AMS 261: Probability Theory (Fall 2017)

Homework 2 solutions

1. Let $\{A_n : n = 1, 2, ...\}$ be a countable sequence of subsets of a sample space Ω.
(a) Assume that $\{A_n : n = 1, 2, ...\}$ is an increasing sequence, that is, $A_n \subseteq A_{n+1}$, for all $n \geq 1$. Show that $\lim_{n\to\infty} A_n$ exists, and $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$.
(b) Assume that $\{A_n : n = 1, 2, ...\}$ is a decreasing sequence, that is, $A_{n+1} \subseteq A_n$, for all $n \geq 1$. Show that $\lim_{n\to\infty} A_n$ exists, and $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.
Solution: (a) Let $B = \bigcup_{n=1}^{\infty} A_n$. Based on the assumption, we have $1_{A_n}(\omega) \leq 1_{A_{n+1}}(\omega)$, for all $\omega \in \Omega$, and for all n. Therefore, for each $\omega \in \Omega$, $\{1_{A_n}(\omega) : n = 1, 2, ...\}$ is an increasing sequence of reals, which is bounded from above by 1. Hence, for each $\omega \in \Omega$, $\lim_{n\to\infty} 1_{A_n}(\omega)$ exists, and thus, by its definition, $\lim_{n\to\infty} A_n$ exists. Let $A = \lim_{n\to\infty} A_n$.
Consider a fixed $\omega \in \Omega$. If ω does not belong to any of the A_n, we have $1_B(\omega) = 0$ as well as $1_{A_n}(\omega) = 0$ for all n, which yields $\lim_{n\to\infty} 1_{A_n}(\omega) = 0$. Next, assume that $\omega \in A_n$ for at least one n. Because $1_{A_n}(\omega) \leq 1_{A_{n+1}}(\omega)$, for all n, there must exist some k (that depends on ω) such that $1_{A_n}(\omega) = 1$ for all $n \geq k$. Hence, $\lim_{n\to\infty} 1_{A_n}(\omega) = 1$, and $1_B(\omega) = \max_n\{1_{A_n}(\omega)\} = 1$.
Therefore, we have shown that for all $\omega \in \Omega$, $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_B(\omega)$, i.e., $1_A(\omega) = 1_B(\omega)$, and thus $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$. The approach is similar for part (b).

2. Consider countable sequences, $\{A_n : n = 1, 2, ...\}$, $\{B_n : n = 1, 2, ...\}$ and $\{C_n : n = 1, 2, ...\}$, of subsets of the same sample space Ω. Assume that $A_n \subseteq B_n \subseteq C_n$, for all $n \geq K$ for some sufficiently large positive integer K. Moreover, suppose that $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} A_n$. Prove that each of $\lim_{n\to\infty} A_n$, $\lim_{n\to\infty} B_n$ and $\lim_{n\to\infty} C_n$ exists, and that all three limits are the same.
Solution: From the assumption $A_n \subseteq B_n$, for all $n \geq K$, we obtain $\cup_{n=K}^{\infty} A_n \subseteq \cup_{n=K}^{\infty} B_n$, and taking intersection over $K = 1, 2, ...$ on both sides, we have

$$\limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} B_n. \tag{2.1}$$

Using similar arguments, it can be shown that:

$$\limsup_{n\to\infty} B_n \subseteq \limsup_{n\to\infty} C_n; \qquad \liminf_{n\to\infty} A_n \subseteq \liminf_{n\to\infty} B_n; \qquad \liminf_{n\to\infty} B_n \subseteq \liminf_{n\to\infty} C_n. \tag{2.2}$$

Now, combining the assumption $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} A_n$ with (2.1) and the first result in (2.2), we obtain that $\limsup_{n\to\infty} A_n \subseteq \liminf_{n\to\infty} A_n$. Hence, $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$, and therefore $A = \lim_{n\to\infty} A_n$ exists. From the assumption $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} A_n$ and the second and third results in (2.2), $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} C_n$, and therefore $\liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n = C$ and $C = \lim_{n\to\infty} C_n$ exists. In addition, from (2.1) and the first result in (2.2), $\limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} C_n$, and therefore $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} C_n$, which, along with the assumption $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} A_n$, implies that $\limsup_{n\to\infty} C_n = \liminf_{n\to\infty} A_n$, that is, $A = C$. Using similar arguments, it can be shown that $B = \lim_{n\to\infty} B_n$ exists and that $A = B = C$.

3. Consider a measurable space $(\Omega, \mathcal{F})$ and a set function P: $\mathcal{F} \longrightarrow [0,1]$, which satisfies $P(\Omega) = 1$, and $P(A \cup B) = P(A) + P(B)$ for any A and B in $\mathcal{F}$ with $A \cap B = \emptyset$. Moreover, assume that P is continuous, that is, $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$, for any sequence $\{A_n : n = 1, 2, ...\}$ of sets in $\mathcal{F}$ for which $\lim_{n\to\infty} A_n$ exists. Prove that P is a probability measure on $(\Omega, \mathcal{F})$.
Solution: We basically need to prove that P is countably additive if it is continuous and finitely additive. Let $\{A_n : n = 1, 2, ...\}$ be a countable pairwise disjoint sequence of events in $\mathcal{F}$. Define $B_n = \bigcup_{k=1}^{n} A_k$, for $n \geq 1$. This is an increasing sequence of events with $\lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. Hence,

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = P(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} P(B_n),$$

using the assumption of continuity. Now, $\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k)$, using the assumption of finite additivity. (Finite additivity for general finite n results by induction from the assumption, which involves the case with $n = 2$.) Hence the result is established noting that, by definition, $\lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{n=1}^{\infty} P(A_n)$.

4. Prove that any non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is measurable. (Assume the usual Borel σ-field on $\mathbb{R}$.)
Solution: Let f denote the non-decreasing function. First, since the collection of intervals $\{(-\infty, b] : b \in \mathbb{R}\}$ generates the Borel σ-field on the real line, it suffices to show that $f^{-1}((-\infty, b]) = \{\omega \in \mathbb{R} : f(\omega) \leq b\}$ is a Borel subset on the real line. This is fairly straightforward to check *graphically* by considering the different possible shapes that f could have, e.g., strictly increasing and continuous; non-decreasing and continuous; non-decreasing with discontinuities.
Alternatively, let α be the least upper bound of $f^{-1}((-\infty, b])$. Therefore, $\omega \leq \alpha$ for all $\omega \in f^{-1}((-\infty, b])$, and hence $f^{-1}((-\infty, b]) \subseteq (-\infty, \alpha]$. Moreover, using the definition for least upper bounds, we have that for each $\epsilon > 0$, there exists some $\omega \in f^{-1}((-\infty, b])$ such that $\omega > \alpha - \epsilon$, and thus $f(\alpha - \epsilon) \leq f(\omega) \leq b$. That is, for each $\epsilon > 0$, $\alpha - \epsilon \in f^{-1}((-\infty, b])$, which yields that $(-\infty, \alpha) \subseteq f^{-1}((-\infty, b])$. Hence, $(-\infty, \alpha) \subseteq f^{-1}((-\infty, b]) \subseteq (-\infty, \alpha]$, which implies that $f^{-1}((-\infty, b])$ must be either $(-\infty, \alpha)$ or $(-\infty, \alpha]$ both of which are Borel sets on the real line.

5. Let $(\Omega_j, \mathcal{F}_j)$, $j = 1, 2, 3$, be measurable spaces. Consider measurable functions $X : \Omega_1 \to \Omega_2$ and $Y : \Omega_2 \to \Omega_3$, and define the composition function $Y \circ X : \Omega_1 \to \Omega_3$ by $Y \circ X(\omega_1) = Y(X(\omega_1))$, for any $\omega_1 \in \Omega_1$. Show that $Y \circ X$ is a measurable function.
Solution: Let $B \in \mathcal{F}_3$. We need to show that $Y \circ X^{-1}(B) = \{\omega_1 \in \Omega_1 : Y \circ X(\omega_1) \in B\} \in \mathcal{F}_1$. Because Y is measurable, $Y^{-1}(B) = \{\omega_2 \in \Omega_2 : Y(\omega_2) \in B\} \in \mathcal{F}_2$. Now, because X is measurable, $X^{-1}(Y^{-1}(B))$ $\in \mathcal{F}_1$, and this establishes the result, since $X^{-1}(Y^{-1}(B)) = \{\omega_1 \in \Omega_1 : X(\omega_1) \in Y^{-1}(B)\} = \{\omega_1 \in \Omega_1 : Y(X(\omega_1)) \in B\} = Y \circ X^{-1}(B)$.

6. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Let C be the set of $\omega \in \Omega$ such that $\{X_n(\omega) : n = 1, 2, ...\}$ is a convergent numerical sequence. Prove that $C \in \mathcal{F}$.
Solution: Recall that a characterization of convergence for a numerical sequence is through the Cauchy criterion, specifically, sequence $\{a_n : n = 1, 2, ...\}$ converges to some limit if and only if it is a Cauchy sequence, that is, for any $\varepsilon > 0$, there exists N such that for all $n, m > N$, $|a_n - a_m| < \varepsilon$. Therefore,

$$C = \{\omega \in \Omega : \{X_n(\omega) : n = 1, 2, ...\} \text{ Cauchy}\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m>N} B_{nm,k},$$

where $B_{nm,k} = \{\omega \in \Omega : |X_n(\omega) - X_m(\omega)| < k^{-1}\}$. Since $|X_n - X_m| = \max\{X_n - X_m, X_m - X_n\}$ is a random variable, $B_{nm,k} = |X_n - X_m|^{-1}((-\infty, k^{-1})) \in \mathcal{F}$, for all n, m, k, and thus, $C \in \mathcal{F}$.
Note that working with the Cauchy criterion avoids the need to refer to the limit of the sequence, which does not necessarily correspond to a well-defined function on Ω (there may be many ω for which the limit $\lim_{n\to\infty} X_n(\omega)$ does not exist).

7. Let X and Y be $\mathbb{R}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$, and consider the subset of Ω defined by $A = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\}$.
(a) Prove that A is an event in $\mathcal{F}$.
(b) Assume that $P(A) = 0$. Prove that $P(X^{-1}(B)) = P(Y^{-1}(B))$ for any Borel subset B of $\mathbb{R}$ (in which case, we say that the distributions of X and Y are equal).
Solution: (a) Let Q be the (countable) set of rational numbers. We can write $A = A_1 \cup A_2$, where $A_1 = \bigcup_{q\in Q} (\{\omega \in \Omega : X(\omega) < q\} \cap \{\omega \in \Omega : Y(\omega) > q\})$, and $A_2 = \bigcup_{q\in Q} (\{\omega \in \Omega : Y(\omega) < q\} \cap \{\omega \in \Omega : X(\omega) > q\})$. (This is based on the *Archimedean Property* of the real numbers: for any real numbers a and b with $a < b$, there exists a rational number q such that $a < q < b$.) Because A_1 and A_2 are expressed through countable operations on events, we have that $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F}$, and thus $A \in \mathcal{F}$.
(b) Consider a Borel subset B of $\mathbb{R}$ and let $D_1 = X^{-1}(B)$ and $D_2 = Y^{-1}(B)$, both of which are events in $\mathcal{F}$. We have $D_1 = (D_1 \cap A) \cup (D_1 \cap A^c)$, and hence $P(D_1) = P(D_1 \cap A^c)$ (note that $P(D_1 \cap A) = 0$, since $P(D_1 \cap A) \leq P(A) = 0$). Similarly, we can show that $P(D_2) = P(D_2 \cap A^c)$. Now $P(D_1 \cap A^c) = P(\{\omega \in \Omega : X(\omega) \in B \text{ and } X(\omega) = Y(\omega)\})$, $P(D_2 \cap A^c) = P(\{\omega \in \Omega : Y(\omega) \in B \text{ and } X(\omega) = Y(\omega)\})$, and thus $P(D_1 \cap A^c) = P(D_2 \cap A^c) = P(\{\omega \in \Omega : X(\omega) \in B \text{ and } Y(\omega) \in B \text{ and } X(\omega) = Y(\omega)\})$. Therefore $P(D_1) = P(D_2)$, i.e., $P(X^{-1}(B)) = P(Y^{-1}(B))$.