Robustness of the nonlinear PI control method to ignored actuator dynamics by Haris E. Psillakis [1]

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***Abstract*— For sector bounded nonlinear systems with unknown control direction, a nonlinear PI or Nussbaum gain controller is not sufficient for stability. This paper shows that combining the nonlinear PI and Nussbaum gain will result in a stable system.**

I. Introduction

This paper explores a control problem where the sign of the control, or control direction, is unknown. The common way to handle such a problem is through the use of Nussbaum functions as control gains. Nussbaum functions are defined as continuous functions $N : \mathbb{R} \to \mathbb{R}$ for which the properties of (1) and (2) hold.

$$\limsup_{\zeta \to \pm\infty} \frac{1}{\zeta} \int_0^{\zeta} N(s)\, ds = +\infty \tag{1}$$

$$\liminf_{\zeta \to \pm\infty} \frac{1}{\zeta} \int_0^{\zeta} N(s)\, ds = -\infty \tag{2}$$

The author has previously shown that combining a nonlinear PI controller with a Nussbaum gain can stabilize a perturbed linear system such as (3), by using the control scheme represented in (4) and (5) as long as $\max\left\{\epsilon\lambda, \epsilon(\alpha+\lambda)\right\} < 1$ and $\kappa(\cdot)$ is a Nussbaum function.

$$\left\{ \begin{array}{c} \dot{x} = \alpha x + bu \\ \varepsilon \dot{y} = x - y \end{array} \right\} \tag{3}$$

$$u = \kappa(z)\, y \tag{4}$$

$$z = \frac{1}{2} y^2 + \lambda \int_0^t y^2(s)\, ds \tag{5}$$

This paper extends the results of this work to explore the robustness of the controller to ignored actuator dynamics as seen in Figure 2 of [1]. Such a system is modeled in (6).

$$\left\{ \begin{array}{c} \dot{y} = f(y) + bu \\ \varepsilon \dot{u} = u_{nom} - u \end{array} \right\} \tag{6}$$

where u_{nom} is modeled for the unperturbed plant seen in (7).

$$\dot{y} = f(y) + b u_{nom} \tag{7}$$

In the following section, it will be shown that combining nonlinear PI with a Nussbaum control gain, results in a controller that is more robust than either technique by itself.

A. Nonlinear PI control: nominal case

Let the sector bounded non-linearity $f(y)$ be defined as follows.

$$f(y) = \alpha(y)\, y \tag{8}$$

$$\alpha_1 \le \alpha(y) \le \alpha_1 \quad \forall y \in R \tag{9}$$

Lemma 1 Let the system be (7) with nonlinearity (8), (9).

$$\dot{y} = f(y) + b u_{nom}$$

$$f(y) = \alpha(y)\, y \quad \alpha_1 \le \alpha(y) \le \alpha_1 \quad \forall y \in R$$

Consider also the nonlinear PI controller of the form

$$u_{nom} = \kappa(z)\, y \tag{10}$$

$$z = \frac{1}{2} y^2 + \lambda \int_0^t y^2(s)\, ds \tag{11}$$

($\lambda > 0$) with PI gain $\kappa(z) \equiv \beta(z)\cos(z)$ and $\beta(\cdot)$ a class $\mathcal{K}_\infty$ function. Then, for the closed loop system we have that z, y, u_{nom} are bounded and $\lim_{x\to\infty} y(t) = \lim_{t\to\infty} u_{nom}(t) = 0$. A function $\beta(\cdot)$ belongs to class $\mathcal{K}_\infty$ if it is continuous, strictly increasing with $\beta(0) = 0$ and $\lim_{x\to+\infty} \beta(z) = +\infty$.

Proof: The proof is given in section 1.1 of [2]. ■

II. Nonlinear PI control: ignored actuator dynamics case

Theorem 1 Let the closed-loop system be given by (6), (10), (11) with sector-bounded nonlinearity given by (8), (9).

$$\left\{ \begin{array}{c} \dot{y} = f(y) + bu \\ \varepsilon \dot{u} = u_{nom} - u \end{array} \right\} \quad u_{nom} = \kappa(z)\, y$$

$$z = \frac{1}{2} y^2 + \lambda \int_0^t y^2(s)\, ds$$

$$f(y) = \alpha(y)\, y \quad \alpha_1 \le \alpha(y) \le \alpha_1 \quad \forall y \in R$$

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If,

1) $\varepsilon(\lambda+\alpha_2)<1$
2) $\kappa(z)=\beta(z)\cos(z)$ with $\beta(\cdot)$ a $\mathcal{K}_\infty$ function having the property

$$\lim_{z\to+\infty}\left[\frac{\beta(z+\varepsilon)}{z}-c\beta(z)\right]=+\infty \tag{12}$$

then all closed loop signal are bounded and

$$\lim_{t\to\infty} y(t)=\lim_{t\to\infty} u(t)=\lim_{t\to\infty} u_{nom}(t)=0.$$

Proof:
We begin with (5) and (7) (repeated below).

$$z=\tfrac{1}{2}y^2+\lambda\int_0^t y^2(s)\,ds \quad \dot{y}=f(y)+bu$$

We now differentiate z, substituting the expression for $\dot{y}$.

$$\dot{z}=y\dot{y}+\lambda y^2=y\big(f(y)+bu\big)+\lambda y^2$$

We then factor common terms.

$$\dot{z}=byu+\big(f(y)+\lambda y\big)y$$

We now use the expression for $f(y)$.

$$f(y)=\alpha(y)\,y$$

Substituting into the previous equation we arrive at $\dot{z}$.

$$\dot{z}=byu+\big(\alpha(y)+\lambda\big)y^2 \tag{13}$$

We now define the function S to be the following.

$$S\equiv\frac{\varepsilon}{2}u^2+\frac{\varepsilon(\alpha_2+\lambda)}{b}uy+\frac{\ell}{2}y^2 \tag{14}$$

We then compute $\dot{S}$ as follows, first using the product rule.

$$\dot{S}=\varepsilon u\dot{u}+\frac{\varepsilon(\alpha_2+\lambda)}{b}\left[\dot{u}y+\dot{y}u\right]+\ell y\dot{y}$$

We now substitute the expressions for $\dot{u},\dot{y}$ from (6).

$$\begin{aligned}&\dot{S}=\varepsilon u\tfrac{u_{nom}-u}{\varepsilon}\\&+\tfrac{\varepsilon(\alpha_2+\lambda)}{b}\left[\tfrac{u_{nom}-u}{\varepsilon}y+\big(f(y)+bu\big)u\right]\\&+\ell y\big(f(y)+bu\big)\end{aligned}$$

We factor in ε to cancel common factors.

$$\begin{aligned}&\dot{S}=u(u_{nom}-u)\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[(u_{nom}-u)y+\varepsilon\big(f(y)+bu\big)u\right]\\&+\ell y\big(f(y)+bu\big)\end{aligned}$$

We substitute the expression for u_{nom} from (10).

$$\begin{aligned}&\dot{S}=u\big(\kappa(z)y-u\big)\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[\big(\kappa(z)y-u\big)y+\varepsilon\big(\alpha(y)y+bu\big)u\right]\\&+\ell y\big(\alpha(y)y+bu\big)\end{aligned}$$

We then expand terms.

$$\begin{aligned}&\dot{S}=\kappa(z)uy-u^2\\&+\tfrac{(\alpha_2+\lambda)}{b}\left[\kappa(z)y^2-uy+\varepsilon\alpha(y)uy+\varepsilon bu^2\right]\\&+\ell y\big(\alpha(y)y+bu\big)\end{aligned}$$

We group common factors of u^2 and uy.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2+\ell\alpha(y)y^2+b\ell uy\end{aligned}$$

We then add and subtract $\ell\dot{z}$ using (13).

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\ell\dot{z}-\ell\left[byu+\big(\alpha(y)+\lambda\big)y^2\right]\\&+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2+\ell\alpha(y)y^2+b\ell uy\end{aligned}$$

We now cancel common terms.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy\\&+\ell\dot{z}-\ell\lambda y^2+\tfrac{1}{b}(\alpha_2+\lambda)\kappa(z)y^2\end{aligned}$$

Rearranging (13), we have an expression for λy^2.

$$\dot{z}=byu+\big(\alpha(y)+\lambda\big)y^2\Rightarrow\lambda y^2=\dot{z}-byu-\alpha(y)y^2$$

We substitute this expression in to arrive at the following.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2+\kappa(z)uy\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy+\ell\dot{z}-\ell\lambda y^2\\&+\tfrac{1}{b}\kappa(z)\alpha_2y^2+\tfrac{1}{b}\kappa(z)\left[\dot{z}-byu-\alpha(y)y^2\right]\end{aligned}$$

Simplifying, we arrive at $\dot{S}$.

$$\begin{aligned}&\dot{S}=-\left[1-(\alpha_2+\lambda)\varepsilon\right]u^2\\&-\tfrac{1}{b}(\alpha_2+\lambda)\big(1-\varepsilon\alpha(y)\big)uy-\lambda\ell y^2\\&+\tfrac{1}{b}\big(\alpha_2-\alpha(y)\big)\kappa(z)y^2+\ell\dot{z}+\tfrac{1}{b}\kappa(z)\dot{z}\end{aligned} \tag{15}$$

The above equation for $\dot{S}$ may be written as a linear equation.

$$\begin{aligned}&\tfrac{d}{dt}\left[S-\tfrac{1}{b}\int_0^{z(t)}\big(\kappa(s)+b\ell\big)ds\right]\\&=-w^T\Lambda(y)w+\tfrac{1}{b}\big(\alpha_2-\alpha(y)\big)\kappa(z)y^2\end{aligned} \tag{16}$$

Here, w is the state vector of (u,y).

$$w=\begin{bmatrix}u & y\end{bmatrix}^T$$

$\Lambda(y)$ is defined to be the following matrix.

$$\Lambda(y)=\begin{bmatrix}1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b}\\ * & \lambda\ell\end{bmatrix} \tag{17}$$

Where * denotes that the matrix is symmetric w.r.t. the main diagonal.
We show the linear equation is correct by expanding the first term.

$$-w^T\Lambda(y)w=-\begin{bmatrix}u & y\end{bmatrix}\cdot\begin{bmatrix}1-\varepsilon(\lambda+\alpha_2) & \frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)\\ * & \lambda\ell\end{bmatrix}\cdot\begin{bmatrix}u\\ y\end{bmatrix}$$

We compute the first product.

$$-w^T\Lambda(y)w=\begin{bmatrix}-u\big(1-\varepsilon(\lambda+\alpha_2)\big)-y\frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)\\ -u\frac{1}{2b}(\lambda+\alpha_2)\big(1-\varepsilon\alpha(y)\big)-y\lambda\ell\end{bmatrix}^T\cdot\begin{bmatrix}u\\ y\end{bmatrix}$$

We now compute the second.

$$
\begin{aligned}
&-w^T\Lambda(y)\,w = \\
&-u^2\left(1-\varepsilon(\lambda+\alpha_2)\right) - uy\tfrac{1}{2b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) \\
&-uy\tfrac{1}{2b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) - y^2\lambda\ell
\end{aligned}
$$

Simplifying, we arrive at the following.

$$
\begin{aligned}
&-w^T\Lambda(y)\,w = \\
&-u^2\left(1-\varepsilon(\lambda+\alpha_2)\right) - uy\tfrac{1}{b}(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right) - y^2\lambda\ell
\end{aligned}
$$

Adding terms, we arrive at the desired expression.

$$
\begin{aligned}
&\tfrac{d}{dt}\left[S(u,y) - \tfrac{1}{b}\int\limits_0^{z(t)}\left(\kappa(s)+b\ell\right)ds\right] \\
&= -w^T\Lambda(y)\,w + \tfrac{1}{b}\left(\alpha_2-\alpha(y)\right)\kappa(z)\,y^2
\end{aligned}
$$

We want to constrain ℓ such that $\Lambda(y)$ is positive definite.

$$
\begin{aligned}
&\Lambda(y) = \\
&\begin{bmatrix} 1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} \\ \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} & \lambda\ell \end{bmatrix} > 0
\end{aligned}
$$

For this to be true the first principle minor must be positive. This yields the following constraint.

$$
1-\varepsilon(\lambda+\alpha_2) > 0 \Rightarrow \varepsilon(\lambda+\alpha_2) < 1
$$

The second principle minor, computed via the determinant, must also be positive.

$$
\begin{vmatrix} 1-\varepsilon(\lambda+\alpha_2) & \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} \\ \frac{(\lambda+\alpha_2)\left(1-\varepsilon\alpha(y)\right)}{2b} & \lambda\ell \end{vmatrix} > 0
$$

The determinant is computed below.

$$
\left(1-\varepsilon(\lambda+\alpha_2)\right)(\lambda\ell) - \frac{1}{4b^2}(\lambda+\alpha_2)^2\left(1-\varepsilon\alpha(y)\right)^2 > 0
$$

We take the second term to the right hand side.

$$
\left(1-\varepsilon(\lambda+\alpha_2)\right)(\lambda\ell) > \frac{1}{4b^2}(\lambda+\alpha_2)^2\left(1-\varepsilon\alpha(y)\right)^2
$$

We arrive at one constraint on ℓ below.

$$
\ell > \left(\frac{\lambda+\alpha_2}{b}\right)^2 \frac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}
$$

We now write S in terms of a linear equation using $\Lambda'(y)$.

$$
\Lambda'(y) = \begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix}
$$

We then test this is valid by expanding the following.

$$
\begin{aligned}
&S = w^T\Lambda'(y)\,w \\
&= \begin{bmatrix} u & y \end{bmatrix} \cdot \begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix} \cdot \begin{bmatrix} u \\ y \end{bmatrix}
\end{aligned}
$$

We compute the first product.

$$
w^T\Lambda'(y)\,w = \begin{bmatrix} \frac{\varepsilon}{2}u + \frac{1}{2b}\varepsilon(\lambda+\alpha_2)\,y \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2)\,u + \frac{\ell}{2}y \end{bmatrix}^T \cdot \begin{bmatrix} u \\ y \end{bmatrix}
$$

Then compute the second product.

$$
\begin{aligned}
&w^T\Lambda'(y)\,w \\
&= \tfrac{\varepsilon}{2}u^2 + \tfrac{1}{2b}\varepsilon(\lambda+\alpha_2)\,uy + \tfrac{1}{2b}\varepsilon(\lambda+\alpha_2)\,uy + \tfrac{\ell}{2}y^2
\end{aligned}
$$

We finally arrive at the desired expression.

$$
w^T\Lambda'(y)\,w = \frac{\varepsilon}{2}u^2 + \frac{1}{b}\varepsilon(\lambda+\alpha_2)\,uy + \frac{\ell}{2}y^2
$$

For S to be positive definite, $\Lambda'(y)$ must be also.

$$
\begin{bmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{bmatrix} > 0
$$

Since the first principle minor is positive by definition, we examine the second.

$$
\begin{vmatrix} \frac{\varepsilon}{2} & \frac{1}{2b}\varepsilon(\lambda+\alpha_2) \\ \frac{1}{2b}\varepsilon(\lambda+\alpha_2) & \frac{\ell}{2} \end{vmatrix} > 0
$$

We then compute the determinant.

$$
\frac{\varepsilon\ell}{4} - \frac{1}{4b^2}\varepsilon^2(\lambda+\alpha_2)^2 > 0 \Rightarrow \frac{\varepsilon\ell}{4} > \frac{1}{4b^2}\varepsilon^2(\lambda+\alpha_2)^2
$$

We arrive at a second condition on ℓ.

$$
\ell > \frac{1}{b^2}(\lambda+\alpha_2)^2\varepsilon
$$

We now have two constraints on ℓ, both upper bounds.

$$
\ell > \tfrac{1}{b^2}(\lambda+\alpha_2)^2\varepsilon \quad \ell > \left(\tfrac{\lambda+\alpha_2}{b}\right)^2 \tfrac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}
$$

Therefore, ℓ must be greater than their maximum.

$$
\ell > \left(\frac{\lambda+\alpha_2}{b}\right)^2 \max\left\{\varepsilon, \frac{\left(1-\varepsilon\alpha(y)\right)^2}{4\lambda\left(1-\varepsilon(\lambda+\alpha_2)\right)}\right\} \tag{18}
$$

We now integrate (16) to remove the derivative.

$$
\begin{aligned}
&S - S(0) - \tfrac{1}{b}\int\limits_0^{z(t)}\left(\kappa(s)+b\ell\right)ds \\
&= \int\limits_0^t \begin{bmatrix} -w^T(a)\,\Lambda(y)\,w(a) \\ +\frac{1}{b}\left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right)y^2(s) \end{bmatrix} ds
\end{aligned}
$$

We break up terms.

$$
\begin{aligned}
&S - S(0) - \tfrac{1}{b}\int\limits_0^{z(t)}\left(\kappa(s)\right)ds - \ell z(t) \\
&= -\int\limits_0^t w^T(s)\,\Lambda(y)\,w(s)\,ds \\
&+\tfrac{1}{b}\int\limits_0^t \left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right)y^2(s)\,ds
\end{aligned}
$$

Rearranging terms, we have the following.

$$
\begin{aligned}
&S - \int\limits_0^t w^T(s)\,\Lambda(y)\,w(s)\,ds \\
&= S(0) + \ell z(t) + \tfrac{1}{b}\int\limits_0^{z(t)}\left(\kappa(s)\right)ds \\
&+\tfrac{1}{b}\int\limits_0^t \left(\alpha_2-\alpha(y)\right)\kappa\left(z(s)\right)y^2(s)\,ds
\end{aligned}
$$

We then rewrite this equation as an inequality.

$$\begin{aligned}
&S-\int_0^t w^T(s)\,\Lambda(y)\,w(s)\,ds \\
&\le S(0)+\ell z(t)+\tfrac{1}{b}\int_0^{z(t)}\big(\kappa(s)\big)\,ds \\
&+\tfrac{1}{b}\int_0^t\big(\alpha_2-\alpha(y)\big)\,\kappa\big(z(s)\big)\,y^2(s)\,ds
\end{aligned} \tag{19}$$

We define sequences $t_{2k}, t_{1k}, z_{1k}, z_{2k}$ as follows.

$$t_{2k}\equiv\inf\big\{t\in R: z(t)=z_{2k}\big\} \tag{20}$$

$$t_{1k}\equiv\sup\big\{t\in[0,t_{2k}): z(t)=z_{1k}\big\} \tag{21}$$

$$z_{1k}\equiv 2\pi k+\left(\frac{\pi}{2}\right)\big(1+\mathrm{sgn}(b)\big)-\frac{\pi}{2} \tag{22}$$

$$z_{2k}\equiv 2\pi k+\left(\frac{\pi}{2}\right)\big(1+\mathrm{sgn}(b)\big)+\frac{\pi}{4} \tag{23}$$

We see from these definitions that $z\in[z_{1k},z_{2k}]$ whenever $t\in[t_{1k},t_{2k}]$. To prove the boundedness of z, we first assume unboundedness and show a contradiction. From the unboundedness assumption on z, we know there exists a $k_0>0\in Z$ such that all the above sequences have infinite cardinality.

$$\{z_{1k}\}_{k=k_0}^{\infty}\quad\{z_{2k}\}_{k=k_0}^{\infty}\quad\{t_{1k}\}_{k=k_0}^{\infty}\quad\{t_{2k}\}_{k=k_0}^{\infty}$$

Based on (22) and (23):

$$\cos(z_{1k})=\cos\left(2\pi k+\left(\frac{\pi}{2}\right)\big(1+\mathrm{sgn}(b)\big)-\frac{\pi}{2}\right)$$

$$\cos(z_{2k})=\cos\left(2\pi k+\left(\frac{\pi}{2}\right)\big(1+\mathrm{sgn}(b)\big)+\frac{\pi}{4}\right)$$

If $b<0$:

$$\cos(z_{1k})=\cos\left(2\pi k-\frac{\pi}{2}\right)=0\quad\forall k$$

$$\cos(z_{2k})=\cos\left(2\pi k+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\quad\forall k$$

If $b>0$:

$$\cos(z_{1k})=\cos\left(2\pi k+\frac{\pi}{2}\right)=0\quad\forall k$$

$$\cos(z_{2k})=\cos\left(2\pi k+\frac{5\pi}{4}\right)=\frac{-\sqrt{2}}{2}\quad\forall k$$

From this, the definition of $\kappa(z)$ and the fact that $\beta\in\mathcal{K}_\infty$, we can conclude that when $z\in[z_{1k},z_{2k}]$. This implies that $\frac{1}{b}\kappa(z)$ is less than or equal to zero.

$$\beta\in\mathcal{K}_\infty\Rightarrow\frac{1}{b}\kappa(z)\le 0\quad\forall z\in[z_{1k},z_{2k}]$$

We know that for all $t\in[0,t_{2k}]$ such that $z(t)\ge z_{1k}$, this relationship also holds.

$$\frac{1}{b}\kappa(z)\le 0\quad\forall\ \begin{matrix}t\in[0,t_{2k}]\\ z(t)\ge z_{1k}\end{matrix}$$

We can apply the above to the last term of (19) to establish the upper bound when $t=t_{1k}$.

$$\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds$$

We know that $\alpha_2-\alpha\big(y(s)\big)>0\quad\forall s$ and $y^2(s)>0\quad\forall s$ and from our above conclusion

$$\frac{1}{b}\kappa\big(z(t)\big)\le 0\quad\mathrm{for\,z\,(t)\ge z_{1k}}$$

We know that when $z(t)\ge z_{1k}$, the integral will be negative. Thus, if we look at $z(t)\le z_{1k}$, we will have an upper bound.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\int_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We can further refine the upper bound by using the fact that

$$\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\le\frac{\alpha_2-\alpha_1}{|b|}$$

We first substitute the expression for $\kappa(z(s))$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\int_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\beta\big(z(s)\big)\cos\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We then bound $\beta\big(z(s)\big)\cos\big(z(s)\big)$ with $\beta\big(z(s)\big)$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\int_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\beta\big(z(s)\big)\,y^2(s)\,ds\end{aligned}$$

We bound $\beta\big(z(s)\big)$ with the supremum and factor it out. We also use the bound for $\alpha(y(s))$.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\frac{\alpha_2-\alpha_1}{|b|}\sup_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\Big\{\beta\big(z(t)\big)\Big\}\int_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}y^2(s)\,ds\end{aligned}$$

We next solve the expression for z for the integral.

$$z=\frac{1}{2}y^2+\lambda\int_0^t y^2(s)\,ds\Rightarrow\lambda\int_0^t y^2(s)\,ds=z-\frac{1}{2}y^2$$

We substitute this in the bound.

$$\begin{aligned}&\int_0^{t_{1k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\frac{\alpha_2-\alpha_1}{|b|}\sup_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\Big\{\beta\big(z(t)\big)\Big\}\left[\frac{z(t)}{\lambda}-\frac{y^2(t)}{2\lambda}\right]_{\substack{t\in[0,t_{1k}]\\ z(t)\le z_{1k}}}\end{aligned}$$

Since the second term is positive, we can remove it and maintain our upper bound. Plugging in the maximum value of $z(t)$ yields an upper bound.

$$\begin{aligned}&\int_0^{t_{2k}}\frac{\alpha_2-\alpha\big(y(s)\big)}{b}\kappa\big(z(s)\big)\,y^2(s)\,ds\\ &\le\frac{\alpha_2-\alpha_1}{\lambda|b|}\beta(z_{1k})\,z_{1k}\end{aligned} \tag{24}$$

When $t = t_{2k}$, the upper bound remains the same, since when $t > t_{1k}$, $z \geq z_{1k}$ and $\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0$, so the integral is negative and we can remove it to create an upper bound similar to the above work.

$$\begin{aligned}&\textstyle\int_0^{t_{2k}} \frac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\\ \leq &\int_{\substack{t\in[0,t_{1k}]\\ z(t)\leq z_{1k}}} \frac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\end{aligned}$$

We use the same logic to arrive at the upper bound.

$$\begin{aligned}&\textstyle\int_0^{t_{2k}} \frac{\alpha_2-\alpha\left(y(s)\right)}{b}\kappa\left(z(s)\right)y^2(s)\,ds\\ &\leq \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta\left(z_{1k}\right)z_{1k}\end{aligned} \tag{25}$$

We can now choose $t = t_{2k}$ and apply (25) to (19) to get the following.

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \tfrac{1}{b}\textstyle\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &+\tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta\left(z_{1k}\right)z_{1k}\end{aligned} \tag{26}$$

We next examine the second term in the bound in (26).

$$\frac{1}{b}\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds$$

For

$$z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$$

When $b < 0$, then we know from before that $\cos\left(z_{1k}\right) = 0$ and we can see that:

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \cos\left(2\pi k - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \geq 0$ and:

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0$$

When $b > 0$, we know from before that $\cos\left(z_{1k}\right) = 0$ and we can see that the following relationship holds.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \cos\left(2\pi k + \frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \leq 0$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq 0 \quad \forall z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$$

For $z \in \left[z_{2k}-\frac{\pi}{2}, z_{2k}\right]$, when $b < 0$, then we know from before that $\cos\left(z_{2k}\right) = \frac{\sqrt{2}}{2}$ and from above that the following is true.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \geq \left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$$

When $b > 0$, then, again, we know from before that $\cos\left(z_{2k}\right) = \frac{-\sqrt{2}}{2}$ and from above that the following is true.

$$\cos\left(z_{2k}-\frac{\pi}{2}\right) = \frac{-\sqrt{2}}{2} \quad \forall k$$

Thus $\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$ and we can bound $\operatorname{sgn}(b)\,\kappa\left(z(t)\right)$.

$$\operatorname{sgn}(b)\,\kappa\left(z(t)\right) \leq -\left(\frac{1}{\sqrt{2}}\right)\beta\left(z_{2k}-\frac{\pi}{2}\right)$$

Using this, we can break up the integral of the term we are interested in. We know that for $z \in \left[z_{1k}, z_{2k}-\frac{\pi}{2}\right]$ the value is negative and can be ignored to create an upper bound and we can easily integrate the $b\ell$ term.

$$\begin{aligned}&\tfrac{1}{b}\textstyle\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &\leq \ell z_{2k} + \tfrac{1}{b}\textstyle\int_0^{z_{1k}}\kappa(s)\,ds + \tfrac{1}{b}\int_{z_{2k}-\pi/2}^{z_{2k}}\kappa(s)\,ds\end{aligned}$$

We can then substitute the above bounds to derive the following upper bound.

$$\begin{aligned}&\tfrac{1}{b}\textstyle\int_0^{z_{2k}}\left(\kappa(s)+b\ell\right)ds\\ &\leq \ell z_{2k} + \tfrac{1}{|b|}\beta\left(z_{1k}\right)z_{1k} - \tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{2k}-\tfrac{\pi}{2}\right)\end{aligned} \tag{27}$$

We first substitute (27) into (26).

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \ell z_{2k} + \tfrac{1}{|b|}\beta\left(z_{1k}\right)z_{1k} - \tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{2k}-\tfrac{\pi}{2}\right)\\ &+\tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta\left(z_{1k}\right)z_{1k}\end{aligned}$$

We have the following relationship between z_{1k} and z_{2k}.

$$z_{2k} = z_{1k} + 3\pi/4$$

Substituting this, we have the following.

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \ell\left[z_{1k}+3\pi/4\right] + \tfrac{1}{|b|}\beta\left(z_{1k}\right)z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(\left[z_{1k}+\tfrac{3\pi}{4}\right]-\tfrac{\pi}{2}\right) + \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta\left(z_{1k}\right)z_{1k}\end{aligned}$$

We then expand and cancel common terms.

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \ell\tfrac{3\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\beta\left(z_{1k}\right)z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right) + \tfrac{\alpha_2-\alpha_1}{\lambda|b|}\beta\left(z_{1k}\right)z\end{aligned}$$

Rearranging we arrive at the final expression.

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \tfrac{3\ell\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\left(1+\tfrac{\alpha_2-\alpha_1}{\lambda}\right)\beta\left(z_{1k}\right)z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right)\end{aligned} \tag{28}$$

We can rewrite (12) as follows, first reversing the sign and then multiplying by z.

$$\begin{aligned}&\lim_{z\to+\infty}\left[\tfrac{\beta(z+\varepsilon)}{z} - c\beta(z)\right] = +\infty\\ &\Rightarrow \lim_{z\to+\infty}\left[c\beta(z) - \tfrac{\beta(z+\varepsilon)}{z}\right] = -\infty\\ &\Rightarrow \lim_{z\to+\infty}\left[c\beta(z)\,z - \beta(z+\varepsilon)\right] = -\infty\end{aligned}$$

We examine (28), repeated below.

$$\begin{aligned}&S\left(t_{2k}\right) \leq S(0) + \tfrac{3\ell\pi}{4} + \ell z_{1k} + \tfrac{1}{|b|}\left(1+\tfrac{\alpha_2-\alpha_1}{\lambda}\right)\beta\left(z_{1k}\right)z_{1k}\\ &-\tfrac{\pi}{2\sqrt{2}|b|}\beta\left(z_{1k}+\tfrac{\pi}{4}\right)\end{aligned}$$

We divide out the factor of $\frac{\pi}{2\sqrt{2}|b|}$. We see that the last two terms approach $-\infty$, forcing the left hand side to be negative. However, we know that $S(t_{2k})$ is positive definite. Therefore, we have a contradiction, and z is thus bounded.

$$\frac{S(t_{2k})}{\frac{\pi}{2\sqrt{2}|b|}} \leq \frac{S(0)}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\frac{3\ell\pi}{4}}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\ell z_{1k}}{\frac{\pi}{2\sqrt{2}|b|}} + \frac{\frac{1}{|b|}\left(1+\frac{\alpha_2-\alpha_1}{\lambda}\right)}{\frac{\pi}{2\sqrt{2}|b|}}\beta\left(z_{1k}\right)z_{1k} - \beta\left(z_{1k}+\frac{\pi}{4}\right)$$

Since $z \in \mathcal{L}_\infty$ looking at (11), repeated below, we can conclude that $y \in \mathcal{L}_\infty \cap \mathcal{L}_2$.

$$z = \frac{1}{2}y^2 + \lambda \int_0^t y^2(s)\,ds \in \mathcal{L}_\infty \Rightarrow y \in \mathcal{L}_\infty \cap \mathcal{L}_2$$

From (28), we know that $S \in \mathcal{L}_\infty$. Examining the definition of S, we see that $u \in \mathcal{L}_\infty \cap \mathcal{L}_2$.

$$S = w^T \Lambda'(y)\, w \in \mathcal{L}_\infty, w = \begin{bmatrix} u & y \end{bmatrix}^T \Rightarrow y \in \mathcal{L}_\infty \cap \mathcal{L}_2$$

The boundedness of u and y together with (6) implies that $\dot{u}, \dot{y} \in \mathcal{L}_\infty$.

$$\left\{\begin{array}{c} f(y) \in \mathcal{L}_\infty \Rightarrow \dot{y} = f(y) + bu \in \mathcal{L}_\infty \\ u_{nom} \in \mathcal{L}_\infty \Rightarrow \varepsilon\dot{u} = u_{nom} - u \in \mathcal{L}_\infty \end{array}\right\}$$

Together, this allows us to apply Barbalat's Lemma and conclude that $y(t), u(t)$ go to zero.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} u(t) = 0$$

And finally, given the definition of u_{nom}, we see that u_{nom} also goes to zero.

$$\begin{array}{l} \lim\limits_{t\to\infty} y(t) = 0, \lim\limits_{t\to\infty} z(t) = 0 \\ \Rightarrow \lim\limits_{t\to\infty} u_{nom} = \lim\limits_{t\to\infty} \kappa\left(z(t)\right) y(t) = 0 \end{array}$$

This completes the proof of Theorem 1, showing the all signals are bounded and converge to zero. ■

Corollary 1 Let the closed-loop system described by the linear system with ignored fast actuator dynamics

$$\left\{\begin{array}{c} \dot{y} = \alpha y + bu \\ \varepsilon\dot{u} = u_{nom} - u \end{array}\right\} \tag{29}$$

and controller (10), (11).

$$u_{nom} = \kappa(z)\, y$$

$$z = \frac{1}{2}y^2 + \lambda \int_0^t y^2(s)\,ds$$

If $\varepsilon(\lambda + \alpha_2) < 1$, and $\kappa(\cdot)$ is a Nussbaum function then, all closed-loop signals are bounded and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} u(t) = \lim_{t\to\infty} u_{nom}(t) = 0.$$

Proof: In the case of a linear system, we have the following.

$$\alpha(y) = \alpha_1 = \alpha_2 = \alpha$$

In this case, the last terms in (19) and (26) will cancel and they will become the following.

$$S - \int_0^t w^T(s)\,\Lambda(y)\, w(s)\, ds \leq S(0) + \ell z(t) + \frac{1}{b}\int_0^{z(t)} \left(\kappa(s)\right) ds$$

$$S(t_{2k}) \leq S(0) + \frac{1}{b}\int_0^{z_{2k}} \left(\kappa(s) + b\ell\right) ds$$

The derivation of (24) and (25) is no longer required. The proof otherwise proceeds the same. ■

III. Simulation examples

A. Linear system

A simulation was performed on the linear system with ignored actuator dynamics described in (29). Three systems were compared: a Nussbaum gain controller described in (30), a nonlinear PI controller (10) and (11) with a non-Nussbaum gain ($\kappa(z) = z\cos(z)$), and a nonlinear PI controller with a Nussbaum gain ($\kappa(z) = z^2\cos(z)$).

$$\left\{\begin{array}{c} u_{nom} = \zeta^2\cos(\zeta)\, y \\ \dot{\zeta} = \lambda y^2 \end{array}\right\} \tag{30}$$

As seen in Figure 4 of [1], only the nonlinear PI controller with a Nussbaum gain provides convergent solutions.

B. Nonlinear system

A simulation of the nonlinear system (6) was also performed with parameters conforming the assumptions of Theorem 1. As seen in Figure 5 of [1], $y(t)$, $u(t)$, and $u_{nom}(t)$ are all bounded and converge to zero as expected.

IV. Conclusions

As seen in Theorem 1 and the accompanying simulations, using a nonlinear PI controller with an appropriately selected Nussbaum gain will allow for robust control of sector bounded nonlinear systems with unknown control directions.

References

[1] Psillakis, Haris E. "Robustness of the nonlinear PI control method to ignored actuator dynamics." arXiv preprint arXiv:1408.3229 (2014).

[2] Psillakis HE. An extension of the Georgiou-Smith example: boundedness and attractivity in the presence of unmodelled dynamics via nonlinear PI control,(e-print, arXiv:1407.7213 [cs.SY]).