(x+y)^1 &= \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} \\ &= (x^0 + y^1) + (x^1 + y^0) = (x+y)\end{aligned}$$

2. Second, suppose that $r \in \mathbb{Z}^+$, then, by inductive hypothesis the statement is true for $n = r$.

$$(x+y)^r = \sum_{k=0}^{r} \binom{r}{k} x^k y^{r-k}.$$

3. Now, we need to prove that the statement is valid for $n = r+1$, by using the inductive assumption. Which is:

$$(x+y)^{r+1} = \sum_{k=0}^{r+1} \binom{r+1}{k} x^k y^{r+1-k}.$$

So, we know that

$$\begin{aligned}(x+y)^{r+1} &= (x+y)^r \cdot (x+y) \\ &= \sum_{k=0}^{r} \binom{r}{k} x^k y^{r-k} \cdot (x+y) \quad \text{by the inductive hypothesis} \\ &= \sum_{k=0}^{r} \binom{r}{k} x^{k+1} y^{r-k} + \sum_{k=0}^{r} \binom{r}{k} x^k y^{r-k+1} \quad \text{by the distributive property;}\end{aligned}$$

Now, taking the first term of the sum in the above equation and setting $k+1 = s$, we get:

$$\sum_{k=0}^{r} \binom{r}{k} x^{k+1} y^{r-k} = \sum_{s=1}^{r+1} \binom{r}{s-1} x^s y^{r-s+1}$$

And, replacing s by k as it is just a name, we get:

$$\begin{aligned}
(x+y)^{r+1} &= \sum_{k=1}^{r+1}\binom{r}{k-1}x^k y^{r-k+1} + \sum_{k=0}^{r}\binom{r}{k}x^k y^{r-k+1} \\
&= \sum_{k=1}^{r}\binom{r}{k-1}x^k y^{r-k+1} + \binom{r}{r+1-1}x^{r+1}y^{r-(r+1)+1} + \\
&\quad + \binom{r}{0}x^0 y^{r-0+1} + \sum_{k=1}^{r}\binom{r}{k}x^k y^{r-k+1} \\
&= \sum_{k=1}^{r}\left[\binom{r}{k-1} + \binom{r}{k}\right]x^k y^{r-k+1} + x^{r+1} + y^{r+1}
\end{aligned}$$

Now, we may use the pascal's rule, the combinatorial identity about binomial coefficients, which states that for $1 \leq k \leq r$, we have:

$$\binom{r}{k-1} + \binom{r}{k} = \binom{r+1}{k}$$

Also, we may use the fact that:

$$x^{r+1} = \binom{r+1}{r+1}x^{r+1}y^0 \quad \text{and} \quad y^{r+1} = \binom{r+1}{0}x^0 y^{r+1}$$

which correspond to the case of $k = r+1$ and $k = 0$ in the sum respectively.

Therefore, we see that:

$$(x+y)^{r+1} = \sum_{k=0}^{r+1}\binom{r+1}{k}x^k y^{r+1-k}.$$

which ends the inductive conclusion and the proof is complete.

$\square$

References

DeGroot, M. H., & Schervish, M. J. (2012). Probability and statistics. Pearson Education.