Winter 18 – AMS206B Homework 3

Due: Feb 7th (W).

1. **(Review Math Stat)** Consider three independent random variables X_1, X_2 and X_3 such that $X_i \stackrel{iid}{\sim} \text{Gamma}(a_i, b)$. Let
$$\mathbf{Y} = (Y_1, Y_2, Y_3) = \left(\frac{X_1}{X_1 + X_2 + X_3}, \frac{X_2}{X_1 + X_2 + X_3}, \frac{X_3}{X_1 + X_2 + X_3} \right).$$
   (a) Show that $\mathbf{Y} \sim \text{Dirichlet}(a_1, a_2, a_3)$, a Dirichlet distribution.

   (b) How can this result be used to generate random variables according to a Dirichlet distribution? Write a simple function in `R` or `Matlab` (your choice) that takes as inputs n, the number of trivariate vectors to be generated, and $\mathbf{a} = (a1, a2, a3)$ and generates as an output a matrix of size $n \times 3$ whose rows correspond to independent samples from a Dirichlet distribution with parameter $(a1, a2, a3)$. (*Note:* the value of b is not important as long as the three X have the same value for b)

2. **(Review Math Stat)** Y follows an inverse Gamma distribution with shape parameter a and scale parameter b ($Y \sim \text{IG}(a, b)$) if $Y = 1/X$ with $X \sim \text{Gamma}(a, b)$ (assume the Gamma distribution is parameterized so that $E(X) = ab$).

   (a) Find the density of Y.

   (b) Compute $E(Y^k)$. Do you need to impose any constrain on the problem for this expectation to exists?

   (c) Compare $E(Y^k)$ to $1/E(X^k)$ (hint: look at the ratio of the two quantities).

3. Let $L(\theta, a) = \omega(\theta)(\theta - a)^2$, with $\omega(\theta)$ a non-negative function, be the weighted quadratic loss. Show that $\delta^B(x)$, the estimator that minimizes the Bayesian expected loss $\rho(\pi, \theta)$ has the form
$$\delta^B(x) = \frac{E(\omega(\theta)\theta|x)}{E(\omega(\theta)|x)}.$$
   Hint: Show that any other estimator has a larger Bayesian expected loss.

4. Consider $x \mid \theta \sim \text{N}(\theta, 1)$, $\theta \sim \text{N}(0, 1)$ and the loss
$$L(\theta, a) = e^{3\theta^2/4}(\theta - a)^2.$$
   (a) Show that the estimator that minimizes the Bayesian expected posterior loss in this case is $\delta(x) = 2x$. Hint: use the previous exercise.

   (b) Show that $\delta_0(x) = x$ dominates $\delta(x)$.

5. Assume you have to guess a secret number θ. You know that θ is an integer. You can perform an experiment that would yield either the number before it or the number after it, with equal probability. You perform the experiment twice. More formally, let x_1 and x_2 be independent observations from
$$f(x = \theta - 1 \mid \theta) = f(x = \theta + 1 \mid \theta) = 1/2.$$

Consider the 0-1 loss function, i.e.,

$$L(\theta, a) = \begin{cases} 1 & a \neq \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the risks of the estimators $\delta_0(x_1, x_2) = \frac{x_1+x_2}{2}$ and $\delta_1(x_1, x_2) = x_1 + 1$.

(b) Find the estimator $\delta^B(x_1, x_2)$ that minimizes the Bayesian expected loss.

6. Consider a point estimation problem in which you observe $x_1, \ldots, x_n$ as i.i.d. random variables of the Poisson distribution with parameter θ. Assume a squared error loss and a prior of the form $\theta \sim \text{Gamma}(\alpha, \beta)$.

   (a) Show that the Bayes estimator is $\delta^B(x) = a + b\bar{x}$ where $a > 0$, $b \in (0, 1)$ and $\bar{x} = \sum_{i=1}^n x_i/n$. You may use the fact that the distribution of $\sum_i x_i$ is Poisson with parameter θn without proof.

   (b) Find the MLE for θ (Note: to remind how to find MLEs, read Casella and Burger, Section 7.2.2– see Def 7.2.4).

   (c) Compute and graph the frequentist risks of $\delta^B(x)$ and that of the MLE.

   (d) Compute the Bayes risk of $\delta^B(x)$.

   (e) Suppose that an investigator wants to collect a sample that is large enough that the Bayes risk after the experiment is half of the Bayes risk before the experiment. Find that sample size.

7. A loss function investigated by Zellner (1986) is the LINEX (LINear-EXponential) loss, a loss function that can handle asymmetries in a smooth way. The LINEX loss is given by

$$L(\theta, a) = e^{c(a-\theta)} - c(a - \theta) - 1,$$

   where c is a positive constant. As the constant c varies, the loss function varies from very asymmetric to almost symmetric.

   Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and suppose that θ has the noninformative prior, $\pi(\theta) \propto 1$. Show that the Bayes estimator of θ under LINEX loss is given by

$$\delta^B(\bar{X}) = \bar{X} - \frac{c\sigma^2}{2n}.$$