Final Exam
AMS 206B — Winter 2018
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Question 1

The data is about quality control measurements from 6 machines in a factory. Some summaries of the data will be given.

First, it can noticed by the boxplot in Figure 1, that the distribution of the machines appear to have different averages for the measures of quality, with some of them having close values. It may be noted that there are some outliers.

Figure 1: Boxplot for the machines data.

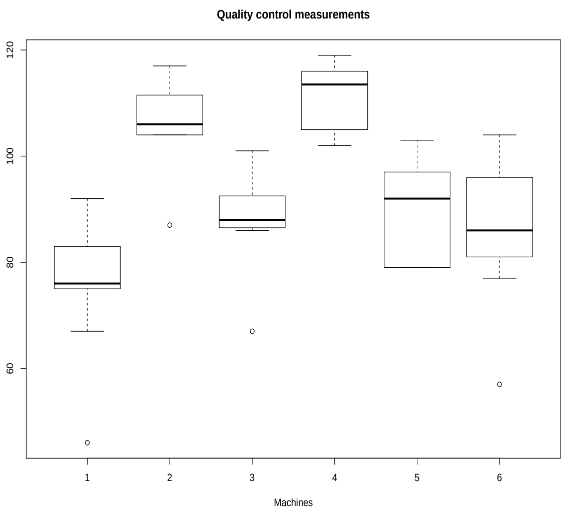


Table 1 shows the summaries of each machine, and it can be seem that the numbers of machines varies, and machine 1 has the lowest mean, and machine 4 has the highest mean. It is also noticed that machine 4 has the lowest variance and machine 6 has the highest variance.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
m	9	8	7	6	6	8
sample mean	76.0000	105.8750	87.7142	111.5000	90.3333	85.8750
sample variance	193.0000	80.4107	109.2381	45.9000	93.4666	211.5535

Table 1: Summaries of measurements for each machine.

By the descriptive analysis there are indications that the means between the machines are different. Other types of analysis are then required.

Question 2

Model 1:

$$y_{ij}|\theta_i, \sigma^2 \overset{indep}{\sim} N(\theta_i, \sigma^2)$$

$$\theta_i|\mu, \tau^2 \overset{iid}{\sim} N(\mu, \tau^2)$$

Assuming that, $\sigma^2|\nu_0, s_0^2 \sim \text{IG}(\nu_0/2, s_0^2/2)$, $\mu \sim \text{N}(\mu_0, \omega^2)$ and $\tau^2 \sim \text{IG}(a_\tau, b_\tau)$, where ν_0, s_0^2, μ_0, ω^2, a_τ and b_τ are fixed.

(a)

The joint posterior distribution up to proportionality has expression given by:

$$\begin{aligned}
\pi(\theta_i, \sigma^2, \mu, \tau^2|y_{ij}) &= \prod_{i=1}^{n}\prod_{j=1}^{m_i}\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\} \times \\
&\times \prod_{i=1}^{n}\left(\frac{1}{2\pi\tau^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} \times \\
&\times \frac{\left(\frac{s_0^2}{2}\right)^{\nu_0/2}}{\Gamma(\nu_0/2)}(\sigma^2)^{-\frac{\nu_0}{2}-1} \exp\left\{-\frac{s_0^2}{2}\frac{1}{\sigma^2}\right\} \times \\
&\times \left(\frac{1}{2\pi\omega^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{(\mu-\mu_0)^2}{2\omega^2}\right\} \times \frac{(b_\tau)^{a_\tau}}{\Gamma(a_\tau)}(\sigma^2)^{-\frac{\nu_0}{2}-1}(\tau^2)^{-a_\tau-1}\exp\left\{-b_\tau\frac{1}{\tau^2}\right\} \\
\pi(\theta_i, \sigma^2, \mu, \tau^2|y_{ij}) &\propto \left(\frac{1}{\sigma^2}\right)^{\frac{nm}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\} \times \\
&\times \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} \times \\
&\times (\sigma^2)^{-\frac{\nu_0}{2}-1} \exp\left\{-\frac{s_0^2}{2}\frac{1}{\sigma^2}\right\} \times \\
&\times \exp\left\{-\frac{(\mu-\mu_0)^2}{2\omega^2}\right\} \times (\tau^2)^{-a_\tau-1}\exp\left\{-b_\tau\frac{1}{\tau^2}\right\}
\end{aligned}$$

(b)

Now, it can be show the full conditionals for all parameters, which are:

- For θ_i

$$\pi(\theta_i|\sigma^2,\mu,\tau^2,y_{ij}) \propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}^2-2y_{ij}\theta_i+\theta_i^2)\right\}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i^2-2\theta_i\mu+\mu^2)\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(-2y_{ij}\theta_i+\theta_i^2)-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i^2-2\theta_i\mu)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(-2y_{ij}\theta_i+\theta_i^2)-\frac{1}{\tau^2}\sum_{i=1}^{n}(\theta_i^2-2\theta_i\mu)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}\theta_i^2\left(\frac{m_i}{\sigma^2}+\frac{1}{\tau^2}\right)-2\sum_{i=1}^{n}\theta_i\left(\frac{1}{\sigma^2}\sum_{j=1}^{m_i}y_{ij}+\frac{\mu}{\tau^2}\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}\theta_i^2-2\sum_{i=1}^{n}\theta_i\left(\frac{\frac{1}{\sigma^2}\sum_{j=1}^{m_i}y_{ij}+\frac{\mu}{\tau^2}}{\left(\frac{m_i}{\sigma^2}+\frac{1}{\tau^2}\right)}\right)\right]\frac{1}{\left(\frac{m_i}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}}\right\}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\theta_i|\sigma^2,\mu,\tau^2,y_{ij}) \sim \mathrm{N}\left(\left(\frac{\frac{1}{\sigma^2}\sum_{j=1}^{m_i}y_{ij}+\frac{\mu}{\tau^2}}{\left(\frac{m_i}{\sigma^2}+\frac{1}{\tau^2}\right)}\right),\left(\frac{m_i}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}\right)$$

- For σ^2

$$\pi(\sigma^2|\theta_i,\mu,\tau^2,y_{ij}) \propto \left(\sigma^2\right)^{-\dfrac{nm}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\}\times$$

$$\times \left(\sigma^2\right)^{-\dfrac{\nu_0}{2}-1}\exp\left\{-\frac{s_0^2}{2}\frac{1}{\sigma^2}\right\}$$

$$
\begin{aligned}
&\propto \quad \left(\sigma^2\right)^{-\frac{nm}{2}-\frac{\nu_0}{2}-1} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 - \frac{s_0^2}{2}\frac{1}{\sigma^2}\right\} \\
&\propto \quad \left(\sigma^2\right)^{-\left(\frac{nm}{2}+\frac{\nu_0}{2}\right)-1} \exp\left\{-\frac{1}{\sigma^2}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 + \frac{s_0^2}{2}\right]\right\}
\end{aligned}
$$

Which is a kernel of a Inverse Gamma distribution with the following parameters:

$$
\pi(\sigma^2|\theta_i,\mu,\tau^2,y_{ij}) \quad \sim \quad \mathrm{IG}\left(\frac{nm}{2}+\frac{\nu_0}{2}, \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 + \frac{s_0^2}{2}\right)
$$

- For μ

$$
\begin{aligned}
\pi(\mu|\theta_i,\sigma^2,\tau^2,y_{ij}) \quad &\propto \quad \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} \exp\left\{-\frac{(\mu-\mu_0)^2}{2\omega^2}\right\} \\
&\propto \quad \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i^2-2\theta_i\mu+\mu^2)\right\} \exp\left\{-\frac{(\mu^2-2\mu\mu_0+\mu_0^2)}{2\omega^2}\right\} \\
&\propto \quad \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(-2\theta_i\mu+\mu^2)\right\} \exp\left\{-\frac{(\mu^2-2\mu\mu_0)}{2\omega^2}\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\left[\frac{1}{\tau^2}\sum_{i=1}^{n}(-2\theta_i\mu+\mu^2) + \frac{\mu^2-2\mu\mu_0}{\omega^2}\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\left[\mu^2\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right) - 2\mu\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)\right]\right\}
\end{aligned}
$$

$$
\propto \quad \exp\left\{-\frac{1}{2}\left[\mu^2 - 2\mu\frac{\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)}\right]\frac{1}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)^{-1}}\right\}
$$

Which is a kernel of a Normal distribution with the following moments:

$$
\pi(\mu|\theta_i,\sigma^2,\tau^2,y_{ij}) \quad \sim \quad \mathrm{N}\left(\frac{\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)}, \left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)^{-1}\right)
$$

- For τ^2

$$\begin{aligned}
\pi(\tau^2|\theta_i,\sigma^2,\mu,y_{ij}) &\propto (\tau^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\}(\tau^2)^{-a_\tau-1}\exp\left\{-b_\tau\frac{1}{\tau^2}\right\} \\
&\propto (\tau^2)^{-\frac{n}{2}-a_\tau-1} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2 - b_\tau\frac{1}{\tau^2}\right\} \\
&\propto (\tau^2)^{-\left(\frac{n}{2}+a_\tau\right)-1} \exp\left\{-\frac{1}{\tau^2}\left[\frac{1}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2 + b_\tau\right]\right\}
\end{aligned}$$

Which is a kernel of a Inverse Gamma distribution with the following parameters:

$$\pi(\tau^2|\theta_i,\sigma^2,\mu,y_{ij}) \sim \text{IG}\left(\frac{n}{2}+a_\tau, \frac{1}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2+b_\tau\right)$$

(c)

The values for the fixed hyperparameters were chosen to be non-informative prior as possible. See Table 2.

ν_0	s_0^2	μ_0	ω^2	a_τ	b_τ
6	3	90	100	3	3

Table 2: Values for the hyperparameters for Model 1.

For the MCMC was used the algorithm of Gibbs sampling using the conditional posterior distribution found in letter (b). In Table 3 are given the summary for the posterior distribution of the parameters $\{\theta_i, \sigma^2, \mu, \tau^2\}$, the inferences are posterior mean and credible intervals.

	θ_1	θ_2	θ_3
post. mean	83.1194	99.3654	89.7139
95% CI	(73.2861, 93.9509)	(88.9505, 109.2171)	(82.2835, 96.3128)
	θ_4	θ_5	θ_6
post. mean	101.5591	91.1543	88.6897
95% CI	(88.9387, 114.0836)	(84.0062, 98.1978)	(81.0594, 95.3112)
	σ^2	μ	τ^2
post. mean	158.1364	92.0962	42.0881
95% CI	(82.5014, 304.1111)	(86.0684, 98.1725)	(0.6368, 150.9656)

Table 3: Summary of posterior inference on parameters for the Model 1.

(d)

Now it will compared the results of the posterior estimates of θ_i and σ^2 with the sample estimates.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
sample mean	76.0000	105.8750	87.7142	111.5000	90.3333	85.8750
sample variance	193.0000	80.4107	109.2381	45.9000	93.4666	211.5535

Table 4: Comparison sample mean and sample variance (machine specified).

The results for the sample means (Table 4) are close to the values found for the posterior mean (Table 3), with the specific values within the credibility intervals. Regarding the overall sample variance it was get $\sigma^2 = 267.6845$, this value not being within the credibility interval for σ^2.

Question 3

Model 2:

$$y_{ij}|\theta_i, \sigma_i^2 \overset{indep}{\sim} N(\theta_i, \sigma_i^2)$$

$$\theta_i|\mu, \tau^2 \overset{iid}{\sim} N(\mu, \tau^2)$$

Assuming that, $\sigma_i^2|\nu_0, s_0^2 \sim \text{IG}(\nu_0/2, s_0^2/2)$, $s_0^2 \sim \text{Gamma}(a_s, b_s)$ $\mu \sim \text{N}(\mu_0, \omega^2)$ and $\tau^2 \sim \text{IG}(a_\tau, b_\tau)$, where ν_0, a_s, b_s, μ_0, ω^2, a_τ and b_τ are fixed.

(a)

The joint posterior distribution up to proportionality has expression given by:

$$\begin{aligned}
\pi(\theta_i, \sigma_i^2, \mu, \tau^2|y_{ij}) &= \prod_{i=1}^{n}\prod_{j=1}^{m_i}\left(\frac{1}{2\pi\sigma_i^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_i^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\} \times \\
&\times \prod_{i=1}^{n}\left(\frac{1}{2\pi\tau^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} \times \\
&\times \prod_{i=1}^{n}\frac{\left(\frac{s_0^2}{2}\right)^{\nu_0/2}}{\Gamma(\nu_0/2)}(\sigma_i^2)^{-\frac{\nu_0}{2}-1}\exp\left\{-\frac{s_0^2}{2}\frac{1}{\sigma_i^2}\right\} \times \\
&\times \left(\frac{1}{2\pi\omega^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{(\mu-\mu_0)^2}{2\omega^2}\right\} \times \frac{(b_\tau)^{a_\tau}}{\Gamma(a_\tau)}(\sigma^2)^{-\frac{\nu_0}{2}-1}(\tau^2)^{-a_\tau-1}\exp\left\{-b_\tau\frac{1}{\tau^2}\right\}
\end{aligned}$$

$$
\begin{aligned}
\pi(\theta_i, \sigma_i^2, s_0^2, \mu, \tau^2 | y_{ij}) \quad \propto \quad & \prod_{i=1}^{n} \left(\frac{1}{\sigma_i^2}\right)^{\frac{m_i}{2}} \exp\left\{-\sum_{i=1}^{n} \frac{1}{2\sigma_i^2} \sum_{j=1}^{m_i} (y_{ij} - \theta_i)^2\right\} \times \\
& \times \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2} \sum_{i=1}^{n} (\theta_i - \mu)^2\right\} \times \\
\times \quad & \prod_{i=1}^{n} (\sigma_i^2)^{-\frac{\nu_0}{2}-1} \exp\left\{-\frac{s_0^2}{2} \sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right\} \times \\
\times \quad & (s_0^2)^{a_s - 1} \exp\left\{-s_0^2 b_s\right\} \times \\
\times \quad & \exp\left\{-\frac{(\mu - \mu_0)^2}{2\omega^2}\right\} \times (\tau^2)^{-a_\tau - 1} \exp\left\{-b_\tau \frac{1}{\tau^2}\right\}
\end{aligned}
$$

(b)

Now, it can be show the full conditionals for all parameters, which are:

- For θ_i

$$
\begin{aligned}
\pi(\theta_i | \sigma_i^2, s_0^2, \mu, \tau^2, y_{ij}) \quad & \propto \quad \exp\left\{-\sum_{i=1}^{n} \frac{1}{2\sigma_i^2} \sum_{j=1}^{m_i} (y_{ij} - \theta_i)^2\right\} \exp\left\{-\frac{1}{2\tau^2} \sum_{i=1}^{n} (\theta_i - \mu)^2\right\} \\
& \propto \quad \exp\left\{-\sum_{i=1}^{n} \frac{1}{2\sigma_i^2} \sum_{j=1}^{m_i} (y_{ij}^2 - 2y_{ij}\theta_i + \theta_i^2)\right\} \exp\left\{-\frac{1}{2\tau^2} \sum_{i=1}^{n} (\theta_i^2 - 2\theta_i\mu + \mu^2)\right\} \\
& \propto \quad \exp\left\{-\sum_{i=1}^{n} \frac{1}{2\sigma_i^2} \sum_{j=1}^{m_i} (-2y_{ij}\theta_i + \theta_i^2) - \frac{1}{2\tau^2} \sum_{i=1}^{n} (\theta_i^2 - 2\theta_i\mu)\right\} \\
& \propto \quad \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \sum_{j=1}^{m_i} (-2y_{ij}\theta_i + \theta_i^2) - \frac{1}{\tau^2} \sum_{i=1}^{n} (\theta_i^2 - 2\theta_i\mu)\right]\right\} \\
& \propto \quad \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n} \theta_i^2 \left(\frac{m_i}{\sigma_i^2} + \frac{1}{\tau^2}\right) - 2\sum_{i=1}^{n} \theta_i \left(\frac{1}{\sigma_i^2} \sum_{j=1}^{m_i} y_{ij} + \frac{\mu}{\tau^2}\right)\right]\right\} \\
& \propto \quad \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n} \theta_i^2 - 2\sum_{i=1}^{n} \theta_i \left(\frac{\frac{1}{\sigma_i^2} \sum_{j=1}^{m_i} y_{ij} + \frac{\mu}{\tau^2}}{\left(\frac{m_i}{\sigma_i^2} + \frac{1}{\tau^2}\right)}\right)\right] \frac{1}{\left(\frac{m_i}{\sigma_i^2} + \frac{1}{\tau^2}\right)^{-1}}\right\}
\end{aligned}
$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\theta_i|\sigma_i^2, s_0^2, \mu, \tau^2, y_{ij}) \sim \mathrm{N}\left(\left(\frac{\frac{1}{\sigma_i^2}\sum_{j=1}^{m_i} y_{ij} + \frac{\mu}{\tau^2}}{\left(\frac{m_i}{\sigma_i^2} + \frac{1}{\tau^2}\right)}\right), \left(\frac{m_i}{\sigma_i^2} + \frac{1}{\tau^2}\right)^{-1}\right)$$

- For σ^2

$$\begin{aligned}
\pi(\sigma_i^2|\theta_i, s_0^2, \mu, \tau^2|y_{ij}) &\propto \prod_{i=1}^{n} \left(\sigma_i^2\right)^{-\frac{m_i}{2}} \exp\left\{-\sum_{i=1}^{n}\frac{1}{2\sigma_i^2}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2\right\} \times \\
&\times \prod_{i=1}^{n}(\sigma_i^2)^{-\frac{\nu_0}{2}-1}\exp\left\{-\frac{s_0^2}{2}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right\} \times \\
&\propto \prod_{i=1}^{n} \left(\sigma_i^2\right)^{-\frac{m_i}{2}-\frac{\nu_0}{2}-1} \exp\left\{-\sum_{i=1}^{n}\frac{1}{2\sigma_i^2}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 - \frac{s_0^2}{2}\frac{1}{\sigma_i^2}\right\} \\
&\propto \prod_{i=1}^{n} \left(\sigma_i^2\right)^{-\left(\frac{m_i}{2}+\frac{\nu_0}{2}\right)-1} \exp\left\{-\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\left[\frac{1}{2}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 + \frac{s_0^2}{2}\right]\right\}
\end{aligned}$$

Which is a kernel of a Inverse Gamma distribution with the following parameters:

$$\pi(\sigma_i^2|\theta_i, \mu, \tau^2, y_{ij}) \sim \mathrm{IG}\left(\frac{m_i}{2} + \frac{\nu_0}{2}, \frac{1}{2}\sum_{j=1}^{m_i}(y_{ij}-\theta_i)^2 + \frac{s_0^2}{2}\right)$$

- For s_0^2

$$\begin{aligned}
\pi(s_0^2|\theta_i, \sigma_i^2, \mu, \tau^2, y_{ij}) &\propto \exp\left\{-\frac{s_0^2}{2}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right\}(s_0^2)^{a_s-1}\exp\left\{-s_0^2 b_s\right\} \\
&\propto (s_0^2)^{a_s-1}\exp\left\{-s_0^2\left(\sum_{i=1}^{n}\frac{1}{2\sigma_i^2} + b_s\right)\right\}
\end{aligned}$$

Which is a kernel of a Gamma distribution with the following parameters:

$$\pi(\tau^2|\theta_i, \sigma^2, \mu, y_{ij}) \sim \mathrm{Gamma}\left(a_s, \left(\sum_{i=1}^{n}\frac{1}{2\sigma_i^2} + b_s\right)\right)$$

- For μ

$$
\begin{aligned}
\pi(\mu|\theta_i, \sigma_i^2, s_0^2, \tau^2, y_{ij}) &\propto \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} \quad \exp\left\{-\frac{(\mu-\mu_0)^2}{2\omega^2}\right\} \\
&\propto \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i^2-2\theta_i\mu+\mu^2)\right\} \exp\left\{-\frac{(\mu^2-2\mu\mu_0+\mu_0^2)}{2\omega^2}\right\} \\
&\propto \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(-2\theta_i\mu+\mu^2)\right\} \exp\left\{-\frac{(\mu^2-2\mu\mu_0)}{2\omega^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{\tau^2}\sum_{i=1}^{n}(-2\theta_i\mu+\mu^2)+\frac{\mu^2-2\mu\mu_0}{\omega^2}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\mu^2\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)-2\mu\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\mu^2-2\mu\frac{\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)}\right]\frac{1}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)^{-1}}\right\}
\end{aligned}
$$

Which is a kernel of a Normal distribution with the following moments:

$$
\pi(\mu|\theta_i, \sigma^2, s_0^2, \tau^2, y_{ij}) \sim \mathrm{N}\left(\frac{\left(\frac{1}{\tau^2}\sum_{i=1}^{n}\theta_i+\frac{\mu_0}{\omega^2}\right)}{\left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)}, \left(\frac{n}{\tau^2}+\frac{1}{\omega^2}\right)^{-1}\right)
$$

- For τ^2

$$
\begin{aligned}
\pi(\tau^2|\theta_i, \sigma_i^2, s_0^2, \mu, y_{ij}) &\propto \left(\frac{1}{\tau^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2\right\} (\tau^2)^{-a_\tau-1}\exp\left\{-b_\tau\frac{1}{\tau^2}\right\} \\
&\propto (\tau^2)^{-\frac{n}{2}-a_\tau-1} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\theta_i-\mu)^2-b_\tau\frac{1}{\tau^2}\right\} \\
&\propto (\tau^2)^{-\left(\frac{n}{2}+a_\tau\right)-1} \exp\left\{-\frac{1}{\tau^2}\left[\frac{1}{2}\sum_{i=1}^{n}(\theta_i-\mu)^2+b_\tau\right]\right\}
\end{aligned}
$$

Which is a kernel of a Inverse Gamma distribution with the following parameters:

$$\pi(\tau^2|\theta_i, \sigma^2, s_0^2, \mu, y_{ij}) \sim \text{IG}\left(\frac{n}{2} + a_\tau, \frac{1}{2}\sum_{i=1}^{n}(\theta_i - \mu)^2 + b_\tau\right)$$

(c)

The values for the fixed hyperparameters were chosen to be non-informative prior as possible See Table 5.

ν_0	s_0^2	μ_0	ω^2	a_τ	b_τ	a_τ	b_τ
6	3	90	100	3	3	3	3

Table 5: Values for the hyperparameters for Model 2.

For the MCMC was used the algorithm of Gibbs sampling using the conditional posterior distribution found in letter (b). In Table 6 are given the summary for the posterior distribution of the parameters $\{\theta_i, \sigma_i^2, \mu, \tau^2\}$, the inferences are posterior mean and credible intervals.

	θ_1	θ_2	θ_3
post. mean	79.1830	104.7009	88.3919
95% CI	(72.1257, 87.3549)	(99.4259, 109.4951)	(82.7437, 94.2251)
	θ_4	θ_5	θ_6
post. mean	110.2361	90.6326	87.3059
95% CI	(105.1868, 114.4287)	(85.1903, 96.2457)	(79.9689, 94.7887)
	σ_1	σ_2	σ_3
post. mean	136.7588	52.8705	66.0206
95% CI	(61.6617, 300.0546)	(23.3767, 118.2893)	(28.4260, 150.2813)
	σ_4	σ_5	σ_6
post. mean	27.9416	52.34551	135.4606
95% CI	(11.0442, 68.9796)	(21.4208, 121.8862)	(60.8871, 293.5781)
	μ	τ^2	
post. mean	93.0143	80.2619	
95% CI	(85.6112, 100.0047)	(28.0964, 200.0602)	

Table 6: Summary of posterior inference on parameters for the Model 2.

(d)

Now it will compared the results of the posterior estimates of θ_i and σ^2 with the sample estimates.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
sample mean	76.0000	105.8750	87.7142	111.5000	90.3333	85.8750
sample variance	193.0000	80.4107	109.2381	45.9000	93.4666	211.5535

Table 7: Comparison sample mean and sample variance (machine specified).

The results for the sample means and sample variance (Table 7) are close to the values found for the posterior mean (Table 6) for the parameters, with the specific values within the credibility intervals.

Question 4

Evaluating the results in Table 3 and 6 for the posterior for each parameter that: Model 1 - $\{\theta_1, \sigma^2, \mu$ and $\tau^2\}$ and for Model 2 $\{\theta_1, \sigma_i^2, \mu$ and $\tau^2.\}$ The results when it is used a different prior for the variance for each θ_i (for each machine) the posterior mean seem to be closer to the true values for θ_i (for each machine). In Model 1 the the credibility interval for the variance σ^2 does not contain the true value. Nevertheless, in Model 2 the posterior mean for the variances σ_i^2 also are closer to the true values.

However, it is necessary to carry out a comparison test between models in order to check more effectively the best one. Therefore, it was perform the deviance information criterion (DIC) which is a criterion that use the information from the posterior distribution of the parameters.

Table 8: Criterion values for the model comparison.

Criterion		Model 1	Model 2
DIC		423.9719	365.8499
	Goodness of Fit	349.1356	344.1599
	Effective Size	74.8363	21.6900

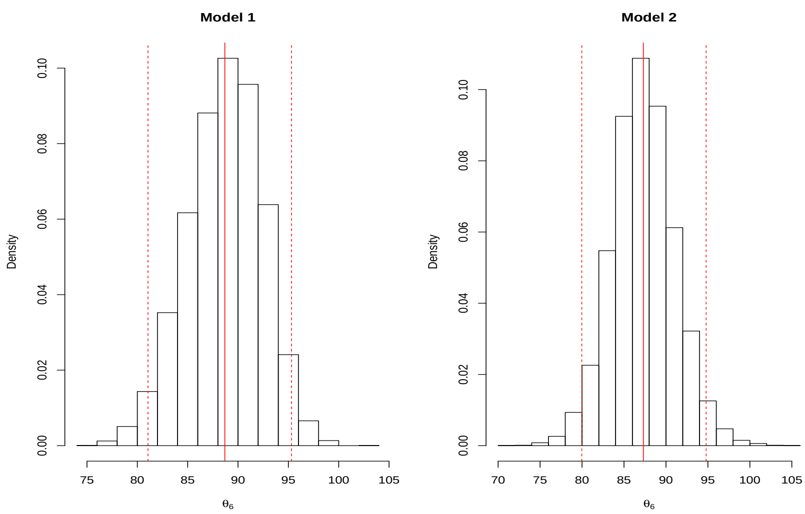
Then, we can notice that this criteria favor Model 2, but if we evaluate the goodness of fit, it can be seen that it favor Model 1, and for the Effective size (penalty) Model 2 has the lower value.

Question 5

It will be show the histogram for each request part.

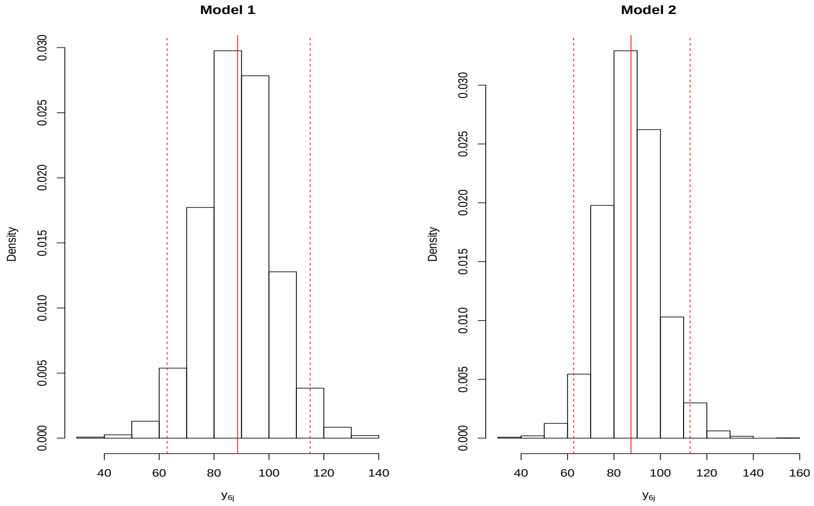
- (i) The posterior distribution of the mean of the quality measurements of the sixth machine.

Figure 2: Posterior distribution of the mean for sixth machine in both models.



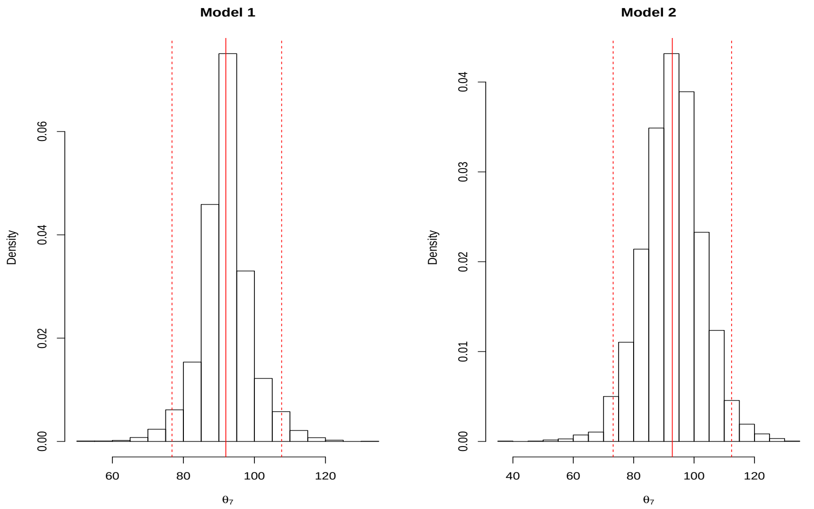
- (ii) The predictive distribution for another quality measurements of the sixth machine.

Figure 3: Posterior distribution of the mean for sixth machine in both models.



- (iii) The predictive distribution of the mean of the quality measurements of the seventh machine.

Figure 4: Posterior distribution of the mean for seventh machine in both models.



The distributions appear to have the similar behavior under the models.

R Code

See below all the R Code used to perform the analysis.

```
rm(list=ls(all=TRUE))
library(dplyr)

setwd("C:\\Users\\WYARAVMS\\Google Drive\\phd-ucsc\\winter-2018\\ams-206b\\final")

load("Machine1.RData")

y<- Machine
y<-(as.data.frame(y))

# some summaries for sample mean and sample variance
aggregate(y, by=list(y$machine), FUN=mean, na.rm=TRUE)
aggregate(y, by=list(y$machine), FUN=var, na.rm=TRUE)

# boxplot graph
boxplot(measurements ~ machine, main="Quality control measurements", xlab="Machines")

n <- 6
mm <- nrow(y)

# number of each measurement
m1=9; m2=8; m3=7; m4=6;
m5=6; m6=8;

m <- c(m1,m2,m3,m4,m5,m6)

# Question 2
# (c)

#Gibbs Sampling

## functions to sample each parameter from its full conditionals

### update theta_i
fn_update_theta <- function(y, m, mu, sig2, tau2)
{
  theta<-NULL
  for(j in 1:length(m))
  {
    theta[j] <- rnorm(1, (sum(y$measurements[y$machine==j])/sig2 +
    mu/tau2)/((m[j]/sig2)+(1/tau2)), sqrt(1/((m[j]/sig2)+(1/tau2))))
  }
  return(theta)
}
```

```
## update sig2
fn_update_sig2 <- function(mm, v0, s0, y_theta_sq)
{
  sig2 <- 1.0/rgamma(1, (mm/2)+(v0/2), (y_theta_sq)/2 + (s0^2/2))  ## mean a/b
  return(sig2)

}

## udpate mu
fn_update_mu <- function(theta_s, tau2, mu0, omega2, n)
{
  va <- 1/(n/tau2 + 1/omega2)
  mme <- va*((theta_s/tau2) + (mu0/omega2))

  mu <- rnorm(1, mme, sqrt(va))
  return(mu)

}

## update tau2
fn_update_tau2 <- function(n, at, y_theta_mu_sq, bt)
{
  tau2 <- 1.0/rgamma(1, (n/2)+at, y_theta_mu_sq/2 + bt)
  return(tau2)

}

# hyperparamenters generating higher variance on the distribution of the priors,
# being so a vague prior

## hyperparameter
hyper <- NULL
hyper$v0 <- 6
hyper$s0 <- 3

hyper$mu0 <- 90
hyper$omega2 <- 100

hyper$at <- 3
hyper$bt <- 3

## initial values
par_sam <- NULL
par_sam$theta <- rep(mean(y$measurements), n)
par_sam$sig2 <- 10^3
par_sam$mu <- 90
par_sam$tau <- 10^3
```

```
## variables for the MCMC
ns <- 40000

## save simulated para
MCMC_sam_M1 <- NULL
MCMC_sam_M1$theta <- array(NA, dim=c(n, ns))
MCMC_sam_M1$sig2 <- rep(NA, ns)
MCMC_sam_M1$mu <- rep(NA, ns)
MCMC_sam_M1$tau <- rep(NA, ns)

## MCMC modeling

for(i_iter in 1:ns)
{
  if((i_iter%%1000)==0)
  {
    print(paste("i.iter=", i_iter))
    print(date())
  }

  ## udpate theta
  par_sam$theta <- fn_update_theta(y, m, par_sam$mu, par_sam$sig2, par_sam$tau)

  #theta_vector <- function(par_theta,m){
    for(i in 1:length(m)){
      if(i==1){
        par_sam$theta_list<-rep(par_sam$theta[i],m[i])
        } else{
      p_the <- rep(par_sam$theta[i],m[i])
      par_sam$theta_list<-c(par_sam$theta_list,p_the)}
    }
    #return()}

  y$theta_list <- par_sam$theta_list
  #sum_dif <- sapply(y$machine, function(machine) {
  y$measurements[y$machine==machine]-par_sam$theta } )
  ## update sig2
  par_sam$sig2 <- fn_update_sig2(mm, hyper$v0, hyper$s0,
  (sum((y$measurements-y$theta_list)^2)))

  ## udpate mu
  par_sam$mu <- fn_update_mu(sum(par_sam$theta), par_sam$tau, hyper$mu0, hyper$omega2,
  n)

  ## update tau2
  par_sam$tau <- fn_update_tau2(n, hyper$at,  sum((par_sam$theta - par_sam$mu)^2),
  hyper$bt)

  ## save cur_sam
```

```
  MCMC_sam_M1$theta[,i_iter] <- par_sam$theta
  MCMC_sam_M1$sig2[i_iter] <- par_sam$sig2
  MCMC_sam_M1$mu[i_iter] <- par_sam$mu
  MCMC_sam_M1$tau[i_iter] <- par_sam$tau

}

n_bur <- 1000
thin <- 3

#posterior mean for the parameters
apply(MCMC_sam_M1$theta[,seq(n_bur+1, ns,by=thin)],1,mean)

mean(MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)])
mean(MCMC_sam_M1$mu[seq(n_bur+1, ns,by=thin)])
mean(MCMC_sam_M1$tau[seq(n_bur+1, ns,by=thin)])

# credibility intervals for the parameters
apply(MCMC_sam_M1$theta[,seq(n_bur+1,ns,by=thin)], 1, quantile, probs=c(0.025, 0.975))

quantile(MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))
quantile(MCMC_sam_M1$mu[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))
quantile(MCMC_sam_M1$tau[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))

# Question 3

rm(list=setdiff(ls(), "MCMC_sam_M1"))
library(dplyr)

setwd("C:\\Users\\WYARAVMS\\Google Drive\\phd-ucsc\\winter-2018\\ams-206b\\final")

load("Machine1.RData")

y<- Machine
y<-(as.data.frame(y))

n <- 6
mm <- nrow(y)

# number of each measurement
m1=9; m2=8; m3=7; m4=6;
m5=6; m6=8;

m <- c(m1,m2,m3,m4,m5,m6)


# (c)

#Gibbs Sampling
```

```
## functions to sample each parameter from its full conditionals

### update theta_i
fn_update_theta <- function(y, m, mu, sig2, tau2)
{
  theta<-NULL
  for(j in 1:length(m))
  {
    theta[j] <- rnorm(1, (sum(y$measurements[y$machine==j])/sig2[j] +
    mu/tau2)/((m[j]/sig2[j])+(1/tau2)), sqrt(1/((m[j]/sig2[j])+(1/tau2))))
  }
  return(theta)
}

## update sig2
fn_update_sig2 <- function(m, v0, s0, y_theta_sq_s)
{
  sigma<-NULL
  for(j in 1:length(m))
  {
    sigma[j] <- 1.0/rgamma(1, (m[j]/2)+(v0/2), (y_theta_sq_s[j])/2 + (s0^2/2))
  }
  return(sigma)

}

## update s02
fn_update_s02 <- function(n, as, sig_sum, bs)
{
  s02 <- 1.0/rgamma(1, as, (1/(sig_sum*2)) + bs)
  return(s02)

}

## udpate mu
fn_update_mu <- function(theta_s, tau2, mu0, omega2, n)
{
  va <- 1/(n/tau2 + 1/omega2)
  mme <- va*((theta_s/tau2) + (mu0/omega2))

  mu <- rnorm(1, mme, sqrt(va))
  return(mu)

}

## update tau2
fn_update_tau2 <- function(n, at, y_theta_mu_sq, bt)
{
```

```
  tau2 <- 1.0/rgamma(1, (n/2)+at, y_theta_mu_sq/2 + bt)
  return(tau2)

}

# hyperparamenters generating higher variance on the distribution of the priors,
# being so a vague prior

## hyperparameter
hyper <- NULL
hyper$v0 <- 6
hyper$s0 <- 3

hyper$mu0 <- 90
hyper$omega2 <- 100

hyper$at <- 3
hyper$bt <- 3

hyper$as <- 3
hyper$bs <- 3

## initial values
par_sam <- NULL
par_sam$theta <- rep(mean(y$measurements), n)
par_sam$sig2 <- rep(var(y$measurements), n)
par_sam$s02 <- 10^3
par_sam$mu <- 90
par_sam$tau <- 10^3

## variables for the MCMC
ns <- 40000

## save simulated para
MCMC_sam_M2 <- NULL
MCMC_sam_M2$theta <- array(NA, dim=c(n, ns))
MCMC_sam_M2$sig2 <- array(NA, dim=c(n, ns))
MCMC_sam_M2$s02 <- rep(NA, ns)
MCMC_sam_M2$mu <- rep(NA, ns)
MCMC_sam_M2$tau <- rep(NA, ns)

## MCMC modeling

for(i_iter in 1:ns)
{
  if((i_iter%%1000)==0)
  {
    print(paste("i.iter=", i_iter))
    print(date())
```

```
  }

  ## udpate theta
  par_sam$theta <- fn_update_theta(y, m, par_sam$mu, par_sam$sig2, par_sam$tau)

  ## creating a vector of theta to evaluate sigma^2
  for(ii in 1:length(m)){
    if(ii==1){
      par_sam$theta_list<-rep(par_sam$theta[ii],m[ii])
    } else{
      p_the <- rep(par_sam$theta[ii],m[ii])
      par_sam$theta_list<-c(par_sam$theta_list,p_the)}
  }

  y$theta_list <- par_sam$theta_list
  y_theta_dif<-NULL
  for(jj in 1:length(m)){
    y_theta_dif[jj] <- sum((y$measurements[y$machine==jj] -
    y$theta_list[y$machine==jj])^2)
  }

  par_sam$sig2 <- fn_update_sig2(m, hyper$v0, hyper$s0, y_theta_dif)

  par_sam$s02 <- fn_update_s02(n, hyper$as, sum(par_sam$sig2), hyper$bs)

  ## udpate mu
  par_sam$mu <- fn_update_mu(sum(par_sam$theta), par_sam$tau, hyper$mu0, hyper$omega2,
  n)

  ## update tau2
  par_sam$tau <- fn_update_tau2(n, hyper$at, sum((par_sam$theta - par_sam$mu)^2),
  hyper$bt)

  ## save cur_sam
  MCMC_sam_M2$theta[,i_iter] <- par_sam$theta
  MCMC_sam_M2$sig2[,i_iter] <- par_sam$sig2
  MCMC_sam_M2$mu[i_iter] <- par_sam$mu
  MCMC_sam_M2$tau[i_iter] <- par_sam$tau
  MCMC_sam_M2$s02[i_iter] <- par_sam$s02

}

n_bur <- 1000
thin <- 3

#posterior mean for the parameters
apply(MCMC_sam_M2$sig2[,seq(n_bur+1, ns,by=thin)],1,mean)
apply(MCMC_sam_M2$theta[,seq(n_bur+1, ns,by=thin)],1,mean)
```

```
mean(MCMC_sam_M2$mu[seq(n_bur+1, ns,by=thin)])
mean(MCMC_sam_M2$tau[seq(n_bur+1, ns,by=thin)])
mean(MCMC_sam_M2$s02[seq(n_bur+1, ns,by=thin)])

# credibility intervals for the parameters
apply(MCMC_sam_M2$theta[,seq(n_bur+1, ns,by=thin)], 1, quantile,  probs=c(0.025, 0.975))
apply(MCMC_sam_M2$sig2[,seq(n_bur+1, ns,by=thin)], 1, quantile,  probs=c(0.025, 0.975))

quantile(MCMC_sam_M2$mu[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))
quantile(MCMC_sam_M2$tau[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))
quantile(MCMC_sam_M2$s02[seq(n_bur+1, ns,by=thin)], probs=c(0.025, 0.975))

# Question 4

# Computing deviance information criteria
ndic = length(MCMC_sam_M2$mu[seq(n_bur+1, ns,by=thin)])
l <- vector("list", 6)
for(i in 1:6){
  theta_m1 = matrix(rep(MCMC_sam_M1$theta[i,seq(n_bur+1, ns,by=thin)],m[i]),
  nrow = ndic, ncol = m[i])
  l[[i]] = theta_m1
}
theta_m1 = do.call("cbind", l)
sigma_m1 = (MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)])

dev.m1 = matrix(0, ndic, nrow(y))
for (i in 1:nrow(y)){
  dev.m1[,i] = dnorm(y$measurements[i], theta_m1[,i], sqrt(sigma_m1), log = TRUE)
}

l <- vector("list", 6)
for(i in 1:6){
  theta_m2 = matrix(rep(MCMC_sam_M2$theta[i,seq(n_bur+1, ns,by=thin)],m[i]),
  nrow = ndic, ncol = m[i])
  l[[i]] = theta_m2
}
theta_m2 = do.call("cbind", l)

l <- vector("list", 6)
for(i in 1:6){
  sigma_m2 = matrix(rep(MCMC_sam_M2$sig2[i,seq(n_bur+1, ns,by=thin)],m[i]),
  nrow = ndic, ncol = m[i])
  l[[i]] = sigma_m2
}
sigma_m2 = do.call("cbind", l)

dev.m2 = matrix(0, ndic, nrow(y))
for (i in 1:nrow(y)){
  dev.m2[,i] = dnorm(y$measurements[i], theta_m2[,i], sqrt(sigma_m2[,i]), log = TRUE)
```

```
}

dev.m1 = -2*apply(dev.m1, 1, sum)
m1_DIC = mean(dev.m1) + var(dev.m1)/2

dev.m2 = -2*apply(dev.m2, 1, sum)
m2_DIC = mean(dev.m2) + var(dev.m2)/2

Gfs = c(mean(dev.m1),mean(dev.m2))
Gfs
pDs = c(var(dev.m1)/2, var(dev.m2)/2)
pDs

DICs = c(m1_DIC,m2_DIC)
DICs


# Question 5

par(mfrow=c(1,2))
### posterior distribution of the mean of sixth machine
hist(MCMC_sam_M1$theta[6,seq(n_bur+1, ns,by=thin)], freq=FALSE, main="Model 1",
xlab=expression(paste(theta[6])))
abline(v=mean(MCMC_sam_M1$theta[6,seq(n_bur+1, ns,by=thin)]), col=2, lty=1)
abline(v=quantile(MCMC_sam_M1$theta[6,seq(n_bur+1, ns,by=thin)],
probs=c(0.025, 0.975)), col=2, lty=2)

hist(MCMC_sam_M2$theta[6,seq(n_bur+1, ns,by=thin)], freq=FALSE, main="Model 2",
xlab=expression(paste(theta[6])))
abline(v=mean(MCMC_sam_M2$theta[6,seq(n_bur+1, ns,by=thin)]), col=2, lty=1)
abline(v=quantile(MCMC_sam_M2$theta[6,seq(n_bur+1, ns,by=thin)],
probs=c(0.025, 0.975)), col=2, lty=2)

### posterior predictive distribution for another quality of the sixth machine
par(mfrow=c(1,2))
y_pred_1 <- (rnorm(5000, MCMC_sam_M1$theta[6,seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)])))
hist(y_pred_1, main="Model 1", freq=FALSE, xlab=expression(paste(y[6][j])))
abline(v=mean(y_pred_1), col=2, lty=1)
abline(v=quantile(y_pred_1, probs=c(0.025, 0.975)), col=2, lty=2)

y_pred_2 <- (rnorm(5000, MCMC_sam_M2$theta[6,seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M2$sig2[6,seq(n_bur+1, ns,by=thin)])))
hist(y_pred_2, main="Model 2", freq=FALSE, xlab=expression(paste(y[6][j])))
abline(v=mean(y_pred_2), col=2, lty=1)
abline(v=quantile(y_pred_2, probs=c(0.025, 0.975)), col=2, lty=2)

par(mfrow=c(1,2))
### posterior predictive distribution for a seventh machine
```

```
theta_pred_7th_1 <- (rnorm(5000, MCMC_sam_M1$mu[seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M1$tau[seq(n_bur+1, ns,by=thin)])))
hist(theta_pred_7th_1, main="Model 1", freq=FALSE, xlab=expression(paste(theta[7])))
abline(v=mean(theta_pred_7th_1), col=2, lty=1)
abline(v=quantile(theta_pred_7th_1, probs=c(0.025, 0.975)), col=2, lty=2)

theta_pred_7th_2 <- (rnorm(5000, MCMC_sam_M2$mu[seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M2$tau[seq(n_bur+1, ns,by=thin)])))
hist(theta_pred_7th_2, main="Model 2", freq=FALSE, xlab=expression(paste(theta[7])))
abline(v=mean(theta_pred_7th_2), col=2, lty=1)
abline(v=quantile(theta_pred_7th_2, probs=c(0.025, 0.975)), col=2, lty=2)

# prediction for a new quality of the sixth machine
theta_6th_M1 = MCMC_sam_M1$theta[6,seq(n_bur+1, ns,by=thin)]
sigma_6th_M1 = MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)]

theta_6th_M2 = MCMC_sam_M2$theta[6,seq(n_bur+1, ns,by=thin)]
sigma_6th_M2= MCMC_sam_M2$sig2[6,seq(n_bur+1, ns,by=thin)]

nsp = length(theta_6th_M1)
y_pred_1 = matrix(0,nsp,m6)
y_pred_2 = matrix(0,nsp,m6)

for (i in 1:nsp){
  y_pred_1[i,] = (rnorm(rep(1,m6), theta_6th_M1[i], sqrt(sigma_6th_M1[i])))
  y_pred_2[i,] = (rnorm(rep(1,m6), theta_6th_M2[i], sqrt(sigma_6th_M2[i])))
}

y_6th_M1 = (apply(y_pred_1,2,mean))
y_6th_M2 = (apply(y_pred_2,2,mean))

# prediction for a  seventh machine
theta_pred_7th_1 = (rnorm(2750, MCMC_sam_M1$mu[seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M1$tau[seq(n_bur+1, ns,by=thin)])))
theta_pred_7th_2 = (rnorm(2750, MCMC_sam_M2$mu[seq(n_bur+1, ns,by=thin)],
sqrt(MCMC_sam_M2$tau[seq(n_bur+1, ns,by=thin)])))

sigma_7th_M1= MCMC_sam_M1$sig2[seq(n_bur+1, ns,by=thin)]

s02_7th_M2 = MCMC_sam_M2$s02[seq(n_bur+1, ns,by=thin)]
sigma_7th_M2 = 1/rgamma(length(s02_7th_M2),1/2,s02_7th/2)

nsp = length(theta_pred_7th_1)
y_pred_7th_1 = matrix(0,nsp,m6)
y_pred_7th_2 = matrix(0,nsp,m6)

for (i in 1:nsp){
  y_pred_7th_1[i,] = (rnorm(rep(1,m6), theta_pred_7th_1[i], sqrt(sigma_7th_M1[i])))
  y_pred_7th_2[i,] = (rnorm(rep(1,m6), theta_pred_7th_2[i], sqrt(sigma_7th_M2[i])))
```

```
}

y_7th_M1 = (apply(y_pred_7th_1,2,mean))
y_7th_M2 = (apply(y_pred_7th_2,2,mean))

plot(y_7th_M1, pch=16,ylim=c(min(y_7th_M2),max(y_7th_M2)), ylab="",
main="Prediciton for a seventh machine")
points(y_7th_M2,pch=16, col=2)
legend(1,max(y_7th_M2),pch=16,col=c(1,2),legend=c("M1","M2"))
```