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1. (a) $r = 1$

$$\int_{C[0,1]} \frac{dz}{z^2 - 2z - 8} = \frac{1}{6}\int_{C[0,1]} \frac{dz}{z-4} - \frac{1}{6}\int_{C[0,1]} \frac{dz}{z+2}$$

Let's choose branch cuts for each integral that point away from our path since the centers of each branch but lie outside its enclosed region. Then each function is holomorphic on $C[0,1] \cup \partial C[0,1]$, so by Cauchy-Gourset we have

$$\frac{1}{6}\int_{C[0,1]} \frac{dz}{z-4} - \frac{1}{6}\int_{C[0,1]} \frac{dz}{z+2} = 0$$

(b) $r = 3$

$$\frac{1}{6}\int_{C[0,3]} \frac{dz}{z-4} - \frac{1}{6}\int_{C[0,3]} \frac{dz}{z+2}$$

By the same line of reasoning above, we have

$$\frac{1}{6}\int_{C[0,3]} \frac{dz}{z-4} = 0.$$

Now we only need to evaluate the remaining integral. We know $C[0,3]$ is a simple loop enclosing $G = (C[0,3] \cup \partial C[0,3]) \setminus \{-2\}$ and $\frac{1}{z+2}$ is holomorphic on G. Then by the Cauchy Integral formula, we have

$$\frac{1}{6}\int_{C[0,3]} \frac{1}{z+2}dz = \frac{\pi i}{3} \implies \int_{C[0,3]} \frac{dz}{z^2 - 2z - 8} = -\frac{\pi i}{3}.$$

(c) $r = 5$

$$\frac{1}{6}\int_{C[0,4]} \frac{dz}{z-4} - \frac{1}{6}\int_{C[0,4]} \frac{dz}{z+2}$$

The path now encloses the singularities of both intergrands. Similar to the case $r = 3$, we have

$$\frac{1}{6}\int_{C[0,4]} \frac{dz}{z-4} - \frac{1}{6}\int_{C[0,4]} \frac{dz}{z+2} = \frac{\pi i}{3} - \frac{\pi i}{3} = 0$$

2. problem 5.1(b)

   By applying theorem 5.1 from page 73, if we let $f(z) = e^{3z}, f'(z) = 3e^{3z}, w = \pi i$, we then have

$$f'(w) = \frac{1}{2\pi i}\int_{\square} \frac{f(z)}{(z-w)^2}dz$$
$$3e^{3\pi i} = \frac{1}{2\pi i}\int_{\square} \frac{e^{3z}}{(z-\pi i)^2}dz$$
$$-3 = \frac{1}{2\pi i}\int_{\square} \frac{e^{3z}}{(z-\pi i)^2}dz$$
$$-6\pi i = \int_{\square} \frac{e^{3z}}{(z-\pi i)^2}dz$$

3. Integrate the following functions over the circle $C[0,3]$:

   (a) $\mathrm{Log}(z-4i)$ Integrating $\mathrm{Log}(z-4i)$ over the given circle is the same as integrating $\mathrm{Log}(z)$ over the circle $C[-4,3]$. So the circle is entirely outside the branch cut given by the principle logarithm. We also know that Log is holomorphic and continuous on $C[-4,3] \cup \partial C[-4,3]$, so we may apply the Cauchy-Goursat theorem, giving us

$$\int_{C[0,3]} \mathrm{Log}(z-4i)dz = \int_{C[-4,3]} \mathrm{Log}(z)dz = 0.$$

   (b) i^{z-3}

$$\int_{C[0,3]} i^{z-3}dz = i\int_{C[0,3]} i^{z}dz = i\int_{C[0,3]} e^{z\mathrm{Log}(i)}dz$$
$$= i\int_{C[0,3]} e^{z(\ln|i|+i\mathrm{Arg}(i))}dz = i\int_{C[0,3]} e^{zi\pi/2}dz$$

   Since e^z is holomorphic and continuous everywhere and $i\pi/2$ is just a constant, then $e^{zi\pi/2}$ is also everywhere continuous and holomorphic. Then by Cauchy-Goursat, we have

$$i\int_{C[0,3]} e^{zi\pi/2}dz = 0.$$

   (c) $\frac{1}{(z+4)(z^2+1)}$

$$\int_{C[0,3]} \frac{1}{(z+4)(z^2+1)}dz$$

$$= -\frac{1}{17}\int_{C[0,3]}\frac{1/2-2i}{z+i}dz - \frac{1}{17}\int_{C[0,3]}\frac{1/2+2i}{z-i}dz + \frac{1}{17}\int_{C[0,3]}\frac{1}{z-4}$$

Now we apply Cauchy Integral formula to each term, giving

$$= -\frac{1}{17}(2\pi i(1/2-2i)) - \frac{1}{17}(2\pi i(1/2+2i)) + \frac{1}{17}(2\pi i)$$
$$= \frac{-(4+i)\pi - (-4+i)\pi + 2\pi i}{17} = \frac{(2-2i)\pi}{17}$$

4. $\gamma(t) = (1-t^2, t), -2 \le t \le 1$

   We know $z^{-1/2}$ is dependent on Log which has a branch cut along the non-positive real line. Thus $z^{-1/2}$ will have an antiderivative along γ as γ does not cross this boundary. Assume usage of $\sqrt{z}$ indicates principle logarithm branch. Applying FTC gives

   $$\int_\gamma \frac{1}{z^{1/2}}\ dz = [2\sqrt{z}]_{\gamma(-2)}^{\gamma(1)}$$

   which gives

   $$= 2\sqrt{\sqrt{13}e^{i(\arctan(2/3)-\pi)}} - 2\sqrt{e^{i(\pi/2)}} = \sqrt[4]{13}e^{i(\frac{\arctan(2/3)-\pi}{2})} - 2e^{i(\pi/4)}$$

5. Problem 7.25

   (a) $\frac{1}{1+4z}$

   We know

   $$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}\ \forall |z| < 1.$$

   Substituting z with $-4z$ yields

   $$\sum_{k=0}^{\infty}(-4z)^k = \frac{1}{1+4z}\ \forall |z| < \frac{1}{4},$$

   producing the desired power series.

   (b) $\frac{1}{3-\frac{z}{2}} = \frac{1}{3}\frac{1}{1-\frac{z}{6}}$

   By a method similar to above, we modify

   $$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}\ \forall |z| < 1$$

by replacing z with $z/6$ and multiplying the resultant series by $1/3$. This gives

$$\frac{1}{3-\frac{z}{2}} = \frac{1}{3}\,\frac{1}{1-\frac{z}{6}} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{6}\right)^k = \sum_{k=0}^{\infty}\frac{z^k}{2^k 3^{k+1}}\ \forall |z| < 6$$

(c) $\frac{z^2}{(4-z)^2}$

We know

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Taking the derivative of the summed term maintains radius of convergence, giving

$$\sum_{k=1}^{\infty} kz^{k-1} = \frac{1}{(1-z)^2}.$$

This then naturally produces follows:

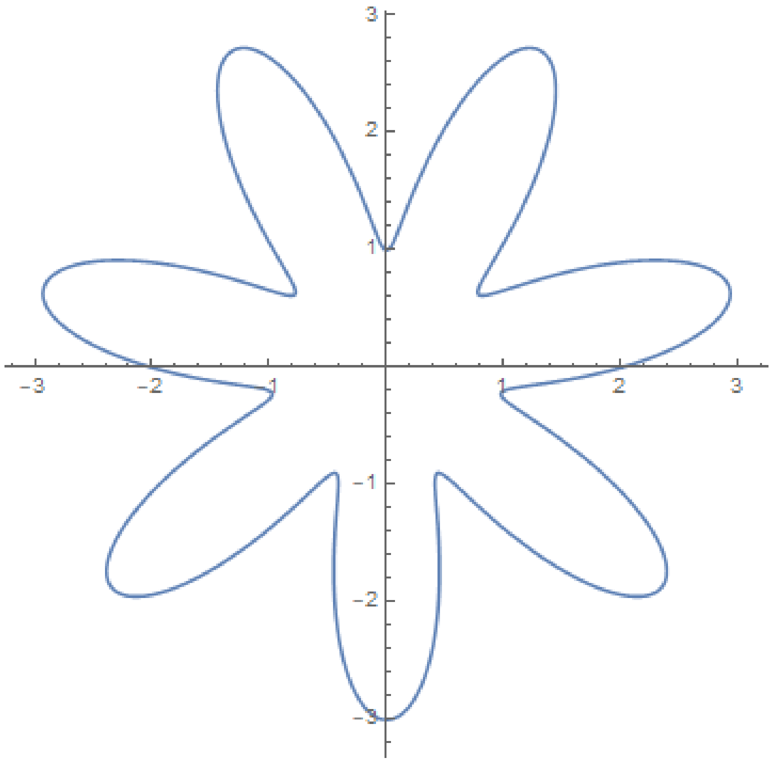
$$\frac{z^2}{(4-z)^2} = \frac{z^2}{4^2(1-(z/4))^2} = \frac{z^2}{4^2}\sum_{k=1}^{\infty} k\left(\frac{z}{4}\right)^{k-1}$$

$$= \sum_{k=1}^{\infty} k\left(\frac{z}{4}\right)^{k+1}\ \forall |z| < 4$$

6. (a) Sketch the curve γ.

```
ParametricPlot[{(2 + Sin[7t]) Cos[t], (2 + Sin[7 t]) Sin[t]},
{t, 0, 2 Pi}]
```

This code plots the curve $\gamma(t)$ taking $t \in [0, 2\pi]$

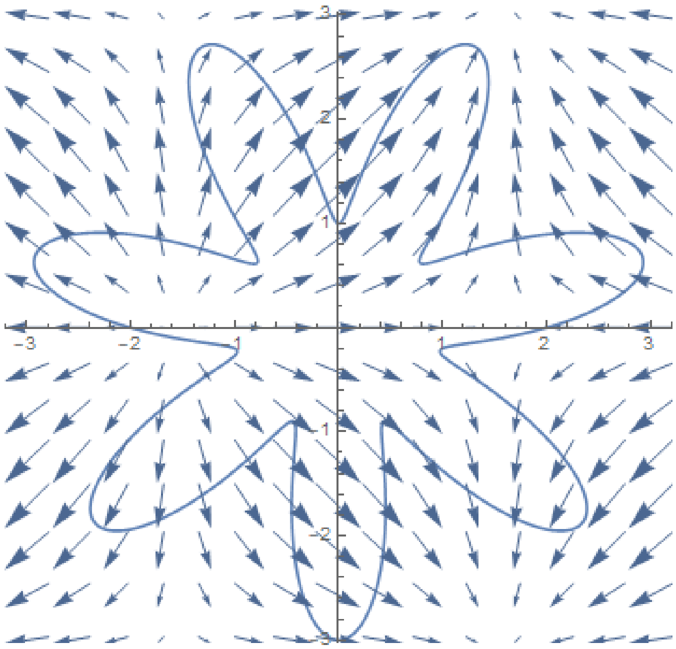


It makes a fun flower-looking thing! Also, there are seven spokes coming from the $\sin(7t)$ term.

(b) Sketch the curve γ and the vector field f on the same graph.

```
vtfield = VectorPlot[{Cos[x], Sin[y]}, {x, -3, 3}, {y, -3, 3}]
gamma = ParametricPlot[{(2 + Sin[7 t]) Cos[t], (2 + Sin[7 t]) Sin[t]}, {t, 0, 2 Pi}]
Show[gamma, vtfield, PlotRange -> All]
```

Sets up plotting the vector field and curve γ on on the same graph, which produces the following figure:



(c) What is the Polya vector field of f?

$$f(z) = \cos(x) + i\sin(y) \implies \overline{f}(z) = \cos(x) - i\sin(y)$$

(d) Find an approximation for the complex integral $\int_\gamma \overline{f}(z)dz$.

```
n = 100
t[k_] := 2 *Pi* k/n
z[k_] := (2 + Sin[7 t[k]]) Cos[t[k]] + I (2 + Sin[7 t[k]])
Sin[t[k]]
f[z_] := Cos[Re[z]] - I Sin[Im[z]]
N[Sum[f[z[k]]*(z[k + 1] - z[k]), {k, 0, n - 1}]]
```

Output: `0.0714602 + 5.91499 I`

This code mimics the provided Mathematica guide by approximating the integral with 100 sample points along γ, being sure to use $\overline{f}$ instead of f.

(e) Find an approximation for the work done by f along γ, and the flux of f across γ.

We know

$$W[f,\gamma] + iF[f,\gamma] = \int_\gamma \overline{f}(z)\ dz \approx\ 0.0714602 + i5.91499$$

$$\implies W[f,\gamma] \approx\ 0.0714602 \text{ and } F[f,\gamma] \approx\ 5.91499$$