Wyatt Whiting

1. Problem 4.18, page 70

   $\gamma := (4+0i) + t(-4+4i) = 4 - 4t + 4it, t \in [0,1]$

   $\gamma' = -4 + 4i$

   (a)

$$\int_\gamma \frac{z+1}{z} dz = [\mathrm{Log}(z) + z]_4^{4i} = (\mathrm{Log}(4i) + 4i) - (\mathrm{Log}(4) + 4)$$

$$(\ln|4i| + i\mathrm{Arg}(4i) + 4i) - (\ln|4| + i\mathrm{Arg}(4) + 4) = 4 + i\left(\frac{\pi}{2} + 4\right)$$

   (b)

$$\int_\gamma \frac{dz}{z^2+z} = [\mathrm{Log}(z) - \mathrm{Log}(z+1)]_4^{4i} =$$

$$= [\mathrm{Log}(4i) - \mathrm{Log}(1+4i)] - [\mathrm{Log}(4) - \mathrm{Log}(5)]$$

$$= [\ln|4i| + i\mathrm{Arg}(4i) - \ln|1+4i| - i\mathrm{Arg}(1+4i)] - [\ln|4| + i\mathrm{Arg}(4) - \ln|5| - i\mathrm{Arg}(5)]$$

$$= i\mathrm{Arg}(4i) - \ln(\sqrt{17}) - i\mathrm{Arg}(1+4i) + \ln(5)$$

$$= \ln\left(\frac{5}{\sqrt{17}}\right) + i\left(\frac{\pi}{2} - \arctan(4))\right)$$

   (c)

$$\int_\gamma z^{-\frac{1}{2}} dz = \left[2z^{\frac{1}{2}}\right]_4^{4i}$$

$$= \left[2(4i)^{\frac{1}{2}}\right] - \left[2(4)^{\frac{1}{2}}\right]$$

$$= 2(e^{\frac{1}{2}\ln 4 + \frac{1}{2}i\frac{\pi}{2}}) - 4$$

$$= 2(2e^{i\frac{\pi}{4}}) - 4$$

$$= 4e^{i\frac{\pi}{4}} - 4 = 4(e^{i\frac{\pi}{4}} - 1)$$

   (d)

$$\int_\gamma \sin^2(z) dz = \left[\frac{z}{2} - \frac{\sin(z)\cos(z)}{2}\right]_4^{4i}$$

$$= \left[\frac{4i}{2} - \frac{\sin(4i)\cos(4i)}{2}\right] - \left[\frac{4}{2} - \frac{\sin(4)\cos(4)}{2}\right]$$

$$= 2i - i\frac{\sinh(4)\cosh(4)}{2} - 2 + \frac{\sin(4)\cos(4)}{2}$$
$$= \frac{\sin(4)\cos(4)}{2} - 2 + i\left(2 - \frac{\sinh(4)\cosh(4)}{2}\right)$$

2. Problem 4.19, page 70

(a) $\gamma_1(t) = e^{it}, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
$\gamma_1'(t) = ie^{it}$

$$\int_{\gamma_1} z^i dz = \int_{\gamma_1} e^{i\mathrm{Log}(z)} dz$$
$$= \int_{-\pi/2}^{\pi/2} e^{i(\ln|e^{it}| + i\mathrm{Arg}(e^{it}))} ie^{it} dt = \int_{-\pi/2}^{\pi/2} e^{i(it)} ie^{it} dt$$
$$= \int_{-\pi/2}^{\pi/2} ie^{it-t} dt = \left[\frac{e^{it-t}}{2} - \frac{ie^{it-t}}{2}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{e^{i\frac{\pi}{2}-\frac{\pi}{2}}}{2} - \frac{ie^{i\frac{\pi}{2}-\frac{\pi}{2}}}{2} - \frac{e^{-i\frac{\pi}{2}+\frac{\pi}{2}}}{2} + \frac{ie^{-i\frac{\pi}{2}+\frac{\pi}{2}}}{2}$$
$$= \frac{ie^{-\pi/2}}{2} + \frac{e^{-\pi/2}}{2} + \frac{ie^{\pi/2}}{2} + \frac{e^{\pi/2}}{2}$$
$$= \frac{e^{-\pi/2} + e^{\pi/2}}{2} + i\frac{e^{-\pi/2} + e^{\pi/2}}{2}$$

(b) $\gamma_2(t) = e^{it}, \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$
For this problem, we must redefine the Log function since taking the principle logarithm over the given interval would prove unfruitful. I will denote the new branch of the logarithm with $\mathrm{Log}^*(z) = \ln|z| + \mathrm{Arg}^*(z)$ where $\mathrm{Arg}^*(z) \in [0, 2\pi)$. Then in a fashion like the previous part,

$$\int_{\gamma_2} z^i dz = \int_{\gamma_2} e^{i\mathrm{Log}^*(z)} dz$$
$$= \int_{\pi/2}^{3\pi/2} e^{i(\ln|e^{it}| + i\mathrm{Arg}^*(e^{it}))} ie^{it} dt = \int_{\pi/2}^{3\pi/2} e^{i(it)} ie^{it} dt$$
$$= \int_{\pi/2}^{3\pi/2} ie^{it-t} dt = \left[\frac{e^{it-t}}{2} - \frac{ie^{it-t}}{2}\right]_{\pi/2}^{3\pi/2}$$
$$= \frac{e^{i\frac{3\pi}{2}-\frac{3\pi}{2}}}{2} - \frac{ie^{i\frac{3\pi}{2}-\frac{3\pi}{2}}}{2} - \frac{e^{i\frac{\pi}{2}-\frac{\pi}{2}}}{2} + \frac{ie^{i\frac{3\pi}{2}-\frac{3\pi}{2}}}{2}$$

$$= -\frac{ie^{\frac{-3\pi}{2}}}{2} - \frac{e^{\frac{-3\pi}{2}}}{2} - \frac{ie^{\frac{-\pi}{2}}}{2} - \frac{e^{\frac{-\pi}{2}}}{2}$$
$$= \frac{-e^{\frac{-3\pi}{2}} - e^{\frac{-\pi}{2}}}{2} + i\frac{-e^{\frac{-3\pi}{2}} - e^{\frac{-\pi}{2}}}{2}$$

3. Evaluate the following integrals where the circle is positively oriented,

   (a) This function does not have an antiderivative along the specified path, so we will apply Cauchy's Integral Formula. Let $\gamma := C_2(-1)$ with positive orientation.

$$\int_\gamma \frac{z^2}{4-z^2}dz = \int_\gamma \frac{1}{z+2} - \frac{1}{z-2} - 1dz = \int_\gamma \frac{1}{z+2}dz - \int_\gamma \frac{1}{z-2}dz - \int_\gamma dz$$

By Cauchy's Integral formula we have

$$\int_\gamma \frac{1}{z+2}dz = 2\pi i$$

It should also be clear that by the Cauchy-Goursat theorem,

$$\int_\gamma 1dz = 0.$$

Since we know that

$$\int_\gamma \frac{1}{z-2}dz = \log(z-2) + c$$

where log has some branch cut, we can choose any branch cut that doesn't not intersect γ which is possible since the "center" of the cut lies outside the circle. Then $\frac{1}{z-2}$ is holomorphic and continuous on $G = \gamma \cup \partial\gamma$, so by Cauchy-Goursat we have

$$\int_\gamma \frac{1}{z-2}dz = 0.$$

So then we have

$$\int_\gamma \frac{z^2}{4-z^2}dz = 2\pi i$$

(b) We know $C_1(0)$ is a simple loop with positive orientation. $\sin(z)$ is holomorphic on $\mathbf{C}$ and 0 is contained within $C_1(0)$. Thus we can apply Cauchy's Integral formula, giving

$$\int_\gamma \frac{\sin(z)}{z-0}dz = 2\pi i \sin(0) = 2\pi i(0) = 0.$$

4. $\partial_{xx}u + \partial_{yy}u = 0 \implies u$ is harmonic.

(a) Let f be holomorphic and f is expressible as $f(z) = u(z) + iv(z)$. We then know

$$\partial_x u = \partial_y v \implies \partial_{xx}u = \partial_{yx}v$$

and

$$\partial_y u = -\partial_x v \implies \partial_{yy}u = -\partial_{xy}v.$$

We also know that $\partial_{yx}v = \partial_{xy}v$, so

$$\partial_{xx}u = \partial_{yx}v = -\partial_{yy}u \implies \partial_{xx}u = -\partial_{yy}u \implies \partial_{xx}u + \partial_{yy}u = 0,$$

so u is harmonic. Likewise we also have

$$\partial_{xy}u = \partial_{yy}v$$

$$\partial_{yx}u = -\partial_{xx}v,$$

so

$$\partial_{yy}v = \partial_{xy}u = \partial_{yx}u = -\partial_{xx}v$$

$$\implies \partial_{yy}v = -\partial_{xx}v \implies \partial_{xx}v + \partial_{yy}v = 0,$$

so v is harmonic as well.

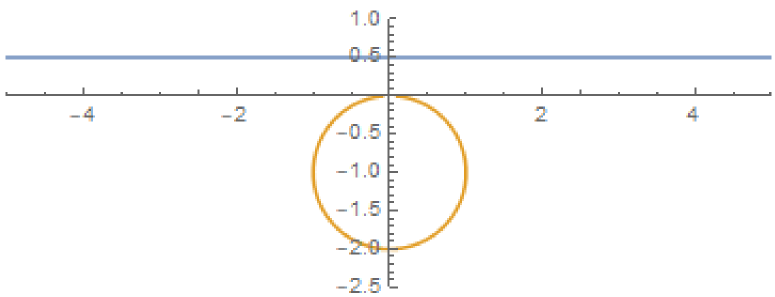
(b) Consider the function $g(z) = x = \mathrm{Re}(z)$. Suppose there exits some function $f(z)$ such that $f'(z) = g(z) = x$. Since $f'(z) = (u(z) + iv(z))' = \partial_x u + i\partial_x v$, then it must be the case that $\partial_x u(z) = x \implies u(z) = \frac{x^2}{2} + c$ and $\partial_x v(z) = 0 \implies v(z) = x + c$. However then we have $\partial_{xx}u = 1$ and $\partial_{yy}u = 0$, so $\partial_{xx}u - \partial_{yy}u = 1 \neq 0$, which contradicts the proof from (a). Thus, there is no function $f(z)$ such that $f'(z) = g(z)$, so g has no antiderivatives on $\mathbf{C}$.

5. Consider the function $f(z) = \frac{1}{z}$.

(a) Sketch the horizontal line $y = 1/2$ together with its image under f.

```
ParametricPlot[{ReIm[t + 0.5 I], ReIm[1/(t + 0.5 I)]}, {t,
-10, 10}, AxesOrigin -> {0, 0}, PlotRange -> {{-5, 5}, {-2.5,
1}}]
```

This code plots both the line $y = 1/2$ and $f(y = 1/2)$, with some extra commands to make the graph how I like it.



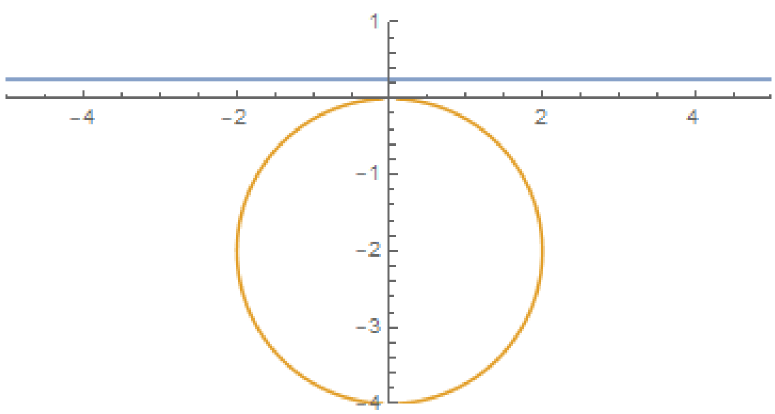
It is interesting to note that an everywhere-positive line becomes everywhere-negative after passing through $f(z)$ as this is not the way inverses would work on the real line. We should also note that the circle does not fully reach the origin. As you trace the horizontal line farther towards positive or negative infinity, the image approaches 0 asymptotically.

(b) Verify that the image of line $y = b > 0$ is a circle. What are its center and radius?

Below I show that various horizontal lines become circles:

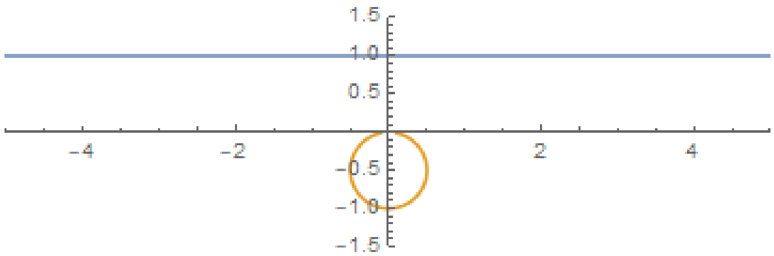
```
ParametricPlot[{ReIm[t + 0.25 I], ReIm[1/(t + 0.25 I)]}, {t,
-10, 10}, AxesOrigin -> {0, 0}, PlotRange -> {{-5, 5}, {-4,
1}}]
```

Same code as before but with modified line height and graphing range



This shows that halving the value of y from 1/2 to 1/4 causes the radius of the circle to double from 1 to 2.

```
ParametricPlot[{ReIm[t + 0 + I], ReIm[1/(t + I)]}, {t, -10,
10}, AxesOrigin -> {0, 0}, PlotRange -> {{-5, 5}, {-1.5, 1.5}}]
```
Same code, modified once again.

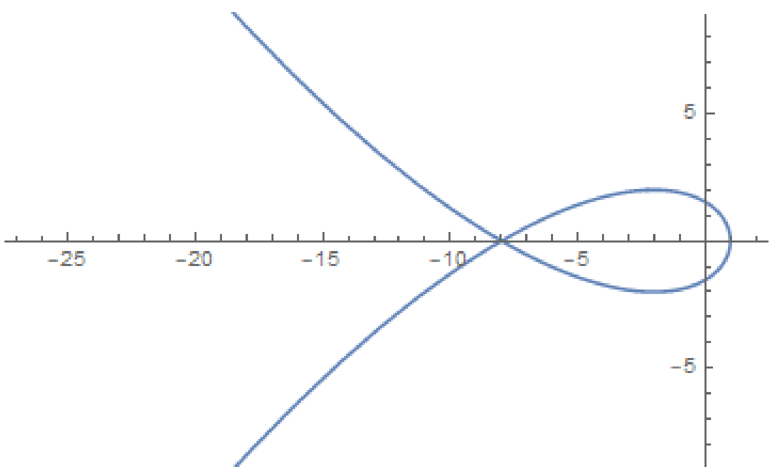


Doubling the height of the line from its initial value halves the radius. The picture now becomes clear.
Given a line $y = b > 0$, the image of that line will be a circle with radius $\frac{1}{2b}$ centered at $0 - i\frac{1}{2b}$.

(c) The image of the half plane is the lower half of the complex plane without the open disc with radius 1 centered at -i.

6. Consider the function $f(z) = z^3$

```
ParametricPlot[ReIm[(1 + I t)^3], {t, -3, 3}]
```
This plots the real and imaginary component of the function along the specified line taking values of y from -3 to 3. It generates the following graph:



7. Take $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i\sqrt{3}$. Then

$$f(z_1) = (1 + i\sqrt{3})^3 = -8 - (1 - i\sqrt{3})^3 = f(z_2)$$

8. $f(z) = z^3 \implies f'(z) = 3z^2$

$$f'(1+i\sqrt{3}) = 3(1+i\sqrt{3})^2 = -6 + i6\sqrt{3}$$
$$f'(1-i\sqrt{3}) = 3(1-i\sqrt{3})^2 = -6 - i6\sqrt{3}$$

9. Find the angle at which the path intersects itself.
   The angle at which the image intersects itself is $\frac{2\pi}{3}$.