Wyatt Whiting

1. Find all complex solutions of the following equations
   (a) $z + 2\overline{z} = 1 \implies (a+bi) + 2(a-bi) = 1 \implies 3a - bi = 1 + 0i \implies a = \frac{1}{3}, b = 0 \implies z = \frac{1}{3} + 0i$
   (b) $2z^2 + (i-1)z + 5i = 0 \implies z_1 = \frac{1+\sqrt{21}}{4} - \frac{1+\sqrt{21}}{4}i, z_2 = -\frac{\sqrt{21}-1}{4} + \frac{\sqrt{21}-1}{4}i$
   (c) $z^2 + 2z^{-2} = -2 \implies z^2 + 2 + 2z^{-2} = 0 \implies \frac{z^4+2z+2}{z^2} = 0 \implies z^4 + 2z^2 + 2 = 0 \implies z_1 = 2^{1/4}e^{i\frac{5\pi}{8}}, z_2 = 2^{1/4}e^{\frac{-3\pi}{8}}, z_3 = 2^{1/4}e^{\frac{-5\pi}{8}}, z_4 = 2^{1/4}e^{\frac{3\pi}{8}}$
   (d) $z^3 + iz^2 + 7z - 5i = 0 \implies z_1 = i, z_2 = 0 + (\sqrt{6}-1)i, z_3 = 0 - (\sqrt{6}+1)i$
2. Find all complex roots:
   (a) $(i+1)^{1/3} = z \implies z^3 = i + 1 = \sqrt{2}\text{cis}(\frac{\pi}{4}) \implies z_1 = 2^{1/6}\text{cis}(\frac{\pi}{12}), z_2 = 2^{1/6}\text{cis}(\frac{3\pi}{4}), z_3 = 2^{1/6}\text{cis}(\frac{17\pi}{12})$
   (b) $i^{1/4} = z \implies i = z^4 = \text{cis}(\frac{\pi}{2}) \implies z_1 = \text{cis}(\frac{\pi}{8}), z_2 = \text{cis}(\frac{5\pi}{8}), z_3 = \text{cis}(\frac{9\pi}{8}), z_4 = \text{cis}(\frac{13\pi}{8})$
   (c) $(-1)^{1/5} = z \implies -1 = z^5 = \text{cis}(\pi) \implies z_1 = \text{cis}(\frac{\pi}{5}), z_2 = \text{cis}(\frac{3\pi}{5}), z_3 = \text{cis}(\pi), z_4 = \text{cis}(\frac{7\pi}{5}), z_5 = \text{cis}(\frac{9\pi}{5})$
   (d) $(1+2i)^{3/2} = z \implies z^{2/3} = 1 + 2i \implies z^2 = (1+2i)^3 = -11 - 2i = 5\sqrt{5}\text{cis}(\arctan(2/11)) \implies z_1 = 5^{3/4}\text{cis}(\frac{\arctan\frac{2}{11}}{2}), z_1 = 5^{3/4}\text{cis}(\frac{\arctan\frac{2}{11}}{2})$
3. Use de Moivre's formula to express $\cos(3x)$ and $\sin(3x)$ in terms of $\cos(x)$ and $\sin(x)$.

$$\text{cis}^n(x) = \text{cis}(nx) \implies \text{cis}^3(x) = \text{cis}(3x)$$
$$(\cos^2 x - \sin^2 x + 2i\cos x \sin x)(\cos x + isinx) = \cos 3x + i \sin 3x$$
$$\cos^3 x - 3\sin^2 x \cos x + i(3\cos^2 x \sin x - \sin^3 x) = \cos 3x + i \sin 3x$$
$$\sin 3x = 3\cos^2 x \sin x - \sin^3 x$$

4. Determine if each of the following statements is true. If it is, prove it. If it is not, give a counterexample.

(a) *Proof*: Let $z, w \in \mathbf{C}$ be arbitrary. Then $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$. We then have $\overline{z + w} = \overline{(a + c) + i(b + d)} = (a+c)-i(b+d) = (a-ib)+(c-id) = \overline{z}+\overline{w}$. This proves $\overline{z + w} = \overline{z}+\overline{w}$.

(b) *Proof*: Let $z, w \in \mathbf{C}$ be arbitrary. Then $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$. We then have $\overline{zw} = \overline{(a + bi)(c + di)} = \overline{(ac - bd) + i(ad + bc)} = (ac - bd) - i(ad + bc) = ac - iad - bd - ibc = a(c - id) - bi(c - id) = (a - bi)(c - di) = \overline{z}\,\overline{w}$. This proves $\overline{zw} = \overline{z}\,\overline{w}$.

(c) *Counterexample*: Let $z = 1 + 0i$ and $w = 0 + i$. Then $|z + w| = |1 + i| = \sqrt{2}$. But $|z| = 1$ and $|w| = 1$ so $|z| + |w| = 2 \neq \sqrt{2} = |z + w|$.

(d) *Proof*: Let $z, w \in \mathbf{C}$ be arbitrary. Then $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ where $r_1, r_2 \in \mathbf{R}$ and $\theta_1, \theta_2 \in (-\pi, \pi]$. We then have $\text{Arg}(zw) = \text{Arg}(r_1 r_2 e^{i\theta_1 + i\theta_2}) = \text{Arg}(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \theta_1 + \theta_2 = \text{Arg}(z) + \text{Arg}(w)$. In the case that $\text{Arg}(z)+\text{Arg}(w) \notin (-\pi, \pi]$, then taking this quantity modulo 2π will ensure it is within the range $(-\pi, \pi]$. This proves $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$ modulo 2π.

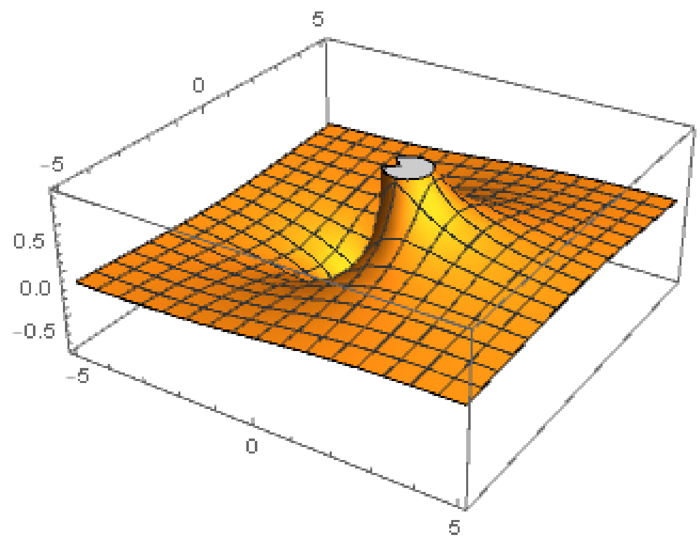
(e) *Proof*: Let $z, w \in \mathbf{C}$ be arbitrary. Then $z = r_1 e^{i\theta_1}$ and $z = r_2 e^{i\theta_2}$. We then have $\text{Arg}(zw) = r_2 r_2 e^{i(\theta_1+\theta_2)}$. If both $\theta_1, \theta_2 \in (\frac{-\pi}{4}, \frac{\pi}{4}]$, then $\theta_1 + \theta_2 \in (-\pi, \pi]$. If the sum is outside the range $(-\pi, \pi]$, then taking the quantity modulo 2π ensures it is within the range. This proves $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)(\text{modulo } 2\pi)$.

(f) *Counterexample*: Let $z = w = e^{i\pi}$. Then $\text{Arg}(zw) = \text{Arg}(e^{i2\pi}) = 0$, but $\text{Arg}(z) + \text{Arg}(w) = \pi + \pi = 2\pi \neq 0$.

(g) *Counterexample*: Let $z = w = e^{i(\pi/2)}$. Then $\arg(z+w) = \arg(2e^{i(\pi/2)}) = \frac{\pi}{2}$ modulo 2π. But $\arg(z) + \arg(w) = \arg(e^{i(\pi/2)}) + \arg(e^{i(\pi/2)}) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ modulo 2π. Thus $\arg(z + w) \neq \arg(z) + \arg(w)$.

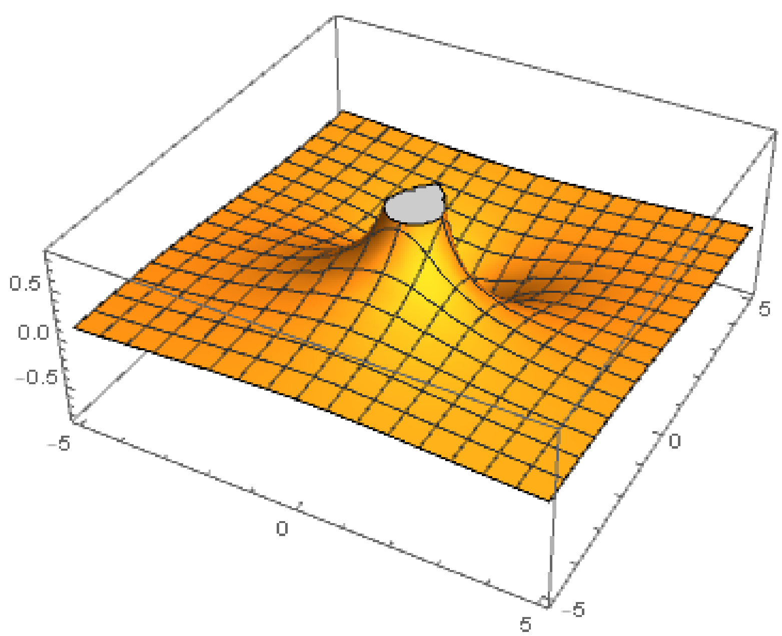
5. $f(z) = 1/z$

   (a) $\text{Re}[f(z)] \rightarrow$ `Plot3D[Re[1/(x + I y)], {x, -5, 5}, {y, -5, 5}]`
   This command graphs the real component of the inverse of a complex number.



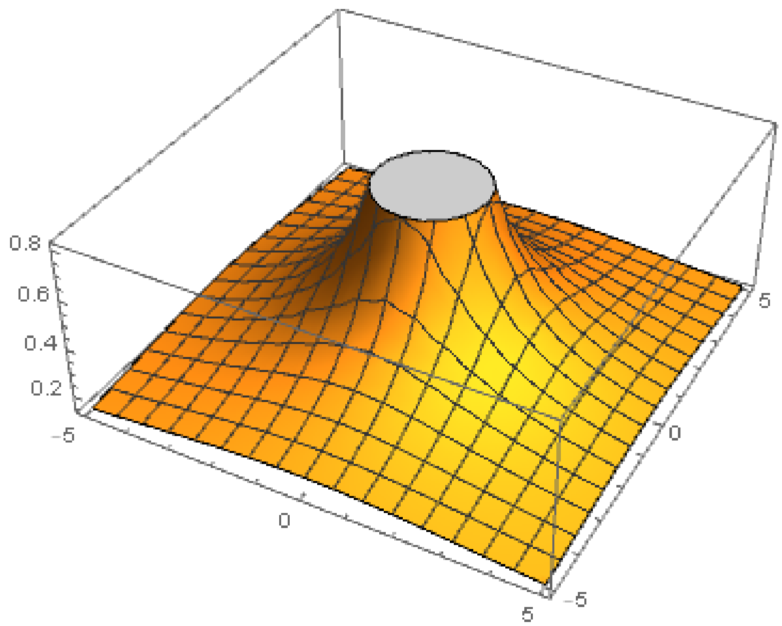
The above figure shows that the function $f(z) = 1/z$ acts the way we expect along the real line. We can also see that the function is defined everywhere except for $z = 0$, which makes good sense.

(b) $\text{Im}[f(z)] \rightarrow$ `Plot3D[Im[1/(x + I y)], {x, -5, 5}, {y, -5, 5}]` This code is identical to that above, but instead of plotting the real component of the inverse of z, we plot the imaginary component.



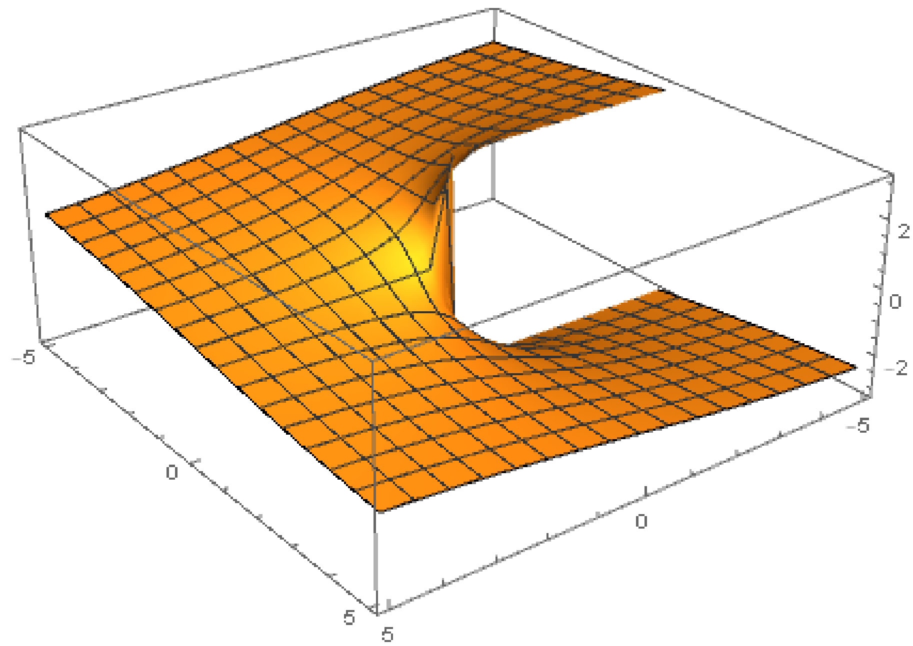
We can see from this graph that the it is identical to the real component except for it being rotated around the z-axis. This feels intuitive as on the complex plane, the imaginary numbers are "rotated" relative to the real line.

(c) $|f(z)|$ $\rightarrow$ `Plot3D[Abs[1/(x + I y)], {x, -5, 5}, {y, -5, 5}]` Now we plot the magnitude of the inverses using the `Abs[]` function in Mathematica.



Nothing seen here is unexpected. Numbers closer to 0 have extremely large inverses which naturally have a large positive magnitude. This causes the peak forming around the center of the figure.

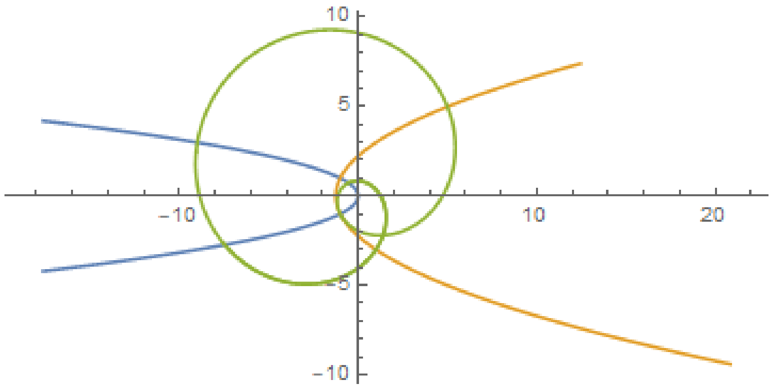
(d) Arg$f(z)$ $\rightarrow$`Plot3D[Arg[1/(x + I y)], {x, -5, 5}, {y, -5, 5}]` This code shows the argument of the inverse of a complex numbers as the height of the graph.



From this we are able to easily imagine the periodicity of the function by imagining this figure stacked on top of itself an infinite number of times. If you were to take a perfectly vertical line, it would intersect such a graph an infinite number of teams each corresponding to a 2π multiple of the argument. However, this graph only shows the principle branch of the argument which is inside the ragne $(-\pi, \pi]$.

6. $f(z) = z^2 - z$

   (a) $x = 1, y - 1,$ circle centered at $(1,1)$ $f(z) = z^2 - z$ can be rewritten as $f(x,y) = (x^2 - y^2, 2xy) - (x,y) = (x^2 - y^2 - x, 2xy - y)$. The curves $x = 1, y = 1,$ and a circle of radius 2 centered at $(1,1)$ can be rewritten as $(1,t),(t,1)$, and $(2\cos t - 1, 2\sin t - 1)$ respectively. By plugging these in to $f(x,y)$, we obtain the Mathematica code:
```
ParametricPlot[{{-t^2,t},{t^2-t-1,2t-1},{(2Cos[t]+1)^2-(2
Sin[t]+1)^2-(2Cos[t]+1),2(2Cos[t]+1)(2Sin[t]+1)-(2Sin[t]+1)}},
{t,-4.2,4.2}]
```



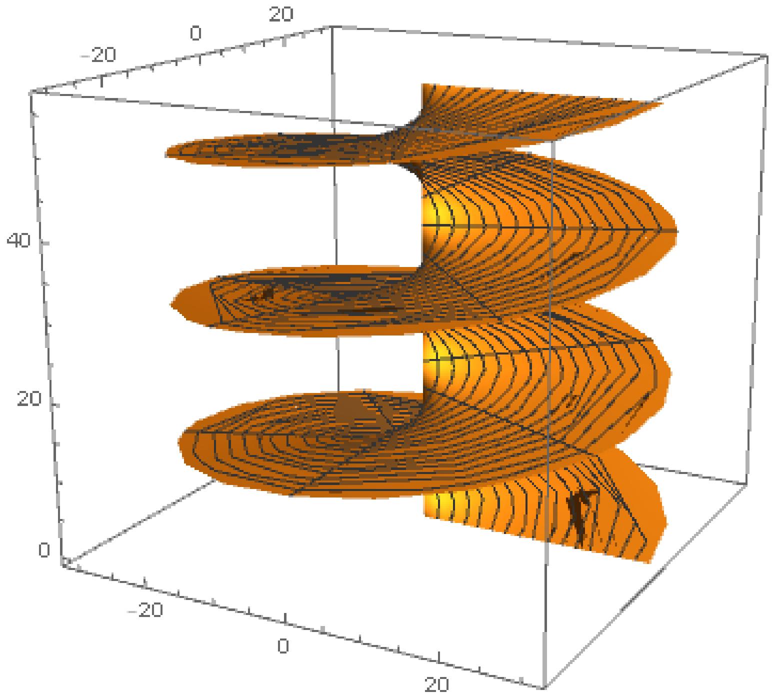
From this figure we can see that straight lines have been transformed into parabolas, which makes sense since the function has a z^2 term. What I find surprising is the image of the circle, shown above in green. It is still round, but has a crimp on one side where it meets the origin.

(b) The angles between intersections of graphs before and after the transformation are preserved. Thus, this function is a conformal map.

7. Visualizing multi-valued function arg z.

(a) $re^{i\theta} = r\cos\theta + i\, r\sin\theta \implies x = r\cos\theta$ and $y = r\sin\theta$.

(b) `ParametricPlot3D[{rCos[:theta:],rSin[:theta:],3:theta:},{:theta:,0,6 Pi},{r,0,10 Pi}]` produces the figure below. The multiplication of :theta: by 6 in the first set of curly brackets stretch the graph out to make the folds more visible.



Here it is clear to see the periodicity of the non-principle arg$z$ function. If we were able to graph the function with parameters out to infinity, the entire space would be filled with this spiral around the center, with arms stretching out horizontally to an infinite distance. It would be a very disorienting place indeed.