AMS 261: Probability Theory (Fall 2017)

Homework 4 solutions

1. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is (pointwise) increasing, that is, for all n and for each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega)$. Moreover, assume that $\mathrm{E}(X_1) > -\infty$. Denote by X the pointwise limit of $\{X_n : n = 1, 2, ...\}$, that is, for each $\omega \in \Omega$, $X(\omega) = \lim_{n\to\infty} X_n(\omega)$. Prove that $\mathrm{E}(X) = \lim_{n\to\infty} \mathrm{E}(X_n)$.
Solution: Since $\{X_n : n = 1, 2, ...\}$ is a (pointwise) increasing sequence of random variables, it is easy to show that the sequence of the corresponding positive parts, $\{X_n^+ : n = 1, 2, ...\}$, is increasing with limit given by X^+. Therefore, applying the MCT to the $\overline{\mathbb{R}}^+$-valued random variables X_n^+, we obtain

$$\lim_{n\to\infty} \mathrm{E}(X_n^+) = \mathrm{E}(X^+). \tag{1.1}$$

Similarly, note that $\{-X_n^- : n = 1, 2, ...\}$ is an increasing sequence of $\overline{\mathbb{R}}^-$-valued random variables. Since $X_1 \leq X_2$ and $\mathrm{E}(X_1) > -\infty$, we have that $\mathrm{E}(X_2)$ exists and $-\infty < \mathrm{E}(X_1) \leq \mathrm{E}(X_2)$ (Fristedt & Gray, 1997, Chapter 4, Theorem 9(iv)). Applying the same argument, we get that $\mathrm{E}(X_n) > -\infty$, for each n, as well as that $\mathrm{E}(X) > -\infty$, which implies that $\mathrm{E}(X_n^-) < \infty$, for all n, as well as $\mathrm{E}(X^-) < \infty$. Next, since $\mathrm{E}(X_1^-) < \infty$, we conclude that X_1^- is almost surely finite, that is, $-X_1^- > -\infty$, almost surely, and thus $c = \inf\{-X_1^-(\omega) : \omega \in \Omega\} > -\infty$. Now, $\{-X_n^- - c : n = 1, 2, ...\}$ is an increasing sequence of $\overline{\mathbb{R}}^+$-valued random variables, and the MCT yields

$$\lim_{n\to\infty} \mathrm{E}(X_n^-) = \mathrm{E}(X^-). \tag{1.2}$$

The result can now be obtained by combining (1.1) and (1.2), noting that $\lim_{n\to\infty}(\mathrm{E}(X_n^+) - \mathrm{E}(X_n^-))$ is well defined because $\mathrm{E}(X_n^-) < \infty$, for all n.

2. Let $\{X_n : n = 1, 2, ...\}$ be a countable sequence of $\overline{\mathbb{R}}^+$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, and assume that $\mathrm{E}(\sum_{n=1}^{\infty} X_n) < \infty$. Show that $\mathrm{E}\left(\sum\limits_{n=1}^{\infty} X_n\right) = \sum\limits_{n=1}^{\infty} \mathrm{E}(X_n)$.
Solution: For $n = 1, 2, ...$, define $Y_n = \sum_{j=1}^{n} X_j$. Then the sequence of $\overline{\mathbb{R}}^+$-valued random variables $\{Y_n : n = 1, 2, ...\}$, defined on $(\Omega, \mathcal{F}, P)$, is increasing, since each of the X_j is $\overline{\mathbb{R}}^+$-valued. Denote by Y the pointwise limit of the Y_n, i.e., for each $\omega \in \Omega$, $Y(\omega) = \lim_{n\to\infty} \sum_{j=1}^{n} X_j(\omega) = \sum_{n=1}^{\infty} X_n(\omega)$. Then, using the MCT and additivity of expectation,

$$\mathrm{E}\left(\sum_{n=1}^{\infty} X_n\right) = \mathrm{E}(Y) = \lim_{n\to\infty} \mathrm{E}(Y_n) = \lim_{n\to\infty} \sum_{j=1}^{n} \mathrm{E}(X_j) = \sum_{n=1}^{\infty} \mathrm{E}(X_n).$$

(Note that the assumption $\mathrm{E}(\sum_{n=1}^{\infty} X_n) < \infty$ implies that $\sum_{n=1}^{\infty} X_n$ is an almost surely finite random variable, but is not strictly needed.)

3. Let $\{X_n : n = 1, 2, ...\}$, $\{Y_n : n = 1, 2, ...\}$, and $\{Z_n : n = 1, 2, ...\}$ be sequences of $\mathbb{R}$-valued random variables (all the random variables are defined on the same probability space). Assume that: (a) $\mathrm{E}(X_n)$ and $\mathrm{E}(Z_n)$ exist for all n and are finite; (b) each of the three sequences converges almost surely (denote by X, Y, and Z the respective almost sure limits); (c) $\mathrm{E}(X)$, $\mathrm{E}(Y)$, and $\mathrm{E}(Z)$ exist and are finite; (d) $X_n \leq Y_n \leq Z_n$ almost surely; (e) $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(Z_n) = \mathrm{E}(Z)$. Show that $\lim_{n\to\infty} \mathrm{E}(Y_n) = \mathrm{E}(Y)$.
Solution: Consider the sequence of random variables $\{Z_n - Y_n : n = 1, 2, ...\}$. Based on assumption (d), $Z_n - Y_n \geq 0$, almost surely, and, therefore, using the Fatou lemma,

$$\mathrm{E}(\liminf_{n\to\infty}(Z_n - Y_n)) \leq \liminf_{n\to\infty} \mathrm{E}(Z_n - Y_n). \tag{3.1}$$

Using assumption (b), we obtain that the almost sure limit of the sequence $\{Z_n - Y_n : n = 1, 2, ...\}$ is given by $Z - Y$, and so $\liminf_{n\to\infty}(Z_n - Y_n) = \lim_{n\to\infty}(Z_n - Y_n) = Z - Y$, almost surely. Therefore, using properties of the lim inf for numerical sequences, (3.1) yields

$$\begin{aligned} \mathrm{E}(Z - Y) &\leq \liminf\nolimits_{n\to\infty} \mathrm{E}(Z_n - Y_n) = \liminf\nolimits_{n\to\infty}\{\mathrm{E}(Z_n) - \mathrm{E}(Y_n)\} \\ &= \liminf\nolimits_{n\to\infty} \mathrm{E}(Z_n) + \liminf\nolimits_{n\to\infty}\{-\mathrm{E}(Y_n)\} = \mathrm{E}(Z) - \limsup\nolimits_{n\to\infty} \mathrm{E}(Y_n), \end{aligned}$$

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since $E(Z) = \lim_{n\to\infty} E(Z_n) = \liminf_{n\to\infty} E(Z_n)$ (assumption (e)). Rearranging terms in the above inequality, we have $E(Y) \geq \limsup_{n\to\infty} E(Y_n)$.
Analogously, consider the sequence $\{Y_n - X_n : n = 1, 2, ...\}$, which is, almost surely, non-negative, and converges, almost surely, to $Y - X$, based on assumptions (d) and (b), respectively. Hence, $E(Y) - E(X) = E(Y - X) = E(\lim_{n\to\infty}(Y_n - X_n)) = E(\liminf_{n\to\infty}(Y_n - X_n))$, and, thus, using, again, the Fatou lemma and properties of the lim inf, $E(Y) - E(X) \leq \liminf_{n\to\infty} E(Y_n - X_n) = \liminf_{n\to\infty}\{E(Y_n) - E(X_n)\} = \liminf_{n\to\infty} E(Y_n) - E(X)$, since $\liminf_{n\to\infty} E(X_n) = \lim_{n\to\infty} E(X_n) = E(X)$ from assumption (e). Hence, $E(Y) \leq \liminf_{n\to\infty} E(Y_n)$, which, combined with $E(Y) \geq \limsup_{n\to\infty} E(Y_n)$, proves the result.

4. Let $\{X_n : n = 1, 2, ...\}$ be a countable sequence of $\mathbb{R}$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that there exist finite real constants $p > 1$ and $K > 0$ such that $\sup_n E(|X_n|^p) \leq K$. Show that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.
Solution: For any $c > 0$, we can write $E(|X_n| 1_{(|X_n| \geq c)}) = E(|X_n|^p |X_n|^{1-p} 1_{(|X_n|\geq c)}) \leq c^{1-p} E(|X_n|^p)$, since $p > 1$. Therefore, $\sup_n E(|X_n| 1_{(|X_n|\geq c)}) \leq c^{1-p} \sup_n E(|X_n|^p) \leq K c^{1-p}$, using the assumption. Hence, finally, $\lim_{c\to\infty} \sup_n E(|X_n| 1_{(|X_n|\geq c)}) \leq \lim_{c\to\infty}(Kc^{1-p}) = 0$, proving the result.

5. Let X be an $\mathbb{R}$-valued random variable, defined on probability space $(\Omega, \mathcal{F}, P)$, with finite expectation $\mu = E(X)$ and finite standard deviation $\sigma = \{\mathrm{Var}(X)\}^{1/2}$. Prove that for any $0 \leq z \leq \sigma$,
$$P(\{\omega \in \Omega : |X(\omega) - \mu| \geq z\}) \geq \frac{\sigma^4\{1 - (z/\sigma)^2\}^2}{E(|X - \mu|^4)}.$$
Solution: Let $Y = |X - \mu|^2$. We have $E(Y) = E(|X - \mu|^2) = \mathrm{Var}(X) < \infty$, by assumption. If $E(Y^2) = E(|X - \mu|^4) = \infty$, the inequality holds true (the right hand side is 0 in this case). The case $E(Y^2) = 0$ is not of interest for the inequality (the right hand side is not well defined in this case); note that if $E(Y^2) = 0$ (and since $E(Y) < \infty$), Y is almost surely equal to a finite constant. Therefore, consider the case $0 < E(Y^2) < \infty$. The result is obtained by applying to random variable Y the inequality that can be viewed as a complement to Chebyshev inequality (Fristedt & Gray, 1997, Corollary 5.5; proved in class). In particular, setting $\lambda = z^2/\sigma^2$, for any $0 \leq z \leq \sigma$, we have (note that $\lambda \in [0, 1]$)
$$P(\{\omega \in \Omega : |X(\omega) - \mu|^2 \geq z^2\sigma^{-2}E(|X - \mu|^2)\}) \geq \left(1 - \frac{z^2}{\sigma^2}\right)^2 \frac{\{E(|X - \mu|^2)\}^2}{E(|X - \mu|^4)},$$
which yields the result noting that $\sigma^2 = E(|X - \mu|^2) < \infty$.

6. Let $\{X_n : n = 1, 2, ...\}$ be a sequence of $\mathbb{R}$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Suppose there exists an $\mathbb{R}^+$-valued random variable Y, defined on $(\Omega, \mathcal{F}, P)$, such that $E(Y) < \infty$ and $|X_n| \leq Y$, almost surely, for all n. Show that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.
Solution: Fix $c > 0$. Because $|X_n| \leq Y$, almost surely, for all n, we have $1_{(|X_n|\geq c)} \leq 1_{(Y\geq c)}$, almost surely, for all n. By combining the above inequalities, $|X_n| 1_{(|X_n|\geq c)} \leq Y 1_{(Y\geq c)}$, almost surely, for all n. Therefore, $E(|X_n| 1_{(|X_n|\geq c)}) \leq E(Y 1_{(Y\geq c)})$, for all n, and so $\sup_n E(|X_n| 1_{(|X_n|\geq c)}) \leq E(Y 1_{(Y\geq c)})$. Next, $\lim_{c\to\infty} \sup_n E(|X_n| 1_{(|X_n|\geq c)}) \leq \lim_{c\to\infty} E(Y 1_{(Y\geq c)}) = 0$, and thus $\lim_{c\to\infty} \sup_n E(|X_n| 1_{(|X_n|\geq c)}) = 0$. (Note that the result $\lim_{c\to\infty} E(Y 1_{(Y\geq c)}) = 0$ was proved in class, using the assumptions that $Y \geq 0$ and $E(Y) < \infty$, and applying the DCT to the sequence $Z_k = Y 1_{(Y\geq k)} \leq Y$.)

7. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$-valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and an increasing function $G : [0, \infty) \to [0, \infty)$, which satisfies $\lim_{t\to\infty}\{t^{-1}G(t)\} = \infty$ and $0 < \sup_n E\{G(|X_n|)\} < \infty$. Prove that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.
Solution: Fix $\varepsilon > 0$ and let $A = \varepsilon^{-1} \sup_n E\{G(|X_n|)\}$ (we have $0 < A < \infty$, by assumption). Because $\lim_{t\to\infty}\{t^{-1}G(t)\} = \infty$, we can find large c (which depends on ε) such that
$$t^{-1}G(t) \geq A, \quad \forall t \geq c. \tag{7.1}$$
For $n = 1, 2, ...,$ let $Y_n = |X_n| 1_{(|X_n|\geq c)}$. For any $\omega \in \Omega$ with $|X_n(\omega)| \geq c$, we have $Y_n(\omega) \geq c$, and using (7.1), $G(Y_n(\omega)) \geq AY_n(\omega)$. Moreover, for any $\omega \in \Omega$ with $|X_n(\omega)| < c$, we have $Y_n(\omega) = 0$, and since $G(0) \geq 0$, the inequality $G(Y_n(\omega)) \geq AY_n(\omega)$ is still valid. Therefore, for any $n = 1, 2, ...,$ $A|X_n| 1_{(|X_n|\geq c)} \leq G(|X_n| 1_{(|X_n|\geq c)}) \leq G(|X_n|)$, using the assumption that G is increasing. Taking expectations, $E(|X_n| 1_{(|X_n|\geq c)}) \leq A^{-1}E\{G(|X_n|)\}$, and therefore, $\sup_n E(|X_n| 1_{(|X_n|\geq c)}) \leq A^{-1} \sup_n E\{G(|X_n|)\} = \varepsilon$, which provides the result, since the inequality above holds true for any $\varepsilon > 0$ and any $c' > c$.

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