Winter 18 – AMS206B Homework 2 Solution

Note: The questions are taken from Chapters 1 and 2 from JB.

1. An insurance company is faced with taking one of the following 3 actions:

   - a_1: increase sales force by 10%
   - a_2: maintaining present sales force
   - a_3: decrease sales force by 10%.

   Depending upon whether or not the economy is good (θ_1), mediocre (θ_2), or bad (θ_3), the company would expect to lose the following amounts of money in each case:

State of	Action Taken		
Economy	a_1	a_2	a_3
θ_1	-10	-5	-3
θ_2	-5	-5	-2
θ_3	1	0	-1

   (a) Determine if each action is admissible or inadmissible.

   (b) The company believes that θ has the probability distribution $\pi(\theta_1) = 0.2$, $\pi(\theta_2) = 0.3$, $\pi(\theta_3) = 0.5$. Order the actions according to their Bayesian expected loss (equivalent to Bayes risk, here), and state the Bayes action.

   (c) Order the actions according to the minimax principle and the find the minimax nonrandomized action.

   Solution:

   (a) We say that an action is *inadmissible* if there exists an R-better decision rule for each θ_i. This means that an action a_1 is inadmissible if there exists another action a_2 for which $\mathrm{L}(\theta_i, a_2) \leq \mathrm{L}(\theta_i, a_1)$. Thus in this question, all three actions are *admissible.*

   (b) The *Bayes risk* of a decision rule δ with respect to a prior θ is defined as

$$\rho(\pi^*, a) = E^\pi[L(\theta, a)]$$

   In this example,

$$\begin{aligned}
E^\pi[L(\theta, a_1)] &= (0.2 \times -10) + (0.3 \times -5) + (0.5 \times 1) = -3 \\
E^\pi[L(\theta, a_2)] &= (0.2 \times -5) + (0.3 \times -5) + (0.5 \times 0) = -2.5 \\
E^\pi[L(\theta, a_3)] &= (0.2 \times -3) + (0.3 \times -2) + (0.5 \times -1) = -1.7
\end{aligned}$$

   So $E^\pi[L(\theta, a_1)] < E^\pi[L(\theta, a_2)] < E^\pi[L(\theta, a_3)]$, and a_1 is the Bayes action.

(c) The *minimax principle* states that one decision rule (δ_1^*) is preferred to another (δ_2^*) if its worse case is better than the other's worst case. i.e.

$$\sup_{\theta \in \Theta} R(\theta, \delta_1^*) < \sup_{\theta \in \Theta} R(\theta, \delta_2^*)$$

We can see from the table that

$$(\sup \ L(\theta, a_1) = \max\{-10, -5, 1\} = 1) > (\sup L(\theta, a_2) = 0) > (\sup L(\theta, a_3) = -1)$$

and a_3 is the minimax nonrandomized action.

2. The owner of a ski shop must order skis for the upcoming season. Orders must be placed in quantities of 25 pairs of skis. The cost *per pair* of skis is \$50 if 25 are ordered, \$45 if 50 are ordered, and \$40 if 75 are ordered. The skis will be sold at \$75 per pair. Any skis left over at the end of the year can be sold (for sure) at \$25 a pair. If the owner runs out of skis during the season he will suffer a loss of "goodwill" among unsatisfied customers. he rates this loss at \$5 per unsatisfied customer. For simplicity, the owner feels that demand for the skis will be 30, 40, 50 or 60 pair of skis, with probabilities 0.2, 0.4, 0.2, and 0.2, respectively.

   (a) Describe $\mathcal{A}$, Θ and $L(\theta, a)$.

   (b) Which actions are admissible?

   (c) What is the Bayes action?

   (d) What is the minimax nonrandomized action?

   Solution:

   (a) The action space $\mathcal{A}$ consists of

   - a_1: The owner purchases 25 skis for \$50 each
   - a_2: The owner purchases 50 skis for \$45 each
   - a_3: The owner purchases 75 skis for \$40 each

   The states Θ are

   - θ_1: The demand will be 30 skis
   - θ_2: The demand will be 40 skis
   - θ_3: The demand will be 50 skis
   - θ_4: The demand will be 60 skis

   Given $\mathcal{A}$ and Θ, the loss function $L(\theta, a)$ is

	a_1	a_2	a_3
θ_1	-600	-500	-375
θ_2	-550	-1000	-875
θ_3	-500	- 1500	-1375
θ_4	-450	-1450	-1875

   The prior distribution is $\pi(\theta_1) = 0.2$, $\pi(\theta_2) = 0.4$, $\pi(\theta_3) = 0.2$ and $\pi(\theta_4) = 0.2$.

   (b) Given the definition of *admissibility* from the previous problem, all three actions are admissible

(c) The Bayesian expected loss of each action is

$$\begin{aligned}\rho(\pi, a_1) &= 0.2 \times -600 + 0.4 \times -550 + 0.2 \times -500 + 0.2 \times -450 = -530 \\ \rho(\pi, a_2) &= 0.2 \times -500 + 0.4 \times -1000 + 0.2 \times -1500 + 0.2 \times -1450 = -1090 \\ \rho(\pi, a_3) &= 0.2 \times -375 + 0.4 \times -875 + 0.2 \times -1375 + 0.2 \times -1875 = -1075\end{aligned}$$

So the Bayes action is a_2.

(d) Given the minimax principle from the previous problem, we find the minimum maximum loss on the loss table (-450, -500, or -375). It belongs to a_2, so a_2 is also the minimax nonrandomized action.

3. An automobile company is about to introduce a new type of car into the market. It must decide how many of these new cars to produce. Let a denote the number of cars decided upon. A market survey will be conducted, with information obtained pertaining to θ, the proportion of the population which plans on buying a car and would tend to favor the new model. The company has determined that the major outside factor affecting the purchase of automobiles is the state of the economy. Indeed, letting Y denote an appropriate measure of the state of the economy, it is felt that $Z = (1+Y)(10^7)\theta$ cars could be sold. Y is unknown but is thought to have a Unif$(0, 1)$ distribution.

Each car produced will be sold at a profit of \$500, unless that supply (a) exceeds the demand (Z). If $a > Z$, the extra $a - Z$ cars can be sold at a loss of \$300 each. The company's utility function for money is linear (say, $U(m) = m$) over the rage involved. Show that $L(\theta, a)$ is given by

$$L(\theta, a) = \begin{cases} -500a & \text{if } \frac{a}{10^7} \leq \theta, \\ \frac{(4225)(10^7)}{4\theta}\left[\frac{8}{13(10^7)}a - \theta\right]^2 - \frac{(2625)(10^7)\theta}{4} & \text{if } \theta \leq \frac{a}{10^7} \leq 2\theta, \\ 300a - (1200)(10^7)\theta & \text{if } \frac{a}{10^7} \geq 2\theta. \end{cases}$$

Solution: First, in general, $L(\theta, a|Z = z) = -500a\mathbb{1}(a \leq z) + (-500z + 300(a - z))\mathbb{1}(a > z)$. Also, notice that because $Y \sim$ Unif$(0, 1)$, $Z \sim$ Unif$(10^7\theta, 10^7(2\theta))$. Therefore, there are three scenarios depending on the value of a:

i. If $a < 10^7\theta$, then surely $a < z$, meaning that every car produced will be in demand. This yields

$$L(\theta, a) = \mathrm{E}_Z[L(\theta, a \mid Z = z)] = \mathrm{E}_Z[-500a] = -500a.$$

ii. If $a > 10^7(2\theta)$, then $a > Z$. Therefore, exactly Z cars will be sold for profit and $(a - Z)$ for loss. In this case,

$$\begin{aligned}L(\theta, a) &= \mathrm{E}_Z[L(\theta, a \mid Z = z)] = \mathrm{E}_Z[-500z + 300(a - z)] = \mathrm{E}_Z[300a - 800z] \\ &= 300a - 800\mathrm{E}_Z[z] = 300a - 800\left(\frac{10^7\theta + 10^7(2\theta)}{2}\right) \\ &= 300a - 1200(10^7)\theta.\end{aligned}$$

iii. Finally, if $10^7\theta \le a \le 10^7(2\theta)$, then the loss depends on the value of Z. If $a < Z$, then all a cars can be sold; otherwise Z cars will be sold for a profit while the remaining $(a - Z)$ must be sold at a loss. Then

$$\begin{aligned}
L(\theta, a) &= \mathrm{E}_Z[L(\theta, a \mid Z = z)] = \int_{10^7\theta}^{10^7(2\theta)} L(\theta, a|Z = z) f_Z(z)\,\mathrm{d}z \\
&= \frac{1}{10^7\theta}\left[-500a \int_{10^7\theta}^{10^7(2\theta)} \mathbb{1}(a \le z)\,\mathrm{d}z - 800 \int_{10^7\theta}^{10^7(2\theta)} z\mathbb{1}(a > z)\,\mathrm{d}z \right. \\
&\qquad \left. +300a \int_{10^7\theta}^{10^7(2\theta)} \mathbb{1}(a > z)\,\mathrm{d}z\right] \\
&= \frac{1}{10^7\theta}\left[-500a \int_{a}^{10^7(2\theta)} \mathrm{d}z - 800 \int_{10^7\theta}^{a} z\,\mathrm{d}z + 300a \int_{10^7\theta}^{a} \mathrm{d}z\right] \\
&= \frac{1}{10^7\theta}\left[-500a\left(10^7(2\theta) - a\right) - 800\left(\frac{1}{2}a^2 - \frac{1}{2}\left(10^7\theta\right)^2\right) + 300a\left(a - 10^7\theta\right)\right] \\
&= -1000a + \frac{500a^2}{10^7\theta} - \frac{400a^2}{10^7\theta} + 400(10^7)\theta + \frac{300a^2}{10^7\theta} - 300a \\
&= -1300a + 400(10^7)\theta + \frac{400a^2}{10^7\theta} \\
&= \frac{4225(10^7)}{4\theta}\left(\frac{8}{13(10^7)}a - \theta\right)^2 - \frac{2625(10^7\theta)}{4}
\end{aligned}$$

In summary,

$$L(\theta, a) = \begin{cases} -500a & \text{if } \frac{a}{10^7} < \theta, \\ \frac{4225(10^7)}{4\theta}\left(\frac{8}{13(10^7)}a - \theta\right)^2 - \frac{2625(10^7\theta)}{4} & \text{if } \theta \le \frac{a}{10^7} \le 2\theta, \\ 300a - 1200(10^7)\theta & \text{if } \frac{a}{10^7} > 2\theta \end{cases}$$

as desired.

4. Let X be Unif$(\theta, \theta + 1)$, and suppose that it is desired to test $H_0 : \theta = 0$ versus $H_1 : \theta = 0.9$ these being the only two values of θ considered possible). Consider the test which rejects H_0 if $x \ge 0.95$, and accepts H_0 otherwise.

   (a) Calculate the probabilities of Type I and Type II error for this test.

   (b) If $0.9 < x < 1$ is observed, what is the intuitive (conditional) probability of actually making an error in use of the test.

   (c) Determine the likelihood function of θ for each x.

   (d) Interpret the "message" conveyed by the likelihood function for each x.

   Solution:

   (a) If $\theta = 0$, then $X \sim$ Unif$(0, 1)$ and $\mathrm{P}(X \ge 0.95|\theta = 0) = \int_{0.95}^{1} 1\,\mathrm{d}x = 0.05$. Thus, the probability of a Type I error (rejecting H_0 given $H_0 : \theta = 0$ is true) is 0.05.
   If $\theta = 0.9$, then $X \sim$ Unif$(0.9, 1.9)$ and $\mathrm{P}(X < 0.95|\theta = 0.9) = \int_{0.90}^{0.95} 1\,\mathrm{d}x = 0.05$. Thus the probability of a Type II error (accepting H_0 given $H_0 : \theta = 0$ is false) is 0.05.

(b) If $0.9 < x < 1$ is observed, the conditional probability of making an error with the specified test is

$$\begin{aligned}
\mathrm{P}\{\textsf{error}|X \in (0.9,1)\} &= \mathrm{P}\{(\theta = 0 \cap X \geq 0.95) \cup (\theta = 0.9 \cap X < 0.95) \,|\, X \in (0.9,1)\} \\
&= \mathrm{P}\{(\theta = 0 \cap X \geq 0.95) \,|\, X \in (0.9,1)\} + \\
&\quad \mathrm{P}\{(\theta = 0.9 \cap X < 0.95) \,|\, X \in (0.9,1)\} \\
&= \frac{\mathrm{P}\{\theta = 0 \cap X \in [0.95,1)\}}{\mathrm{P}\{X \in (0.9,1)\}} + \frac{\mathrm{P}\{\theta = 0.9 \cap X \in (0.9,0.95)\}}{\mathrm{P}\{X \in (0.9,1)\}} \\
&= \frac{\mathrm{P}\{X \in [0.95,1) \,|\, \theta = 0\}\mathrm{P}\{\theta = 0\}}{\mathrm{P}\{X \in (0.9,1) \,|\, \theta = 0\}\mathrm{P}\{\theta = 0\} + \mathrm{P}\{X \in (0.9,1) \,|\, \theta = 0.9\}\mathrm{P}\{\theta = 0.9\}} + \\
&\quad \frac{\mathrm{P}\{X \in (0.9,0.95) \,|\, \theta = 0.9\}\mathrm{P}\{\theta = 0.9\}}{\mathrm{P}\{X \in (0.9,1) \,|\, \theta = 0\}\mathrm{P}\{\theta = 0\} + \mathrm{P}\{X \in (0.9,1) \,|\, \theta = 0.9\}\mathrm{P}\{\theta = 0.9\}} \\
&= \frac{0.05\,\mathrm{P}\{\theta = 0\}}{0.1\,\mathrm{P}\{\theta = 0\} + 0.1\,\mathrm{P}\{\theta = 0.9\}} + \frac{0.05\,\mathrm{P}\{\theta = 0.9\}}{0.1\,\mathrm{P}\{\theta = 0\} + 0.1\,\mathrm{P}\{\theta = 0.9\}} \\
&= \frac{0.05}{0.10}\left(\frac{\mathrm{P}\{\theta = 0\}}{\mathrm{P}\{\theta = 0\} + \mathrm{P}\{\theta = 0.9\}} + \frac{\mathrm{P}\{\theta = 0.9\}}{\mathrm{P}\{\theta = 0\} + \mathrm{P}\{\theta = 0.9\}}\right) \\
&= \frac{0.05}{0.10} \times 1 = \frac{1}{2}.
\end{aligned}$$

Note: The error rate when $0.9 < X < 1$ is observed, is much greater than Type I error rate or Type II error rate found in part (a)

(c) For a single x, $f_X(x|\theta) = \frac{1}{\theta+1-\theta}\mathbb{1}(\theta \leq x \leq \theta + 1) = \mathbb{1}(\theta \leq x \leq \theta + 1) = \mathcal{L}(\theta; x)$ which is the likelihood function when viewed as a function of θ.

(d) In this case, the likelihood function is simply an indicator function over possible values of θ. Specifically, observing $X = x$ immediately rules out any hypothesized $\theta < x - 1$ as well as any hypothesized $\theta > x$ since such values would violate $\theta \leq x \leq \theta + 1$. For example, observing $x = 0.5$ immediately rules out $H_1 : \theta = 0.9$ because Unif$(0.9, 1.9)$ could not have produced $x = 0.5$. Consequently, the test yields a result that is certain (is 100% accurate) for any $x \in \{[0, 0.9) \cup (1, 1.9]\}$.

Hence, using the conditional approach, we can report more sensible error rates of the decision rule than the type I error rate or the type II error rate in (a) as follows: (i) when $x \in [0, 0.9)$ or $x \in (1, 1.9)$ is observed, the actual error rate is 0%. (ii) when $x \in [0.9, 1]$ is observed, the actual performance of the decision rule is 50%.