

## Prova 2 - Versão 2

Probabilidade

Wyara Vanesa Moura e Silva

Primeiro Semestre

2022

### Questão 1

Seja  $\mathbf{X} \sim \text{Exp}(\theta)$ ;  $\theta > 0$ . Definimos  $\mathbf{T} = \lfloor \mathbf{X} \rfloor$ ; onde  $a \in \mathbb{R}$ ,  $\lfloor a \rfloor = k \Leftrightarrow k \leq a < k + 1$ .  
Calcular

$$\mathbb{P}(\mathbf{T} = k); \quad \mathbb{E}(\mathbf{T})$$

*Solução:*

$$\begin{aligned} \mathbb{P}(\mathbf{T} = k) &= \mathbb{P}(\lfloor \mathbf{X} \rfloor = k) = \mathbb{P}(k \leq \mathbf{X} < k + 1) \\ &= \int_k^{k+1} \theta \cdot e^{-\theta x} dx = \theta \left( -\frac{1}{\theta} \cdot e^{-\theta x} \Big|_k^{k+1} \right) \\ &= -e^{-\theta(k+1)} + e^{-\theta k} = e^{-\theta k} (1 - e^{-\theta}) \end{aligned}$$

em que tal função corresponde a função de probabilidade da distribuição geométrica  $(1 - e^{-\theta})$ .

$$\begin{aligned} \mathbb{E}(\mathbf{T}) &= \sum_{k=0}^{\infty} k \cdot e^{-\theta k} (1 - e^{-\theta}) = (1 - e^{-\theta}) \cdot \sum_{k=0}^{\infty} k \cdot e^{-\theta k} \\ &= (1 - e^{-\theta}) \cdot \left[ \frac{d}{d\theta} \left( -\sum_{k=0}^{\infty} e^{-\theta k} \right) \right] = (1 - e^{-\theta}) \frac{e^{-\theta}}{(1 - e^{-\theta})^2} \\ &= \frac{e^{-\theta}}{(1 - e^{-\theta})} \end{aligned}$$

### Questão 2

Sejam  $\mathbf{X}$  e  $\mathbf{Y}$  i.i.d. geométricas  $(\mathbf{p})$ ;  $0 < \mathbf{p} < 1$ .

$$\mathbb{P}(\mathbf{X} = k) = \mathbf{p}(1 - \mathbf{p})^{k-1}; \quad k = 1, 2, 3, \dots$$

Sejam  $\mathbf{Z} = \mathbf{Y} - \mathbf{X}$  e  $\mathbf{W} = \min \{\mathbf{X}, \mathbf{Y}\}$ . Encontrar a probabilidade conjunta  $\mathbb{P}(\mathbf{W} = j, \mathbf{Z} = k)$ .

*Solução:*

$$\begin{aligned} \mathbb{P}(\mathbf{W} = j, \mathbf{Z} = k) &= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = j, \mathbf{Y} - \mathbf{X} = k, \mathbf{X} \leq \mathbf{Y}) + \\ &\quad + \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = j, \mathbf{Y} - \mathbf{X} = k, \mathbf{X} > \mathbf{Y}) \\ &= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + j, \mathbf{X} \leq \mathbf{Y}) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = j - k, \mathbf{X} > \mathbf{Y}) \\ &= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + j, j \leq k + j) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = j - k, j - k > j) \\ &= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + j, 0 \leq k) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = j - k, k < 0) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{P}(\mathbf{X} = j) \cdot \mathbb{P}(\mathbf{Y} = k + j) \mathbb{1}_{\{0,1,2,\dots\}}(k) + \mathbb{P}(\mathbf{Y} = j) \cdot \mathbb{P}(\mathbf{X} = j - k) \mathbb{1}_{\{-1,-2,\dots\}}(k) \\
&= \mathbf{p}(1 - \mathbf{p})^{j-1} \mathbf{p}(1 - \mathbf{p})^{k+j-1} \cdot \mathbb{1}_{\{0,1,2,\dots\}}(k) + \mathbf{p}(1 - \mathbf{p})^{j-k-1} \mathbf{p}(1 - \mathbf{p})^{j-1} \mathbb{1}_{\{-1,-2,\dots\}}(k) \\
&= \mathbf{p}^2(1 - \mathbf{p})^{k+2j-2} \mathbf{p}(1 - \mathbf{p})^{k+j-1} \cdot \mathbb{1}_{\{0,1,2,\dots\}}(k) + \\
&\quad + \mathbf{p}^2(1 - \mathbf{p})^{-k+2j-2} \mathbf{p}(1 - \mathbf{p})^{j-1} \mathbb{1}_{\{-1,-2,\dots\}}(k) \\
&= \mathbf{p}^2(1 - \mathbf{p})^{2j-2} [(1 - \mathbf{p})^k \mathbb{1}_{\{0,1,2,\dots\}}(k) + (1 - \mathbf{p})^{-k} \mathbb{1}_{\{-1,-2,\dots\}}(k)]
\end{aligned}$$

### Questão 3

$\mathbf{X} \sim \text{Uniforme}[0, a]$ ;  $a > 0$ ,  $\mathbf{Y} \sim \text{Exp}(\theta)$ ;  $\theta > 0$ , independentes. Seja  $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ . Calcular a densidade de  $\mathbf{X}$ .

*Solução:*

$$f_z(z) = \int_0^\infty f_{\mathbf{Y}}(y) \cdot f_{\mathbf{X}}(z - y) dy$$

Logo,

$$\text{se } 0 < z < a$$

$$f_z(z) = \int_0^z \frac{1}{a} \cdot \theta \cdot e^{-\theta x} dy = \frac{1}{a} \left[ \theta \left( -\frac{1}{\theta} \cdot e^{-\theta x} \right) \Big|_0^z \right] = \frac{1}{a} (1 - e^{-\theta z}) \mathbb{1}_{(0,a)}(z)$$

$$\text{se } a < z < \infty$$

$$\begin{aligned}
f_z(z) &= \int_{z-a}^z \frac{1}{a} \cdot \theta \cdot e^{-\theta x} dy = \frac{1}{a} \left[ \theta \left( -\frac{1}{\theta} \cdot e^{-\theta x} \right) \Big|_0^z \right] \\
&= \frac{1}{a} \cdot (e^{-\theta(z-a)}) \\
&= \frac{1}{a} e^{-\theta z} (e^{\theta a} - 1) \mathbb{1}_{(a,\infty)}(z)
\end{aligned}$$

### Questão 4

Dada a densidade conjunta de  $\mathbf{X}, \mathbf{Y}$ .

$$f_{\mathbf{X}, \mathbf{Y}}(x, y) = \frac{\sqrt{3}}{4\pi} \exp \left[ -\frac{1}{2}(x^2 - xy + y^2) \right]; \quad x, y \in \mathbb{R}$$

Calcular  $\mathbb{E}[\mathbf{XY}]$ .

*Solução:*

$$\begin{aligned}
\mathbb{E}[\mathbf{XY}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \cdot f_{\mathbf{XY}}(xy) dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \cdot \frac{\sqrt{3}}{4\pi} \exp \left[ -\frac{1}{2}(x^2 - xy + y^2) \right] dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot x \cdot \exp \left[ -\frac{1}{2}(x^2 - xy + y^2) \right] dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot x \cdot \exp \left[ -\frac{1}{2}(x^2 - xy + y^2) \right] dx dy \\
&= \int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot \int_{-\infty}^{\infty} x \cdot \exp \left[ -\frac{1}{2} \left( x - \frac{y}{2} \right)^2 - \frac{1}{2} \left( \frac{3y^2}{4} \right) \right] dx dy \\
&= \frac{\sqrt{3}}{4\pi} \cdot \int_{-\infty}^{\infty} y \cdot \exp \left[ -\frac{1}{2} \left( \frac{3y^2}{4} \right) \right] \sqrt{2\pi} \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( x - \frac{y}{2} \right)^2 \right] dx dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{4\pi} \cdot \int_{-\infty}^{\infty} y \cdot \exp \left[ -\frac{1}{2} \left( \frac{3y^2}{4} \right) \right] \frac{y}{2} dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{4\pi} \cdot \int_{-\infty}^{\infty} \frac{y^2}{2} \cdot \exp \left[ -\frac{1}{2} \left( \frac{y^2}{\frac{4}{3}} \right) \right] dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{8\pi} \cdot \int_{-\infty}^{\infty} y^2 \cdot \frac{\sqrt{2\pi}\sqrt{\frac{4}{3}}}{\sqrt{2\pi}\sqrt{\frac{4}{3}}} \cdot \exp \left[ -\frac{1}{2} \left( \frac{y^2}{\frac{4}{3}} \right) \right] dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{8\pi} \cdot \sqrt{2\pi} \cdot \sqrt{\frac{4}{3}} = \frac{2}{3}
\end{aligned}$$