AMS 261: Probability Theory (Fall 2017)

Homework 5 solutions

1. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(\Psi, \mathcal{G})$, where $\mathcal{G}$ is the σ-field on space Ψ. Consider the collection $\mathcal{A}$ of subsets of Ω consisting of $X^{-1}(B)$ for all $B \in \mathcal{G}$. Show that $\mathcal{A}$ is a σ-field on Ω.
Solution: First, note that because X is a random variable, we know that $\mathcal{A} \subseteq \mathcal{F}$. To show that $\mathcal{A}$ is a σ-field, we need to verify the three conditions of the definition of a σ-field. First, because $\Psi \in \mathcal{G}$, we have $X^{-1}(\Psi) = \Omega \in \mathcal{A}$. Next, consider $A \in \mathcal{A}$. We have $A = X^{-1}(B)$ for some $B \in \mathcal{G}$. Using properties of inverse images, $X^{-1}(B^c) = (X^{-1}(B))^c = A^c$. Because $\mathcal{G}$ is a σ-field, we have $B^c \in \mathcal{G}$, which implies that $X^{-1}(B^c) \in \mathcal{A}$, and therefore $A^c \in \mathcal{A}$. Finally, let $\{A_n : n = 1, 2, ...\}$ be a countable collection of members of $\mathcal{A}$. For each n, $A_n = X^{-1}(B_n)$ for $B_n \in \mathcal{G}$. Now, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{G}$, since $\mathcal{G}$ is a σ-field. Hence, $X^{-1}(\bigcup_{n=1}^{\infty} B_n) \in \mathcal{A}$, which yields the third condition, since $X^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} X^{-1}(B_n) = \bigcup_{n=1}^{\infty} A_n$, using again properties of inverse images.

2. For $k = 1, 2, ...,$ consider random variables $X_k : (\Omega, \mathcal{F}, P) \to (\Psi_k, \mathcal{G}_k)$ and measurable functions $\varphi_k : (\Psi_k, \mathcal{G}_k) \to (\Theta_k, \mathcal{H}_k)$. Assume that the countable sequence of random variables $\{X_k : k = 1, 2, ...\}$ is independent. Prove that the sequence $\{\varphi_k \circ X_k : k = 1, 2, ...\}$ is independent.
Solution: We are given that $\{X_k : k = 1, 2, ...\}$ is independent, i.e., $\{\sigma(X_k) : k = 1, 2, ...\}$ is independent, i.e., for any finite index set J (with $J \subset \{1, 2, ...\}$), $\{\sigma(X_j) : j \in J\}$ is independent, which implies that for any $B_j \in \mathcal{G}_j$,
$$P(\bigcap_{j \in J} X_j^{-1}(B_j)) = \prod_{j \in J} P(X_j^{-1}(B_j)). \tag{2.1}$$
Consider an arbitrary finite index set J and $C_j \in \mathcal{H}_j$. We have
$$P(\bigcap_{j \in J} (\varphi_j \circ X_j)^{-1}(C_j)) = P(\bigcap_{j \in J} X_j^{-1}(\varphi_j^{-1}(C_j))) = \prod_{j \in J} P(X_j^{-1}(\varphi_j^{-1}(C_j))) = \prod_{j \in J} P((\varphi_j \circ X_j)^{-1}(C_j))$$
using (2.1) (with $B_j = \varphi_j^{-1}(C_j)$). Hence, $\{\sigma(\varphi_j \circ X_j) : j \in J\}$ is independent for any finite index set J, and therefore $\{\sigma(\varphi_k \circ X_k) : k = 1, 2, ...\}$ is independent.

3. Let $\{A_n : n = 1, 2, ...\}$ be a countable independent sequence of events on a probability space $(\Omega, \mathcal{F}, P)$. Prove that $P(\bigcap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n)$. (**Note:** For a countable sequence of reals, $\{b_n : n = 1, 2, ...\}$, the infinite product $\prod_{n=1}^{\infty} b_n$ is defined by $\lim_{n\to\infty} \prod_{k=1}^{n} b_k$, provided this limit exists.)
Solution: Consider the new sequence of events $\{B_n : n = 1, 2, ...\}$, where $B_n = \bigcap_{k=1}^{n} A_k$. This is a decreasing sequence of events with $\lim_{n\to\infty} B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{n} A_k = \bigcap_{n=1}^{\infty} A_n$. Therefore, using continuity of probability measure and the assumption of independence for $\{A_n : n = 1, 2, ...\}$, we have
$$P(\bigcap_{n=1}^{\infty} A_n) = P(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(\bigcap_{k=1}^{n} A_k) = \lim_{n\to\infty} \prod_{k=1}^{n} P(A_k) = \prod_{n=1}^{\infty} P(A_n).$$
Note that for the sequence $q_n = \prod_{k=1}^{n} P(A_k)$ we have $1 \geq q_n \geq q_{n+1} \geq ... \geq 0$, and therefore the infinite product $\prod_{n=1}^{\infty} P(A_n) = \lim_{n\to\infty} q_n$ exists as either a strictly positive constant or 0.

4. Consider two countable sequences of events, $\{A_n : n = 1, 2, ...\}$ and $\{B_n : n = 1, 2, ...\}$, on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that, for each n, A_n and B_n are independent. Moreover, assume that $A = \lim_{n\to\infty} A_n$ and $B = \lim_{n\to\infty} B_n$ exist. Show that A and B are independent.
Solution: We have $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_A(\omega)$ and $\lim_{n\to\infty} 1_{B_n}(\omega) = 1_B(\omega)$, for each $\omega \in \Omega$. Therefore $1_{A\cap B}(\omega) = 1_A(\omega)1_B(\omega) = \lim_{n\to\infty}(1_{A_n}(\omega)1_{B_n}(\omega)) = \lim_{n\to\infty} 1_{A_n \cap B_n}(\omega)$, for each $\omega \in \Omega$, and thus $\lim_{n\to\infty}(A_n \cap B_n) = A \cap B$. Hence,
$$\begin{aligned} P(A \cap B) &= P(\lim_{n\to\infty}(A_n \cap B_n)) = \lim_{n\to\infty} P(A_n \cap B_n) = \lim_{n\to\infty}(P(A_n)P(B_n)) \\ &= (\lim_{n\to\infty} P(A_n))(\lim_{n\to\infty} P(B_n)) = P(\lim_{n\to\infty} A_n)P(\lim_{n\to\infty} B_n) = P(A)P(B) \end{aligned}$$
using continuity of probability measure (twice) and the independence of A_n and B_n, for each n.

5. A sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}$-valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$, is said to converge completely if for any $k = 1, 2, ...$, $\sum_{n=1}^{\infty} P(|X_n| > k^{-1}) < \infty$. Show that if $\{X_n : n = 1, 2, ...\}$ converges completely, then $\lim_{n\to\infty} X_n = 0$ almost surely.
Solution: The assumption of complete convergence yields that

$$P(\limsup_{n\to\infty}\{\omega \in \Omega : |X_n(\omega)| > k^{-1}\}) = 0, \quad \text{for} \quad k = 1, 2, ...$$

using the Borel lemma. Now the result follows using one of the equivalent definitions of almost sure convergence proved in class.

6. Construct a sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}^+$-valued random variables (i.e., $X_n \geq 0$, for all n) that satisfies $\sum_{n=1}^{\infty} P(X_n > k^{-1}) < \infty$, for any $k = 1, 2, ...$, but for which $\lim_{n\to\infty} \mathrm{E}(X_n) \neq 0$.
Solution: For each $n = 1, 2, ...$, define X_n so that it takes value 3^n with probability 2^{-n}, and value 0 with probability $1 - 2^{-n}$. (For example, X_n can be defined on $\Omega = (0, 1]$, with $\mathcal{F}$ the Borel σ-field on $(0, 1]$ and P the uniform distribution, such that $X_n(\omega) = 3^n$, if $\omega \in (0, 2^{-n}]$, and $X_n(\omega) = 0$, otherwise.) Then, for any $k = 1, 2, ...$, $\sum_{n=1}^{\infty} P(X_n > k^{-1}) = \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$, but $\lim_{n\to\infty} \mathrm{E}(X_n) = \lim_{n\to\infty}(3/2)^n = \infty$.

7. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that each random variable X_n is uniformly distributed on $(0, 1)$, hence, $P(c < X_n < d) \equiv P(\{\omega \in \Omega : X_n(\omega) \in (c, d)\}) = d - c$, for any $0 \leq c < d \leq 1$. Show that the sequence $\{1/(n^2 X_n) : n = 1, 2, ...\}$ converges almost surely to 0 as $n \to \infty$.
Solution: We need to show that $P\left(\left\{\omega \in \Omega : \forall k, \exists j, \forall n \geq j, \frac{1}{n^2 X_n(\omega)} < \frac{1}{k}\right\}\right) = 1$, or, equivalently, that

$$P\left(\bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{n=j}^{\infty}\left\{\omega \in \Omega : \frac{1}{n^2 X_n(\omega)} < \frac{1}{k}\right\}\right) = P\left(\bigcap_{k=1}^{\infty} \liminf_{j\to\infty} A_{j,k}\right) = 1,$$

or, equivalently, that

$$0 = P\left(\left(\bigcap_{k=1}^{\infty} \liminf_{j\to\infty} A_{j,k}\right)^c\right) = P\left(\bigcup_{k=1}^{\infty} (\liminf_{j\to\infty} A_{j,k})^c\right) = P\left(\bigcup_{k=1}^{\infty} \limsup_{j\to\infty} A_{j,k}^c\right) \tag{7.1}$$

Here, for each $j = 1, 2, ...$, $k = 1, 2, ...$, $A_{j,k}$ is the event $\{\omega \in \Omega : \frac{1}{j^2 X_j(\omega)} < \frac{1}{k}\}$.

Now, if we fix k, there exists some $M = M(k)$ such that $k/j^2 < 1$, for any $j \geq M$. Then, for any such $j \geq M$,

$$P\left(A_{j,k}^c\right) = P\left(\left\{\omega \in \Omega : \frac{1}{j^2 X_j(\omega)} \geq \frac{1}{k}\right\}\right) = P\left(\left\{\omega \in \Omega : X_j(\omega) \leq \frac{k}{j^2}\right\}\right) = \frac{k}{j^2},$$

since each X_j is uniformly distributed on $(0, 1)$. Hence, the series $\sum_{j=1}^{\infty} P(A_{j,k}^c)$ converges, since $\sum_{j=M}^{\infty} P(A_{j,k}^c) = k\sum_{j=M}^{\infty} j^{-2} < \infty$. Therefore, the Borel lemma yields that $P(\limsup_{j\to\infty} A_{j,k}^c) = 0$, for any k. Finally, (7.1) is established if we note that $P\left(\bigcup_{k=1}^{\infty} \limsup_{j\to\infty} A_{j,k}^c\right) \leq \sum_{k=1}^{\infty} P(\limsup_{j\to\infty} A_{j,k}^c) = 0$.