Winter 18 – AMS206B Homework 6

Due: Friday March 2nd

1. Let $y_t = \rho y_{t-1} + \epsilon_t$, $\epsilon_t \sim^{i.i.d.} N(0, \sigma^2)$. This is a popular model in time series analysis known as the autoregressive model of order one or AR(1).

   (a) Write down the conditional likelihood given y_1, i.e., $f(y_2, \ldots, y_n | y_1, \rho, \sigma^2)$ (we treat y_1 as a given constant).

   (b) Assume a prior of the form $\pi(\rho, \sigma^2) \propto 1/\sigma^2$.

      i. Find the joint posterior $p(\rho, \sigma^2 | y_1, \ldots, y_n)$ based on the conditional likelihood.

      ii. Find $p(\rho | \sigma^2, y_1, \ldots, y_n)$ and $p(\sigma^2 | y_1, \ldots, y_n)$ based on the conditional likelihood.

2. Find the posterior mean and posterior variance for the models below;

   (a) $X \sim \text{Bin}(n, \theta)$ and $\theta \sim \text{Be}(\alpha, \beta)$.

   (b) $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, $i = 1, \ldots, n$ and $\theta \sim \text{IG}(\alpha, \beta)$.

   (c) $X \sim \text{Gamma}(n/2, 2\theta)$ (so that X/θ is χ^2_n) and $\theta \sim \text{IG}(\alpha, \beta)$.

   (d) For (a)–(c), find the MAP (δ) as an estimate of θ and its posterior variance of the estimate (posterior mean square error), that is, $E((\delta - \theta)^2 \mid x)$.

   Hint: From the lecture, we know

$$\begin{aligned} E\{(\delta-\theta)^2 \mid x\} &= E\{(\delta - E(\theta \mid x) + E(\theta \mid x) - \theta)^2 \mid x\} \\ &= E\{(\delta - E(\theta \mid x))^2 \mid x\} + E\{(E(\theta \mid x) - \theta)^2 \mid x\} \\ &= E\{(\delta - E(\theta \mid x))^2 \mid x\} + \underbrace{Var(\theta \mid x)}_{\text{var of post. dist.}} \end{aligned}$$

   Note: The posterior variance of an estimator is a way to assess the posterior uncertainty associated with the estimator. Please check the lecture note from Feb-10^{th}.

3. Let Y_n be the n^{th} order statistic of a random sample of size n from a distribution with p.d.f $f(x \mid \theta) = 1/\theta$, $0 < x < \theta$, zero else where. Take the loss function to be $L(\theta, \delta) = (\theta - \delta)^2$. Consider $\pi(\theta) = \beta\alpha^\beta/\theta^{\beta+1}$, $0 < \theta < \infty$, zero elsewhere, where $\alpha > 0$, $\beta > 0$ are known numbers. (i) Find the distribution of Y_n. (ii) Find the Bayes point estimate of θ.

4. Let X be $\text{N}(0, \sigma^2)$. Assume that the unknown $1/\sigma^2$ has a gamma distribution with parameters $\alpha = r/2$ and $\beta = 2/r$, where r is a positive integer. Show that the marginal distribution of X is a $t-$distribution with r degrees of freedom.