AMS 261: Probability Theory (Fall 2017)

Homework 1 solutions

1. Consider a sample space Ω.

(a) Prove that any intersection of σ-fields (of subsets of Ω) is a σ-field. That is, if $\mathcal{F}_j$, $j \in J$, are σ-fields on Ω (with J an arbitrary index set, countable or uncountable), then show that $\mathcal{F} = \bigcap_{j\in J} \mathcal{F}_j$ is a σ-field.

Solution: First, because each $\mathcal{F}_j$ is a σ-field on Ω, we have that $\Omega \in \mathcal{F}_j$, for all $j \in J$, and therefore $\Omega \in \mathcal{F}$. For the second condition of the definition, consider an $A \in \mathcal{F}$. Then, $A \in \mathcal{F}_j$, for all $j \in J$, and therefore $A^c \in \mathcal{F}_j$, for all $j \in J$ (since the $\mathcal{F}_j$, $j \in J$, are σ-fields), which yields that $A^c \in \mathcal{F}$. Finally, consider a countable collection $\{A_i : i = 1, 2, ...\}$ of members of $\mathcal{F}$. Then, for each $j \in J$, $A_i \in \mathcal{F}_j$, for all i, which implies that for each $j \in J$, $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_j$ (since the $\mathcal{F}_j$, $j \in J$, are σ-fields), and therefore $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

(b) Show by counterexample that a union of σ-fields may not be a σ-field.

Solution: Consider a finite sample space with three sample points, say, $\Omega = \{a, b, c\}$, and two σ-fields on Ω given by $\mathcal{F}_1 = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$ and $\mathcal{F}_2 = \{\emptyset, \Omega, \{b\}, \{a, c\}\}$ (that is, the σ-fields generated by $\{a\}$ and $\{b\}$, respectively). Then, the union of the two σ-fields, $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \Omega, \{a\}, \{b\}, \{b, c\}, \{a, c\}\}$, is not closed under (finite) unions (it contains $\{a\}$ and $\{b\}$ but not $\{a\} \cup \{b\} = \{a, b\}$), and hence it is not a σ-field.

2. Given a sample space Ω and a collection $\mathcal{E}$ of subsets of Ω, the σ-field generated by $\mathcal{E}$, $\sigma(\mathcal{E})$, is defined as the intersection of all σ-fields on Ω that contain $\mathcal{E}$. (As discussed in class, $\sigma(\mathcal{E})$ is the smallest σ-field that contains $\mathcal{E}$.)

(a) Consider two collections $\mathcal{E}_1$ and $\mathcal{E}_2$ of subsets of Ω. Show that if $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.

Solution: Because $\mathcal{E}_1 \subseteq \mathcal{E}_2$, any σ-field on Ω that contains $\mathcal{E}_2$ will also contain $\mathcal{E}_1$. Therefore, to define $\sigma(\mathcal{E}_2)$ we are intersecting over a smaller number of sets relative to $\sigma(\mathcal{E}_1)$, and thus by their definition, we have $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.

(b) As in part (a), let $\mathcal{E}_1$ and $\mathcal{E}_2$ be collections of subsets of the sample space Ω. Prove that if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$, then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

Solution: Using the result from part (a), the assumptions $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$ yield $\sigma(\mathcal{E}_1) \subseteq \sigma(\sigma(\mathcal{E}_2))$ and $\sigma(\mathcal{E}_2) \subseteq \sigma(\sigma(\mathcal{E}_1))$, respectively. Because $\sigma(\mathcal{E}_1)$ and $\sigma(\mathcal{E}_2)$ are σ-fields, we have $\sigma(\sigma(\mathcal{E}_1)) = \sigma(\mathcal{E}_1)$ and $\sigma(\sigma(\mathcal{E}_2)) = \sigma(\mathcal{E}_2)$. Therefore, $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$ and $\sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{E}_1)$, which provides the result.

3. Let $\mathcal{F}$ be a collection of subsets of a sample space Ω.

(a) Suppose that $\Omega \in \mathcal{F}$, and that when $A, B \in \mathcal{F}$ then $A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

Solution: First, $\Omega \in \mathcal{F}$, by assumption. For the second condition, let $A \in \mathcal{F}$. Because $A \in \mathcal{F}$ and $\Omega \in \mathcal{F}$, we have by assumption that $\Omega \cap A^c = A^c \in \mathcal{F}$. Finally, for the third condition, consider $A \in \mathcal{F}$ and $B \in \mathcal{F}$. We have shown that $A^c \in \mathcal{F}$, hence by assumption, $A^c \cap B^c = (A \cup B)^c \in \mathcal{F}$. Now using the second condition, we obtain that $A \cup B = ((A \cup B)^c)^c \in \mathcal{F}$.

(b) Suppose that $\Omega \in \mathcal{F}$, and that $\mathcal{F}$ is closed under the formation of complements and finite pairwise disjoint unions. Show by counterexample that $\mathcal{F}$ need not be a field.

Solution: For a counterexample, consider again a finite sample space, now, with four sample points, say, $\Omega = \{1, 2, 3, 4\}$. Let $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. It is straightforward to check that $\mathcal{F}$ is closed under taking complements and is closed under finite disjoint unions (it is also defined to contain Ω). However, although, say, $\{1, 2\} \in \mathcal{F}$ and $\{1, 3\} \in \mathcal{F}$, their union, $\{1, 2, 3\}$, does not belong to $\mathcal{F}$, and thus $\mathcal{F}$ is not a field.

4. Consider the sample space $\Omega = (0,1]$ and the collection $\mathcal{B}_0$ of all finite pairwise disjoint unions of subintervals of $(0,1]$. That is, any member B of $\mathcal{B}_0$ is of the form $B = \bigcup_{i=1}^{n}(a_i, b_i]$, where n is finite, and for each $i = 1, ..., n$, $0 \le a_i < b_i \le 1$, with $(a_i, b_i] \cap (a_j, b_j] = \emptyset$ for any $i \neq j$.
Show that $\mathcal{B}_0$ augmented by the empty set is a field, but not a σ-field.
Solution: First, taking $n = 1$ with $a_1 = 0$ and $b_1 = 1$, we obtain that $\Omega \in \mathcal{B}_0$. Next, consider $B \in \mathcal{B}_0$, which therefore is of the form $B = \bigcup_{i=1}^{n}(a_i, b_i]$ for disjoint $(a_i, b_i]$. Without loss of generality, assume that $a_1 \le a_2 \le ... \le a_n$. Then, $B^c = (0, a_1] \cup (b_1, a_2] \cup \, ... \, (b_{n-1}, a_n] \cup (b_n, 1]$ which belongs to $\mathcal{B}_0$ (note that some of the intervals in the expression for B^c may be given by the empty set, since we can have $b_{i-1} = a_i$ for some i). Finally, consider $B \in \mathcal{B}_0$ and $C \in \mathcal{B}_0$, say, $B = \bigcup_{i=1}^{n}(a_i, b_i]$ for disjoint $(a_i, b_i]$, and $C = \bigcup_{j=1}^{m}(c_j, d_j]$ for disjoint $(c_j, d_j]$. Note that for the third condition of the definition for a field, rather than proving that $B \cup C \in \mathcal{B}_0$, it is equivalent to show that $B \cap C \in \mathcal{B}_0$. Now, $B \cap C = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \{(a_i, b_i] \cap (c_j, d_j]\}$, where the $(a_i, b_i] \cap (c_j, d_j]$ are disjoint, and each set $(a_i, b_i] \cap (c_j, d_j]$ is either the empty set or an interval of the form $(e, f]$. Hence, $B \cap C \in \mathcal{B}_0$, completing the proof that $\mathcal{B}_0$, along with the empty set, is a field. To show that $\mathcal{B}_0$ is not a σ-field, note that for any $x \in (0,1)$, we can write $\{x\} = \bigcap_{n=1}^{\infty}(x - n^{-1}, x]$, i.e., $\{x\}$ can be expressed as a countable intersection of members of $\mathcal{B}_0$, but $\{x\}$ does not belong to $\mathcal{B}_0$.

5. Let $\Omega = \{\omega_1, \omega_2, ...\}$ be a countable set, $\{p_n : n = 1, 2, ...\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} p_n = 1$, and $\mathcal{F}$ be the collection of all subsets of Ω. For each $A \in \mathcal{F}$, define the set function

$$P(A) = \sum_{\{n:\, \omega_n \in A\}} p_n.$$

Show that $(\Omega, \mathcal{F}, P)$ is a probability space.
Solution: Using directly the definition, $\mathcal{F}$ is a σ-field (the default σ-field for countable sample spaces). Therefore, to establish that $(\Omega, \mathcal{F}, P)$ is a probability space, we need to show that P is a probability measure on $(\Omega, \mathcal{F})$. By definition, $P(\Omega) = \sum_{\{n:\omega_n \in \Omega\}} p_n = \sum_{n=1}^{\infty} p_n$, which is equal to 1 by assumption. Moreover, since the p_n are non-negative, we have $P(A) \ge 0$ for any $A \in \mathcal{F}$; also, for any $A \in \mathcal{F}$, i.e., any $A \subseteq \Omega$, we have $P(A) = \sum_{\{n:\omega_n \in A\}} p_n \le \sum_{\{n:\omega_n \in \Omega\}} p_n = 1$. Finally, to show countable additivity, consider a countable sequence, $\{A_m : m = 1, 2, ...\}$, of pairwise disjoint subsets of Ω. For any $\omega_n \in \bigcup_{m=1}^{\infty} A_m$, we have that $\omega_n \in A_k$ for some k and $\omega_n \notin A_\ell$ for all $\ell \neq k$, since the A_m are pairwise disjoint. Therefore,

$$P\left(\bigcup_{m=1}^{\infty} A_m\right) = \sum_{\{n:\, \omega_n \in \bigcup_{m=1}^{\infty} A_m\}} p_n = \sum_{m=1}^{\infty} \sum_{\{n:\, \omega_n \in A_m\}} p_n = \sum_{m=1}^{\infty} P(A_m).$$