Name: ________________________________

AMS 206B Midterm 2, Monday, March 5th, 2018

You MUST show all your work and justify all steps. **You may use any results from lecture without proving after clearly stating them.**

Consider a regression model of y on x;

$$y_i = x_i\beta + \epsilon_i, \quad i = 1, \ldots, n,$$

where $\epsilon_i \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$ (a regression only with a slope, no intercept). Assume x is fixed and β and σ are unknown. Let $\theta = (\beta, \sigma^2)$ where $\beta \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$.

1. (40 pts) Assume that β and σ^2 are independent a priori. Use the Jeffrey's prior for each parameter and find the corresponding joint noninformative prior of $\theta = (\beta, \sigma^2)$. For simplicity, you may assume $n = 1$ to derive the Jeffrey's prior.

2. (40 pts) From lecture, we can write the likelihood,

$$f(\boldsymbol{y} \mid \boldsymbol{x}, \beta, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\left(s^2 + \sum_{i=1}^n x_i^2(\hat{\beta} - \beta)^2\right)\right\},$$

where $\hat{\beta} = \sum_{i=1}^n x_i y_i / \sum_{i=1}^n x_i^2$ and $s^2 = \sum_{i=1}^n (y_i - \hat{\beta}x_i)^2$. Consider the conjugate prior of β and σ^2,

$$\pi(\beta, \sigma^2) = \pi_1(\beta \mid \sigma^2)\pi_2(\sigma^2) = \mathrm{N}\left(\mu, \frac{\sigma^2}{n_0 \sum_{i=1}^n x_i^2}\right) \mathrm{IG}\left(\frac{\nu_0}{2}, \frac{s_0^2}{2}\right).$$

Show that the marginal posterior density of σ^2 given $(\boldsymbol{y}, \boldsymbol{x})$, $\pi_2(\sigma^2 \mid \boldsymbol{y}, \boldsymbol{x})$ is

$$\mathrm{IG}\left(\frac{n + \nu_0}{2}, \frac{s^2 + s_0^2 + \frac{n_0}{1+n_0}\sum_{i=1}^n x_i^2(\mu - \hat{\beta})^2}{2}\right).$$

note: Inverse-Gamma distribution with α and β

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right),$$

where $\mathrm{E}(X) = \beta/(\alpha - 1)$ for $\alpha > 1$, $\mathrm{Var}(X) = \beta^2/\{(\alpha-1)^2(\alpha-2)\}$ for $\alpha > 2$ and mode $\beta/(\alpha + 1)$.

1. $$f(y \mid x, \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(y - x\beta)^2}{2\sigma^2} \right)$$

$$\Rightarrow \log(f(y \mid x, \beta, \sigma^2)) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(y - x\beta)^2}{2\sigma^2}$$

Due to a priori independence,

$$\pi^J(\beta, \sigma^2) = \pi_1^J(\beta)\, \pi_2^J(\sigma^2).$$

Also, $\pi_1^J(\beta) \propto \sqrt{I(\beta)}$ and $\pi_2^J(\sigma^2) \propto \sqrt{I(\sigma^2)}$

Since $f(y \mid x, \beta, \sigma^2)$ is an exponential family,

$$I(\beta) = -E\left(\frac{\partial^2 \log f(y \mid x, \beta, \sigma^2)}{\partial \beta^2} \right)$$

$$\& \quad I(\sigma^2) = -E\left(\frac{\partial^2 \log f(y \mid x, \beta, \sigma^2)}{\partial (\sigma^2)^2} \right)$$

$$\frac{\partial \log f(y \mid x, \beta, \sigma^2)}{\partial \beta} = \frac{-\cancel{2}(-x)(y - x\beta)}{\cancel{2}\sigma^2} = \frac{x(y - x\beta)}{\sigma^2}$$

$$\frac{\partial^2 \log f(y \mid x, \beta, \sigma^2)}{\partial \beta^2} = \frac{-x^2}{\sigma^2}$$

$$\Rightarrow I(\beta) = -E\left(-\frac{x^2}{\sigma^2} \right) = \frac{x^2}{\sigma^2} \Rightarrow \pi_1^J(\beta) \propto \sqrt{\frac{x^2}{\sigma^2}} \propto 1$$

$$\frac{\partial \log f(y \mid x, \beta, \sigma^2)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{(y - x\beta)^2}{2(\sigma^2)^2}$$

$$\frac{\partial^2 \log f(y \mid x, \beta, \sigma^2)}{\partial (\sigma^2)^2} = + \frac{1}{2(\sigma^2)^2} - \frac{(y - x\beta)^2}{(\sigma^2)^3}$$

$$I(\sigma^2) = -E\left[\frac{1}{2(\sigma^2)^2} - \frac{(y - x\beta)^2}{(\sigma^2)^3} \right] = \frac{-1}{2(\sigma^2)^2} + \frac{\sigma^2}{(\sigma^2)^3}$$

$$= \frac{1}{2(\sigma^2)^2}$$

$$\Rightarrow \quad \pi_2^J(\sigma^2) \propto \sqrt{\frac{1}{2(\sigma^2)^2}} \propto \frac{1}{\sigma^2}$$

$$\Rightarrow \quad \pi^J(\beta, \sigma^2) \propto 1 \times \frac{1}{\sigma^2} = \frac{1}{\sigma^2}$$

2.

$$\pi_2(\sigma^2 \mid y, x) \propto \int_{\mathbb{R}} f(y \mid x, \beta, \sigma^2)\, \pi_1(\beta \mid \sigma^2)\, \pi_2(\sigma^2)\, d\beta$$

$$\propto (\sigma^2)^{-\frac{n}{2}} (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{\nu_0}{2}-1} \exp\left\{ -\frac{S^2}{2\sigma^2} - \frac{S_0^2}{2\sigma^2} \right\}$$

$$\int_{\mathbb{R}} \exp\left\{ -\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} (\beta - \hat{\beta})^2 - \frac{n_0 \sum x_i^2 (\beta - \mu)^2}{2\sigma^2} \right\} d\beta$$

$$= (\sigma^2)^{-\frac{n+1+\nu_0}{2}-1} \exp\left(-\frac{S^2 + S_0^2}{2\sigma^2} - \frac{\sum_{i=1}^{n} x_i^2 \hat{\beta}^2}{2\sigma^2} - \frac{n_0 \sum x_i^2 \mu^2}{2\sigma^2} \right)$$

$$\int_{\mathbb{R}} \exp\left\{ \underbrace{-\frac{\sum x_i^2}{2\sigma^2} \left((1+n_0)\beta^2 - 2(\hat{\beta} + n_0\mu)\beta \right)}_{\text{kernel for } N\left(\frac{\hat{\beta} + n_0\mu}{1+n_0},\ \frac{\sigma^2}{(1+n_0)\sum x_i^2} \right)} \right\} d\beta$$

$$\propto (\sigma^2)^{-\frac{n+1+\nu_0}{2}-1} (\sigma^2)^{\frac{1}{2}} \exp\left(-\frac{S^2 + S_0^2}{2\sigma^2} - \frac{\sum x_i^2 \hat{\beta}^2}{2\sigma^2} - \frac{n_0 \sum x_i^2 \mu^2}{2\sigma^2} \right)$$

$$\times \exp\left(+ \frac{(1+n_0)\sum x_i^2}{2\sigma^2} \left(\frac{\hat{\beta} + n_0\mu}{1+n_0} \right)^2 \right)$$

$$= (\sigma^2)^{-\frac{n+\nu_0}{2}-1} \exp\left(-\frac{S + S_0^2}{2\sigma^2} - \frac{\sum x_i^2}{2\sigma^2} \left(\hat{\beta}^2 + n_0\mu^2 - \frac{(\hat{\beta} + n_0\mu)^2}{1+n_0} \right) \right)$$

$$= (\sigma^2)^{-\frac{n+\nu_0}{2}-1} \exp\left\{ -\frac{S+S_0}{2\sigma^2} - \frac{\Sigma x_i^2}{2\sigma^2(1+n_0)} \left((1+n_0)\hat\beta^2 + n_0(1+n_0)\mu^2 \right.\right.$$

$$\left.\left. - \hat\beta^2 - 2n_0\mu\hat\beta - n_0^2\mu^2 \right) \right\}$$

$$= (\sigma^2)^{-\frac{n+\nu_0}{2}-1} \exp\left\{ \underbrace{-\frac{S+S_0}{2\sigma^2} - \frac{\Sigma x_i^2}{2\sigma^2(1+n_0)} \left(n_0\hat\beta^2 + n_0\mu^2 - 2n_0\mu\hat\beta \right)}_{= -\frac{\left(S+S_0+\frac{\Sigma x_i^2}{1+n_0} n_0(\hat\beta-\mu)^2\right)}{2\sigma^2}} \right\}$$

$$\Rightarrow \text{ a kernel for } IG\left(\frac{n+\nu_0}{2}, \frac{S+S_0+\frac{n_0}{1+n_0}\Sigma x_i^2(\hat\beta-\mu)^2}{2} \right)$$

$$\Rightarrow \pi_2(\sigma^2 \mid y, x) = IG\left(\frac{n+\nu_0}{2}, \frac{S+S_0+\frac{n_0}{1+n_0}\Sigma x_i^2(\hat\beta-\mu)^2}{2} \right)$$