Winter 18 – AMS206B Homework 5 Solution

1. Let (X_1, X_2, X_3) have trinomial distribution with density

$$f(x_1, x_2, x_3|\theta_1, \theta_2) \propto \theta_1^{x_1}\theta_2^{x_2}(1-\theta_1-\theta_2)^{x_3}.$$

Derive Jeffreys prior for (θ_1, θ_2).

Solution:

The log likelihood in this case is given by

$$\ell(\boldsymbol{\theta}) = \log(\mathcal{L}(\boldsymbol{\theta})) \propto x_1\log(\theta_1) + x_2\log(\theta_2) + x_3\log(1-\theta_1-\theta_2).$$

Thus,

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_1} = \frac{x_1}{\theta_1} - \frac{x_3}{1-\theta_1-\theta_2} \text{ and } \frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_2} = \frac{x_2}{\theta_2} - \frac{x_3}{1-\theta_1-\theta_2},$$

and the Fisher information matrix

$$\begin{aligned} I(\boldsymbol{\theta}) &= -E\left(\begin{bmatrix} \frac{\partial \ell(\boldsymbol{\theta})}{\partial\theta_1\partial\theta_1} & \frac{\partial \ell(\boldsymbol{\theta})}{\partial\theta_1\partial\theta_2} \\ \frac{\partial \ell(\boldsymbol{\theta})}{\partial\theta_1\partial\theta_2} & \frac{\partial \ell(\boldsymbol{\theta})}{\partial\theta_2\partial\theta_2} \end{bmatrix}\right) = \begin{bmatrix} \frac{E(X_1)}{\theta_1^2} + \frac{E(X_3)}{(1-\theta_1-\theta_2)^2} & \frac{E(X_3)}{(1-\theta_1-\theta_2)^2} \\ \frac{E(X_3)}{(1-\theta_1-\theta_2)^2} & \frac{E(X_2)}{\theta_2^2} + \frac{E(X_3)}{(1-\theta_1-\theta_2)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{n}{\theta_1} + \frac{n}{1-\theta_1-\theta_2} & \frac{n}{1-\theta_1-\theta_2} \\ \frac{n}{1-\theta_1-\theta_2} & \frac{n}{\theta_2} + \frac{n}{1-\theta_1-\theta_2} \end{bmatrix} \end{aligned}$$

Thus, the Jeffrey's Prior is $\pi^J(\boldsymbol{\theta}) \propto \sqrt{\det I(\boldsymbol{\theta})} \propto \theta_1^{-1/2}\theta_2^{-1/2}(1-\theta_1-\theta_2)^{-1/2}$. That is, a Dirichlet(1/2,1/2,1/2).

2. (Robert Problem 3.9) Let $x \mid \theta \sim \text{Bin}(n, \theta)$ and $\theta \sim \text{Be}(\alpha, \beta)$. Determine whether there exits values of α, β such that $f(\theta \mid x)$ is uniform on [0, 1], even for a single value of x.

Solution:

First, notice that

$$\theta \mid x \sim \text{Be}(x+\alpha, n-x+\beta).$$

since

$$f(\theta \mid x) \propto f(x \mid \theta)\pi(\theta) \propto \theta^{x+\alpha-1}(1-\theta)^{n+\beta-x-1}.$$

Then, the posterior would become uniform whenever $x + \alpha = 1$ and $n + \beta - x = 1$. But notice that x being a non-negative integer, along with $0 < \alpha = 1 - x$, implies that $x = 0$. In that case, $0 < \beta = 1 - n \leq 0$, a contradiction. Thus, there is no combination of $\alpha > 0$ and $\beta > 0$ for the posterior to be the uniform on 0 and 1.

3. (Robert Problem 3.10) Let $x \mid \theta \sim \text{Pa}(\alpha, \theta)$, a Pareto distribution, and $\theta \sim \text{Be}(\mu, \nu)$. Show that if $\alpha < 1$ and $x > 1$, a particular choice of μ and ν gives $f(\theta \mid x)$ as uniform on [0,1].

Solution:

The posterior satisfies

$$f(\theta|x) \propto f(x \mid \theta)\pi(\theta) \propto \theta^{\alpha+\mu-1}(1-\theta)^{\nu-1} I_{[0,1]}(\theta)$$

and, in particular,

$$f(\theta|x>1) \propto \theta^{\alpha+\mu-1}(1-\theta)^{\nu-1}$$

which is a Beta$(\alpha+\mu,\nu)$ that becomes Unif$(0,1)$ whenever $\nu=1$ and $\mu = 1-\alpha$.

4. (Robert Problem 3.31) Consider $x \mid \theta \sim \mathrm{N}(\theta,\theta)$ with $\theta > 0$.

   (a) Determine the Jeffreys prior $\pi^J(\theta)$.

   (b) Say whether the distribution of x belongs to an exponential family and derive the conjugate priors on θ.

   (c) Use Proposition 3.3.14 to relate the hyperparameters of the conjugate priors with the mean of θ.

   Solution:

   (a) The likelihood in this case is given by

   $$f(x \mid \theta) \quad = \quad (2\pi\theta)^{-\frac{1}{2}} \exp\left\{-\frac{(x-\theta)^2}{2\theta}\right\}.$$

   So

   $$\log(f(x \mid \theta)) = -1/2\log(2\pi\theta) - \frac{(x-\theta)^2}{2\theta},$$

   and taking first and second derivatives,

   $$\ell'(\theta) = -\frac{1}{2\theta} - \frac{\theta^2 - x^2}{2\theta^2} \qquad \text{and} \qquad \ell''(\theta) = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$

   Then,

   $$I(\theta) = -\mathrm{E}[\ell''(\theta)] = -\frac{1}{2\theta^2} + \frac{E[X^2 \mid \theta]}{\theta^3} = -\frac{1}{2\theta^2} + \frac{\theta+\theta^2}{\theta^3} = \frac{2\theta+1}{2\theta^2},$$

   and

   $$\pi^J(\theta) \propto \{I(\theta)\}^{1/2} = \left(\frac{2\theta+1}{\theta^2}\right)^{1/2}.$$

   (b)

   $$f(x|\theta) = (2\pi\theta)^{-\frac{1}{2}} \exp\left\{-\frac{(x-\theta)^2}{2\theta}\right\} = (2\pi)^{-\frac{1}{2}} \exp\{x\} \exp\left\{-\frac{x^2}{2}\cdot\frac{1}{\theta} - \frac{1}{2}\left[\ln\theta + \theta\right]\right\}$$

   which is a member of the exponential family with $h(x) = (2\pi)^{-\frac{1}{2}}\exp\{x\}$, $t(x) = \frac{-x^2}{2}$, $\eta(\theta) = \frac{1}{\theta}$ and $\psi(\eta) = \frac{1}{2}\left[\frac{1}{\eta} - \ln\eta\right]$. Then, the conjugate prior satisfies

   $$\pi(\eta|\mu,\lambda) \propto \exp\{\eta\mu - \lambda\psi(\eta)\}$$

   and, transforming back to θ,

   $$\pi(\theta|\mu,\lambda) \propto \exp\left\{\frac{\mu}{\theta} - \frac{\lambda}{2}\left[\theta + \ln\theta\right]\right\}\cdot\frac{1}{\theta^2} = \exp\left\{\mu\frac{1}{\theta} - \frac{\lambda}{2}\theta\right\}\theta^{-\frac{\lambda}{2}-2}$$

(c) Let $z = -\frac{x^2}{2}$, then we can write

$$f(z \mid \eta) = h'(z) \exp\{z\eta - \psi(\eta)\}$$

Define $\xi(\eta) = \mathbb{E}[z \mid \eta]$, then:

$$\xi(\eta) = -\frac{1}{2}\left(\frac{1}{\eta} + \frac{1}{\eta^2}\right)$$

then by proposition 3.3.14 we have that:

$$\mathbb{E}[\xi(\eta) \mid z] = \mathbb{E}\left[-\frac{1}{2}\left(\frac{1}{\eta} + \frac{1}{\eta^2}\right) \mid z\right] = \frac{\mu + z}{\lambda + 1}$$

which implies that

$$\mathbb{E}[\theta + \theta^2 \mid x] = \frac{x^2 - 2\mu}{\lambda + 1}.$$

5. (Berger Problem 3-12) Determine the Jefferys noninformative prior for the unknown parameter in each of the following distributions:

(a) Poi(θ)

(c) NB(m, θ) (m is given)

Solution:

(a) the Poisson likelihood is given by

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

and the log-likelihood

$$\log f(x|\theta) = -\theta + x\log\theta - \log(x!).$$

Taking first and second derivatives,

$$\frac{\partial \log f(x|\theta)}{\partial\theta} = -1 + \frac{x}{\theta} \quad \text{and} \quad \frac{\partial^2 \log f(x|\theta)}{\partial\theta^2} = -\frac{x}{\theta^2},$$

and the Fisher information satisfies

$$I(\theta) = \frac{1}{\theta^2}\mathrm{E}_\theta[x] = \frac{1}{\theta^2} \cdot \theta = \frac{1}{\theta}$$

Therefore, Jeffrey's prior $\pi^J(\theta) \propto \frac{1}{\sqrt{\theta}}$.

(b) Let x denote the number of failures until m successes occur,

$$f(x|m,\theta) = \binom{m+x-1}{x} \theta^m (1-\theta)^x$$

and

$$\log f(x|m,\theta) = \log\binom{m+x-1}{x} + m\log\theta + x\log(1-\theta).$$

Then,

$$\frac{\partial \log f(x|m,\theta)}{\partial\theta} = \frac{m}{\theta} - \frac{x}{1-\theta} \quad \text{and} \quad \frac{\partial^2 \log f(x|m,\theta)}{\partial\theta^2} = \frac{-m}{\theta^2} - \frac{x}{(1-\theta)^2},$$

and the Fisher information is

$$I(\theta) = \frac{m}{\theta^2} + \frac{\mathrm{E}_\theta[x]}{(1-\theta^2)} = \frac{m}{\theta^2} + \frac{\frac{m}{\theta} - m}{(1-\theta)^2} = \frac{m}{\theta^2(1-\theta)}.$$

Therefore, Jeffrey's prior $\pi^J(\theta) \propto \frac{1}{\theta\sqrt{1-\theta}}$.

6. (Berger Problem 3-22) Suppose X, the failure time of an electronic component, has density (on $(0,\infty)$) $f(x \mid \theta) = \theta^{-1}\exp\{-x/\theta\}$. The unknown θ has an $\mathrm{IG}(1, 0.01)$ prior distribution. Calculate the (marginal) probability that the component fails before time 200.

   Solution:

   Note that, since the problem is taken from Berger, we follow the book's parametrization of the Inverse Gamma distribution. In this case the prior density is given by

$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{-(\alpha+1)}e^{-\frac{1}{\beta\theta}}.$$

   The prior predictive

$$m(x) = \int_{\mathbf{R}^+} f(x \mid \theta)\pi(\theta)d\theta = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_{\mathbf{R}^+}\theta^{-(\alpha+2)}e^{-\frac{x+\frac{1}{\beta}}{\theta}}d\theta = \frac{\Gamma(\alpha+1)\left(x+\frac{1}{\beta}\right)^{-(\alpha+1)}}{\Gamma(\alpha)\beta^\alpha}.$$

   Then,

$$P(x \le 200) \;=\; \int_0^{200}\frac{\alpha\left(\frac{1}{\beta}\right)^\alpha}{\left(x+\frac{1}{\beta}\right)^{(\alpha+1)}}dx = -\frac{\alpha\left(\frac{1}{\beta}\right)^\alpha}{\alpha\left(x+\frac{1}{\beta}\right)^\alpha}\Bigg|_0^{200} = 1 - \frac{1}{(1+200\beta)^\alpha} = \frac{2}{3}.$$