Winter 18 – AMS206B Homework 1 Solution

1. In Kokomo, IN, 65% are conservative, 20% are liberals and 15% are independents. Records show that in a particular election, 82% of conservatives voted, 65% of liberals voted and 50% of independents voted. If a person from the city is selected at random and it is learned that he/she did not vote, what is the probability that the person is liberal?

   Solution: Let C = Conservative, L = Liberal, I = Independent, and V = Vote. From the question we elicit that $P(C) = .65$, $P(L) = .20$, $P(I) = .15$, $P(V|C) = .82$, $P(V|L) = .65$, and $P(V|I) = .5$. Thus, $P(V^c|C) = .18$, $P(V^c|L) = .35$, and $P(V^c|I) = .5$

   Thus,

$$P(L|V^c) = \frac{P(V^c, L)}{P(V^c)} = \frac{P(V^c|L)P(L)}{P(V^c|L)P(L) + P(V^c|C)P(C) + P(V^c|I)P(I)} = .2672.$$

2. Suppose $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a sample from uniform distribution, Unif$(0, \theta)$. Let θ have Pareto Pa(θ_0, a) distribution where $\theta_0 > 0$ and $a > 0$ are fixed. That is,

$$\pi(\theta \mid \theta_0, a) = \frac{a}{\theta_0}\left(\frac{\theta_0}{\theta}\right)^{(a+1)}, \text{ for } \theta \geq \theta_0.$$

   Show that the posterior is Pareto, Pa$(\max\{\theta_0, x_1, \ldots, x_n\}, a + n)$.

   Solution: The likelihood function satisfies

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^n \frac{1}{\theta}\mathbb{1}(0 \leq x_i \leq \theta) = \frac{1}{\theta^n}\prod_{i=1}^n \mathbb{1}(0 \leq x_i \leq \theta) = \frac{1}{\theta^n}\mathbb{1}(0 \leq \min\{x_i\}_{i=1}^n)\mathbb{1}(\max\{x_i\}_{i=1}^n \leq \theta)$$

   while the prior can be written as

$$\pi(\theta) = \frac{a}{\theta_0}\left(\frac{\theta_0}{\theta}\right)^{(a+1)} \mathbb{1}(\theta_0 \leq \theta).$$

   Now, the joint distribution for $\boldsymbol{X}$ and θ is

$$\begin{aligned} h(\boldsymbol{x}, \theta) &= f(\boldsymbol{x} \mid \theta)\pi(\theta) \\ &= \frac{1}{\theta^n}\mathbb{1}(0 \leq \min\{x_i\}_{i=1}^n)\mathbb{1}(\max\{x_i\}_{i=1}^n \leq \theta)\frac{a}{\theta_0}\left(\frac{\theta_0}{\theta}\right)^{(a+1)} \mathbb{1}(\theta_0 \leq \theta) \\ &= \frac{a\theta_0^a}{\theta^{a+n+1}}\mathbb{1}(0 \leq \min\{x_i\}_{i=1}^n)\mathbb{1}(\max\{\theta_0, x_1, \ldots, x_n\} \leq \theta). \end{aligned}$$

   To compute the marginal distribution of $\boldsymbol{X}$ let $c = \max\{\theta_0, x_1, \ldots, x_n\}$,

$$\begin{aligned} m(\boldsymbol{x}) &= \int_0^\infty h(\boldsymbol{x}, \theta)d\theta = a\theta_0^a\mathbb{1}(0 \leq \min\{x_i\}_{i=1}^n)\int_c^\infty \theta^{-(a+n+1)}d\theta \\ &= \frac{a\theta_0^a}{(a+n)c^{(a+n)}}\mathbb{1}(0 \leq \min\{x_i\}_{i=1}^n). \end{aligned}$$

Then, the posterior is given by

$$f(\theta \mid \boldsymbol{x}) = \frac{h(\boldsymbol{x}, \theta)}{m(\boldsymbol{x})} = \frac{(a+n)c^{(a+n)}}{\theta^{a+n+1}} \mathbb{1}(c \leq \theta),$$

a Pareto$(c, a+n)$.

Alternatively, is possible to note that the posterior satisfies that

$$f(\theta \mid \boldsymbol{x}) \propto \left(\frac{1}{\theta}\right)^{(a+n+1)} \mathbb{1}(\max\{\theta_0, x_1, \ldots, x_n\} \leq \theta)$$

and identify this kernel as the desired Pareto$(\max\{\theta_0, x_1, \ldots, x_n\}, a+n)$.

3. Let $X \sim \text{Gamma}(n/2, 2\theta)$ (that is, $X/\theta \sim \chi^2_n$) where $E(X) = n\theta$.

$$f(x \mid \theta) = \frac{1}{\Gamma(n/2)(2\theta)^{n/2}} x^{n/2-1} \exp\left\{-\frac{x}{2\theta}\right\}, \qquad x > 0.$$

Let $\theta \sim \text{IG}(\alpha, \beta)$, inverse gamma distribution. Find the posterior distribution of θ.

Solution: The density function for the Inverse Gamma prior satisfies

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} \exp\left\{-\frac{\beta}{\theta}\right\}$$

Therefore, the joint distribution is

$$h(x, \theta) = f(x \mid \theta)\pi(\theta) = \frac{\beta^\alpha x^{\frac{n}{2}-1} \theta^{-(\alpha+\frac{n}{2}+1)} \exp\left\{-\frac{(\beta+\frac{x}{2})}{\theta}\right\}}{\Gamma(\alpha)\Gamma(\frac{n}{2})2^{\frac{n}{2}}}.$$

Now, compute marginal distribution of X

$$\begin{aligned} m(x) &= \int_0^\infty h(x,\theta)d\theta = \frac{\beta^\alpha x^{\frac{n}{2}-1}}{\Gamma(\alpha)\Gamma(\frac{n}{2})2^{\frac{n}{2}}} \int_0^\infty \theta^{-(\alpha+\frac{n}{2}+1)} \exp\left\{-\frac{(\beta+\frac{x}{2})}{\theta}\right\} d\theta \\ &= \frac{\beta^\alpha x^{\frac{n}{2}-1}}{\Gamma(\alpha)\Gamma(\frac{n}{2})2^{\frac{n}{2}}} \frac{\Gamma(\alpha+\frac{n}{2})}{(\beta+\frac{x}{2})^{\alpha+\frac{n}{2}}} \int_0^\infty \frac{(\beta+\frac{x}{2})^{\alpha+\frac{n}{2}}}{\Gamma(\alpha+\frac{n}{2})} \theta^{-(\alpha+\frac{n}{2}+1)} \exp\left\{-\frac{(\beta+\frac{x}{2})}{\theta}\right\} d\theta \\ &= \frac{\beta^\alpha x^{\frac{n}{2}-1}}{\Gamma(\alpha)\Gamma(\frac{n}{2})2^{\frac{n}{2}}} \frac{\Gamma(\alpha+\frac{n}{2})}{(\beta+\frac{x}{2})^{\alpha+\frac{n}{2}}}. \end{aligned}$$

and, thus, the posterior is given by

$$f(\theta \mid x) = \frac{h(x,\theta)}{m(x)} = \frac{(\beta+\frac{x}{2})^{\alpha+\frac{n}{2}}}{\Gamma(\alpha+\frac{n}{2})} \theta^{-\left(\alpha+\frac{n}{2}+1\right)} \exp\left\{-\frac{(\beta+\frac{x}{2})}{\theta}\right\}$$

which is readily identified as an IG $\left(\alpha+\frac{n}{2}, \beta+\frac{x}{2}\right)$.

Alternatively, is possible to note that

$$f(\theta \mid x) \propto \theta^{-\left(\alpha+\frac{n}{2}+1\right)} \exp\left\{-\frac{(\beta+\frac{x}{2})}{\theta}\right\}$$

and identify this expression as the kernel of an IG $\left(\alpha+\frac{n}{2}, \beta+\frac{x}{2}\right)$.

4. Suppose that $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a sample from negative binomial, $\mathrm{NB}(m, \theta)$ distribution, that is, $X_i \stackrel{iid}{\sim} \mathrm{NB}(m, \theta)$, $i = 1, \ldots, n$. Consider a Beta distribution as a prior distribution for θ, $\theta \sim \mathrm{Be}(\alpha, \beta)$. Find the posterior distribution of θ.

Solution: In this case the likelihood function is given by

$$f(\boldsymbol{x} \mid \theta) = \prod_{i=1}^{n} \binom{x_i + m - 1}{x_i} (1-\theta)^m \theta^{x_i} = (1-\theta)^{nm} \theta^{\sum_{i=1}^n x_i} \prod_{i=1}^{n} \binom{x_i + m - 1}{x_i}$$

and the prior

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

Then, the joint is

$$h(\boldsymbol{x}, \theta) \quad = \quad f(\boldsymbol{x} \mid \theta)\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \prod_{i=1}^{n} \binom{x_i + m - 1}{x_i} \theta^{\alpha+\sum_{i=1}^n x_i - 1}(1-\theta)^{\beta+nm-1}$$

while the marginal distribution satisfies

$$\begin{aligned} m(\boldsymbol{x}) &= \int_0^\infty h(\boldsymbol{x}, \theta) d\theta \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \prod_{i=1}^{n} \binom{x_i + m - 1}{x_i} \int_0^\infty \theta^{\alpha+\sum_{i=1}^n x_i - 1}(1-\theta)^{\beta+nm-1} d\theta \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \prod_{i=1}^{n} \binom{x_i + m - 1}{x_i} \frac{\Gamma(\alpha + \sum_{i=1}^n x_i)\Gamma(\beta+nm)}{\Gamma(\alpha + \sum_{i=1}^n x_i + \beta + nm)} \end{aligned}$$

In this case, the posterior density

$$f(\theta \mid \boldsymbol{x}) = \frac{h(\boldsymbol{x}, \theta)}{m(\boldsymbol{x})} = \frac{\Gamma(\alpha + \sum_{i=1}^n x_i + \beta + nm)}{\Gamma(\alpha + \sum_{i=1}^n x_i)\Gamma(\beta+nm)} \theta^{\alpha+\sum_{i=1}^n x_i - 1}(1-\theta)^{\beta+nm-1}$$

is that of a $\mathrm{Be}\left(\alpha + \sum_{i=1}^n x_i, \beta + nm\right)$

Alternatively, the posterior can be written as

$$f(\theta \mid \boldsymbol{x}) \propto \theta^{\alpha+\sum_{i=1}^n x_i - 1}(1-\theta)^{\beta+nm-1}$$

which is the kernel of a $\mathrm{Be}\left(\alpha + \sum_{i=1}^n x_i, \beta + nm\right)$.