AMS 206B - midterm 1

1-(a) From lecture, if $x_i \mid \theta \overset{iid}{\sim} N(\theta, 1)$, $i=1,\dots,n$,

$\bar{x}$ is a sufficient statistic, & we know $\bar{x} \mid \theta \sim N(\theta, \frac{1}{n})$

Due to the sufficiency principle, we know $\pi(\theta \mid x_1, \dots, x_n) = \pi(\theta \mid \bar{x})$

$$\pi(\theta \mid \bar{x}) \propto f(\bar{x} \mid \theta)\, \pi(\theta)$$

$$\propto \exp\left\{ -\frac{n(\bar{x}-\theta)^2}{2} \right\} \cdot \exp\left\{ -\frac{(\theta-\mu)^2}{2\tau^2} \right\}$$

$$\propto \exp\left[-\frac{1}{2}\left\{ \left(n+\frac{1}{\tau^2}\right)\theta^2 - 2\left(n\bar{x}+\frac{\mu}{\tau^2}\right)\theta \right\} \right]$$

$$\Rightarrow \text{ a kernel for } N\left(\left(n+\frac{1}{\tau^2}\right)^{-1}\left(n\bar{x}+\frac{\mu}{\tau^2}\right),\ \left(n+\frac{1}{\tau^2}\right)^{-1} \right)$$

$$\Rightarrow \quad \eta = \left(n+\frac{1}{\tau^2}\right)^{-1}\left(n\bar{x}+\frac{\mu}{\tau^2}\right) = \frac{n\tau^2\bar{x}+\mu}{n\tau^2+1}$$

$$v^2 = \left(n+\frac{1}{\tau^2}\right)^{-1} = \frac{\tau^2}{n\tau^2+1}$$

(b)

$$\rho(\pi, a \mid \bar{x}) = E\left(L(\theta, a) \mid \bar{x}\right)$$

$$= E\left(e^{c(a-\theta)} - c(a-\theta) - 1 \mid \bar{x}\right)$$

$$= e^{ca} E\left(e^{-c\theta} \mid \bar{x}\right) - ca + cE(\theta \mid \bar{x}) - 1$$

$$\Rightarrow \delta^B(\mathbf{x}) = \delta^B(\bar{x}) = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \ \rho(\pi, a \mid \bar{x})$$

$$\frac{\partial \rho(\pi, a \mid \bar{x})}{\partial a} = c \cdot E\left(e^{-c\theta} \mid \bar{x}\right) e^{ca} - c = 0$$

$$\Rightarrow e^{ca} = \frac{1}{E\left(e^{-c\theta} \mid \bar{x}\right)}$$

$$\Rightarrow a = -\frac{1}{c} \log\left(E\left(e^{-c\theta} \mid \bar{x}\right)\right)$$

$$\frac{\partial^2 \rho(\pi, a \mid \bar{x})}{\partial a^2} = c^2 E\left(e^{-c\theta} \mid \bar{x}\right) e^{ca} \geq 0$$

$$-c\theta \mid \bar{x} \sim N(-c\eta, c^2 v^2)$$

$$z = e^{-c\theta} \mid \bar{x} \sim \text{LogNormal}(-c\eta, c^2 v^2)$$

$$E\left(e^{-c\theta} \mid \bar{x}\right) = E(z \mid \bar{x}) = \exp\left(-c\eta + \frac{c^2 v^2}{2}\right)$$

$$\Rightarrow a = -\frac{1}{c} \cdot \left(-c\eta + \frac{c^2 v^2}{2}\right) = \eta - \frac{c v^2}{2} = \frac{n\tau^2 \bar{x} + \mu}{n\tau^2 + 1} - \frac{c\tau^2}{2(n\tau^2 + 1)}$$

(c) - i

$$\delta_1^B(x) = \frac{n\tau^2 \bar{x} + \mu}{n\tau^2 + 1}$$

$$R(\theta, \delta_1^B) = E\left[\left(\theta - \frac{n\tau^2\bar{X} + \mu}{n\tau^2+1}\right)^2 \right]$$

$$= E\left[\frac{\left(n\tau^2 \bar{x} + \mu - (n\tau^2+1)\theta \right)^2}{(n\tau^2+1)^2} \right]$$

$$= \frac{1}{(n\tau^2+1)^2} E\left[\left(n\tau^2(\bar{X} - \theta) - (\mu - \theta) \right)^2 \right]$$

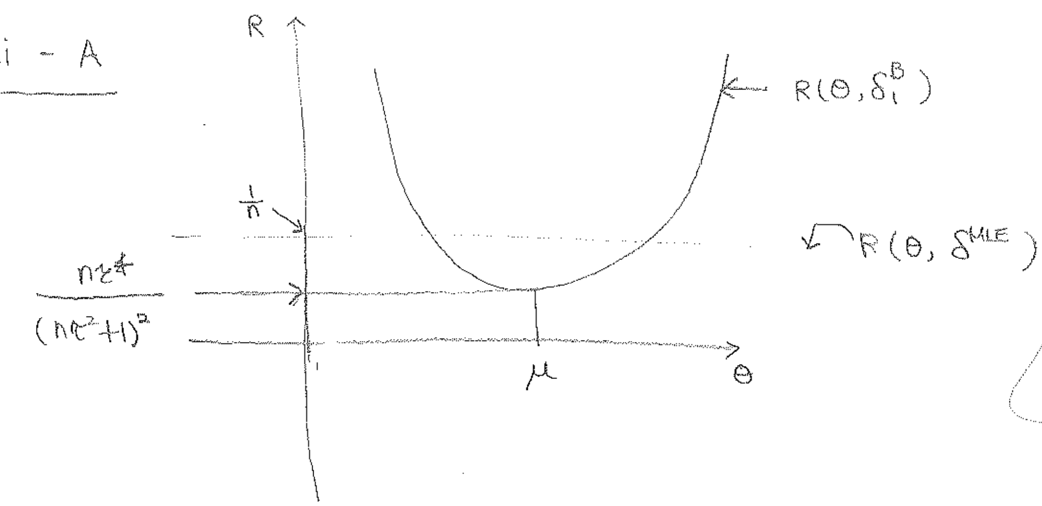
$\bar{X} | \theta \sim N(\theta, \frac{1}{n})$

$$= \frac{1}{(n\tau^2+1)^2} \left[\underbrace{n^2\tau^4 \frac{1}{n}}_{= n\tau^2} + (\mu - \theta)^2 \right]$$

$$\frac{n\tau^4}{(n\tau^2+1)^2} = \frac{1}{\left(\sqrt{n} + \frac{1}{\sqrt{n}\tau^2}\right)^2} < \frac{1}{n}$$

since $\frac{1}{\sqrt{n}\tau^2} > 0$ for $n \geq 1$

ii - A



ii-B From the graph, for some θ, $R(\theta, \delta_1^B) < R(\theta, \delta^{MLE})$

$\Rightarrow$ δ_1^B is admissible

ii - c $\sup_{\theta \in \mathbb{R}} R(\theta, \delta_1^B) = \infty$ & $\sup_{\theta \in \mathbb{R}} R(\theta, \delta^{MLE}) = \frac{1}{n}$

$\Rightarrow$ δ^{MLE} is the minimax