Homework 7

AMS 206B: Intermediate Bayesian Inference

Wyara Vanesa Moura e Silva

March 16, 2018

1 Question 1

Appendix, R code.

2 Question 2

Appendix, R code.

3 Question 3

The posterior for $p(\theta|\sigma^2, \mathbf{x})$ can be found using the distribution in the question by:

$$
\begin{aligned}
p(\theta|\sigma^2,\mathbf{x}) &\propto f(\mathbf{x}|\theta,\sigma^2)\pi(\theta|\sigma^2) \\
&\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \left(\frac{1}{\kappa_0\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2\kappa_0\sigma^2} - \frac{(\theta-\theta_0)^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{\kappa_0(-2\theta\sum_{i=1}^{n}x_i + n\theta^2) + (\theta^2 - 2\theta\theta_0)}{\kappa_0\sigma^2}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{-2\theta\kappa_0\sum_{i=1}^{n}x_i + \kappa_0 n\theta^2 + \theta^2 - 2\theta\theta_0}{\kappa_0\sigma^2}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{\theta^2(\kappa_0 n + 1) - 2\theta(\kappa_0\sum_{i=1}^{n}x_i + \theta_0)}{\kappa_0\sigma^2}\right)\right\}
\end{aligned}
$$

$$\propto \quad \exp\left\{-\frac{1}{2}\left(\theta^2\left(\frac{\kappa_0 n+1}{\kappa_0\sigma^2}\right)-2\theta\left(\frac{\kappa_0\sum_{i=1}^{n}x_i+\theta_0}{\kappa_0\sigma^2}\right)\right)\right\}$$

$$\propto \quad \exp\left\{-\frac{1}{2}\left(\theta^2-2\theta\left(\frac{\kappa_0\sum_{i=1}^{n}x_i+\theta_0}{\kappa_0\sigma^2}\right)\left(\frac{\kappa_0 n+1}{\kappa_0\sigma^2}\right)^{-1}\right)\frac{1}{\left(\frac{\kappa_0 n+1}{\kappa_0\sigma^2}\right)^{-1}}\right\}$$

Then, this is a kernel of a Normal distribution with the following moments:

$$p(\theta|\sigma^2,\mathbf{x}) \quad \sim \quad \mathrm{N}\left(\frac{\kappa_0\sum_{i=1}^{n}x_i+\theta_0}{\kappa_0 n+1},\left(\frac{\kappa_0\sigma^2}{\kappa_0 n+1}\right)\right)$$

And, now the posterior for $p(\sigma^2|\mathbf{x})$ can be found using the distribution in the question by:

$$p(\sigma^2|\mathbf{x}) \quad \propto \quad \int_\theta f(\mathbf{x}|\theta,\sigma^2)\pi(\theta|\sigma^2)\pi(\sigma^2)d\theta$$

$$\propto \quad \int_\theta\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\left(\frac{1}{\kappa_0\sigma^2}\right)^{\frac{1}{2}}\exp\left\{-\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2\kappa_0\sigma^2}-\frac{(\theta-\theta_0)^2}{2\sigma^2}\right\}(\sigma^2)^{-a-1}\exp\{\frac{1}{b\sigma^2}\}d\theta$$

$$\propto \quad (\sigma^2)^{(-a-n/2-1)}\exp\left\{\frac{-\frac{1}{b}-\frac{\sum_{i=1}^{n}x_i^2}{2}-\frac{\theta_0^2}{2\kappa_0}+\frac{\theta_0+\sum_{i=1}^{n}x_i\kappa_0}{2\kappa_0(1+n\kappa_0)}}{\sigma^2}\right\}$$

Then, this is a kernel of a Inverse Gamma distribution with the following parameters:

$$p(\sigma^2|\mathbf{x}) \quad \sim \quad \mathrm{IG}\left(a+n/2,\frac{1}{b}+\frac{\sum_{i=1}^{n}x_i^2}{2}+\frac{\theta_0^2}{2\kappa_0}-\frac{\theta_0+\sum_{i=1}^{n}x_i\kappa_0}{2\kappa_0(1+n\kappa_0)}\right)$$

Appendix, R code.

4 Question 4

4.1 (a)

Appendix, R code.

4.2 (b)

As $W = log(Z)$, the density of W can be found using transformation rule, by:

$$\begin{aligned} f(z|\theta_1,\theta_2) &\propto z^{-3/2}\text{exp}\left\{-\theta_1 z - \frac{\theta_2}{z} + 2\sqrt{\theta_1\theta_2} + \log(\sqrt{2\theta_2})\right\}, z > 0 \\ w &= \log(z) \\ z &= \exp(w) \end{aligned}$$

Then,

$$\begin{aligned} f_W(w) &= f_Z(w)|\tfrac{dz}{dw}| \\ &\propto \exp(w)^{-3/2}\text{exp}\left\{-\theta_1\exp(w) - \frac{\theta_2}{\exp(w)} + 2\sqrt{\theta_1\theta_2} + \log(\sqrt{2\theta_2})\right\}|\exp(w)| \\ &\propto \exp\left\{-\frac{3}{2}w - \theta_1\exp(w) - \frac{\theta_2}{\exp(w)} + w\right\} \\ &\propto \exp\left\{-\frac{1}{2}w - \theta_1\exp(w) - \frac{\theta_2}{\exp(w)}\right\} \end{aligned}$$

Using the log of $f_W(w)$ we get:

$$f(w|\theta_1,\theta_2) \propto -\frac{1}{2}w - \theta_1\exp(w) - \frac{\theta_2}{\exp(w)}$$

In which for Z, we have:

$$f(z|\theta_1,\theta_2) \propto -\frac{1}{2}\log(z) - \theta_1 z - \frac{\theta_2}{z}$$

Appendix, R code.

5 Question 5

$$\begin{aligned} \pi(\nu|\alpha,\beta) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\nu^{\alpha-1}\exp(-\beta\nu), \qquad \alpha = 3, \beta = 1; \\ \pi(\theta|a,b) &= \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta), \qquad a = 2, b = 2; \end{aligned}$$

Then, the joint posterior distribution for ν and θ are given by:

$$\begin{aligned}\pi(\nu,\theta|\mathbf{x}) &\propto \prod_{i=1}^{n}\frac{\theta^{\nu}}{\Gamma(\nu)}x_i^{\nu-1}\exp(-\theta x_i)\nu^{\alpha-1}\exp(-\beta\nu)\theta^{a-1}\exp(-b\theta)\\ &\propto \prod_{i=1}^{n}x_i^{\nu-1}\frac{1}{\Gamma(\nu)^n}\theta^{n\nu}\exp\left(-\theta\sum_{i=1}^{n}x_i\right)\nu^{\alpha-1}\exp(-\beta\nu)\theta^{a-1}\exp(-b\theta)\\ &\propto \prod_{i=1}^{n}x_i^{\nu-1}\frac{1}{\Gamma(\nu)^n}\nu^{\alpha-1}\exp(-\beta\nu)\theta^{n\nu+a-1}\exp\left(-\theta(b+\sum_{i=1}^{n}x_i)\right)\end{aligned}$$

The conditional distribution for θ can be found by:

$$\pi(\theta|\nu,\mathbf{x}) \quad \propto \quad \theta^{n\nu+a-1}\exp\left(-\theta(b+\sum_{i=1}^{n}x_i)\right)$$

This is a kernel of a Gamma with the following parameters:

$$\pi(\theta|\nu,\mathbf{x}) \quad \sim \quad \text{Gamma}\left(n\nu+a,\sum_{i=1}^{n}x_i+b\right)$$

The conditional distribution for ν can be found by:

$$\pi(\nu|\theta,\mathbf{x}) \quad \propto \quad \prod_{i=1}^{n}x_i^{\nu-1}\frac{1}{\Gamma(\nu)^n}\nu^{\alpha-1}\exp(-\beta\nu)\theta^{n\nu}$$

This is not a kernel of a known distribution.

5.1 (a)

Appendix, R code.

5.2 (b)

Appendix, R code.

6 Question 6

The joint likelihood for the model would be:

$$\begin{aligned}f(y_{ij}|u_i,\beta,\sigma^2,\tau^2) &= \prod_{i=1}^{I}\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi\tau^2}}\exp\left\{-\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}\prod_{i=1}^{I}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{u_i^2}{2\sigma^2}\right\}\\ &\propto \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}}\exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}}\exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}\end{aligned}$$

And the joint posterior distribution for the model would be:

$$\begin{aligned}
\pi(u_i, \beta, \sigma^2, \tau^2 | y_{ij}) &\propto f(y_{ij}|u_i, \beta, \sigma^2, \tau^2)\pi(\beta, \sigma^2, \tau^2) \\
&\propto \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}} \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}} \exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}\frac{1}{\sigma^2\tau^2}
\end{aligned}$$

6.1 (a)

- $\pi(u_i|\mathbf{y}, \beta, \sigma^2, \tau^2)$

$$\begin{aligned}
\pi(u_i|\mathbf{y}, \beta, \sigma^2, \tau^2) &\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}\exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left[y_{ij}^2-2y_{ij}(\beta+u_i)+(\beta+u_i)^2\right]}{2\tau^2}-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left[-2y_{ij}\beta-2y_{ij}u_i+\beta^2+3\beta u_i+u_i^2\right]}{2\tau^2}-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left[-2y_{ij}u_i+2\beta u_i+u_i^2\right]}{2\tau^2}-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{2}{\tau^2}\sum_{i=1}^{I}u_i\left(\sum_{j=1}^{J}y_{ij}-\sum_{j=1}^{J}\beta\right)\right]+\sum_{i=1}^{I}u_i^2\left[\frac{J}{2}+\frac{1}{\sigma^2}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{2}{\tau^2}\sum_{i=1}^{I}u_i\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)\right]+\sum_{i=1}^{I}u_i^2\left[\frac{J\sigma^2+\tau^2}{\tau^2\sigma^2}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}u_i^2\left[\frac{J\sigma^2+\tau^2}{\tau^2\sigma^2}\right]-2\sum_{i=1}^{I}u_i\frac{\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)}{\tau^2}\right]\right\}
\end{aligned}$$

Evaluating only for independents u_i(th).

$$
\propto \quad \exp\left\{ -\frac{1}{2} \left[u_i^2 \left[\frac{J\sigma^2 + \tau^2}{\tau^2\sigma^2} \right] - 2u_i \frac{\left(\sum_{j=1}^{J} (y_{ij} - \beta) \right)}{\tau^2} \right] \right\}
$$

$$
\propto \quad \exp\left\{ -\frac{1}{2} \left[u_i^2 - 2u_i \frac{\left(\sum_{j=1}^{J} (y_{ij} - \beta) \right)}{\tau^2} \cdot \left[\frac{\tau^2\sigma^2}{J\sigma^2 + \tau^2} \right] \right] \frac{1}{\left[\frac{J\sigma^2 + \tau^2}{\tau^2\sigma^2} \right]^{-1}} \right\}
$$

Then, this is a kernel of a Normal distribution with the following moments:

$$
\pi(u_i|\mathbf{y}, \beta, \sigma^2, \tau^2) \quad \sim \quad \mathrm{N}\left(\frac{\sigma^2}{J\sigma^2 + \tau^2} \sum_{j=1}^{J} (y_{ij} - \beta), \frac{\tau^2\sigma^2}{J\sigma^2 + \tau^2} \right)
$$

- $\pi(\beta|\mathbf{y}, \mathbf{u}, \beta, \sigma^2, \tau^2)$

$$
\begin{aligned}
\pi(\beta|\mathbf{y},\mathbf{u},\sigma^2,\tau^2) &\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left[y_{ij}-(\beta+u_i)\right]^2}{2\tau^2}\right\} \\
&\propto \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left[y_{ij}^2-2y_{ij}(\beta+u_i)+(\beta+u_i)^2\right]}{2\tau^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}(\beta+u_i)+(\beta+u_i)^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-2y_{ij}\beta-2y_{ij}u_i+\beta^2+2\beta u_i+u_i^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-2y_{ij}\beta+\beta^2+2\beta u_i\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[\beta^2-2\beta(y_{ij}-u_i)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\frac{IJ}{\tau^2}\beta^2-2\beta\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-u_i)}{\tau^2}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\beta^2-2\beta\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-u_i)}{IJ}\right]\frac{1}{\left(\frac{IJ}{\tau^2}\right)^{-1}}\right\}
\end{aligned}
$$

Then, this is a kernel of a Normal distribution with the following moments:

$$
\pi(\beta|\mathbf{y},\mathbf{u},\sigma^2,\tau^2) \sim \mathrm{N}\left(\frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-u_i),\frac{\tau^2}{IJ}\right)
$$

- $\pi(\sigma^2|\mathbf{y},\mathbf{u},\beta,\tau^2)$

$$\pi(\sigma^2|\mathbf{y},\mathbf{u},\beta,\tau^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1} \exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\frac{I}{2}-1} \exp\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{I}\frac{u_i^2}{2}\right\}$$

Then, this is a kernel of a Inverse Gamma distribution with the following parameters:

$$\pi(\sigma^2|\mathbf{y},\mathbf{u},\beta,\tau^2) \sim \text{IG}\left(\frac{I}{2},\sum_{i=1}^{I}\frac{u_i^2}{2}\right)$$

- $\pi(\tau^2|\mathbf{y},\mathbf{u},\beta,\sigma^2)$

$$\pi(\tau^2|\mathbf{y},\mathbf{u},\beta,\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{IJ}{2}+1} \exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}$$

$$\propto (\tau^2)^{-\frac{IJ}{2}-1} \exp\left\{-\frac{1}{\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2}\right\}$$

Then, this is a kernel of a Inverse Gamma distribution with the following parameters:

$$\pi(\tau^2|\mathbf{y},\mathbf{u},\beta,\sigma^2) \sim \text{IG}\left(\frac{IJ}{2},\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2}\right)$$

6.2 (b)

$$\pi(\beta,\sigma^2,\tau^2|\mathbf{y}) \propto \int_{u_i}\pi(u_i,\beta,\sigma^2,\tau^2|\mathbf{y})du_i$$

$$\propto \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\int_{u_i}\exp\left\{-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[y_{ij}-(\beta+u_i)]^2}{2\tau^2}\right\}\exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}du_i$$

$$\propto \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\int_{u_i}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}(\beta+u_i)+(\beta+u_i)^2\right]-\frac{1}{2\sigma^2}\sum_{i=1}^{I}u_i^2\right\}du_i$$

$$
\begin{aligned}
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\int_{u_i}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta-2y_{ij}u_i+\beta^2+2\beta u_i+u_i^2\right]\right\}\times\\
\times\quad & \exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}du_i\\
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
\times\quad & \int_{u_i}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-2u_i(y_{ij}-\beta)+u_i^2\right]\right\}\exp\left\{-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}du_i\\
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
\times\quad & \int_{u_i}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-2u_i(y_{ij}-\beta)\right]-\frac{J}{2\tau^2}\sum_{i=1}^{I}u_i^2-\sum_{i=1}^{I}\frac{u_i^2}{2\sigma^2}\right\}du_i\\
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
\times\quad & \int_{u_i}\exp\left\{-\frac{1}{2}\frac{1}{\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[-2u_i(y_{ij}-\beta)\right]-\frac{1}{2}\sum_{i=1}^{I}u_i^2\left[\frac{J}{\tau^2}+\frac{1}{\sigma^2}\right]\right\}du_i\\
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
\times\quad & \int_{u_i}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}u_i^2\left[\frac{J\sigma^2+\tau^2}{\tau^2\sigma^2}\right]-2\sum_{i=1}^{I}u_i\frac{\sum_{j=1}^{J}\left[(y_{ij}-\beta)\right]}{\tau^2}\right]\right\}du_i\\
\propto\quad & \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
\times\quad & \int_{u_i}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}u_i^2-2\sum_{i=1}^{I}u_i\frac{\sum_{j=1}^{J}\left[(y_{ij}-\beta)\right]}{\tau^2}\left[\frac{\tau^2\sigma^2}{J\sigma^2+\tau^2}\right]\right]\frac{1}{\left(\frac{J\sigma^2+\tau^2}{\tau^2\sigma^2}\right)^{-1}}\right\}du_i
\end{aligned}
$$

$$
\begin{aligned}
&\propto \left(\frac{1}{\tau^2}\right)^{\frac{IJ}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{I}{2}+1}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
&\times \quad \exp\left\{\sum_{i=1}^{I}\frac{\left(\sum_{j=1}^{J}(y_{ij}-\beta)\sigma^2\right)^2}{2(J\sigma^2+\tau^2)^2}\left[\frac{J\sigma^2+\tau^2}{\tau^2\sigma^2}\right]\right\}\times\left(\frac{\tau^2\sigma^2}{J\sigma^2+\tau^2}\right)^{\frac{I}{2}}\\
&\propto \left(\tau^2\right)^{-\frac{IJ}{2}-1+\frac{I}{2}}\left(\sigma^2\right)^{-\frac{IJ}{2}-1+\frac{I}{2}}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta)^2\right\}\times\\
&\times \quad \exp\left\{\sum_{i=1}^{I}\frac{\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)^2\sigma^2}{2(J\sigma^2+\tau^2)}\left[\frac{1}{\tau^2}\right]\right\}\\
&\propto \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta)^2\right\}\times\\
&\times \quad \exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)^2\right\}
\end{aligned}
$$

6.3 (c)

$$
\begin{aligned}
\pi(\sigma^2,\tau^2|\mathbf{y}) &\propto \int_\beta \pi(\beta,\sigma^2,\tau^2|\mathbf{y})d\beta\\
&\propto \int_\beta \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta)^2\right\}\times\\
&\quad\times \quad \exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)^2\right\}d\beta\\
&\propto \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\int_\beta \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left[y_{ij}^2-2y_{ij}\beta+\beta^2\right]\right\}\times\\
&\quad\times \quad \exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left(\sum_{j=1}^{J}(y_{ij}-\beta)\right)^2\right\}d\beta
\end{aligned}
$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\int_\beta \exp\left\{-\frac{1}{2\tau^2}\left(\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2-2\beta\sum_{i=1}^I\sum_{j=1}^J y_{ij}+IJ\beta^2\right)\right\}\times$$

$$\times \quad \exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2-2J\beta\sum_{j=1}^J(y_{ij})+(J\beta)^2\right]\right\}d\beta$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\tau^2}\left(-2\beta\sum_{i=1}^I\sum_{j=1}^J y_{ij}+IJ\beta^2\right)+\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[-2J\beta\sum_{j=1}^J(y_{ij})+(J\beta)^2\right]\right\}d\beta$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\tau^2\dfrac{(J\sigma^2+\tau^2)}{\sigma^2}}\left(\left(\frac{(J\sigma^2+\tau^2)}{\sigma^2}\right)\left(IJ\beta^2-2\beta\sum_{i=1}^I\sum_{j=1}^J y_{ij}\right)+2J\beta\sum_{i=1}^I\left[\sum_{j=1}^J(y_{ij})\right]-I(J\beta)^2\right)\right\}d\beta$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\tau^2\dfrac{(J\sigma^2+\tau^2)}{\sigma^2}}\left(\left(\frac{(J\sigma^2+\tau^2)IJ}{\sigma^2}-IJ^2\right)\beta^2-2\beta\left[\frac{(J\sigma^2+\tau^2)}{\sigma^2}\left(\sum_{i=1}^I\sum_{j=1}^J y_{ij}\right)-J\sum_{i=1}^I\sum_{j=1}^J y_{ij}\right]\right)\right\}d\beta$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\tau^2\dfrac{(J\sigma^2+\tau^2)}{\sigma^2}}\left(\frac{IJ\tau^2}{\sigma^2}\beta^2-2\beta\left[\frac{\tau^2}{\sigma^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}\right]\right)\right\}d\beta$$

$$\propto \quad \left(\tau^2\right)^{-\dfrac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\dfrac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^I\left[\left(\sum_{j=1}^J(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\tau^2\dfrac{\sigma^2}{IJ\tau^2}\dfrac{(J\sigma^2+\tau^2)}{\sigma^2}}\left(\beta^2-2\beta\left[\frac{\sigma^2}{IJ\tau^2}\frac{\tau^2}{\sigma^2}\sum_{i=1}^I\sum_{j=1}^J y_{ij}\right]\right)\right\}d\beta$$

$$\propto \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left[\left(\sum_{j=1}^{J}(y_{ij})\right)^2\right]\right\}\times$$

$$\times\int_\beta \exp\left\{-\frac{1}{2\frac{(J\sigma^2+\tau^2)}{IJ}}\left(\beta^2-2\beta\left[\frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}\right]\right)\right\}d\beta$$

$$\propto \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left[\left(\sum_{j=1}^{J}(y_{ij})\right)^2\right]\right\}\times$$

$$\times\left(\frac{(J\sigma^2+\tau^2)}{IJ}\right)^{\frac{1}{2}}\exp\left\{\frac{\left(\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}\right)^2}{2IJ}\frac{IJ}{(J\sigma^2+\tau^2)}\right\}$$

$$\propto \left(\tau^2\right)^{-\frac{I(J-1)}{2}-1}\left(\sigma^2\right)^{-1}\left(J\sigma^2+\tau^2\right)^{-\frac{I}{2}+\frac{1}{2}}\exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}^2\right\}\exp\left\{\frac{\sigma^2}{2\tau^2(J\sigma^2+\tau^2)}\sum_{i=1}^{I}\left[\left(\sum_{j=1}^{J}(y_{ij})\right)^2\right]\right\}\times$$

$$\times\quad\exp\left\{\frac{1}{2IJ(J\sigma^2+\tau^2)}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij}\right)^2\right\}$$

7 Question 7

Given that, $y_i \sim$ Poisson (θ), for $i=\{1,\cdots,m\}$ and $y_i \sim$ Poisson (ϕ), for $i=\{m+1,\cdots,n\}$ and using a model with independent prior using as distributions $\theta \sim$ Gamma (α,β), $\phi \sim$ Gamma (γ,δ) and $m \sim$ uniformly over $\{1,\cdots,n\}$.

Then, the joint posterior distribution for this model would be:

$$\begin{aligned}
p(\theta,\phi,m|\mathbf{y}) &\propto p(\mathbf{y}|\theta,m)p(\mathbf{y}|\phi,m)\pi(\theta)\pi(\phi)\pi(m)\\
p(\theta,\phi,m|\mathbf{y}) &\propto \prod_{i=1}^{m}\frac{\theta^{y_i}exp\{-\theta\}}{y_i!}\prod_{i=m+1}^{n}\frac{\phi^{y_i}\text{exp}\{-\phi\}}{y_i!}\theta^{\alpha-1}\text{exp}\{-\beta\theta\}\phi^{\gamma-1}\text{exp}\{-\delta\phi\}\\
&\propto \theta^{\sum_{i=1}^{m}y_i}exp\{-m\theta\}\phi^{\sum_{i=m+1}^{n}y_i}\text{exp}\{-(n-m)\phi\}\theta^{\alpha-1}\text{exp}\{-\beta\theta\}\phi^{\gamma-1}\text{exp}\{-\delta\phi\}\\
&\propto \theta^{\alpha-1+\sum_{i=1}^{m}y_i}exp\{-(m+\beta)\theta\}\phi^{\gamma-1+\sum_{i=m+1}^{n}y_i}\text{exp}\{-(n-m+\delta)\phi\}
\end{aligned}$$

Thus, the conditional for each parameters are:

$$p(\theta|m,\mathbf{y}) \quad \propto \quad \theta^{\alpha-1+\sum_{i=1}^{m} y_i} exp\{-(m+\beta)\theta\}\}$$

Which is a kernel of a Gamma distribution, with parameters, $p(\theta|m,\mathbf{y}) \sim$Gamma $\left(\alpha+\sum_{i=1}^{m} y_i, m+\beta\right)$

$$p(\phi|m,\mathbf{y}) \quad \propto \quad \phi^{\gamma-1+\sum_{i=m+1}^{n} y_i} \exp\{-(n-m+\delta)\phi\}$$

Which is a kernel of a Gamma distribution, with parameters, $p(\phi|m,\mathbf{y}) \sim$Gamma $\left(\gamma+\sum_{i=m+1}^{n} y_i, n-m+\delta\right)$

and for m:

$$p(m|\theta,\phi,\mathbf{y}) \quad \propto \quad \theta^{\alpha-1+\sum_{i=1}^{m} y_i} exp\{-(m+\beta)\theta\}\phi^{\gamma-1+\sum_{i=m+1}^{n} y_i} \exp\{-(n-m+\delta)\phi\}$$

We can noticed that the $p(\phi|m,\mathbf{y})$ and $p(m|\theta,\phi,\mathbf{y})$ are conditional independent a posteriori. We can try to find the marginal distribution of m to see which form it has, then:

$$\begin{aligned}
p(m|\mathbf{y}) &\propto \int_\phi \int_\theta p(m|\theta,\phi,\mathbf{y}) d\theta d\phi \\
&\propto \int_\phi \int_\theta \theta^{\alpha-1+\sum_{i=1}^{m} y_i} exp\{-(m+\beta)\theta\}\phi^{\gamma-1+\sum_{i=m+1}^{n} y_i} \exp\{-(n-m+\delta)\phi\} d\theta d\phi \\
&\propto \int_\phi \theta^{\alpha-1+\sum_{i=1}^{m} y_i} exp\{-(m+\beta)\theta\} \frac{\Gamma\left(\sum_{i=1}^{m} y_i+\alpha\right)}{\Gamma\left(\sum_{i=1}^{m} y_i+\alpha\right)} \frac{(\beta+m)^{\left(\sum_{i=1}^{m} y_i+\alpha\right)}}{(\beta+m)^{\left(\sum_{i=1}^{m} y_i+\alpha\right)}} \\
&\quad \times \int_\theta \phi^{\gamma-1+\sum_{i=m+1}^{n} y_i} \exp\{-(n-m+\delta)\phi\} d\theta d\phi \\
&\propto \frac{\Gamma\left(\sum_{i=1}^{m} y_i+\alpha\right)}{(\beta+m)^{\left(\sum_{i=1}^{m} y_i+\alpha\right)}} \frac{\Gamma\left(\sum_{i=m+1}^{n} y_i+\gamma\right)}{\Gamma\left(\sum_{i=m+1}^{n} y_i+\gamma\right)} \frac{(n-m+\delta)^{\left(\sum_{i=m+1}^{n} y_i+\gamma\right)}}{(n-m+\delta)^{\left(\sum_{i=m+1}^{n} y_i+\gamma\right)}} \\
&\quad \times \int_\theta \theta^{\alpha-1+\sum_{i=1}^{m} y_i} exp\{-(m+\beta)\theta\} d\phi
\end{aligned}$$

$$\propto \quad \frac{\Gamma\left(\sum_{i=1}^{m} y_i + \alpha\right)}{(\beta+m)^{\left(\sum_{i=1}^{m} y_i + \alpha\right)}} \frac{\Gamma\left(\sum_{i=m+1}^{n} y_i + \gamma\right)}{(n-m+\delta)^{\left(\sum_{i=m+1}^{n} y_i + \gamma\right)}}$$

It will be used the marginal posterior distribution for m given that it does not depend on the parameters. Appendix, R code.

8 Question 8

The likelihood and prior distributions for all the parameters are given by:

$$\pi(y_{ij}|\alpha_i, \beta_i, \sigma^2) \quad \propto \quad \prod_{i=1}^{I}\prod_{j=1}^{n_i} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_{ij} - (\alpha_i + \beta_i t_{ij}))^2}{2\sigma^2}\right\}$$

$$\pi((\alpha_i, \beta_i)'|y_{ij}, \alpha, \beta) \quad \propto \quad \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi(\tau_\alpha^{-1})(\tau_\beta^{-1})}} \exp\left\{-\frac{(\alpha_i - \alpha)^2}{2(\tau_\alpha^{-1})^2} - \frac{(\beta_i - \beta)^2}{2(\tau_\beta^{-1})^2}\right\}$$

$$\pi(\alpha, \beta) \quad \propto \quad \frac{1}{\sqrt{2\pi(P_\alpha^{-1})(P_\beta^{-1})}} \exp\left\{-\frac{\alpha^2}{2(P_\alpha^{-1})} - \frac{\beta^2}{2(P_\beta^{-1})}\right\}$$

$$\pi((\sigma^2)^{-1}) \quad \propto \quad \left[(\sigma^2)^{-1}\right]^{a-1} \exp\{-b(\sigma^2)^{-1}\}$$

$$\pi(\tau_\alpha) \quad \propto \quad [\tau_\alpha]^{a-1} \exp\{-b\tau_\alpha\}$$

$$\pi(\tau_\beta) \quad \propto \quad [\tau_\beta]^{a-1} \exp\{-b\tau_\beta\}$$

Now, we can evaluate the joint posterior distribution of the parameters:

$$\begin{aligned}
\pi((\alpha_i, \beta_i)', \alpha, \beta, \sigma^2, \tau_\alpha, \tau_\beta|y_{ij}) \quad \propto \quad & \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij} - (\alpha_i + \beta_i t_{ij}))^2\right\} \times \\
& \times \left(\frac{1}{(\tau_\alpha^{-1})(\tau_\beta^{-1})}\right)^{\frac{I}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\alpha(\alpha_i - \alpha)^2 + \tau_\beta(\beta_i - \beta)^2\right]\right\} \times \\
& \times \quad \exp\left\{-\frac{1}{2}\left[P_\alpha\alpha^2 + P_\beta\beta^2\right]\right\} \times \\
& \times \left[(\sigma^2)^{-1}\tau_\alpha\tau_\beta\right]^{a-1} \exp\{-b((\sigma^2)^{-1} + \tau_\alpha + \tau_\beta)\}
\end{aligned}$$

Then, the full conditional distributions of the parameters are given by:

- For α

$$\begin{aligned}
\pi(\alpha|(\alpha_i,\beta_i)',\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) &\propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\alpha(\alpha_i-\alpha)^2\right]\right\}\exp\left\{-\frac{1}{2}\left[P_\alpha\alpha^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\tau_\alpha\sum_{i=1}^{I}(\alpha_i^2-2\alpha_i\alpha-\alpha^2)+P_\alpha\alpha^2\right]\right\}
\end{aligned}$$

$$\begin{aligned}
&\propto \exp\left\{-\frac{1}{2}\left[-2\tau_\alpha\alpha\sum_{i=1}^{I}\alpha_i+\tau_\alpha I\alpha^2+P_\alpha\alpha^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-2\alpha\tau_\alpha\sum_{i=1}^{I}\alpha_i+(\tau_\alpha I+P_\alpha)\alpha^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\alpha^2-2\alpha\frac{\tau_\alpha\sum_{i=1}^{I}\alpha_i}{(\tau_\alpha I+P_\alpha)}\right]\frac{1}{(\tau_\alpha I+P_\alpha)^{-1}}\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\alpha|(\alpha_i,\beta_i)',\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \sim \text{Normal}\left(\frac{\tau_\alpha\sum_{i=1}^{I}\alpha_i}{(\tau_\alpha I+P_\alpha)},\frac{1}{(\tau_\alpha I+P_\alpha)}\right)$$

- For β

$$\begin{aligned}
\pi(\beta|(\alpha_i,\beta_i)',\alpha,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) &\propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\beta(\beta_i-\beta)^2\right]\right\}\exp\left\{-\frac{1}{2}\left[P_\beta\beta^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\tau_\beta\sum_{i=1}^{I}(\beta_i^2-2\beta_i\beta-\beta^2)+P_\beta\beta^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-2\tau_\beta\beta\sum_{i=1}^{I}{}_i+\tau_\beta I\beta^2+P_\beta\beta^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-2\beta\tau_\beta\sum_{i=1}^{I}\beta_i+(\tau_\beta I+P_\beta)\beta^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\beta^2-2\beta\frac{\tau_\beta\sum_{i=1}^{I}\beta_i}{(\tau_\beta I+P_\beta)}\right]\frac{1}{(\tau_\beta I+P_\beta)^{-1}}\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\beta|(\alpha_i,\beta_i)',\alpha,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \sim \text{Normal}\left(\frac{\tau_\beta\sum_{i=1}^{I}\beta i}{(\tau_\beta I+P_\beta)},\frac{1}{(\tau_\beta I+P_\beta)}\right)$$

- For τ_α

$$\begin{aligned}
\pi(\tau_\alpha|(\alpha_i,\beta_i)',\alpha,\beta,\sigma^2,\tau_\beta,y_{ij}) &\propto (\tau_\alpha^2)^{\dfrac{I}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left(\tau_\alpha(\alpha_i-\alpha)^2\right)\right\}[\tau_\alpha]^{a-1}\exp\{-b\tau_\alpha\}\\
&\propto (\tau_\alpha^2)^{\dfrac{I}{2}+a-1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\tau_\alpha(\alpha_i-\alpha)^2-b\tau_\alpha\right\}\\
&\propto (\tau_\alpha^2)^{\dfrac{I}{2}+a-1}\exp\left\{-\tau_\alpha\left[\frac{1}{2}\sum_{i=1}^{I}(\alpha_i-\alpha)^2+b\right]\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\tau_\alpha|(\alpha_i,\beta_i)',\alpha,\beta,\sigma^2,\tau_\beta,y_{ij}) \sim \text{Gamma}\left(\frac{I}{2}+a,\frac{1}{2}\sum_{i=1}^{I}(\alpha_i-\alpha)^2+b\right)$$

- For τ_β

$$\begin{aligned}
\pi(\tau_\beta|(\alpha_i,\beta_i)',\alpha,\beta,\sigma^2,\tau_\alpha,y_{ij}) &\propto \left(\tau_\beta^2\right)^{\dfrac{I}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left(\tau_\alpha(\beta_i-\beta)^2\right)\right\}[\tau_\beta]^{a-1}\exp\{-b\tau_\beta\}\\
&\propto \left(\tau_\beta^2\right)^{\dfrac{I}{2}+a-1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\tau_\beta(\beta_i-\beta)^2-b\tau_\beta\right\}\\
&\propto (\tau_\alpha^2)^{\dfrac{I}{2}+a-1}\exp\left\{-\tau_\beta\left[\frac{1}{2}\sum_{i=1}^{I}(\beta_i-\beta)^2+b\right]\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\tau_\beta|(\alpha_i,\beta_i)',\alpha,\beta,\sigma^2,\tau_\alpha,y_{ij}) \sim \text{Gamma}\left(\frac{I}{2}+a,\frac{1}{2}\sum_{i=1}^{I}(\beta_i-\beta)^2+b\right)$$

- For $(\sigma^2)^{-1}$

$$\begin{aligned}
\pi((\sigma^2)^{-1}|(\alpha_i,\beta_i)',\alpha,\beta,\tau_\alpha,\tau_\beta,y_{ij}) &\propto \left((\sigma^2)^{-1}\right)^{\dfrac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2\right\}\times\\
&\times\left[(\sigma^2)^{-1}\right]^{a-1}\exp\{-b((\sigma^2)^{-1})\}
\end{aligned}$$

$$\propto \quad \left((\sigma^2)^{-1}\right)^{\frac{n}{2}+a-1} \exp\left\{-(\sigma^2)^{-1}\left[\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2+b\right]\right\}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi((\sigma^2)^{-1}|(\alpha_i,\beta_i)',\alpha,\beta,\tau_\alpha,\tau_\beta,y_{ij}) \sim \text{Gamma}\left(\frac{n}{2}+a,\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2+b\right)$$

- For α_i

$$\begin{aligned}
\pi(\alpha_i|\beta_i,\alpha,\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \quad &\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\alpha(\alpha_i-\alpha)^2\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}^2-2y_{ij}(\alpha_i+\beta_i t_{ij})+(\alpha_i+\beta_i t_{ij})^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\tau_\alpha\sum_{i=1}^{I}\left[\alpha_i^2-2\alpha_i\alpha+\alpha^2\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(-2y_{ij}\alpha_i+\alpha_i^2+2\alpha_i\beta_i t_{ij})\right\}\exp\left\{-\frac{1}{2}\tau_\alpha\sum_{i=1}^{I}\left[\alpha_i^2-2\alpha_i\alpha+\alpha^2\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(-2y_{ij}\alpha_i+\alpha_i^2+2\alpha_i\beta_i t_{ij})\right\}\exp\left\{-\frac{1}{2}\tau_\alpha\sum_{i=1}^{I}\left[\alpha_i^2-2\alpha_i\alpha\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\left[-2\sum_{i=1}^{I}\alpha_i\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}(y_{ij}-\beta_i t_{ij})\right]+\frac{1}{\sigma^2}\sum_{i=1}^{I}n_i\alpha_i^2+\tau_\alpha\sum_{i=1}^{I}\alpha_i^2-2\tau_\alpha\alpha\sum_{i=1}^{I}\alpha_i\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}\left(\frac{n_i}{\sigma^2}+\tau^2\right)\alpha_i^2-2\sum_{i=1}^{I}\alpha_i\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}(y_{ij}-\beta_i t_{ij})+\tau_\alpha\alpha\right]\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}\alpha_i^2-2\sum_{i=1}^{I}\alpha_i\frac{\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}(y_{ij}-\beta_i t_{ij})+\tau_\alpha\alpha\right]}{\left(\frac{n_i}{\sigma^2}+\tau_\alpha^2\right)}\right]\frac{1}{\left(\frac{n_i}{\sigma^2}+\tau_\alpha^2\right)^{-1}}\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\alpha_i|\beta_i,\alpha,\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \sim \mathrm{N}\left(\frac{\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}(y_{ij}-\beta_i t_{ij})+\tau_\alpha\alpha\right]}{\left(\frac{n_i}{\sigma^2}+\tau_\alpha^2\right)}, \left(\frac{n_i}{\sigma^2}+\tau_\alpha^2\right)^{-1}\right)$$

- For β_i

$$\begin{aligned}
\pi(\beta_i|\alpha_i,\alpha,\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) &\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\beta(\beta_i-\beta)^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}^2-2y_{ij}(\alpha_i+\beta_i t_{ij})+(\alpha_i+\beta_i t_{ij})^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\tau_\alpha\sum_{i=1}^{I}\left[\beta_i^2-2\beta_i\beta+\beta^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(-2y_{ij}\beta_i t_{ij}+2\alpha_i\beta_i t_{ij}+(\beta_i t_{ij})^2)\right\}\exp\left\{-\frac{1}{2}\tau_\alpha\sum_{i=1}^{I}\left[\beta_i^2-2\beta_i\beta\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[-2\sum_{i=1}^{I}\beta_i\left(\frac{\sum_{j=1}^{n_i}y_{ij}t_{ij}}{\sigma^2}-\frac{\sum_{j=1}^{n_i}\alpha_i t_{ij}}{\sigma^2}+\tau_\beta\beta\right)+\sum_{i=1}^{I}\beta_i^2\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}^2+\tau_\beta\right]\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{I}\beta_i^2-2\sum_{i=1}^{I}\beta_i\frac{\left(\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}(y_{ij}-\alpha_i)+\tau_\beta\beta\right)}{\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}^2+\tau_\beta\right]}\right]\frac{1}{\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}^2+\tau_\beta\right]^{-1}}\right\}
\end{aligned}$$

Which is a kernel of a Normal distribution with the following moments:

$$\pi(\alpha_i|\beta_i,\alpha,\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \sim \mathrm{N}\left(\frac{\left(\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}(y_{ij}-\alpha_i)+\tau_\beta\beta\right)}{\left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}^2+\tau_\beta\right]}, \left[\frac{1}{\sigma^2}\sum_{j=1}^{n_i}t_{ij}^2+\tau_\beta\right]^{-1}\right)$$

- For (α_i, β_i)

$$
\begin{aligned}
\pi((\alpha_i,\beta_i)|\alpha,\beta,\sigma^2,\tau_\alpha,\tau_\beta,y_{ij}) \quad &\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-(\alpha_i+\beta_i t_{ij}))^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\sum_{i=1}^{I}\left[\tau_\alpha(\alpha_i-\alpha)^2+\tau_\beta(\beta_i-\beta)^2\right]\right\}\times \\
&\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}^2-2y_{ij}(\alpha_i+\beta_i t_{ij})+(\alpha_i+\beta_i t_{ij})^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\left[\tau_\alpha\sum_{i=1}^{I}\left[\alpha_i^2-2\alpha_i\alpha+\alpha^2\right]+\tau_\beta\left[\beta_i^2-2\beta_i\alpha+\beta^2\right]\right]\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(-2y_{ij}\alpha_i+-2y_{ij}\beta_i t_{ij}+\alpha_i^2+2\beta_i^2-2\beta_i\beta+\beta^2\right\}\times \\
&\times \quad \exp\left\{-\frac{1}{2}\left[\tau_\alpha\sum_{i=1}^{I}\left[\alpha_i^2-2\alpha_i\alpha\right]+\tau_\beta\left[\beta_i^2-2\beta_i\alpha\right]\right]\right\}
\end{aligned}
$$

9 Appendix

All the codes and results are in the next page.