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1. Evaluate the following limits.

   (a)
   $$\lim_{z\to i}\frac{z^3+i}{z-i}=\lim_{z\to i}\frac{3z^2}{1}\text{ (by L'Hopital's Rule) }=\frac{3(-1)}{1}=-3.$$

   (b)
   $$\lim_{z\to 0}\frac{\operatorname{Log}(z+i)-\operatorname{Log}(i)}{z}=\lim_{z\to 0}\frac{\operatorname{Log}(z+i)-\frac{i\pi}{2}}{z}=\text{ (L'H) }\lim_{z\to 0}\frac{\frac{1}{z+i}}{1}=$$
   $$\lim_{z\to 0}\frac{1}{z+i}=\frac{1}{i}=-i$$

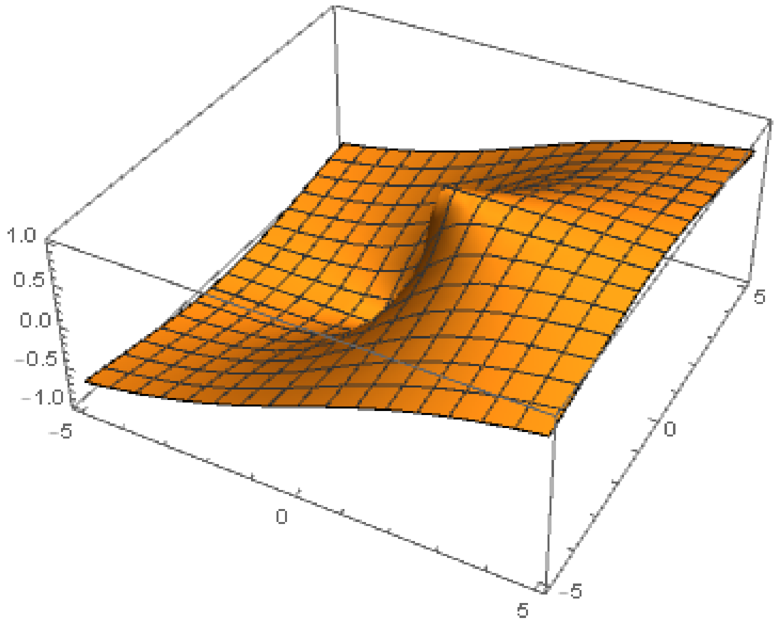
   (c)
   $$\lim_{z\to 0}\frac{e^z-1}{\operatorname{Log}(z+1)}=\text{ (L'H) }\lim_{z\to 0}\frac{e^z}{\frac{1}{z+1}}=\lim_{z\to 0}e^z(z+1)=1(1)=1$$

   (d)
   $$\lim_{z\to\infty}z\sin\left(\frac{1}{z}\right),w=1/z$$
   $$\lim_{z\to\infty}1/z=\lim_{z\to\infty}w=0=\lim_{w\to 0}w\implies\lim_{z\to\infty}f(1/z)=\lim_{w\to 0}f(w)$$
   $$\lim_{w\to 0}\frac{\sin(w)}{w}=\text{ (L'H) }=\lim_{w\to 0}\frac{\cos(w)}{1}=1$$

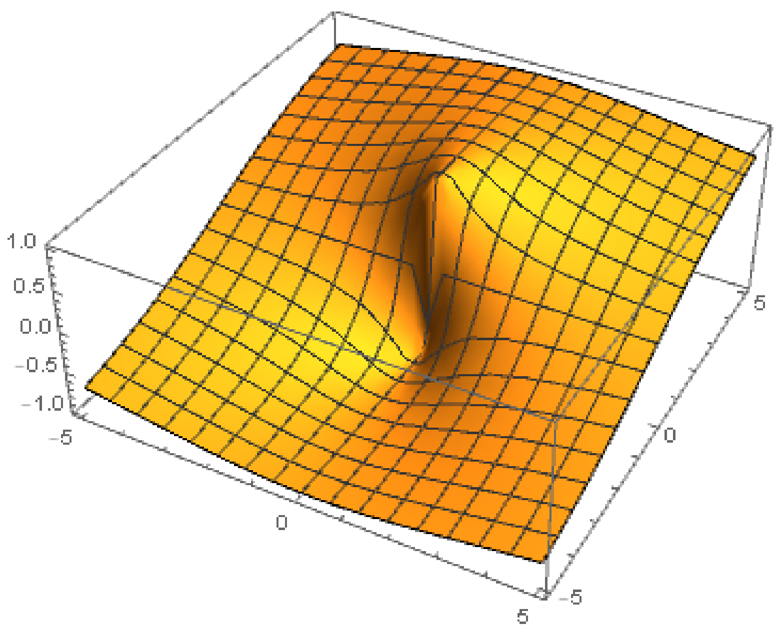
2. Consider the function $f(z)=z/|z|$, where $z=x+yi$.

   (a)
   $$f(a+bi)=\left(\frac{x}{\sqrt{x^2+y^2}}\right)+i\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

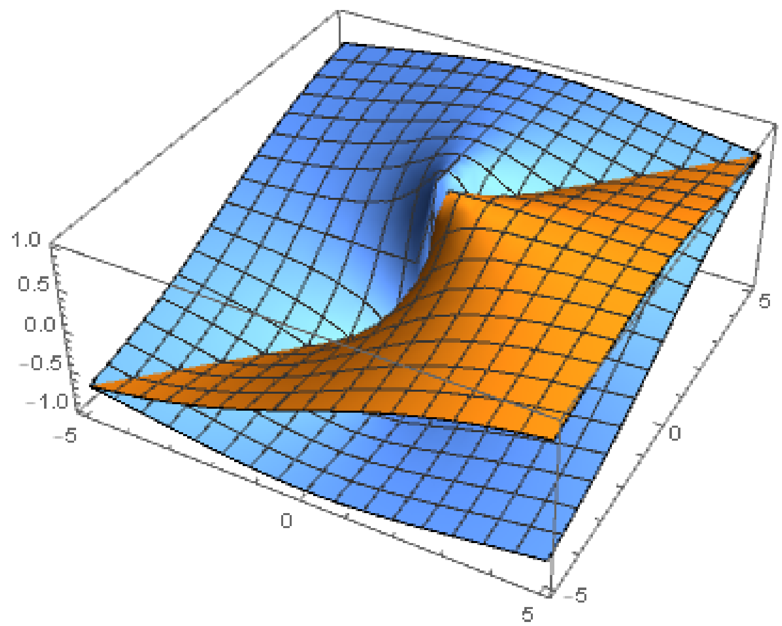
   (b) Use Mathematica to plot u and v. `Plot3D[Re[(x + I y)/Abs[x + I y]], {x, -5, 5}, {y, -5, 5}]`



This is the real component of $f(z)$ `Plot3D[Im[(x + I y)/Abs[x + I y]], {x, -5, 5}, {y, -5, 5}]`



This is the imaginary component of $f(z)$.



Here are the two figures in the same graph, where the real is in orange and the imaginary in blue.

(c) Find the limit of $f(z)$ as z approaches 0 along each of the following paths:

i.
$$\lim_{t\to 0^-} f(t+0i) = \lim_{t\to 0^-} \frac{t}{\sqrt{t^2}} = \lim_{t\to 0^-} \frac{t}{|t|} = -1$$

ii.
$$\lim_{t\to 0^+} f(t+0i) = \lim_{t\to 0^+} \frac{t}{\sqrt{t^2}} = \lim_{t\to 0^+} \frac{t}{|t|} = 1$$

iii.
$$\lim_{t\to 0^-} f(0+ti) = \lim_{t\to 0^-} i\frac{t}{\sqrt{t^2}} = \lim_{t\to 0^-} i\frac{t}{|t|} = -i$$

iv.
$$\lim_{t\to 0^+} f(0+ti) = \lim_{t\to 0^+} i\frac{t}{\sqrt{t^2}} = \lim_{t\to 0^+} i\frac{t}{|t|} = i$$

(d) Find the limit of $f(z)$ as z approaches ∞ along each of the following paths:

i.
$$\lim_{t\to\infty} f(t+0i) = \lim_{t\to\infty} \frac{t}{\sqrt{t^2}} = \lim_{t\to\infty} \frac{t}{|t|} = 1$$

ii.
$$\lim_{t\to\infty} f(0+ti) = \lim_{t\to\infty} i\frac{t}{\sqrt{t^2}} = \lim_{t\to\infty} i\frac{t}{|t|} = i$$

3. $f(z) = \text{Log}(z) = \ln|z| + i\text{Arg}(z) \implies u = \ln|z|$ and $v = \text{Arg}(z)$. To satisfy the Cauchy-Riemann equations, it must be the case that:
$$\frac{\partial \ln|z|}{\partial x} = \frac{1}{z} = \frac{\partial \text{Arg}(z)}{\partial y} \quad (1)$$
$$\frac{\partial \ln|z|}{\partial y} = \frac{i}{z} = -\frac{\partial \text{Arg}(z)}{\partial x} \quad (2)$$
$$(1) \implies \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \begin{cases} \frac{x}{x^2+y^2} & \text{if } x>0 \\ \frac{xy}{\sqrt{\frac{y^2}{x^2+y^2}}(x^2+y^2)^{3/2}} & \text{if } y>0 \\ -\frac{xy}{\sqrt{\frac{y^2}{x^2+y^2}}(x^2+y^2)^{3/2}} & \text{if } y<0 \end{cases}$$
$$\implies \frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} \text{ when } x>0.$$
$$\frac{x}{x^2+y^2} = \frac{xy}{\sqrt{\frac{y^2}{x^2+y^2}}(x^2+y^2)^{3/2}} \implies 1 = \frac{y}{\sqrt{\frac{y^2}{x^2+y^2}}\sqrt{x^2+y^2}} = \frac{y}{\sqrt{y^2}} = \frac{y}{|y|},$$

so $1 = \dfrac{y}{|y|}$ when $y>0$. Since $x \le 0$ and $y \neq 0$, then this excludes$\{z = t+0i : t \in \mathbf{R}_{\le 0}\}$
$$\frac{x}{x^2+y^2} = -\frac{xy}{\sqrt{\frac{y^2}{x^2+y^2}}(x^2+y^2)^{3/2}} \implies 1 = -\frac{y}{\sqrt{\frac{y^2}{x^2+y^2}}\sqrt{x^2+y^2}} = -\frac{y}{\sqrt{y^2}} = -\frac{y}{|y|}$$

so $1 = -\dfrac{y}{|y|}$ when $y<0$, which excludes the same region as above.

Thus the first half of the Cauchy-Riemann equation is satisfied on $\mathbf{C}\backslash\mathbf{R}_{\le 0}$
$$(2) \implies \frac{i}{x+iy} = \frac{y+ix}{x^2+y^2} = \begin{cases} \frac{y}{x^2+y^2} & \text{if } x>0 \\ \frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}} & \text{if } y>0 \\ -\frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}} & \text{if } y<0 \end{cases}$$

$$\implies \frac{y}{x^2+y^2} = \frac{y}{x^2+y^2} \text{ when } x > 0$$

$$\frac{y}{x^2+y^2} = \frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}} = \frac{|y|}{x^2+y^2} \text{ when } y > 0$$

This excludes $z = 0 + 0i$ since this point causes a division-by-zero error.

$$\frac{y}{x^2+y^2} = -\frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}} = -\frac{|y|}{x^2+y^2} \text{ when } y < 0.$$

This excludes the origin, just as above.

Therefore, the Cauchy-Riemann equations are satisfied on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$. It then follows that $f(z) = Logz$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$.

4.

$$F(z) = z\mathrm{Log}(z) - z$$

$$F'(z) = \frac{d}{dz}[z\mathrm{Log}(z) - z] = \frac{d}{dz}[z\mathrm{Log}(z)] - \frac{d}{dz}[z]$$

$$= \frac{d}{dz}[z]\mathrm{Log}(z) + z\frac{d}{dz}[\mathrm{Log}(z)] - 1 = \mathrm{Log}(z) + z\frac{d}{dz}[Log(z) - 1]$$

We now apply chain rule. $\frac{d}{dz}[Log(z)] = \frac{d\mathrm{Log}(u)}{du}\frac{du}{dz}$ where $u = z$ and $\frac{d}{du}[Log(u)] = \frac{1}{u}$, giving

$$= \mathrm{Log}(z) + z\frac{\frac{d}{dz}[z]}{z} - 1 = \mathrm{Log}(z) + z\frac{1}{z} - 1 = \mathrm{Log}(z) + 1 - 1 = \mathrm{Log}(z).$$

This shows that $F'(z) = \mathrm{Log}(z) = f(z) \implies F(z)$ is an antiderivative of $f(z)$

5. Consider the function $f(z) = \mathrm{Log}(z) + \mathrm{Log}(iz - i)$

   (a) We know that $\mathrm{Log}(z)$ on its own is holomorphic on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$. Next we know $\mathrm{Log}(iz + i) = \mathrm{Log}(i(z - 1))$. So we must exclude from the region of holomorphism all $i(z-1) = -a + 0i \implies z - 1 = 0 + ia \implies z = 1 + ia$ where $a \geq 0$. This shows we have to remove $\{z = 1 + ia \in \mathbf{C} : a \geq 0\}$ from the region of holomorphism on $\mathbf{C}$. We may then conclude that $f(z) = \mathrm{Log}(z) + \mathrm{Log}(iz + i)$ is $\mathbf{C} \setminus (\mathbf{R}_{\leq 0} \cup \{z = 1 + ia \in \mathbf{C} : a \geq 0\})$

(b) Determine all antiderivatives on the region $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$

$$\int f(z)dz = \int \mathrm{Log}(z)+\mathrm{Log}(iz-i)dz = \int \mathrm{Log}(z)dz+\int \mathrm{Log}(iz-i)dz$$

$$= [z\mathrm{Log}(z) - z + c_1] + [z\mathrm{Log}(iz-i) - z - \mathrm{Log}(1-z) + c_2]$$

$$= z(\mathrm{Log}(z) + \mathrm{Log}(iz-i) - 2) - \mathrm{Log}(1-z) + c$$

where $c = c_1 + c_2$ is an arbitrary constant.

6. The upper semicircle $C_2(0)$ oriented counter-clockwise is parametrized by

$$\gamma := 2\cos(t) + 2i\sin(t) = 2e^{it}, t \in [0, \pi]$$

$$\gamma' := -2\sin(t) + 2i\cos(t) = 2ie^{it}$$

(a) $f(z) = z + \overline{z} = a + bi + (a - bi) = 2a = 2\mathrm{Re}(z)$

$$\int_\gamma f(z)dz = \int_0^\pi f(2\cos(t) + 2i\sin(t))(-2\sin(t) + 2i\cos(t))dt$$

$$= \int_0^\pi 4\cos(t)(-2\sin(t)+2i\cos(t))dt = \int_0^\pi -8\cos(t)\sin(t)+8i\cos^2(t)dt$$

$$= 8\int_0^\pi \cos(t)\sin(t) + 8i\int_0^\pi \cos^2(t)dt = 0 + 8i\left[\frac{\pi}{2}\right] = 4i\pi$$

(b) $f(z) = z^2 - 2z + 3$

$$\int_\gamma f(z)dz = \int_0^\pi f(2e^{it})2ie^{it}dt = \int_0^\pi ((2e^{it})^2 - 4e^{it} + 3)(2ie^{it})dt$$

$$\int_0^\pi (2e^{2it} - 4e^{it} + 3)(2ie^{it})dt = \int_0^\pi 4ie^{3it} - 8ie^{2it} + 6ie^{it}dt$$

$$= \left[\frac{4}{3}e^{3it} - 4e^{2it} + 6e^{it}\right]_0^\pi = \left(-\frac{4}{3} - 4 - 6\right) - \left(\frac{4}{3} - 4 + 6\right)$$

$$= -\frac{8}{3} - 12 = -\frac{44}{3}$$

(c) $f(z) = xy = \mathrm{Re}(z)\mathrm{Im}(z)$

$$\int_\gamma f(z)dz = \int_0^\pi f(2\cos(t) + 2i\sin(t))(-2\sin(t) + 2i\cos(t))dt$$

$$= \int_0^\pi (4\cos(t)\sin(t))(-2sin(t) + 2i\cos(t))dt$$

$$= -8\int_0^\pi \cos(t)\sin^2(t)dt + 8i\int_0^\pi \cos^2(t)\sin(t)dt$$

$$= 0 + 8i\left[-\frac{1}{3}\cos^3(t)\right]_0^\pi = \frac{16i}{3}$$

(d) $f(z) = \frac{1}{z^4} = z^{-4}$

$$\int_\gamma f(z)dz = \int_0^\pi f(2e^{it})(2ie^{it})dt = \int_0^\pi (2e^{it})^{-4}(2ie^{it})$$

$$= \int_0^\pi 2e^{-4it}(2ie^{it})dt = \int_0^\pi 4ie^{-3it}dt = \left[-\frac{4}{3}e^{-3it}\right]_0^\pi$$

$$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$