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1. Find the radius of convergence and the region of convergence of the following power series:

   (a) $\sum_{n=1}^{\infty} \frac{(2z)^n}{1+3^n}$
   Applying the ratio test:

   $$\lim_{n\to\infty}\left|\frac{(2z)^{n+1}}{1+3^{n+1}}\frac{1+3^n}{(2z)^n}\right| = \lim_{n\to\infty}\left|\frac{2z(1+3^n)}{1+3^{n+1}}\right| = \frac{2|z|}{3}$$

   We demand that $\frac{2|z|}{3} < 1$, so then $|z| < \frac{3}{2}$, so the radius of convergence is $\frac{3}{2}$. We can then conclude that the region of convergence is $D_{3/2}(0)$

   (b) $\sum_{n=1}^{\infty} \frac{z^n}{(\sqrt{3}+i)^n} = \sum_{n=1}^{\infty}(\frac{z}{\sqrt{3}+i})^n$
   The form of the summand make applying the root test quite easy:

   $$\lim_{n\to\infty}\sqrt[n]{\left(\frac{z}{\sqrt{3}+i}\right)^n} = \frac{z}{\sqrt{3}+i}$$

   To converge, we demand $\frac{|z|}{|\sqrt{3}+i|} < 1 \implies \frac{|z|}{2} < 1 \implies |z| < 2$, so the radius of convergence is 2. Then the region of convergence is $D_4(0)$.

2. Find three different Laurent series representations (about 0) for the function

   $$f(z) = \frac{3}{-z^2+z+2} = \frac{1}{z+1} - \frac{1}{z-2}$$

   We can see that the function has two singularities at $z = -1$ and $z = 2$. These split the plane into three regions with varying radius. Let's begin with the region $|z| < 1$.

   The function is holomorphic on $D_1(0)$ since it contains no singularities. Then our Laurent series ends up being a regular Taylor series given by

   $$f(z) = \frac{3}{-z^2+z+2} = \frac{1}{1+z} + \frac{1}{2-z} = \sum_{k=0}^{\infty}(-z)^k + \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$$

   $$= \sum_{k=0}^{\infty}(-z)^k + \frac{z^k}{2^{k+1}},$$

which holds when $|z| < 1$.

Now we focus on the annulus $1 < |z| < 2$. We can manipulate the terms as follows:

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{\frac{1}{z}+1} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^k} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}},$$

which converges when $|z| > 1$. We can also expand the other term, giving

$$\frac{1}{2-z} = \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}},$$

which holds for $|z| < 2$. Combining these gives

$$f(z) = \frac{1}{z+1} + \frac{1}{2-z} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} + \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} + \frac{z^k}{2^{k+1}},$$

which holds for $1 < |z| < 2$.

Finally, let's examine the case $|z| > 2$. We can recycle the manipulation of $\frac{1}{1+z}$ from above since its valid for $|z| > 1$. For the other term, we have

$$\frac{1}{2-z} = \frac{1}{z}\frac{1}{\frac{2}{z}-1} = -\frac{1}{z}\frac{1}{1-\frac{2}{z}} = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{2^k}{z^k} = -\sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}},$$

which holds for $|z| > 2$. Combining the two series gives

$$\begin{aligned} f(z) = \frac{1}{z+1} + \frac{1}{2-z} &= \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} - \sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k - 2^k}{z^{k+1}}, \end{aligned}$$

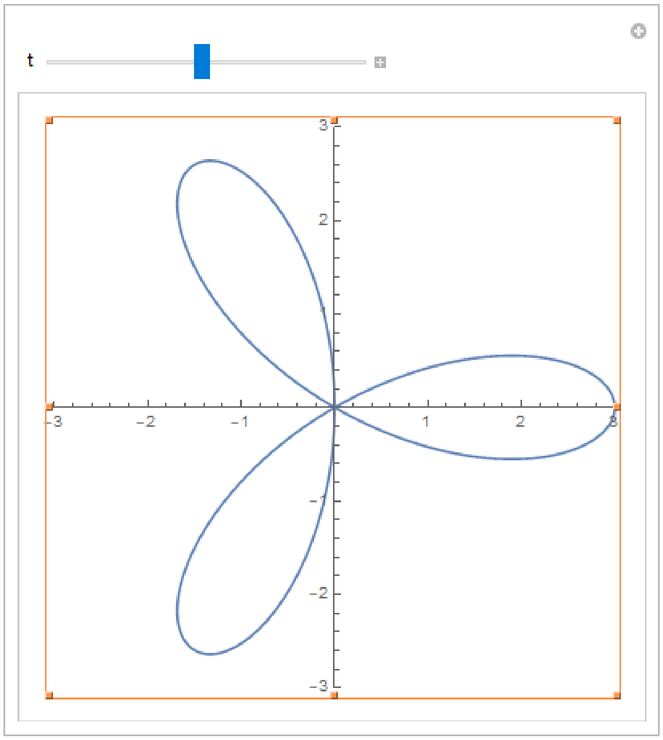
which holds for all $|z| > 2$. Thus, these are three different Laurent series for $f(z) = \frac{3}{-z^2+z+2}$.

3. (a) Sketch γ.

```
p[s_] := ParametricPlot[{3 Cos[t] Cos[3 t], 3 Sin[t] Cos[3
t]}, {t, 0, s}, PlotRange -> {{-3, 3}, {-3, 3}}]
```

```
Manipulate[p[t], {t, 0.1, 2 Pi}]
```

This code allows me to use the slider to watch what happens to the graph of γ as the range of t is varied. It's from this that we see that each lobe has positive orientation as t is increasing.



We can see that the figure completes an entire loop and touches its endpoint when $t = \pi$. Thus, since $t \in [0, 2\pi]$, γ traces itself twice.

(b) Compute the exact value of $\int_\gamma f(z)\, dz$

In order the compute this integral, we will split up γ into its three lobes and evaluate them separately. Let the rightmost lobe be γ_1 and working counterclockwise, the next two lobes be γ_2 and γ_3. By watching the manipulation of γ with Mathematica, we can each region is positively oriented.

Let's begin with γ_1. We can see that γ_1 encloses the point $a = 1$. We can then apply Cauchy's integral formula by rearranging the integrand as follows:

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_1} \frac{1}{(z^2+2z+2)(z-1)}\, dz = \int_{\gamma_1} \frac{\frac{1}{(z^2+2z+2)}}{z-1}\, dz$$

$$= 2\pi i \frac{1}{1+2+2} = \frac{2\pi i}{5}$$

Now let's examine γ_2 in a simliar fashion. We can see the point $a = -1 + i$ is contained within γ_2, which is also a root of the second degree polynomial. We can then rearrange the integral as follows:

$$\int_{\gamma_2} \frac{1}{(z^2+2z+2)(z-1)}\,dz = \int_{\gamma_2} \frac{\frac{1}{(z+1+i)(z-1)}}{z+1-i}\,dz$$
$$= 2\pi i \frac{1}{((-1+i)-1)((-1+i)+1+i)} = 2\pi i \frac{1}{(-2+i)(2i)}$$
$$= \frac{2\pi i}{-2-4i} = \frac{\pi i}{-1-2i}$$

Finally, let's evaluate γ_3 in a similar fashion. I omit explanation for obvious reasons.

$$\int_{\gamma_3} \frac{1}{(z^2+2z+2)(z-1)}\,dz = \int_{\gamma_3} \frac{\frac{1}{(z-1)(z+1-i)}}{z+1+i}$$
$$= 2\pi i \frac{1}{((-1-i)-1)((-1-i)+1-i)} = 2\pi i \frac{1}{(-2-i)(-2i)}$$
$$= \frac{2\pi i}{-2+4i} = \frac{\pi i}{-1+2i}$$

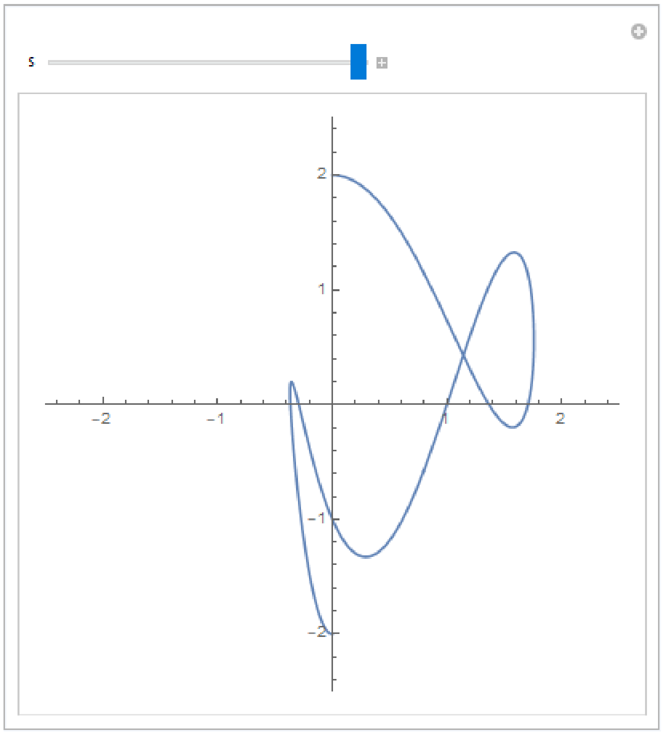
Now we add the integrals together:

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$
$$= \frac{2\pi i}{5} + \frac{\pi i}{-1-2i} + \frac{\pi i}{-1+2i} = 0$$

4. (a) Sketch γ.

```
p[s_] := ParametricPlot[{Sin[t] + Sin[2 t], Cos[t] + Cos[5
t]}, {t, 0, s}, PlotRange -> {{-2.5, 2.5}, {-2.5, 2.5}}]
Manipulate[p[s], {s, 0.1, Pi}]
```

Similar to the previous code, this enters γ as a parametrized curve and then allows me to view how γ is traced as t varies from 0 to π.



We can see that γ takes a very strange path. It is not "nice" as in the previous problem, which makes integration over γ a somewhat more formidable task.

(b) Find a Laurent series representation of $f(z)$ about 0.

Luckily for us, the form of this function makes finding a proper Laurent series exceedingly easy. Observe:

$$f(z) = \frac{e^z}{z^3} = z^{-3}e^z$$

We already know a series expansion for e^z, giving

$$z^{-3}e^z = z^{-3}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...\right) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + ...$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-3}}{n!}$$

(c) Use part (b) to find an approximation for $\int_\gamma f(z)\, dz$.

For this section I just mimicked the example in the supplementary document, evaluating terms up to $k = 3$ by hand. This gave me the mathematica code

```
n = 100
-1/(2 (2 I)^2) + 1/(2 (-2 I)^2) - 1/(2 I) - 1/(2 I) + (I Pi/
2) +  N[Sum[((2 I)^(k - 2) - (-2 I)^(k - 2))/(k!  (k - 2)),
{k, 3, n}]]
```

When evaluated, this gives the solution

$$\int_\gamma f(z)\, dz \approx 0 + 3.1955i.$$