Example 2.4.12, Robert (2007)

Wyara Vanesa Moura e Silva July 20, 2018

Consider a Bernoulli observation, $x \sim \text{Ber}(\theta)$ with $\theta \in \{0.1, 0.5\}$. Four nonrandomized estimators are available,

$$\delta_1(x) = 0.1, \qquad \delta_2(x) = 0.5,$$
$$\delta_3(x) = 0.1\mathbb{I}_{\{x=0\}} + 0.5\mathbb{I}_{\{x=1\}}, \qquad \delta_4(x) = 0.5\mathbb{I}_{\{x=0\}} + 0.1\mathbb{I}_{\{x=1\}}.$$

Then, we assume in addition that the penalty for a wrong answer is 2 when $\theta = 0.1$ and 1 when $\theta = 0.5$.

Now, the risk values, $R(\theta, \delta(x))$, are:

1. $R(0.1, \delta_1(x)) = \mathbf{0}$ and $R(0.5, \delta_1(x)) = \mathbf{1}$; given that $\delta_1(x) = 0.1$ for any value of $x = \{0, 1\}$.

2. $R(0.1, \delta_2(x)) = \mathbf{2}$ and $R(0.5, \delta_2(x)) = \mathbf{0}$; given that $\delta_2(x) = 0.5$ for any value of $x = \{0, 1\}$.

3. $R(0.1, \delta_3(x))$, for this case we have that:

   if $x = 0$ and $\theta = 0.1$, then $p(x|\theta) = 0.9$, and $\delta_3(x) = 0.1$ and the loss is 0;
   if $x = 1$ and $\theta = 0.1$, then $p(x|\theta) = 0.5$, and $\delta_3(x) = 0.5$ and the loss is 2;

   Now, we can use the definition of frequency risk, which is:

   $$R(\theta, \delta(x)) = \mathbb{E}_\theta[L(\theta, \delta(x))] \tag{1}$$

   Therefore, using Equation (1), $R(0.1, \delta_3(x)) = 0 \cdot 0.9 + 2 \cdot 0.1 = \mathbf{0.2}$.

   Now, the same for $\theta = 0.5$, $R(0.5, \delta_3(x))$, for this case we have that:

   if $x = 0$ and $\theta = 0.5$, then $p(x|\theta) = 0.5$, and $\delta_3(x) = 0.1$ and the loss is 1;
   if $x = 1$ and $\theta = 0.5$, then $p(x|\theta) = 0.5$, and $\delta_3(x) = 0.5$ and the loss is 0;

   Therefore, using Equation (1), $R(0.5, \delta_3(x)) = 1 \cdot 0.5 + 0 \cdot 0.5 = \mathbf{0.5}$.

4. $R(0.1, \delta_4(x))$, for this case we have that:

   if $x = 0$ and $\theta = 0.1$, then $p(x|\theta) = 0.9$, and $\delta_4(x) = 0.5$ and the loss is 2;
   if $x = 1$ and $\theta = 0.1$, then $p(x|\theta) = 0.5$, and $\delta_4(x) = 0.1$ and the loss is 0;

   Therefore, using Equation (1), $R(0.1, \delta_4(x)) = 2 \cdot 0.9 + 0 \cdot 0.1 = \mathbf{1.8}$.

   Now, the same for $\theta = 0.5$, $R(0.5, \delta_4(x))$, for this case we have that:

   if $x = 0$ and $\theta = 0.5$, then $p(x|\theta) = 0.5$, and $\delta_4(x) = 0.5$ and the loss is 0;
   if $x = 1$ and $\theta = 0.5$, then $p(x|\theta) = 0.5$, and $\delta_4(x) = 0.1$ and the loss is 1;

   Therefore, using Equation (1), $R(0.5, \delta_4(x)) = 0 \cdot 0.5 + 1 \cdot 0.5 = \mathbf{0.5}$.

It is straightforward to see that the risk vector of any randomized estimator is a convex combination of these four vectors or, equivalently, that the risk set, $\mathcal{R}$ is the convex hull of the above four vectors (Figure 2.4.3, page 71).

References

Robert, C. (2007). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.