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1. Write the following complex numbers in standard form. Use the principal branch of the logarithm if necessary.

   (a) $4^i = e^{i\text{Log}4} = \cos\text{Log}(4) + i\sin\text{Log}(4)$

   (b) $(1+i\sqrt{3})^{i/2} = e^{i/2\text{Log}(1+i\sqrt{3})} = e^{i/2(\ln|2|+i\arctan\sqrt{3})} = e^{-\pi/6+i(\ln|2|/2)} = e^{-\pi/6}\cos\frac{\ln|2|}{2} + ie^{-\pi/6}\sin\frac{\ln|2|}{2}$

   (c) $(-1)^{\sqrt{2}} = e^{\sqrt{2}(\text{Log}(-1))} = e^{\sqrt{2}(\ln|-1|+i\pi)} = e^{\sqrt{2}\ln 1+i\pi\sqrt{2}} = \cos\pi\sqrt{2} + i\sin\pi\sqrt{2}$

   (d) $(i-1)^{2i+3} = e^{(2i+3)(\text{Log}(i-1))} = e^{(2i+3)(\ln\sqrt{2}+i(3\pi/4))} = e^{(3\ln\sqrt{2}-(3\pi/2))+i(2\ln\sqrt{2}+(9\pi/4))} = e^{3\ln\sqrt{2}-(3\pi/2)}e^{2\ln\sqrt{2}+(9\pi/4)} = 2\sqrt{2}e^{-3\pi/2}e^{i(2\ln\sqrt{2}+(9\pi/4))}$
   $= 2\sqrt{2}e^{-3\pi/4}\cos(2\ln\sqrt{2}+(\pi/4)) + i2\sqrt{2}e^{-3\pi/4}\sin(2\ln\sqrt{2}+(\pi/4))$

   (e) $e^{\sin i} = e^{i\sinh 1} = \cos(\sinh 1) + i\sin(\sinh 1)$

   (f) $\cos(-2+i) = \frac{e^{i(-2+i)}+e^{-i(-2+i)}}{2} = \frac{1}{2}(e^{-1-2i}+e^{1+2i})$
   $= \frac{1}{2}(e^{-1}\cos(-2) + ie^{-1}\sin(-2) + e\cos(2) + ie\sin(2))$
   $= \frac{1}{2}(e^{-1}\cos(-2) + e\cos(2)) + i\frac{1}{2}(e^{-1}\sin(2) + e\sin(2))$

   (g) $\sin(\pi/4+i) = \frac{e^{i((\pi/4)+i)}-e^{-i((\pi/4)+i)}}{2i} = \frac{e^{(-1+i(\pi/4))}-e^{(1-i(\pi/4))}}{2i}$
   $= \frac{e^{-1}\cos(\pi/4)+e^{-1}i\sin(\pi/4)-e\cos(-\pi/4)-ei\sin(-\pi/4)}{2i}$
   $= \frac{e^{-1}\cos(\pi/4)+e^{-1}i\sin(\pi/4)-e\cos(\pi/4)+ei\sin(\pi/4)}{2i}$
   $= \frac{\cos(\pi/4)(e^{-1}-e)+i\sin(\pi/4)(e^{-1}+e)}{2i}$
   $= \frac{1}{2}\sin(\pi/4)(e^{-1}+e) - \frac{1}{2}i\cos\pi/4(e^{-1}-e)$
   $= \frac{\cosh(1)}{\sqrt{2}} + i\frac{\sinh(1)}{\sqrt{(2)}}$

   (h) $\tan\left(\frac{\pi+i}{2}\right) = \frac{\sin((\pi/2)+(i/2))}{\cos((\pi/2)+(i/2))} = \frac{\frac{e^{i((\pi/2)+(i/2))}-e^{-i((\pi/2)+(i/2))}}{2i}}{\frac{e^{i((\pi/2)+(i/2))}+e^{-i((\pi/2)+(i/2))}}{2}}$
   $= \frac{e^{(-1/2)+i(\pi/2)}-e^{(1/2)-i(\pi/2)}}{i(e^{(-1/2)+i(\pi/2)}+e^{(1/2)-i(\pi/2)})}$
   $= \frac{e^{-1/2}\cos(\pi/2)+ie^{-1/2}\sin(\pi/2)-e^{1/2}\cos(-\pi/2)-ie^{1/2}\sin(-\pi/2)}{i(e^{-1/2}\cos(\pi/2)+ie^{-1/2}\sin(\pi/2)-e^{1/2}\cos(-\pi/2)-ie^{1/2}\sin(-\pi/2))}$
   $= \frac{ie^{-1/2}+ie^{1/2}}{i(ie^{-1/2}-ie^{1/2})}$
   $= \frac{e^{-1/2}+e^{1/2}}{ie^{-1/2}-ie^{1/2}} = i\coth(1/2)$

   (i) $\cosh(1-i(\pi/4)) = \cosh(1)\cos(-\pi/4) + i\sinh(1)\sin(-\pi/4)$
   $= \frac{\cosh(1)}{\sqrt{2}} - i\frac{\sinh(1)}{\sqrt{2}}$

(j) $\sinh(1+i\pi) = \sinh(1)\cos(\pi) + i\cosh(1)\sin(\pi)$
$= -\sinh(1) + 0i$

2. Find all complex values of the following:

(a) $\log(\sqrt{3}-i) = \ln|2| + i\arg(\sqrt{3}-i) = \ln|2| + i(-\frac{\pi}{6} + 2\pi k), k \in \mathbf{Z}$

(b) $\log(-ie) = \ln|e| + i\arg(-ie) = 1 + i(-\frac{\pi}{2} + 2\pi k), k \in \mathbf{Z}$

(c) $\arcsin(1+i) = \frac{1}{i}\ln(i(1+i) + |1-(1+i)^2|^{1/2}e^{(i/2)(\arg(1-(1+i)^2)})$
$= \frac{1}{i}\ln((-1+i) + \sqrt{5}^{1/2}e^{(i/2)\arg(1-2i)})$
$= \frac{1}{i}\ln((-1+i) + \sqrt{5}^{1/2}e^{(i/2)(\arctan(2)+2\pi k)})$
$= \frac{1}{i}\ln((-1+i) + 5^{1/4}e^{i(\frac{\arctan(2)+2\pi k}{2})})$
$= -i\ln((-1+i) + \sqrt{1-2i})$

3. Determine if each of the following statements is true. If it is, prove it. If it is not, give a counterexample.

(a) *Counterexample*: Let $z = -\frac{1}{\sqrt{5}} - i\frac{2}{\sqrt{5}}$. Then we have:

$$\sqrt{-\frac{1}{\sqrt{5}} - i\frac{2}{\sqrt{5}}} = e^{\frac{1}{2}(\ln|1| + i(\arctan(2)-\pi))}$$

$$= e^{\frac{1}{2}(i\arctan(2) - i\pi)}$$

$$= e^{-i\frac{\pi}{2}}e^{\frac{1}{2}(i\arctan(2))}$$

$$= -ie^{\frac{1}{2}(i\arctan(2))}$$

$$= -i\sqrt{\frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}} = -i\sqrt{-z} \neq i\sqrt{z}$$

(b) *Proof*: Let $z \in \mathbf{C}$ be arbitrary. Then $\sin^2(z) + cos^2(z) = (\frac{e^{iz}-e^{-iz}}{2i})^2 + (\frac{e^{iz}+e^{-iz}}{2})^2 = \frac{e^{2iz}-2+e^{-2iz}}{-4} + \frac{e^{2iz}+2+e^{-2iz}}{4} = \frac{4}{4} = 1$. This proves $\sin^2(z) + \cos^2(z) = 1$.

(c) *Counterexample*: Let $z = i$. Then we have $|\cos(i)|^2 + |\sin(i)|^2 = |\frac{e^{-1}+e^1}{2}|^2 + |\frac{e^{-1}-e^1}{2i}|^2 =$. It's is clear that $|\frac{e^{-1}+e}{2}|^2 = \frac{e^{-2}+2+e^2}{4} \geq 1$, so the sum is already greater than 1. Thus $|\cos(i)|^2 + |\sin(i)|^2 \neq 1$.

(d) *Proof*: Let $z = a + bi \in \mathbf{C}$ be arbitrary. Then we have $\sin(a + bi + 2\pi) = \sin(a + 2\pi)\cosh(b) + i\cos(a + 2\pi)\sinh(b) = \sin(a)\cosh(b) + i\cos(a)\sinh(b) = \sin(a + bi) = sin(z)$. This proves $\sin(z + 2\pi) = \sin(z)$.

(e) *Proof*: Let $z = a + bi \in \mathbf{C}$ be arbitrary. Then we have $\sin(\pi - a - bi) = \sin(\pi - a)\cosh(-b) + i\cos(\pi - a)\sinh(-b) = \sin(a)\cosh(-b) + i\cos(a)\sinh(-b) = \sin(a)\cosh(b) + i\cos(a)\sinh(b) = \sin(a + bi) = sin(z)$. This proves $\sin(\pi - z) = \sin(z)$.

(f) *Proof*: Let $z = a + bi \in \mathbf{C}$ be arbitrary. Then $e^{\overline{z}} = e^{\overline{a+bi}} = e^{a-bi} = e^a\cos(-b) + e^a i\sin(-b) = e^a\cos(b) - e^a i\sin(b) = \overline{e^a\cos(b) + e^a i\sin(b)} = \overline{e^{a+bi}} = \overline{e^z}$. This proves $e^{\overline{z}} = \overline{e^z}$

4. $f(z) = z^{1/4}, f(1) = 1^{1/4} = -i$

   $f(1) = e^{\frac{1}{4}(\ln|1| + i\arg(1))} = e^{\frac{1}{4}(i(0+2\pi k))} = e^{i\frac{k\pi}{2}0} = -i$

   $e^{i\frac{k\pi}{2}0} = -i = e^{i\frac{-\pi}{2}} \implies k = -1$

   So we have $f(x) = e^{i\frac{-\pi}{2}}z^{1/4}$ where $z^{1/4}$ takes the principle value.

   So thus $f(x) = e^{i\frac{-\pi}{2}}(4i)^{1/4} = e^{i\frac{-\pi}{2}}e^{\frac{1}{4}(\ln|4i| + i\mathrm{Arg}(4i))} = e^{i\frac{-\pi}{2}}e^{\frac{\ln 2}{2} + i\frac{\pi}{8}} = e^{-i\frac{3\pi}{8} + \frac{\ln 2}{2}}$
   $= \sqrt{2}(\cos(-3\pi/8) + i\sin(-3\pi/8))$,

5. $f(z) = \sqrt{z-1}\sqrt[3]{z-i}$. Since the domain of the root functions depend on the domain of $\mathrm{Log}(z)$, then we have that $z - 1 \leq 0$ and $z - i \leq 0$ must be excluded from the domain of f(z). Thus $z \leq 1$ and $z \leq i$ must be excluded. Although $\mathbf{C}$ is not an ordered field, we take this notation to mean that the rays pointing in the negative real direction from both 1 and i must be excluded from the domain of f(z).

6. (a) $\log(z^5)$

   We will use the principle argument when defining this logarithm. So we cut the plane along $\{z : \mathrm{Arg}(z) = -\pi\} \cup \{0\}$, defining the domain of the logarithm. So then z^5 cannot a negative real number or 0. The argument of such a number is $-\pi$, so the argument of z cannot be $-\frac{\pi}{5}$. So we must omit from the domain of $\log(z^5)$ the ray emanating from the origin with principle argument $-\frac{\pi}{5}$ including the origin itself, giving us the function f(z)=Log(z^5).

$$f(1 + 2i) = \mathrm{Log}(1 + 2i) = \ln|25\sqrt{5}| + i\mathrm{Arg}(41 - 38i)$$

$$= \ln|25\sqrt{5}| + i\arctan(-\frac{38}{41})$$

$$f(-2+i) = \text{Log}(-2+i) = \ln|25\sqrt{5}| + i\text{Arg}(38+41i)$$

$$= \ln|25\sqrt{5}| + i\arctan(\frac{38}{41})$$

$$\text{So } f(1+2i) = \overline{f(-2+i)}$$

(b) $\log(z^2+1)$

We will use the same cutting method as above for Log, so we have that z^2+1 cannot be a negative real number or 0. This is to say that $\text{Arg}(z^2+1) = \text{Arg}((z+i)(z-i)) = \text{Arg}(z+i) + \text{Arg}(z-i)(\text{mod } 2\pi) \neq -\pi$. We may then choose the argument of our cuts from coming from i and $-i$ to be arbitrary as long as their sum of their principle arguments mod 2π to be -pi. Thus our cuts will be along the ray extending from i with principle argument 0 and along the ray extending from $-i$ with argument $-\pi$.

$$f(1+2i) = \text{Log}(1+2i) = \ln|2\sqrt{5}| + i\text{Arg}(-2-4i)$$

$$= \ln|2\sqrt{5} + i(\text{Arctan}(2) - \pi)$$

$$f(-2+i) = \text{Log}(-2+i) = \ln|2\sqrt{5}| + i\text{Arg}(4-4i)$$

$$= \ln|2\sqrt{5} + i(\text{Arctan}(-1))$$

7. (a)
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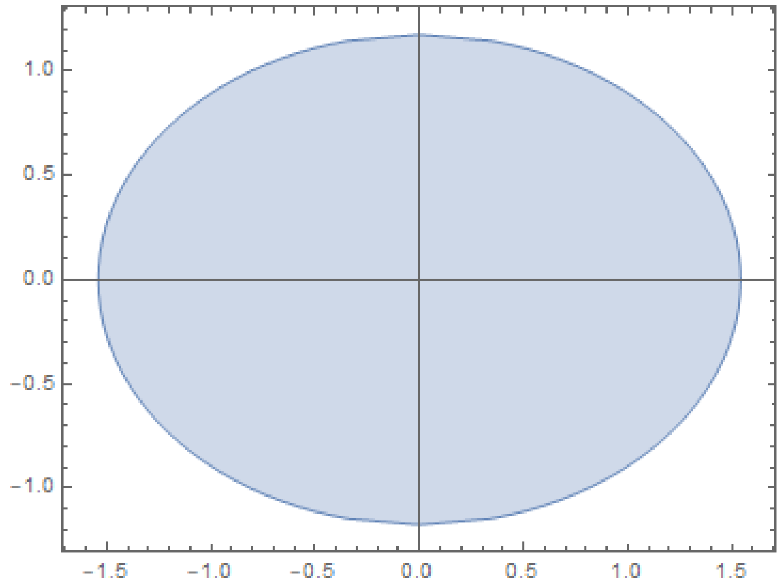
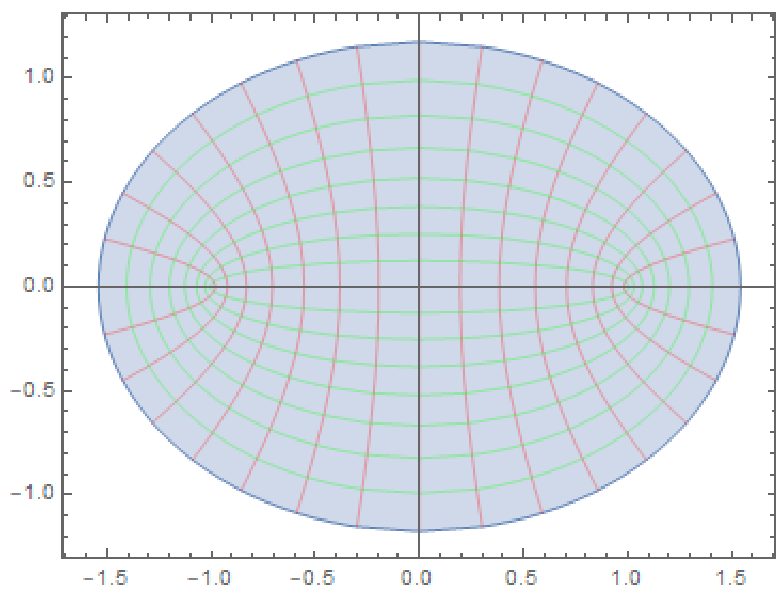
ParametricPlot[{Re[Sin[x + I y]], Im[Sin[x + I y]]}, {x, -Pi/2,
Pi/2}, {y, -1, 1}]
```

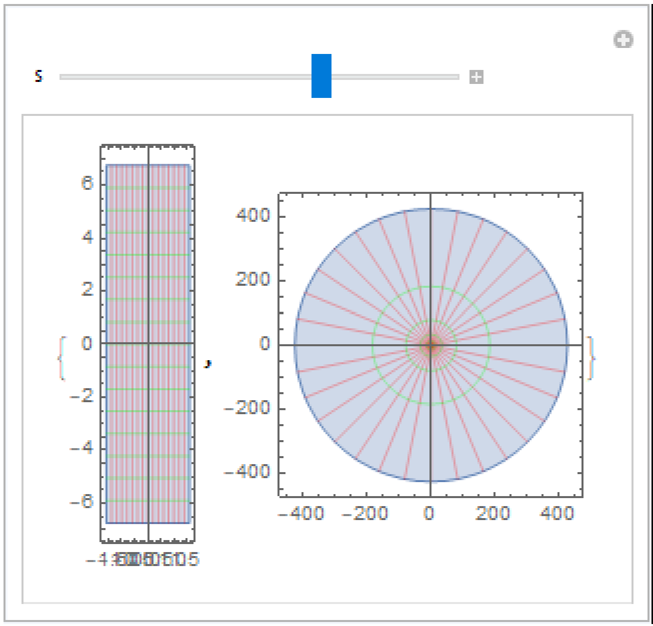


(b) `ParametricPlot[{Re[Sin[x + I y]], Im[Sin[x + I y]]}, {x, -Pi]/2, Pi]/2}, {y, -1, 1}, PlotRange -> Full, Mesh -> Automatic, MeshStyle -> {Red, Green}]`



Based on this picture, it appear that the function maps the lines $x = \frac{\pi}{2}$ to $\{z \in \mathbf{C} : z = a + 0i, a \geq 1\}$ and $x = -\frac{\pi}{2}$ to $\{z \in \mathbf{C} : z = a + 0i, a \leq -1\}$.

(c) Let's examine side-by-side figures of the rectangle and its image.



We can see that having a taller rectangle causes the image to become wider and wider without overlaps because the background blue remains the same color. Therefore the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\infty, \infty]$ maps onto the entire complex plane. This is not one-to-one as the line $x = \frac{\pi}{2}$ must fold onto itself. i.e. $\sin((\pi/2) + i) = \sin((\pi/2) - i)$.

(d) If we now consider the open vertical strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\infty, \infty]$, then the problematic lines $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are no longer an issue, so this will map onto $\mathbf{C} \setminus \{z \in \mathbf{C} : z = a + 0i, |a| \geq 1\}$. With those rays removed, the mapping is now one-to-one over the open rectangle.