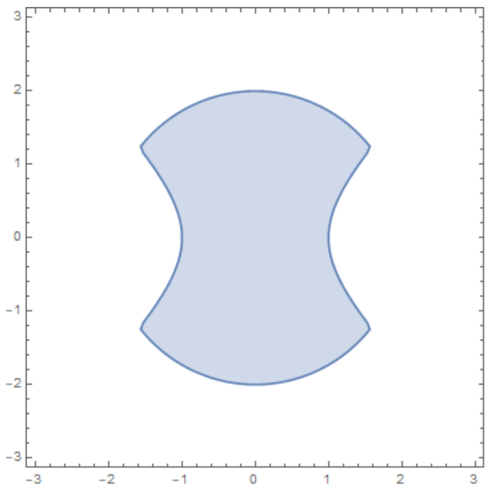
Wyatt Whiting

1. $G = \{z \in \mathbf{C} : |z| < 2 \text{ and } \mathrm{Re}(z^2) \leq 1\}$

   (a) `RegionPlot[Abs[x + I y] < 2 && Re[(x + I y)^2] <= 1, {x, -3, 3}, {y, -3, 3}]`

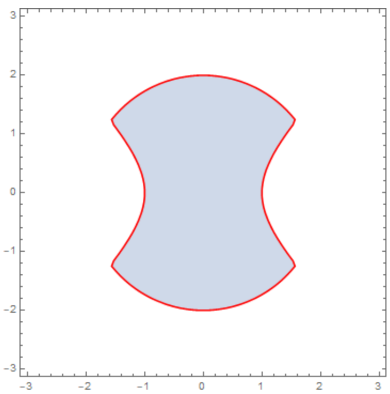


   The region does not contain the upper or lower convex curves or their endpoints, but does contain the concave curves on the sides of the region.

   (b) Interior points: $\{z \in \mathbf{C} : |z| < 2 \text{ and } \mathrm{Re}(z^2) < 1\}$

   (c) Boundary points: I wasn't able to come up with a closed set to describe the boundary, but it is the line surrounding the region including the corners. The following figure highlights the set of boundary points in red.

   `RegionPlot[Abs[x + I y] < 2 && Re[(x + I y)^2] <= 1, {x, -3, 3}, {y, -3, 3}, BoundaryStyle -> Directive[Red, Bold]]`



(d) G is neither open nor closed. We may construct a sequence of complex number which converge to $z = 2i$, but $z = 2i$ is not in the region, so it cannot be closed. It is not open because $z = 1 + 0i$ is in G, but no circle around this point will be entirely within G. Thus, G is neither open nor closed.

2. To each of the following functions, determine the region of continuity.

(a) $f(z) = \overline{z}$
$f(z) = \overline{z} = \overline{a + bi} = a - bi$. So then given an arbitrary $z_0 \in \mathbf{C}$,

$$\lim_{z \to z_0} f(z) = \lim_{a \to a_0} a - \lim_{b \to b_0} bi = a_0 - b_0 i.$$

Each component is continuous on $\mathbf{C}$, so $f(z) = \overline{z}$ is continuous on $\mathbf{C}$ as well.

(b) $f(z) = |z|$
We have

$$\lim_{z \to z_0} f(z) = \lim_{r \to r_0} \lim_{\theta \to \theta_0} |re^{i\theta}| = \lim_{r \to r_0} r = r_0$$

It intuitively makes sense that this function is continuous on all of $\mathbf{C}$ since the modulus function has no jump discontinuities; the modulus of a complex number varies smoothly between any two points.

(c) $f(z) = \sinh(z)$
We have $f(z) = \sinh(a + bi) = \sinh(a)\cos(b) + i\cosh(a)\sin(b)$. So then

$$f(z) = \sinh(a + bi) = \frac{e^a - e^{-a}}{2}\cos(b) + i\left(\frac{e^a + e^{-a}}{2}\sin(b)\right)$$

We know that sin and cos are continuous for all real values, and we also know that both $\frac{e^a - e^{-a}}{2}$ and $\frac{e^a - e^{-a}}{2}$ are continuous for all real values as well. Thus the products $\frac{e^a - e^{-a}}{2}\cos(b)$ and $\frac{e^a + e^{-a}}{2}\sin(b)$ are both continuous for any $a, b \in \mathbf{R}$. Therefore, we may conclude that $f(z) = \sinh(z)$ is continuous on $\mathbf{C}$.

(d) $f(z) = (z+1)^{1/2}$ We know that $f(z) = (z+1)^{1/2} = e^{\ln|z+1| + i\mathrm{Arg}(z+1)}$, and the region of continuity of the Arg function is $\mathbf{C}\setminus\mathbf{R}_{\leqslant 0}$. By adding 1 to the argument, the region of continuity is effectively shifted in the negative real direction by 1 unit. Thus, the region of continuity of $f(z) = (z+1)^{1/2}$ is $\mathbf{C} \setminus \mathbf{R}_{\leqslant -1}$.

(e) $f(z) = \mathrm{Log}(z-i) + \mathrm{Log}(z+i)$

Each Log term has its own region of continuity. We are using Log and not log, so we use the principle branch of the function. Subtracting i from the z has the effect of moving the region of continuity of $\mathrm{Log}(z-i)$ by $+i$ units relative to $\mathrm{Log}(z)$, so the region of continuity of $Log(z-i)$ is $\{z = a + bi \in \mathbf{C} : a \in \mathbf{R}_{\leqslant 0}, b = 1\}$. By a parallel argument, the region of continuity of $\mathrm{Log}(z+i)$ is $\{z = a + bi \in \mathbf{C} : a = \mathbf{R}_{\leqslant 0}, b = -1\}$. Thus, the region of continuity of $f(z) = \mathbf{C} \setminus \{z = a + bi \in \mathbf{C} : a \leqslant 0 \text{ and } (b = 1 \text{ or } b = -1)\}$

3. Find a parametrization for each of the following curves:

(a) The circle centered at $1 + i$ with radius 3

$$\{(3\cos(t) + 1) + i(3\sin(t) + 1) \in \mathbf{C} : 0 \leqslant t \leq 2\pi\}$$

(b) The line segment from $-1 - i$ to $2i$.

$$\{(-1 + t) + i(-1 + 3t) \in \mathbf{C} : 0 \leqslant t \leqslant 1\}$$

(c) The infinite line passing through $1 - 2i$ and $2 + i$

$$\{(1 + t) + i(-2 + 3t) \in \mathbf{C} : -\infty < t < \infty\}$$

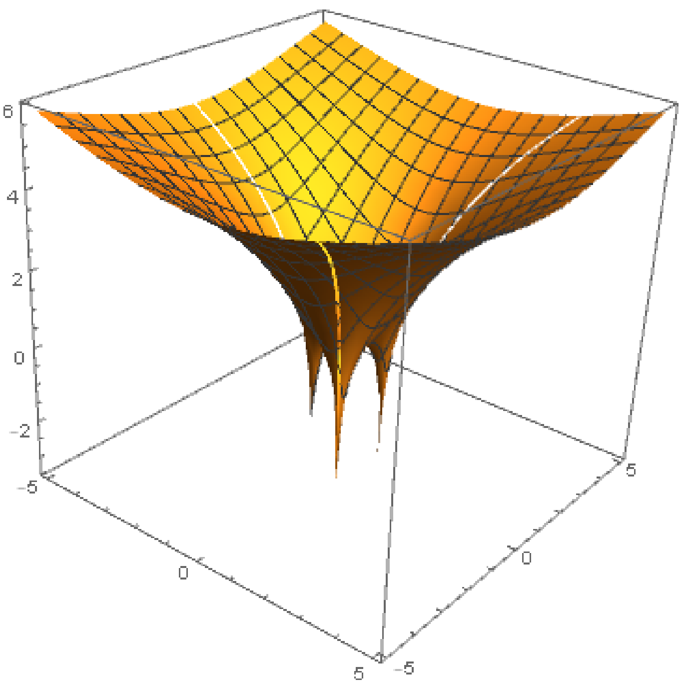
(d) The upper half of the circle centered at $-1 + i$ with radius 2 oriented clockwise.

$$\{(-2\cos(t) - 1) + i(2\sin(t) + 1) \in \mathbf{C} : 0 \leqslant t \leqslant \pi\}$$

5. Consider the multivalued function $g(z) = \log(z^3 + 1)$

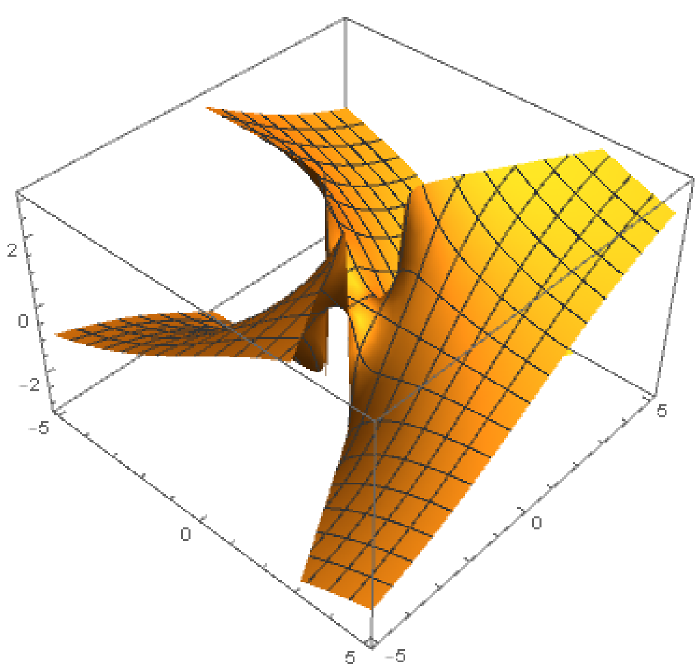
(a) $\log(z^3 + 1) = \ln|z^3 + 1| + i\arg(z^3 + 1)$, so $\mathrm{Re}(g(z)) = \ln|z^3 + 1|$ and $\mathrm{Im}(g(z)) = \arg(z^3 + 1)$.

(b)
```
g[z_] := Log[z^3 + 1]
ParametricPlot3D[{x, y, Re[g[x + I y]]}, {x, -5, 5}, {y, -5,
5}]
```



(c) $F(z) = \mathrm{Arg}(z^3 + 1)$

(d) `ParametricPlot3D[{x, y, Arg[(x + I y)^3 + 1]}, {x, -5, 5}, {y, -5, 5}]`



$$F(1+i) = \text{Arg}((1+i)^3+1) = Arg(-1+2i) = \arctan(-2)+\pi$$

(e) The branch cuts in the complex plane are three rays pointing radially away from the origin with endpoints evenly distributed around the origin such that the rays are symmetric over the real axis. The branch points are the endpoints of the rays.

(f) Let B denote the set of branch points. Then,

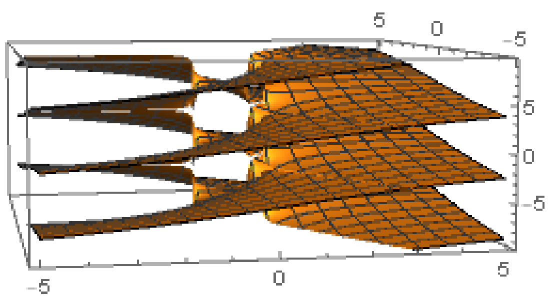
$$B = \{e^{-i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, e^{i\pi}\}$$

(g) The branch cuts are represented by the paths:

$$\gamma_1 := te^{-i\frac{\pi}{3}};$$
$$\gamma_2 := te^{i\frac{\pi}{3}};$$
$$\gamma_3 := te^{i\pi};$$

where $t \geq 1$ in each path.

(h) `p[k_] := Plot3D[Arg[(x + I y)^3 + 1] + Pi*2*k, {x, -5, 5}, {y, -5, 5}] Show[p[-1], p[0], p[1], PlotRange -> All]`



The figure was rotated from its default view in order to show the multivalued property of the function $f(z)$.