# Winter 18 – AMS206B Homework 3 Solution

1. Consider three independent random variables $X_1, X_2$ and $X_3$ such that $X_i \overset{iid}{\sim} \text{Gamma}(a_i, b)$. Let

$$\mathbf{Y} = (Y_1, Y_2, Y_3) = \left( \frac{X_1}{X_1 + X_2 + X_3}, \frac{X_2}{X_1 + X_2 + X_3}, \frac{X_3}{X_1 + X_2 + X_3} \right).$$

   (a) Show that $\mathbf{Y} \sim \text{Dirichlet}(a_1, a_2, a_3)$, a Dirichlet distribution.

   (b) How can this result be used to generate random variables according to a Dirichlet distribution? Write a simple function in `R` or `Matlab` (your choice) that takes as inputs $n$, the number of trivariate vectors to be generated, and $\mathbf{a} = (a1, a2, a3)$ and generates as an output a matrix of size $n \times 3$ whose rows correspond to independent samples from a Dirichlet distribution with parameter $(a1, a2, a3)$. (*Note:* the value of $b$ is not important as long as the three $X$ have the same value for $b$)

**Solution:**

(a) First, we consider the joint density of the three independent Gamma-distributed RVs:

$$p(x_1, x_2, x_3) = \prod_{i=1}^{3} p(x_i) = \prod_{i=1}^{3} \frac{x_i^{a_i - 1} e^{-x_i/b}}{\Gamma(a_i) b^{a_i}} = \frac{e^{\sum_{i=1}^3 x_i/b} \prod_{i=1}^3 x_i^{a_i-1}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)}.$$

Since $Y_i = X_i / \sum_{i=1}^3 X_i$, we find

$$\begin{aligned} X_1 &= Y_1 Z \\ X_2 &= Y_2 Z \\ X_3 &= Y_3 Z = (1 - Y_1 - Y_2) Z, \end{aligned}$$

where $Z = \sum_{i=1}^3 X_i$. To obtain the joint distribution of $(Y_1, Y_2, Z)$, we find the Jacobian for this change of variables. The matrix is

$$J = \begin{pmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} & \frac{dx_1}{d_z} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} & \frac{dx_2}{d_z} \\ \frac{dx_3}{dy_1} & \frac{dx_3}{dy_2} & \frac{dx_3}{d_z} \end{pmatrix} = \begin{pmatrix} Z & 0 & Y_1 \\ 0 & Z & Y_2 \\ -Z & -Z & (1 - Y_1 - Y_2) \end{pmatrix}$$

So, the Jacobian, $|J|$ is $Z^2$.

$$p(Y_1, Y_2, Z) = \frac{(y_1 z)^{a_1 - 1} e^{-y_1 z/b} (y_2 z)^{a_2 - 1} e^{-y_2 z/b} \{(1 - y_1 - y_2) z\}^{a_3 - 1} e^{-(1 - y_1 - y_2) z/b}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)} z^2,$$

where $0 < y_1, y_2 < 1$, $y_1 + y_2 < 1$ and $0 < z$.
By letting $y_3 = 1 - y_1 - y_2$,

$$p(Y_1, Y_2, Z) = \frac{y_1^{a_1-1} y_2^{a_2-1} y_3^{a_3-1} z^{\sum_{i=1}^3 a_i - 1} e^{z \overbrace{\sum_{i=1}^3 y_i}^{=1} /b}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)}.$$

We now integrate out $z$ to obtain $p(y_1, y_2, y_3)$.

$$
\begin{aligned}
p(y_1, y_2, y_3) &= \int_{\mathbb{R}} \frac{y_1^{a_1-1} y_2^{a_2-1} y_3^{a_3-1} \quad \overbrace{z^{\sum_{i=1}^3 a_i - 1} e^{-z/b}}^{\text{kernel for Gamma}(\sum_{i=1}^3 a_i,\, b)}}{b^{\sum_{i=1}^3 a_i} \prod_{i=1}^3 \Gamma(a_i)} dz \\
&= \frac{\Gamma(\sum_{i=1}^3 a_i)}{\prod_{i=1}^3 \Gamma(a_i)} y_1^{a_1-1} y_2^{a_2-1} y_3^{a_3-1},
\end{aligned}
$$

(b)
```
#a is a vector of length p; a=(a_1, a_2, ..., a_p)
#n is the sample size

> dirichlet <- function(a, n){
        p <- length(a)
        y <- array(NA, dim=c(n, p)) #Each row of y is iid sample from Dir(a)

        for (i in 1:n) {
              tmp <- rgamma(p, a, 1)
              y[i, ] <- tmp / sum(tmp)
        }

        return(y)
  }
```

2. $Y$ follows an inverse Gamma distribution with shape parameter $a$ and scale parameter $b$ ($Y \sim \text{IG}(a, 1/b)$) if $Y = 1/X$ with $X \sim \text{Gamma}(a, b)$ (assume the Gamma distribution is parameterized so that $E(X) = ab$).

   (a) Find the density of $Y$.

   (b) Compute $E(Y^k)$. Do you need to impose any constrain on the problem for this expectation to exists?

   (c) Compare $E(Y^k)$ to $1/E(X^k)$ (hint: look at the ratio of the two quantities).

**Solution:**

(a) Since $X \sim \text{Gamma}(a, b)$,

$$f(x) = \frac{1}{\Gamma(a) b^a} x^{a-1} \exp\left\{-\frac{x}{b}\right\}, \text{for } x > 0.$$

Let $y = 1/x$. Then $x = 1/y$ and $\frac{dx}{dy} = -1/y^2$. Therefore,

$$p(y) = \frac{1}{\Gamma(a) b^a} \left(\frac{1}{y}\right)^{a-1} \exp\left\{-\frac{1}{by}\right\} \left|-\frac{1}{y^2}\right| = \frac{(1/b)^a}{\Gamma(a)} y^{-(a+1)} \exp\left\{-\frac{(1/b)}{y}\right\}$$

which is an inverse Gamma with shape $a$ and rate $1/b$.

(b)

$$\begin{aligned} \mathrm{E}(Y^k) &= \int_{\mathbb{R}+} y^k \frac{(1/b)^a}{\Gamma(a)} y^{-(a+1)} \exp\left\{-\frac{(1/b)}{y}\right\} dy \\ &= \frac{(1/b)^a}{\Gamma(a)} \int_{\mathbb{R}+} y^{-(a-k+1)} \exp\left\{-\frac{(1/b)}{y}\right\} dy = \frac{(1/b)^a}{\Gamma(a)} \frac{\Gamma(a-k)}{(1/b)^{a-k}} \end{aligned}$$

That is, $\mathrm{E}(Y^k) = \Gamma(a-k)/(b^k\Gamma(a))$ for $a - k > 0$.

(c)

$$\begin{aligned} \mathrm{E}(X^k) &= \int_{\mathbb{R}+} x^k \frac{1}{\Gamma(a)b^a} x^{a-1} \exp\left\{-\frac{x}{b}\right\} dx \\ &= \frac{1}{\Gamma(a)b^a} \int_{\mathbb{R}+} x^{a+k-1} \exp\left\{-\frac{x}{b}\right\} dx = \frac{1}{\Gamma(a)b^a} \Gamma(a+k) b^{a+k}. \end{aligned}$$

Then,

$$\zeta \equiv \frac{1/\mathrm{E}(X^k)}{\mathrm{E}(Y^k)} = \frac{\Gamma(a)\Gamma(a)}{\Gamma(a+k)\Gamma(a-k)}$$

which implies that $\zeta = 1 \Leftrightarrow k = 0$. That is, none of the moments of $X$ is invariant to the reciprocal transformation.

3. Let $L(\theta, a) = \omega(\theta)(\theta - a)^2$, with $\omega(\theta)$ a non-negative function, be the weighted quadratic loss. Show that $\delta^B(x)$, the estimator that minimizes the Bayesian expected loss $\rho(\pi, \theta)$ has the form

$$\delta^B(x) = \frac{E(\omega(\theta)\theta|x)}{E(\omega(\theta)|x)}.$$

Hint: Show that any other estimator has a larger Bayesian expected loss.

**Solution:**

Let $\delta(x)$ be any decision rule,

$$
\begin{aligned}
\rho(\pi,\delta) &= \int_\Theta w(\theta)(\theta-\delta)^2 f(\theta \mid x)d\theta = \int_\Theta w(\theta)(\theta-\delta^B+\delta^B-\delta)^2 f(\theta \mid x)d\theta \\
&= \int_\Theta w(\theta)\left[(\theta-\delta^B)^2 - 2(\theta-\delta^B)(\delta^B-\delta)+(\delta^B-\delta)^2\right] f(\theta \mid x)d\theta \\
&= \int_\Theta w(\theta)(\theta-\delta^B)^2 f(\theta \mid x)d\theta - 2\int_\Theta w(\theta)(\theta-\delta^B)(\delta^B-\delta) f(\theta \mid x)d\theta \\
&+ \int_\Theta (\delta^B-\delta)^2 w(\theta) f(\theta \mid x)d\theta \\
&= \int_\Theta w(\theta)(\theta-\delta^B)^2 f(\theta \mid x)d\theta + \int_\Theta (\delta^B-\delta)^2 w(\theta) f(\theta \mid x)d\theta \\
&- 2(\delta^B-\delta)\left[\int_\Theta w(\theta)\theta f(\theta \mid x)d\theta - \delta^B \int_\Theta w(\theta) f(\theta \mid x)d\theta\right] \\
&= \int_\Theta w(\theta)(\theta-\delta^B)^2 f(\theta \mid x)d\theta + \int_\Theta (\delta^B-\delta)^2 w(\theta) f(\theta \mid x)d\theta \\
&- 2(\delta^B-\delta)\left[E(\omega(\theta)\theta|x) - \delta^B E(\omega(\theta)|x)\right] \\
&= \int_\Theta w(\theta)(\theta-\delta^B)^2 f(\theta \mid x)d\theta + \int_\Theta (\delta^B-\delta)^2 w(\theta) f(\theta \mid x)d\theta \\
&- 2(\delta^B-\delta)\cancelto{0}{\left[E(\omega(\theta)\theta|x) - \frac{E(\omega(\theta)\theta|x)}{E(\omega(\theta)|x)}E(\omega(\theta)|x)\right]}
\end{aligned}
$$

and notice that the first term does not depend on $\delta$. Thus, the rule that minimizes $\rho(\pi,\delta)$ is that which makes the second term equal to zero, namely $\delta = \delta^B$.

4. Consider $x \mid \theta \sim \mathrm{N}(\theta, 1)$, $\theta \sim \mathrm{N}(0,1)$ and the loss

$$L(\theta, a) = e^{3\theta^2/4}(\theta - a)^2.$$

   (a) Show that the estimator that minimizes the Bayesian expected posterior loss in this case is $\delta(x) = 2x$. Hint: use the previous exercise.

   (b) Show that $\delta_0(x) = x$ dominates $\delta(x)$.

**Solution:**

   (a) First notice that if $X \mid \theta \sim \mathrm{N}(\theta,1)$ and $\theta \sim \mathrm{N}(0,1)$ then $\theta \mid X \sim \mathrm{N}\left(\frac{x}{2}, \frac{1}{2}\right)$. Now, taking $w(\theta) = e^{3\theta^2/4}$, form the previous exercise is known that $\delta^B(x) = \frac{E(\omega(\theta)\theta|x)}{E(\omega(\theta)|x)}$.

In this case

$$\begin{aligned}
\mathrm{E}(w(\theta)\theta \mid x) &= \int_{\mathbf{R}} \theta \exp\left\{\frac{3\theta^2}{4}\right\} \frac{1}{\sqrt{\pi}} \exp\left\{-\left(\theta - \frac{x}{2}\right)^2\right\} d\theta \\
&= \int_{\mathbf{R}} \theta \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\theta^2 + \theta x - \frac{x^2}{4}\right\} d\theta \\
&= \exp\left\{\frac{3x^2}{4}\right\} \int_{\mathbf{R}} \theta \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left(\theta^2 - 2\theta 2x - 4x^2\right)\right\} d\theta \\
&= 2\exp\left\{\frac{3x^2}{4}\right\} \int_{\mathbf{R}} \theta \frac{1}{\sqrt{2\pi 2}} \exp\left\{-\frac{1}{2\cdot 2}(\theta - 2x)^2\right\} d\theta \\
&= 2\exp\left\{\frac{3x^2}{4}\right\} \mathrm{E}[\psi] = 4x \exp\left\{\frac{3x^2}{4}\right\}
\end{aligned}$$

where $\psi \sim \mathrm{N}(2x, 2)$.
Similarly,

$$\begin{aligned}
\mathrm{E}(w(\theta) \mid x) &= 2\exp\left\{\frac{3x^2}{4}\right\} \cancelto{1}{\int_{\mathbf{R}} \frac{1}{\sqrt{2\pi 2}} \exp\left\{-\frac{1}{2\cdot 2}(\theta - 2x)^2\right\} d\theta} \\
&= 2\exp\left\{\frac{3x^2}{4}\right\}
\end{aligned}$$

Thus, $\delta^B = 2x$.

(b) Notice that $R(\delta, \theta)$ satisfies

$$\begin{aligned}
R(\delta, \theta) &= \mathrm{E}(w(\theta)(\theta - 2x)^2) = w(\theta)\mathrm{E}(\theta^2 - 4\theta x + 4x^2) \\
&= w(\theta)\{\theta^2 - 4\theta^2 + 4(1 + \theta^2)\} = w(\theta)(\theta^2 + 4).
\end{aligned}$$

while $R(\delta_0, \theta)$ is given by

$$\begin{aligned}
R(\delta_0, \theta) &= \mathrm{E}(w(\theta)(\theta - x)^2) = w(\theta)\mathrm{E}(\theta^2 - 2\theta x + x^2) \\
&= w(\theta)\{\theta^2 - 2\theta^2 + (1 + \theta^2)\} = w(\theta).
\end{aligned}$$

Thus, $R(\delta_0, \theta) < R(\delta, \theta)$ for all $\theta$, which implies that $\delta_0$ dominates $\delta$.

5. Assume you have to guess a secret number $\theta$. You know that $\theta$ is an integer. You can perform an experiment that would yield either the number before it or the number after it, with equal probability. You perform the experiment twice. More formally, let $x_1$ and $x_2$ be independent observations from

$$f(x = \theta - 1 \mid \theta) = f(x = \theta + 1 \mid \theta) = 1/2.$$

Consider the 0-1 loss function, i.e.,

$$L(\theta, a) = \begin{cases} 1 & a \neq \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the risks of the estimators $\delta_0(x_1, x_2) = \frac{x_1 + x_2}{2}$ and $\delta_1(x_1, x_2) = x_1 + 1$.

(b) Find the estimator $\delta^B(x_1, x_2)$ that minimizes the Bayesian expected loss.

**Solution:**

(a) By the definition,

$$R(\delta_0, \theta) = \mathrm{E}\left[L\left(\frac{X_1 + X_2}{2}, \theta\right)\right] = 0P(X_1 \neq X_2) + 1P(X_1 = X_2) = \frac{1}{2}.$$

since $X_1$ and $X_2$ are independent. Similarly,

$$R(\delta_1, \theta) = \mathrm{E}[L(X_1 + 1, \theta)] = 0P(X_1 = \theta - 1) + 1P(X_1 = \theta + 1) = \frac{1}{2}.$$

(b) To obtain the posterior distribution of $\theta$, we consider the following two cases;

i. if $x_1 \neq x_2$

$$p\left(\theta = \frac{x_1 + x_2}{2} \;\middle|\; x_1, x_2\right) = 1 \quad \text{and} \quad p(\theta = \theta_0 \mid x_1, x_2) = 0 \text{ for } \theta_0 \neq \frac{x_1 + x_2}{2}.$$

In words, the posterior probability is a mass point at $\theta = (x_1 + x_2)/2$. Thus, in this case, $\delta^B(X_1, X_2) = \frac{X_1+X_2}{2}$ since $\mathrm{E}(L(\delta^B, \theta) \mid X_1 \neq X_2) = 0$.

ii. if $x_1 = x_2 = x$

$$\begin{aligned}
p(\theta = x + 1 \mid x_1, x_2) &= \frac{p(\theta = x + 1 \mid x_1, x_2)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{p(x \mid \theta = x + 1)\pi(\theta = x + 1)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{\frac{1}{2}\pi(\theta = x + 1)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{\frac{1}{2}\pi(\theta = x + 1)}{\frac{1}{2}\pi(\theta = x - 1) + \frac{1}{2}\pi(\theta = x + 1)} \\
&= \frac{\pi(\theta = x + 1)}{\pi(\theta = x - 1) + \pi(\theta = x + 1)}
\end{aligned}$$

$$\begin{aligned}
p(\theta = x - 1 \mid x_1, x_2) &= \frac{p(\theta = x - 1 \mid x_1, x_2)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{p(x \mid \theta = x - 1)\pi(\theta = x - 1)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{\frac{1}{2}\pi(\theta = x - 1)}{p(\theta = x - 1 \mid x_1, x_2) + p(\theta = x + 1 \mid x_1, x_2)} \\
&= \frac{\frac{1}{2}\pi(\theta = x - 1)}{\frac{1}{2}\pi(\theta = x - 1) + \frac{1}{2}\pi(\theta = x + 1)} \\
&= \frac{\pi(\theta = x - 1)}{\pi(\theta = x - 1) + \pi(\theta = x + 1)}
\end{aligned}$$

$$p(\theta = x + j \mid x_1, x_2) = 0 \quad \forall j \in \mathbb{Z} - \{-1, 1\}$$

In words, the posterior supports two values, $\{x+1, x-1\}$. Since $\theta - 1$ and $\theta + 1$ are equally likely, the posterior probabilities of $\theta$ being $x+1$ and $x-1$ are proportional to their prior probabilities.
Consider first $\delta^B(X_1, X_2) = x + 1$. The resulting expected loss is

$$\mathrm{E}(L(\delta^B, \theta) \mid x_1 = x_2 = x) = \frac{\pi(\theta = x-1)}{\pi(\theta = x-1) + \pi(\theta = x+1)}.$$

Next, we consider $\delta^B(X_1, X_2) = x - 1$. The resulting expected loss is

$$\mathrm{E}(L(\delta^B, \theta) \mid x_1 = x_2 = x) = \frac{\pi(\theta = x+1)}{\pi(\theta = x-1) + \pi(\theta = x+1)}.$$

Therfore, the Bayes estimator is

$$\delta = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 \neq x_2, \\ x+1 & \text{if } x_1 = x_2 = x \text{ and } \pi(\theta = x-1) < \pi(\theta = x+1) \\ x-1 & \text{if } x_1 = x_2 = x \text{ and } \pi(\theta = x-1) \geq \pi(\theta = x+1). \end{cases}$$

6. Consider a point estimation problem in which you observe $x_1, \ldots, x_n$ as i.i.d. random variables of the Poisson distribution with parameter $\theta$. Assume a squared error loss and a prior of the form $\theta \sim \mathrm{Gamma}(\alpha, \beta)$.

   (a) Show that the Bayes estimator is $\delta^B(x) = a + b\bar{x}$ where $a > 0$, $b \in (0,1)$ and $\bar{x} = \sum_{i=1}^n x_i/n$. You may use the fact that the distribution of $\sum_i x_i$ is Poisson with parameter $\theta n$ without proof.

   (b) Find the MLE for $\theta$ (Note: to remind how to find MLEs, read Casella and Berger, Section 7.2.2– see Def 7.2.4).

   (c) Compute and graph the frequentist risks of $\delta^B(x)$ and that of the MLE.

   (d) Compute the Bayes risk of $\delta^B(x)$.

   (e) Suppose that an investigator wants to collect a sample that is large enough that the Bayes risk after the experiment is half of the Bayes risk before the experiment. Find that sample size.

**Solution:**

   (a) The sufficient statistics, $T = \sum_{i=1}^n X_i \sim \mathrm{Poi}(n\theta)$. So

$$\pi(\theta \mid t) \propto e^{-n\theta}(n\theta)^t \theta^{\alpha-1} e^{-\beta\theta}.$$

   That is, $\theta \mid t \sim \mathrm{Gamma}(\alpha + t, \beta + n)$. We know that under squared error loss, the Bayes estimator is the posterior mean, in this case

$$\delta^B(x) = \frac{\alpha + t}{\beta + n} = a + b\bar{x}$$

   where $a = \alpha/(\beta + n)$ and $b = n/(\beta + n)$.

(b) $f(\boldsymbol{x} \mid \theta) \propto e^{-n\theta}\theta^t$ so $\log(f(\boldsymbol{x} \mid \theta)) \propto -n\theta + t\log(\theta)$.
Then,

$$\frac{\partial \log(f(\boldsymbol{x} \mid \theta))}{\partial \theta} = -n + \frac{t}{\theta}$$

$\Rightarrow \hat{\theta} = t/n = \bar{x}$.

(c) First notice that $\mathrm{E}(\bar{x}) = \frac{1}{n}\mathrm{E}(t) = \theta$, $\mathrm{Var}(\bar{x}) = \frac{1}{n^2}\mathrm{Var}(t) = \frac{\theta}{n}$ and $\mathrm{E}(\bar{x}^2) = \frac{\theta}{n} + \theta^2$. Therefore,

$$R(\hat{\theta}, \theta) = \mathrm{E}((\bar{x} - \theta)^2) = \mathrm{E}(\bar{x}^2) - 2\theta\mathrm{E}(\bar{x}) + \theta^2 = \frac{\theta}{n}$$

while

$$\begin{aligned} R(\delta, \theta) &= \mathrm{E}((a + b\bar{x} - \theta)^2) = \mathrm{E}(a^2 + 2a(b\bar{x} - \theta) + (b\bar{x} - \theta)^2) \\ &= (b-1)^2\theta^2 + \left[2a(b-1) + \frac{b^2}{n}\right]\theta + a^2 \\ &= \frac{1}{(\beta + n)^2}\left(\beta^2\theta^2 + (-2\alpha\beta + n)\theta + \alpha^2\right). \end{aligned}$$

(d) Since $\theta \sim \mathrm{Gamma}(\alpha, \beta)$, we have $\mathrm{E}(\theta) = \alpha/\beta$ and $\mathrm{E}(\theta^2) = \alpha/\beta^2 + \alpha^2/\beta^2$.

$$\begin{aligned} r(\pi, \delta) &= \mathrm{E}\left[\frac{1}{(\beta + n)^2}\left(\beta^2\theta^2 + (-2\alpha\beta + n)\theta + \alpha^2\right)\right] \\ &= \frac{\alpha}{\beta(\beta + n)}. \end{aligned}$$

(e) The Bayes risk before the experiment under squared error loss is the prior variance,that is, $\alpha/\beta^2$. We want to find $n$ such that

$$\frac{\alpha}{\beta(\beta + n)} < \frac{\alpha}{2\beta^2}.$$

Now we solve for $n$ and get $n > \beta$.

7. A loss function investigated by Zellner (1986) is the LINEX (LINear-EXponential) loss, a loss function that can handle asymmetries in a smooth way. The LINEX loss is given by

$$L(\theta, a) = e^{c(a-\theta)} - c(a - \theta) - 1,$$

where $c$ is a positive constant. As the constant $c$ varies, the loss function varies from very asymmetric to almost symmetric.

Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\theta, \sigma^2)$, where $\sigma^2$ is known, and suppose that $\theta$ has the noninformative prior, $\pi(\theta) \propto 1$. Show that the Bayes estimator of $\theta$ under LINEX loss is given by

$$\delta^B(\bar{X}) = \bar{X} - \frac{c\sigma^2}{2n}.$$

**Solution:**

In general, for any estimator $a$

$$\rho(\pi, a) = \mathrm{E}[L(\theta, a) \mid X] = \mathrm{E}[e^{c(a-\theta)} - c(a - \theta) - 1 \mid X] = e^{ca}\mathrm{E}[e^{-c\theta} \mid X] - ca + cE[\theta \mid X] - 1.$$

Then

$$\frac{d\rho(\pi, a)}{da} = ce^{ca}\mathrm{E}[e^{-c\theta} \mid X] - c = 0 \quad \Leftrightarrow \quad a^B = \frac{-\ln(\mathrm{E}[e^{-c\theta} \mid X]}{c}$$

and

$$\frac{d^2\rho(\pi, a)}{da^2} = c^2 e^{ca}\mathrm{E}[e^{-c\theta} \mid X] > 0$$

so, $a^B$ is a minimizer.

Now, in particular, $\theta \mid \mathbf{X} \sim \mathrm{N}\left(\bar{x}, \frac{\sigma^2}{n}\right)$ which implies that $e^{-c\theta} \mid \mathbf{X} \sim \mathrm{LogNormal}\left(-c\bar{x}, \frac{c^2\sigma^2}{n}\right)$.
Therefore, $\mathrm{E}[e^{-c\theta} \mid X] = e^{-c\bar{x}+\frac{c^2\sigma^2}{2n}}$ and $a^B = \bar{x} - \frac{c\sigma^2}{2n}$ as desired.