# Winter 18 – AMS206B Homework 4

**Due: Friday Feb 16th.**

**This hw includes more questions on the topics covered in the previous two hws. These questions will give you more chance to practice the concepts in different settings.**

1. Let $X_1, \ldots, X_n$ be an i.i.d. sample such that $X_i \mid \theta, \sigma^2 \sim \mathrm{N}(\theta, \sigma^2)$, where $\sigma^2$ is known and $\mu$ is unknown. Also, let your prior for $\theta$ be a mixture of conjugate priors, i.e.,

$$\pi(\theta) = \sum_{\ell=1}^{K} w_\ell \phi(\theta|\mu_\ell, \tau^2),$$

   where $\phi(\theta \mid \mu, \tau^2)$ denotes the Gaussian density with mean $\mu$ and variance $\tau^2$ ($K$ is fixed). Assume $w_\ell \geq 0$ and $\sum_{\ell=1}^{K} w_\ell = 1$.

   (a) Find the posterior distribution distribution for $\theta$ based on this prior.

   (b) Find the posterior mean.

   (c) Find the prior predictive distribution associated with this model (that is, the marginal distribution of $X$).

   (d) Find the posterior predictive distribution associated with this model.

2. Let $X_1, \ldots, X_n$ be an i.i.d. sample such that $X_i \mid \theta \sim \mathrm{N}(\theta, 1)$. Suppose that you know that $\theta > 0$, and you want your prior to reflect that fact. Hence, you decide to set $\pi(\theta)$ to be a normal distribution with mean $\mu$ and variance $\tau^2$, truncated to be positive, i.e.,

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\tau\Phi(\mu/\tau)} \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\} \mathrm{I}_{[0,\infty)}(\theta).$$

   (a) Find the posterior distribution distribution for $\theta$ based on this prior. Is this a conjugate prior?

   (b) Find the prior predictive distribution (that is, the marginal distribution of $X$).

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample such that each $X_i$ comes from a truncated normal with unknown mean $\theta$ and variance 1,

$$f(X_i \mid \theta) = \frac{1}{\sqrt{2\pi}\Phi(\theta)} \exp\{-\frac{1}{2}(X_i - \theta)^2\} \mathrm{I}_{[0,\infty)}(X_i).$$

   If $\theta \sim \mathrm{N}(\mu, \tau^2)$, find the posterior for $\theta$. Is this a conjugate prior for this problem? How is this problem different from the previous one?

4. Suppose that $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a sample from negative binomial, $\mathrm{NB}(m, \theta)$ distribution, that is, $X_i \overset{iid}{\sim} \mathrm{NB}(m, \theta)$, $i = 1, \ldots, n$.

   (a) Show that the pdf of $\boldsymbol{X}$ is in the exponential family. Find the natural parameterization and the natural parameters.

(b) Find a conjugate family of distribution using the natural parameterization.

5. Consider $X \mid \theta \sim \text{Gamma}(\theta, \beta)$ where $E(X) = \theta/\beta$ (note: $\beta$ is a rate parameter!). We assume that $\beta$ is fixed. That is,

$$f(x \mid \theta) = \frac{\beta^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\beta x}, \qquad x > 0.$$

   (a) Show that the pdf of $\boldsymbol{X}$ is in the exponential family. Find the natural parameterization and the natural parameters.

   (b) Find a conjugate family of distribution using the natural parameterization.

6. Let $\boldsymbol{X} = (X_1, \ldots, X_k)'$ be a random vector with a multinomial distribution with index $n$ and probabilities $\theta_1, \ldots, \theta_k$ such that $\sum_{i=1}^k X_i = n$ and $\sum_{i=1}^k \theta_i = 1$.

   (a) Show that the pdf of $\boldsymbol{X}$ (the multinomial pdf) is in the exponential family. Find the natural parameterization and the natural parameters.

   (b) Find a conjugate family of distributions using the natural parameterization.