AMS 261: Probability Theory (Fall 2017)

Homework 3 solutions

1. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}^+$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that all random variables X_n have the same distribution, with distribution function given by $F(x) = 1 - \exp(-x)$, $x \in \mathbb{R}^+$.
   - Show that
   $$P\left(\liminf_{n\to\infty} \{\omega \in \Omega : X_n(\omega) \le (1+\delta)\log(n)\}\right) = 1,$$
   for any fixed $\delta > 0$.

   Solution: Fix $\delta > 0$, and for each n, let $A_n = \{\omega \in \Omega : X_n(\omega) > (1+\delta)\log(n)\}$. (Note that each $A_n \in \mathcal{F}$, since each X_n is a random variable.) We need to show that $P(\liminf_{n\to\infty} A_n^c) = 1$. We have
   $$P(A_n) = 1 - P(X_n \le (1+\delta)\log(n)) = \exp\{-(1+\delta)\log(n)\} = \frac{1}{n^{(1+\delta)}}$$
   using the form of the distribution function for each X_n, $P(X_n \le x) = 1 - \exp(-x)$, $x \in \mathbb{R}^+$. Hence,
   $$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n^{(1+\delta)}} < \infty,$$
   since $\delta > 0$, and the Borel lemma yields $P(\limsup_{n\to\infty} A_n) = 0$, i.e., $P(\liminf_{n\to\infty} A_n^c) = 1$.

2. Let F and G be distribution functions on $\mathbb{R}$ such that $G(t) \le F(t)$, for all $t \in \mathbb{R}$ (in which case, G is said to be *stochastically larger* than F).
   - Construct two $\mathbb{R}$-valued random variables X and Y, defined on the same probability space $(\Omega, \mathcal{F}, P)$, such that the distribution function of X is G, the distribution function of Y is F, and $P(X \ge Y) = 1$.

   Solution: Consider $\Omega = (0, 1)$, $\mathcal{F}$ the Borel σ-field on $(0, 1)$, and P the uniform distribution on $(0, 1)$. Next, define $\mathbb{R}$-valued functions X and Y on $(\Omega, \mathcal{F}, P)$ by $X(\omega) = \inf\{t \in \mathbb{R} : G(t) \ge \omega\}$ and $Y(\omega) = \inf\{t \in \mathbb{R} : F(t) \ge \omega\}$, for each $\omega \in (0, 1)$. Then, X and Y are random variables with distribution functions G and F, respectively. Moreover, because $G(t) \le F(t)$ for any $t \in \mathbb{R}$, we have that $\{t \in \mathbb{R} : G(t) \ge \omega\} \subseteq \{t \in \mathbb{R} : F(t) \ge \omega\}$ for any $\omega \in (0, 1)$, and thus, $\inf\{t \in \mathbb{R} : G(t) \ge \omega\} \ge \inf\{t \in \mathbb{R} : F(t) \ge \omega\}$ for any $\omega \in (0, 1)$. Therefore, $X(\omega) \ge Y(\omega)$ for any $\omega \in (0, 1)$, which yields the result.

3. Consider a simple random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$, and let F be its distribution function. Denote by $F(x^-) = \lim_{y \nearrow x} F(y)$ (or equivalently, $F(x^-) = \lim_{n\to\infty} F(x_n)$ for an arbitrary increasing sequence $\{x_n : n = 1, 2, ...\}$ converging to x).
   - Show that the expectation of X can be written in the form
   $$\mathrm{E}(X) = \sum_{x\in\mathbb{R}} x\{F(x) - F(x^-)\}.$$

   Solution: Because X is a simple random variable, it can be expressed in the form $\sum_{j=1}^{k} c_j 1_{C_j}$, where the c_j are the distinct values of X and the C_j partition Ω, in particular, $C_j = \{\omega \in \Omega : X(\omega) = c_j\}$. Denote by Q_X the probability measure induced on $\mathbb{R}$ by X. Consider an arbitrary increasing sequence $\{x_n : n = 1, 2, ...\}$ converging to c_j. If we define $A_n = (-\infty, x_n]$, $n = 1, 2, ...$, we have $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = (-\infty, c_j)$. Hence, using continuity of probability measure, we obtain $\lim_{n\to\infty} Q_X(A_n) = Q_X((-\infty, c_j))$, i.e., $\lim_{n\to\infty} F(x_n) = P(\{\omega \in \Omega : X(\omega) < c_j\})$, and thus $F(c_j^-) = P(\{\omega \in \Omega : X(\omega) < c_j\})$ (by the definition of $F(c_j^-)$). Now, $F(c_j) = P(\{\omega \in \Omega : X(\omega) \le c_j\}) = P(\{\omega \in \Omega : X(\omega) < c_j\}) + P(C_j)$, and hence $P(C_j) = F(c_j) - F(c_j^-)$. Therefore, $\mathrm{E}(X) = \sum_{j=1}^{k} c_j P(C_j) = \sum_{j=1}^{k} c_j\{F(c_j) - F(c_j^-)\}$, which yields the result, since for any $x \in \mathbb{R}$ that is not a value of X (i.e., not one of the c_j) we have $F(x) - F(x^-) = P(\{\omega \in \Omega : X(\omega) = x\}) = 0$.

4. Let X be a simple random variable (taking both negative and positive values) defined on some probability space $(\Omega, \mathcal{F}, P)$.
   • Show that expectation definitions 1 (for simple random variables) and 3 (for general random variables taking values on the extended real line) are equivalent.
   Solution: We can write X in the form $\sum_{j=1}^{k} c_j 1_{C_j}$, where the c_j are the distinct values of X ($c_j \in \mathbb{R}$), and $\{C_j : j = 1, ..., k\}$ is a finite measurable partition of Ω (in particular, $C_j = \{\omega \in \Omega : X(\omega) = c_j\}$). Hence, $X^+ = \sum_{j \in J_1} c_j 1_{C_j}$, and $X^- = -\sum_{j \in J_2} c_j 1_{C_j}$, where $J_1 = \{j : c_j \geq 0\}$, and $J_2 = \{j : c_j < 0\}$. Because X^+ and X^- are both simple random variables, based on definition 1, we have $\mathrm{E}_1(X^+) = \sum_{j \in J_1} c_j P(C_j)$, and $\mathrm{E}_1(X^-) = -\sum_{j \in J_2} c_j P(C_j)$, both finite. Moreover, as shown in class, expectation definitions 1 and 2 are equivalent, and thus $\mathrm{E}_2(X^+) = \mathrm{E}_1(X^+)$ and $\mathrm{E}_2(X^-) = \mathrm{E}_1(X^-)$. Therefore, the expectation of X according to definition 3 exists and is given by $\mathrm{E}_3(X) = \mathrm{E}_2(X^+) - \mathrm{E}_2(X^-) = \sum_{j \in J_1} c_j P(C_j) + \sum_{j \in J_2} c_j P(C_j) = \sum_{j=1}^{k} c_j P(C_j) = \mathrm{E}_1(X)$.

5. Consider an $\overline{\mathbb{R}}^+$-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$, and assume that $\mathrm{E}(X) < \infty$. Let $A = \{\omega \in \Omega : X(\omega) = +\infty\}$, and note that, based on the general definition for $\overline{\mathbb{R}}^+$-valued measurable functions, we have $A \in \mathcal{F}$.
   • Show that X is almost surely finite, that is, $P(A) = 0$.
   Solution: Consider the countable sequence of random variables $\{X_n : n = 1, 2, ...\}$, with X_n defined by $X_n(\omega) = n 1_A(\omega)$, $\omega \in \Omega$. By construction, $X_n(\omega) \leq X(\omega)$, for all n and for all $\omega \in \Omega$. Note also that each X_n is a simple random variable with $\mathrm{E}(X_n) = nP(A)$. Therefore, we have that, for all n, $nP(A) = \mathrm{E}(X_n) \leq \mathrm{E}(X) < \infty$, which can only occur if $P(A) = 0$.

6. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}^+$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is (pointwise) increasing, that is, for all n and for each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega)$. Denote by X the pointwise limit of $\{X_n : n = 1, 2, ...\}$, that is, for each $\omega \in \Omega$, $X(\omega) = \lim_{n\to\infty} X_n(\omega)$, and assume that $\mathrm{E}(X) < \infty$. Define the *variance* for X by $\mathrm{Var}(X) = \mathrm{E}\{(X - \mathrm{E}(X))^2\}$, and similarly, for each n, $\mathrm{Var}(X_n) = \mathrm{E}\{(X_n - \mathrm{E}(X_n))^2\}$. (In general, the variance for a random variable Y with finite expectation $\mathrm{E}(Y)$ is given by $\mathrm{Var}(Y) = \mathrm{E}\{(Y - \mathrm{E}(Y))^2\}$, whether finite or infinite.)
   • Prove that $\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(X_n)$.
   Solution: Let $\mu = \mathrm{E}(X)$ and, for $n = 1, 2, ...$, $\mu_n = \mathrm{E}(X_n)$. The increasing structure of the X_n and the assumption $\mu = \mathrm{E}(X) < \infty$ imply that $\mu_n < \infty$ for each n. Hence, $\mathrm{Var}(X)$ and $\mathrm{Var}(X_n)$, for each n, are well defined, and, based on the variance definition, we can write $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$, and $\mathrm{Var}(X_n) = \mathrm{E}(X_n^2) - \mu_n^2$, for $n = 1, 2, ...$ (by expanding the square and using additivity of expectation).
   Next, applying the MCT to the sequence $\{X_n : n = 1, 2, ...\}$, we have $\lim_{n\to\infty} \mu_n = \mu$, and thus $\lim_{n\to\infty} \mu_n^2 = \mu^2$. Note that the sequence $\{X_n^2 : n = 1, 2, ...\}$ also satisfies the MCT assumptions; it is a pointwise increasing sequence with pointwise limit X^2. Therefore, $\lim_{n\to\infty} \mathrm{E}(X_n^2) = \mathrm{E}(X^2)$. Combining the above results/expressions, we obtain

$$\lim_{n\to\infty} \mathrm{Var}(X_n) = \lim_{n\to\infty} \{\mathrm{E}(X_n^2) - \mu_n^2\} = \lim_{n\to\infty} \mathrm{E}(X_n^2) - \lim_{n\to\infty} \mu_n^2 = \mathrm{E}(X^2) - \mu^2 = \mathrm{Var}(X).$$