

## Prova 2

Probabilidade

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### Questão 1

Sejam  $\mathbf{X}$  e  $\mathbf{Y}$  variáveis aleatórias independentes com função de probabilidade geométrica de parâmetros  $\alpha > 0$  e  $\beta > 0$  respectivamente.

$$\mathbb{P}(\mathbf{X} = k) = \alpha(1 - \alpha)^{k-1}, \quad \mathbb{P}(\mathbf{Y} = k) = \beta(1 - \beta)^{k-1}, \quad k = 1, 2, \dots$$

Definimos:

$$\mathbf{U} = \min \{\mathbf{X}, \mathbf{Y}\}, \quad \mathbf{V} = \max \{\mathbf{X}, \mathbf{Y}\}, \quad \mathbf{W} = \mathbf{V} - \mathbf{U}$$

1. Calcular a probabilidade conjunta de  $(\mathbf{U}, \mathbf{V})$

*Solução:*

Probabilidade conjunta de  $(\mathbf{U}, \mathbf{V})$ :

$$\begin{aligned} \mathbb{P}(\mathbf{U} = u, \mathbf{V} = v) &= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = u, \max \{\mathbf{X}, \mathbf{Y}\} = v, \mathbf{X} \geq \mathbf{Y}) + \\ &\quad + \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = u, \max \{\mathbf{X}, \mathbf{Y}\} = v, \mathbf{X} < \mathbf{Y}) \\ &= \mathbb{P}(\mathbf{Y} = u, \mathbf{X} = v, v \geq u) + \mathbb{P}(\mathbf{X} = u, \mathbf{Y} = v, u < v) \\ &= \mathbb{P}(\mathbf{Y} = u)\mathbb{P}(\mathbf{X} = v)\mathbb{1}(v \geq u) + \mathbb{P}(\mathbf{X} = u)\mathbb{P}(\mathbf{Y} = v)\mathbb{1}(u < v) \\ &= \alpha(1 - \alpha)^{v-1}\beta(1 - \beta)^{u-1}\mathbb{1}(v \geq u) + \alpha(1 - \alpha)^{u-1}\beta(1 - \beta)^{v-1}\mathbb{1}(u < v) \\ &= \alpha\beta \left[ (1 - \alpha)^{v-1}(1 - \beta)^{u-1}\mathbb{1}(v \geq u) + (1 - \alpha)^{u-1}(1 - \beta)^{v-1}\mathbb{1}(u < v) \right] \end{aligned}$$

- resultados adicionais:

função de probabilidade do  $\min \{\mathbf{X}, \mathbf{Y}\} = \mathbf{U}$

$$\begin{aligned}
\mathbb{P}(\mathbf{U} = u) &= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = k) \\
&= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = k, \mathbf{X} \leq \mathbf{Y}) + \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = k, \mathbf{X} > \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X} = k, \mathbf{Y} \geq k) + \mathbb{P}(\mathbf{X} > k, \mathbf{Y} = k)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\mathbf{X} > k) &= \mathbb{P}(\mathbf{X} \geq k) - \mathbb{P}(\mathbf{X} = k) \\
&= (1 - \alpha)^{k-1} - \alpha(1 - \alpha)^{k-1} \\
&= (1 - \alpha)^{k-1}(1 - \alpha)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\mathbf{X} \geq k) &= \sum_{k=u}^{\infty} \alpha(1 - \alpha)^{k-1} \\
&= \alpha(1 - \alpha)^{u-1} [1 + (1 - \alpha)^1 + (1 - \alpha)^2 + \dots] \\
&= \alpha(1 - \alpha)^{u-1} \sum_{i=0}^{\infty} (1 - \alpha)^i \\
&= \alpha(1 - \alpha)^{u-1} \frac{1}{1 - (1 - \alpha)} = (1 - \alpha)^{u-1}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\mathbf{U} = u) &= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = k) \\
&= [\alpha(1 - \alpha)^{k-1}(1 - \beta)^{k-1} + (1 - \alpha)^{k-1}(1 - \alpha)\beta(1 - \beta)^{k-1}] \\
&= [(1 - \alpha)(1 - \beta)]^{k-1} [\alpha + \beta(1 - \alpha)] \\
&= [(1 - \alpha)(1 - \beta)]^{k-1} [\alpha + \beta - \alpha\beta]
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{P}(\mathbf{U} = k) &= \sum_{k=1}^{\infty} [(1 - \alpha)(1 - \beta)]^{k-1} [\alpha + \beta(1 - \alpha)] \\
&= [\alpha + \beta - \alpha\beta] \cdot \frac{1}{1 - (1 - \alpha)(1 - \beta)} = 1
\end{aligned}$$

Logo,

$$\mathbb{P}(\mathbf{U} = u) = [(1 - \alpha)(1 - \beta)]^{k-1} [\alpha + \beta - \alpha\beta]$$

Função de probabilidade do  $\max \{\mathbf{X}, \mathbf{Y}\} = \mathbf{V}$

$$\begin{aligned}
\mathbb{P}(\max\{\mathbf{X}, \mathbf{Y}\}) &= \mathbb{P}(\max\{\mathbf{X}, \mathbf{Y}\} = k, \mathbf{X} \leq \mathbf{Y}) + \mathbb{P}(\max\{\mathbf{X}, \mathbf{Y}\} = k, \mathbf{X} > \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X} \leq k, \mathbf{Y} = k) + \mathbb{P}(\mathbf{X} = k, \mathbf{Y} < k) \\
&= \mathbb{P}(\mathbf{X} \leq k)\mathbb{P}(\mathbf{Y} = k) + \mathbb{P}(\mathbf{X} = k)\mathbb{P}(\mathbf{Y} < k)
\end{aligned}$$

$$\mathbb{P}(\mathbf{X} \leq k) = 1 - \mathbb{P}(\mathbf{X} > k) = 1 - [(1 - \alpha)^{k-1}(1 - \alpha)]$$

$$\mathbb{P}(\mathbf{Y} < k) = 1 - \mathbb{P}(\mathbf{X} \geq k) = 1 - [(1 - \beta)^{k-1}]$$

$$\begin{aligned}
\mathbb{P}(\max\{\mathbf{X}, \mathbf{Y}\}) &= \{1 - [(1 - \alpha)^{k-1}(1 - \alpha)]\} \beta(1 - \beta)^{k-1} + \\
&\quad + \{1 - [(1 - \beta)^{k-1}]\} \alpha(1 - \alpha)^{k-1} \\
&= \beta(1 - \beta)^{k-1} - \beta\beta(1 - \beta)^{k-1}(1 - \alpha)^{k-1}(1 - \alpha) + \\
&\quad + \alpha(1 - \alpha)^{k-1} - \alpha(1 - \alpha)^{k-1}(1 - \beta)^{k-1} \\
&= (1 - \beta)^{k-1} [\beta - \beta(1 - \alpha)^{k-1}(1 - \alpha) - \alpha(1 - \alpha)^{k-1}] + \alpha(1 - \alpha)^{k-1} \\
&= \beta(1 - \beta)^{k-1} + \alpha(1 - \alpha)^{k-1} - [(1 - \alpha)(1 - \beta)]^{k-1} [\beta(1 - \alpha) + \alpha]
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{P}(\mathbf{V} = k) &= \sum_{k=1}^{\infty} \left\{ \beta(1 - \beta)^{k-1} + \alpha(1 - \alpha)^{k-1} - [\beta(1 - \alpha) + \alpha] [(1 - \alpha)(1 - \beta)]^{k-1} \right\} \\
&= \beta \sum_{k=1}^{\infty} (1 - \beta)^{k-1} + \alpha \sum_{k=1}^{\infty} (1 - \alpha)^{k-1} - [\beta(1 - \alpha) + \alpha] \cdot \\
&\quad \cdot \sum_{k=1}^{\infty} [(1 - \alpha)(1 - \beta)]^{k-1} \\
&= \beta \cdot \frac{1}{1 - (1 - \beta)} + \alpha \cdot \frac{1}{1 - (1 - \alpha)} - [\beta(1 - \alpha) + \alpha] \cdot \frac{1}{1 - (1 - \alpha)(1 - \beta)} \\
&= 1 + 1 - 1 = 1
\end{aligned}$$

Logo,

$$\mathbb{P}(\mathbf{V} = k) = \beta(1 - \beta)^{k-1} + \alpha(1 - \alpha)^{k-1} - [(1 - \alpha)(1 - \beta)]^{k-1} [\beta(1 - \alpha) + \alpha]$$

2. Provar que  $\mathbf{U}$  e  $\mathbf{V}$  são independentes

*Solução:*

$$\mathbf{U} = \min\{\mathbf{X}, \mathbf{Y}\}, \quad \mathbf{W} = \max\{\mathbf{X}, \mathbf{Y}\} - \mathbf{U}$$

$$\begin{aligned}
\mathbb{P}(\mathbf{U} = u, \mathbf{W} = w) &= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = u, \max \{\mathbf{X}, \mathbf{Y}\} - \min \{\mathbf{X}, \mathbf{Y}\} = w) \\
&= \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = u, \max \{\mathbf{X}, \mathbf{Y}\} - \min \{\mathbf{X}, \mathbf{Y}\} = w, \mathbf{X} \leq \mathbf{Y}) \\
&\quad + \mathbb{P}(\min \{\mathbf{X}, \mathbf{Y}\} = u, \max \{\mathbf{X}, \mathbf{Y}\} - \min \{\mathbf{X}, \mathbf{Y}\} = w, \mathbf{X} > \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X} = u, \mathbf{Y} - \mathbf{X} = w, \mathbf{Y} \geq \mathbf{X}) + \mathbb{P}(\mathbf{Y} = u, \mathbf{X} - \mathbf{Y} = w, \mathbf{X} > \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X} = u, \mathbf{Y} = w + u, w + u \geq u) + \mathbb{P}(\mathbf{Y} = u, \mathbf{X} = w + u, w + u > u) \\
&= \mathbb{P}(\mathbf{X} = u, \mathbf{Y} = w + u, w \geq 0) + \mathbb{P}(\mathbf{Y} = u, \mathbf{X} = w + u, w > 0) \\
&= \alpha(1 - \alpha)^{u-1}\beta(1 - \beta)^{w+u-1}\mathbb{1}_{\{0,1,2,\dots\}}(w) + \\
&\quad + \alpha(1 - \alpha)^{w+u-1}\beta(1 - \beta)^{u-1}\mathbb{1}_{\{1,2,\dots\}}(w) \\
&= \alpha\beta(1 - \alpha)^{u-1}\beta(1 - \beta)^{u-1} \cdot \\
&\quad \cdot [(1 - \beta)^w\mathbb{1}_{\{0,1,2,\dots\}}(w) + (1 - \alpha)^w\mathbb{1}_{\{1,2,\dots\}}(w)]
\end{aligned}$$

Agora, iremos encontrar as marginais,

$$\begin{aligned}
\mathbb{P}(\mathbf{U} = u) &= \sum_{w=0}^{\infty} \alpha\beta(1 - \alpha)^{u-1}(1 - \beta)^{u-1}(1 - \beta)^w + \\
&\quad + \sum_{w=1}^{\infty} \alpha\beta(1 - \alpha)^{u-1}(1 - \beta)^{u-1}(1 - \alpha)^w \\
&= \alpha\beta(1 - \alpha)^{u-1}(1 - \beta)^{u-1} \cdot \\
&\quad \cdot \left[ \sum_{w=0}^{\infty} (1 - \beta)^w + \sum_{w=1}^{\infty} (1 - \alpha)^w \right] \\
&= \alpha\beta [(1 - \alpha)(1 - \beta)]^{u-1} \cdot \\
&\quad \cdot \left[ \frac{1}{1 - (1 - \beta)} + \frac{(1 - \alpha)}{1 - (1 - \alpha)} \right] \\
&= [(1 - \alpha)(1 - \beta)]^{u-1} (\alpha + \beta - \alpha\beta)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\mathbf{W} = w) &= \sum_{u=0}^{\infty} \alpha\beta(1 - \alpha)^{u-1}(1 - \beta)^{u-1} \cdot \\
&\quad \cdot [(1 - \beta)^w\mathbb{1}_{\{0,1,2,\dots\}}(w) + (1 - \alpha)^w\mathbb{1}_{\{1,2,\dots\}}(w)] \\
&= \alpha\beta [(1 - \beta)^w\mathbb{1}_{\{0,1,2,\dots\}}(w) + (1 - \alpha)^w\mathbb{1}_{\{1,2,\dots\}}(w)] \cdot \\
&\quad \cdot \sum_{u=0}^{\infty} [(1 - \alpha)(1 - \beta)]^{u-1} \\
&= \alpha\beta [(1 - \beta)^w\mathbb{1}_{\{0,1,2,\dots\}}(w) + (1 - \alpha)^w\mathbb{1}_{\{1,2,\dots\}}(w)] \cdot \\
&\quad \cdot \frac{1}{1 - (1 - \alpha)(1 - \beta)}
\end{aligned}$$

$$= \alpha\beta \frac{1}{(\alpha + \beta - \alpha\beta)} \cdot [(1 - \beta)^w \mathbf{1}_{\{0,1,2,\dots\}}(w) + (1 - \alpha)^w \mathbf{1}_{\{1,2,\dots\}}(w)]$$

Logo, como  $\mathbb{P}(\mathbf{U} = u, \mathbf{W} = w) = \mathbb{P}(\mathbf{U} = u) \cdot \mathbb{P}(\mathbf{W} = w)$  então,  $\mathbf{U}$  e  $\mathbf{W}$  são independentes, assim está provado.

## Questão 2

Seja o espaço amostral  $\Omega = \{a, b, c\}$  e considere a sigma álgebra como o conjunto das partes. Definimos as probabilidades sobre este espaço por

$$\mathbb{P}(\{\mathbf{a}\}) = \frac{1}{2} \quad \mathbb{P}(\{\mathbf{b}\}) = \frac{1}{4} \quad \mathbb{P}(\{\mathbf{c}\}) = \frac{1}{4}.$$

Sejam as variáveis aleatórias  $\mathbf{X}$  e  $\mathbf{Y}$  definidas por

$$\mathbf{X}(\omega) = \mathbf{I}_{\{a\}}(\omega) - \mathbf{I}_{\{b,c\}}(\omega) \quad \text{e} \quad \mathbf{Y}(\omega) = \mathbf{I}_{\{b\}}(\omega) - \mathbf{I}_{\{c\}}(\omega).$$

Onde o indicador é definido por

$$\mathbf{I}_{\{A\}}(\omega) = \begin{cases} 1 & \text{se } \omega \in A \\ 0 & \text{se caso contrário.} \end{cases}$$

1. Calcular as distribuições de probabilidade de  $\mathbf{X}$  e  $\mathbf{Y}$ .

*Solução:*

$$\mathbf{X}(\{\mathbf{a}\}) = 1, \quad \mathbf{X}(\{\mathbf{b}\}) = -1, \quad \mathbf{X}(\{\mathbf{c}\}) = -1$$

$$\mathbb{P}(\mathbf{X} = 1) = \frac{1}{2}, \quad \mathbb{P}(\mathbf{X} = -1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

assim a distribuição de probabilidade de  $\mathbf{X}$  é:

Table 1: Distribuição de probabilidade de  $\mathbf{X}$

$\mathbf{X}$	-1	1
$\mathbb{P}(\mathbf{X} = k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\mathbf{Y}(\{\mathbf{a}\}) = 0, \quad \mathbf{Y}(\{\mathbf{b}\}) = 1, \quad \mathbf{Y}(\{\mathbf{c}\}) = -1$$

$$\mathbb{P}(\mathbf{Y} = 0) = \frac{1}{2}, \quad \mathbb{P}(\mathbf{Y} = 1) = \frac{1}{4}, \quad \mathbb{P}(\mathbf{Y} = -1) = \frac{1}{4}.$$

assim a distribuição de probabilidade de  $\mathbf{Y}$  é:

Table 2: Distribuição de probabilidade de  $\mathbf{Y}$

$\mathbf{Y}$	-1	0	1
$\mathbb{P}(\mathbf{Y} = k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

2. Calcular  $\mathbb{E}[\mathbf{XY}]$ .  $\mathbf{X}$  e  $\mathbf{Y}$  são independentes?

*Solução:*

Como,

$$\begin{aligned}\mathbb{E}(\mathbf{X}) &= -1 \cdot \mathbb{P}(\mathbf{X} = -1) + 1 \cdot \mathbb{P}(\mathbf{X} = 1) \\ &= -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0 \\ \mathbb{E}(\mathbf{Y}) &= -1 \cdot \mathbb{P}(\mathbf{Y} = -1) + 0 \cdot \mathbb{P}(\mathbf{Y} = 0) + 1 \cdot \mathbb{P}(\mathbf{Y} = 1) \\ &= -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0\end{aligned}$$

Logo,  $\mathbf{X}$  e  $\mathbf{Y}$  são independentes, dado que a probabilidade de um evento em  $\mathbf{X}$  ocorrer não depende de nenhum evento ocorrer em logo,  $\mathbf{X}$  e  $\mathbf{Y}$  são independentes, dado que a probabilidade de um evento em  $\mathbf{Y}$ .

Portanto, pela propriedade da esperança,

$$\mathbb{E}(\mathbf{XY}) = \mathbb{E}(\mathbf{X}) \cdot \mathbb{E}(\mathbf{Y}) = 0$$

### Questão 3

Sejam  $\mathbf{X}$  e  $\mathbf{Y}$  variáveis aleatórias independentes com função densidade de probabilidade uniforme no intervalo  $[0, a]$ . Definimos  $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$

1. Fazer um desenho da função de distribuição de  $\mathbf{Z}$ , para  $a > 0$ .

*Solução:*

2. Calcular a função densidade de  $\mathbf{Z}$ , para  $a > 0$ .

*Solução:*

$$\begin{aligned}f_x(x) &= \frac{1}{a} = f_y(y); \quad a > 0 \\ f_z(z) &= \int_{\mathbf{X}} f_x(x) \cdot f_y(z - x) dx\end{aligned}$$

Logo,

se  $0 < z < a$

$$f_z(z) = \int_0^z \frac{1}{a} \cdot \frac{1}{a} dx = \frac{1}{a^2} \cdot x \Big|_0^z = z \cdot \frac{1}{a^2} \cdot \mathbb{1}_{(0,a)}(z)$$

se  $a < z < 2a$

$$f_z(z) = \int_{z-a}^a \frac{1}{a} \cdot \frac{1}{a} dx = \frac{1}{a^2} \cdot x \Big|_{z-a}^a = (a - z + a) \cdot \frac{1}{a^2} = (2a - z) \cdot \frac{1}{a^2} \cdot \mathbb{1}_{(a, 2a)}(z)$$

## Questão 4

Seja o vetor  $(\mathbf{X}, \mathbf{Y})$  aleatório com função densidade conjunta densidade conjunta dada por

$$f_{\mathbf{XY}}(x, y) = \begin{cases} 2e^{-x-y} & \text{se } 0 < x < y < \infty \\ 0 & \text{se caso contrário.} \end{cases}$$

1. Calcular  $\mathbb{E}[\mathbf{X}]$  e  $\mathbb{E}[\mathbf{Y}]$ .

*Solução:*

$$\begin{aligned} f_x(x) &= \int_x^\infty 2e^{-x-y} dy = 2e^{-x} \int_x^\infty e^{-y} dy \\ &= 2e^{-x} (-e^{-y} \Big|_x^\infty) = 2e^{-x} e^{-x} = 2e^{-2x} \mathbb{1}_{(0, \infty)}(x) \end{aligned}$$

$$\begin{aligned} f_y(y) &= \int_0^y 2e^{-x-y} dx = 2e^{-y} \int_0^y e^{-x} dx \\ &= 2e^{-y} (-e^{-x} \Big|_0^y) = 2e^{-y} (-e^{-y} + 1) = 2e^{-y} (1 - e^{-y}) \mathbb{1}_{(0, \infty)}(y) \end{aligned}$$

Assim,

$$\begin{aligned} \mathbb{E}(\mathbf{X}) &= \int_0^\infty x \cdot 2e^{-2x} dx = 2 \int_0^\infty x e^{-2x} dx \\ &= 2 \left[ x \cdot -\frac{1}{2} e^{-2x} \Big|_0^\infty - \int_0^\infty -\frac{1}{2} e^{-2x} dx \right] = \\ &= 2 \left[ \frac{1}{2} \int_0^\infty e^{-2x} dx \right] = \left( -\frac{1}{2} e^{-2x} \Big|_0^\infty \right) = -\frac{1}{2} \cdot (-1) = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbb{E}(\mathbf{Y}) &= \int_0^\infty y \cdot 2e^{-2y} (1 - e^{-y}) dy = 2 \int_0^\infty (y e^{-2y} - y e^{-3y}) dy \\ &= 2 \left\{ \left[ (y e^{-2y} \Big|_0^\infty) - \int_0^\infty -e^{-2y} dy \right] - \left[ \left( y \cdot -\frac{1}{2} e^{-2y} \Big|_0^\infty \right) - \int_0^\infty -\frac{1}{2} e^{-2y} dy \right] \right\} \\ &= 2 \left[ \int_0^\infty e^{-y} dy - \left( \frac{1}{2} \int_0^\infty -e^{-2y} dy \right) \right] \\ &= 2 \left[ (-e^{-y} \Big|_0^\infty) - \frac{1}{2} \cdot \left( -\frac{1}{2} -e^{-2y} \Big|_0^\infty \right) \right] \\ &= 2 \left[ 1 - \left( \frac{1}{2} \cdot \frac{1}{2} \right) \right] = 2 \left( 1 - \frac{1}{4} \right) = 2 \left( \frac{3}{4} \right) = \frac{3}{2} \end{aligned}$$

2. Calcular a esperança condicional de  $\mathbb{E}[\mathbf{Y}|\mathbf{X}]$ .

*Solução:*

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}] = \int y \cdot f_{\mathbf{Y}|\mathbf{X}}(y|x) dy$$

$$f_{\mathbf{Y}|\mathbf{X}}(y|x) = \frac{2e^{-x-y}}{2e^{-2x}} = e^{-x-y+2x} = e^{x-y}, \quad \text{logo,}$$

$$\begin{aligned}\mathbb{E}[\mathbf{Y}|\mathbf{X} = x] &= \int_x^\infty y \cdot e^{x-y} dy = e^x \cdot \int_x^\infty y \cdot e^{-y} dy \\ &= e^x \left[ y \cdot -e^{-y} \Big|_x^\infty - \int_x^\infty -e^{-y} dy \right] \\ &= e^x \left[ x \cdot e^{-x} + \int_x^\infty e^{-y} dy \right] = e^x [x \cdot e^{-x} + (-e^{-y} \Big|_x^\infty)] \\ &= e^x [x \cdot e^{-x} + e^{-x}] = x + 1\end{aligned}$$

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}] = \mathbf{X} + 1$$