Wyara Vanesa Moura e Silva

Professor Kevin Knuth

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Homework 2 Written

1. A fair 20-sided die

(a) States: {1}, {2}, {3}, {4}, {5}, {6}.

All of them are mutually exclusive because two or more events cannot happen at the same time, and exhaustive because at least one of those outcomes will occur.

(b) As the fair 20-sided die, so all probability of which side is the same (p). Therefore, the sum of probabilities of all state is 1.

$$\sum_{i=1}^{20} p(i = x|I) = 1 \to \sum_{i=1}^{20} p = 1 \to 20p = 1 \to p = \frac{1}{20}$$

So, the probability of $i = 7$ is equal:

$$p(i = 7|I) = \frac{1}{20} = 0.05$$

c) If i is odd $\to$ there are 10 sided odd.

$$p(i\ is\ odd|I) = p(i = 1 \vee i = 3 \vee i = 5 \vee i = 7 \vee i = 9 \vee i = 11 \vee$$

$$i = 13 \vee i = 15 \vee i = 17 \vee i = 19|I)$$

$$p(i\ is\ odd|I) = p(i = 1|I) + p(i = 3|I) + p(i = 5|I) + p(i = 7|I) + p(i = 9|I) +$$

$$p(i = 11|I) + p(i = 13|I) + p(i = 15|I) + p(i = 17|I) + p(i = 19|I)$$

$$p(i\ is\ odd|I) = 10p = 10 \times \frac{1}{20} = \frac{1}{2} = 0.5$$

d) If i is prime $\to$ there are 8 sided prime.

$$p(i\ is\ prime|I) = p(i = 2 \vee i = 3 \vee i = 5 \vee i = 7 \vee$$

$$i = 11 \lor i = 13 \lor i = 17 \lor i = 19|I)$$

$$p(i\ is\ prime|I) = p(i = 2|I) + p(i = 3|I) + p(i = 5|I) + p(i = 7|I)+$$

$$p(i = 11|I) + p(i = 13|I) + p(i = 17|I) + p(i = 19|I)$$

$$p(i\ is\ prime|I) = 8p = 8 \times \frac{1}{20} = \frac{2}{5} = 0.4$$

e) The expected value of i is calculated for:

$$E(i) = \sum_{i=1}^{20} i \times P(x = i|I)$$

$$E(i) = 1 \times \frac{1}{20} + 2 \times \frac{1}{20} + 3 \times \frac{1}{20} + 4 \times \frac{1}{20} + 5 \times \frac{1}{20} + 6 \times \frac{1}{20} + 7 \times \frac{1}{20}+$$

$$8 \times \frac{1}{20} + 9 \times \frac{1}{20} + 10 \times \frac{1}{20} + 11 \times \frac{1}{20} + 12 \times \frac{1}{20} + 13 \times \frac{1}{20} + 14 \times \frac{1}{20}+$$

$$15 \times \frac{1}{20} + 16 \times \frac{1}{20} + 17 \times \frac{1}{20} + 18 \times \frac{1}{20} + 19 \times \frac{1}{20} + 20 \times \frac{1}{20}$$

$$E(i) = 10.5$$

f) No, it is not possible to observe the expected value because it is not an integer, and can be regarded as weighted average.

g)

$$p(i\ is\ \{1, 2, 3, 4, 5\}|I) = p(i = 1 \lor i = 2 \lor i = 3 \lor i = 4 \lor i = 5)$$

$$p(i = 1|I) + p(i = 2|I) + p(i = 3|I) + p(i = 4|I) + p(i = 5|I)$$

$$5p = 5 \times \frac{1}{20} = \frac{1}{4}$$

$$p(i\ is\ \{1, 2, 3, 4, 5\}|I) = 0.25$$

2. Independent Pair of Fair 8-Sided Dice

a) Considering the independence of the dice, the probability of the state j occur does not dependent of the state i occur.

$$p(i|j,I) = \frac{p(i|j,I)}{p(j|I)} = \frac{p(i|I)p(j|I)}{p(j|I)} = p(i|I)$$

b) The two dice are independent, so the first dice have 8 sides with 8 possible events ({1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}).

$$\sum_{i=1}^{8} = 1 \rightarrow \sum_{i=1}^{8} p = 1 \rightarrow 8p = 1 \rightarrow p = \frac{1}{8}$$

Therefore,

$$p(i = 2|I) = p = \frac{1}{8}$$

c)

$$p(i = 2, j = 4|I) = p(i = 2|I) = p(j = 4|i = 2|I)$$

$$\frac{1}{8} \times p(j = 4|I) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$$

d) The expected value of i:

$$E(i) = \sum_{i=1}^{8} i \times P(x = i|I)$$

$$E(i) = 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{8} + 5 \times \frac{1}{8} + 6 \times \frac{1}{8} + 7 \times \frac{1}{8} + 8 \times \frac{1}{8}$$

$$E(i) = \frac{36}{8} = 4.5$$

e) No, it is not possible to observe the expected value because it is not an integer, and can be regarded as weighted average.

f) Expected value of $i + j$:

$$E(i + j) = \sum_{i=1}^{8} \sum_{i=1}^{8} (i + j) \times P(i, j|I)$$

$$E(i + j) = \sum_{i=1}^{8} \sum_{i=1}^{8} (i + j) \times P(i|I) * p(j|I)$$

$$E(i+j) = \sum_{i=1}^{8}\sum_{i=1}^{8}(i+j) \times \frac{1}{8} * \frac{1}{8}$$

$$E(i+j) = \sum_{i=1}^{8}\sum_{i=1}^{8}(i+j) \times \frac{1}{64}$$

All possible sum are in the Table 1 below:

i / j	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	11
4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	13
6	7	8	9	10	11	12	13	14
7	8	9	10	11	12	13	14	15
8	9	10	11	12	13	14	15	16

Table 1: Table with all the possible cases of sum

$$E(i) = 1 \times \frac{2}{64} + 2 \times \frac{3}{64} + 3 \times \frac{4}{64} + 4 \times \frac{5}{64} + 5 \times \frac{6}{64} + 6 \times \frac{7}{64} +$$

$$7 \times \frac{8}{64} + 8 \times \frac{9}{64} + 7 \times \frac{10}{64} + 6 \times \frac{11}{64} + 5 \times \frac{12}{64} + 4 \times \frac{13}{64} +$$

$$3 \times \frac{14}{64} + 2 \times \frac{15}{64} + \frac{16}{64}$$

$$E(i) = \frac{2}{64} + \frac{6}{64} + \frac{12}{64} + \frac{20}{64} + \frac{30}{64} + \frac{42}{64} + \frac{56}{64} + \frac{72}{64} + \frac{70}{64} + \frac{66}{64} + \frac{60}{64}$$

$$+ \frac{52}{64} + \frac{42}{64} + \frac{30}{64} + \frac{16}{64}$$

$$E(i) = \frac{576}{64} = 9$$

g) There are 8 possibilities to happen sum equal 9.

$$p(i+j=9|I) = \frac{8}{64} = 0.125$$

3. Coupled Dice

a) Probabilities of all the possible cases.

$$p(i=1|I)=0 \text{ and } p(j=6|I)=0$$

$$P(i=x,j=x|I)=\underbrace{2p(i=x,j=y|I)}_{adjacent\ \text{faces}}=\underbrace{4p(i=x,j=y|I)}_{opposite\ \text{faces}}$$

$$p=2p_a=4p_o$$

The same faces appear:

$$\sum_{i=2}^{5} p(i=x,j=x|I)$$

$$p(i=2,j=2|I)+p(i=3,j=3|I)+p(i=4,j=4|I)+p(i=5,j=5|I)$$

$$p+p+p+p=4p$$

Adjacent faces appear:

$$p(i=x,j=x|I)=2p(i=x,j=y|I)$$

$$4p=2p_a$$

$$2p=p_a$$

Opposite faces appear:

$$p(i=x,j=x|I)=4p(i=x,j=y|I)$$

$$4p=4p_a$$

$$p=p_a$$

b) Proof of how the probabilities sum to unity:

$$\sum_{i=1}^{6}\sum_{i=1}^{6} p(i=x, j=y|I) = 1$$

$$4 \times 4p + 8 \times 2p + 4 \times p = 1$$

$$36p = 1 \rightarrow p = \frac{1}{36}$$

The same faces appear: $4p = \frac{1}{9}$

Adjacent faces appear: $2p = \frac{1}{18}$

Opposite faces appear: $p = \frac{1}{36}$

All probabilities can be viewed in the Table 2 below:

i / j	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	4p	2p	2p	p	0
3	0	2p	4p	p	2p	0
4	0	2p	p	4p	2p	0
5	0	p	2p	2p	4p	0
6	0	0	0	0	0	0

Table 2: Table with probabilities for all the possible results

c) The expected value of i+j:

$$E(i+j) = \sum_{i=1}^{6}\sum_{i=1}^{6} (i+j) \times P(i, j|I)$$

All possible sums are below in the Table 3:

i / j	1	2	3	4	5	6
0	0	0	0	0	0	0
0	3	4	5	6	7	0
0	4	5	6	7	8	0
0	5	6	7	8	9	0
0	6	7	8	9	10	0
0	0	0	0	0	0	0

Table 3: Table with all the possible cases of sum

$$E(i+j) = \underbrace{4 \times 4p + 6 \times 4p + 10 \times 4p +}_{\text{the same faces appear}}$$

$$\underbrace{5 \times 2p + 5 \times 2p + 6 \times 2p + 8 \times 2p + 8 \times 2p + 9 \times 2p + 9 \times 2p +}_{\text{adjacent faces appear}}$$

$$\underbrace{7 \times p + 7 \times p + 7 \times p + 7 \times p}_{\text{opposite faces appear}}$$

$$E(i + j) = 28 \times 4p + 56 \times 2p + 28 \times p = 112p + 112p + 28p$$

$$E(i + j) = 252p = 252 \times \frac{1}{36}$$

$$E(i + j) = 7$$

d) $p(i|I)$ for all values of i.

$$\sum_{i=1}^{6} p(i|I) = p(1|I) + p(2|I) + p(3|I) + p(4|I) + p(5|I) + p(6|I)$$

$$p(2|I) \rightarrow \text{for which row is the same}$$

$$p(2|I) = 4p + 2p + 2p + p = 9p = 9 \times \frac{1}{36} = \frac{1}{4}$$

e) Expected value of i.

$$E(i) = \sum_{i=1}^{6} i \times P(i|I) = 1 \times 0 + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} + 5 \times \frac{1}{4} + 6 \times 0$$

$$E(i) = \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} = \frac{2 + 3 + 4 + 5}{4} = \frac{14}{4}$$

$$E(i) = 3.5$$