Part One : Metric Spaces

Chapter 1: Calculus Review

1.1 If A is a nonempty subset of $\mathbb{R}$ that is bounded below, show that A has a greatest lower bound.

1.3 Establish the following apparently different (but "fancier") characterization of the supremum. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $s = \sup A$ if and only if (i) s is an upper bound for A, and (ii) for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$. State and prove the corresponding result for the infimum of a nonempty subset of $\mathbb{R}$ that is bounded below.

1.4 Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that there is a sequence (x_n) of elements of A that converges to $\sup A$.

1.6 Prove that every convergent sequence of real numbers is bounded. Moreover, if (a_n) is convergent, show that $\inf_n a_n \leq \lim_{n\to\infty} a_n \leq \sup_n a_n$.

1.13 Let $a_n \geq 0$ for all n, and let $s_n = \sum_{i=1}^{n} a_i$. Show that (s_n) converges if and only if (s_n) is bounded.

1.14 Prove that a convergent sequence is Cauchy, and that any Cauchy sequence is bounded.

1.15 Show that a Cauchy sequence with a convergent subsequence actually converges.

1.17 Given real numbers a and b, establish the following formulas: $|a+b| \leq |a| + |b|$, $||a| - |b|| \leq |a-b|$, $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$, and $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$.

1.21 Let $p \geq 2$ be a fixed integer, and let $0 < x < 1$. If x has a finite-length base p decimal expansion, that is, if $x = a_1/p + \cdots + a_n/p^n$ with $a_n \neq 0$, prove that x has precisely *two* base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique. Characterize the numbers $0 < x < 1$ that have *repeating* base p decimal expansions. How about *eventually repeating*?

1.24 Show that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

1.25 If $\limsup_{n\to\infty} a_n = -\infty$, show that (a_n) diverges to $-\infty$. If $\limsup_{n\to\infty} = +\infty$, show that (a_n) has a *subsequence* that diverges to $+\infty$. What happens if $\liminf_{n\to\infty} a_n = \pm\infty$.

1.26 Prove the characterization of lim sup given above. That is, given a bounded sequence (a_n), show that the number $M = \limsup_{n\to\infty} a_n$ satisfies $(*)$ and, conversely, that any number M satisfying $(*)$ must equal $\limsup_{n\to\infty} a_n$. State and prove the corresponding result for $m = \liminf_{n\to\infty} a_n$.

1.27 Prove that every sequence of real numbers (a_n) has a subsequence (a_{n_k}) that converges to $\limsup_{n\to\infty}$.

1.33 Show that (x_n) converges to $x \in \mathbb{R}$ if and only if every subsequence (x_{n_k}) has a *further* subsequence $(x_{n_{k_l}})$ that converges to x.

1.37 If (E_n) is a sequence of subsets of a fixed set S, we define

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) \quad \text{and} \quad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right).$$

Show that

$$\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n \text{ and that } \quad \liminf_{n\to\infty}\,(E_n^c) = \left(\limsup_{n\to\infty} E_n\right)^c$$

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1.45 Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose that $f(x) = 0$ whenever x is rational. Show that $f(x) = 0$ for every x in $[a,b]$.

1.46 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some open interval $(-a, a)$.

(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≤ 0" replaced by "> 0"? Explain.

Chapter 2: Countable and Uncountable Sets

2.4 Show that any infinite set has a countably infinite subset.

2.6 If A is infinite and B is countable, show that A and $A \cup B$ are equivalent. [Hint: No containment relation between A and B is assumed here.

2.13 Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

2.15 Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable. [Hint: Each one contains a rational!]

2.21 Show that any ternary decimal of the form $0.a_1a_2\cdots a_n11$ (base 3), i.e., any finite-length decimal ending in two (or more) 1s, is *not* an element of Δ.

2.22 Show that Δ contains no (nonempty) open intervals. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $z \in [0,1] \setminus \Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small").

2.23 The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all the points in Δ of the form $a/3^n$ for some integers n and $0 \leq a \leq 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as $0.a_1a_2\cdots a_{n+1}$ (base 3), where each a_k is 0 or 2, except a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2\cdots a_n1, 0.a_1a_2\cdots a_n2)$, where both entries are points of Δ written in base 3.

2.26 Let $f : \Delta \to [0,1]$ be the Cantor function (defined above) and let $x, y \in \Delta$ with $x < y$. Show that $f(x) \leq f(y)$. If $f(x) = f(y)$, show that x has two distinct binary decimal expansions. Finally, show that $f(x) = f(y)$ if and only if x and y are "consecutive" endpoints of the form $x = 0.a_1a_2\cdots a_n1$ and $y = 0.a_1a_2\cdots a_n2$ (base 3).

2.29 Prove that the extended cantor function $f : [0,1] \to [0,1]$ (as defined above) is increasing. [Hint: Consider cases.]

Chapter 3: Metrics and Norms

3.2 If d is a metric on M, show that $|d(x,z) - d(z,y)| \leq d(x,y)$ for any $x, y, z \in M$.

3.5 There are other, albeit less natural, choices for a metric on $\mathbb{R}$. For instance, check that $\rho(a,b) = \sqrt{|a-b|}$, $\sigma(a,b) = |a-b|/(1+|a-b|)$, and $\tau(a,b) = \min\{|a-b|, 1\}$ each define metrics on $\mathbb{R}$. [Hint: To show that σ is a metric, you might first show that the function $F(t) = t/(1+t)$ is increasing and satisfies $F(s+t) \leq F(s) + F(t)$ for $s, t \geq 0$. A similar approach will also work for ρ and τ.]

3.6 If d is any metric on M, show that $\rho(a,b) = \sqrt{d(x,y)}$, $\sigma(a,b) = d(x,y)/(1+d(x,y))$, and $\tau(a,b) = \min\{d(x,y), 1\}$ are also metrics on M. [Hint: $\sigma(x,y) = F(d(x,y))$, where F is as in Exercise 5.]

3.14 We say that a subset A of a metric space M is **bounded** if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.

3.15 We define the **diameter** of a nonempty subset A of M by $\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Show that A is bounded if and only if $\operatorname{diam}(A)$ is finite.

3.18 Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$. Also check that $\|x\|_1 \leq n\|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

3.29 Prove that A is bounded if and only if $\operatorname{diam}(A) < \infty$.

3.30 If $A \subset B$, show that $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

3.32 In a normed vector space $(V, \|\cdot\|)$ show that $B_r(x) = x + B_r(0) = \{x + y : \|y\| < r\}$ and that $B_r(0) = rB_1(0) = \{rx : \|x\| < 1\}$.

3.34 If $x_n \to x$ in (M,d), show that $d(x_n, y) \to d(x,y)$ for any $y \in M$. More generally, if $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x,y)$.

3.36 A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set $\{x_n : n \geq 1\}$ is bounded).

3.37 A Cauchy sequence with a convergent subsequence converges.

3.39 If every subsequence of (x_n) has a further subsequence that converges to x, then (x_n) converges to x.

3.42 Two metrics d and ρ on a set M are said to be **equivalent** if they generate the same convergent sequences; that is, $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. If d is any metric on M, show that the metrics ρ, σ, and τ, defined in Exercise 6, are all equivalent to d.

3.43 Show that the usual metric on $\mathbb{N}$ is equivalent to the discrete metric. Show that any metric on a *finite* set is equivalent to the discrete metric.

3.44 Show that the metrics induced by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are all equivalent. [Hint: See Exercise 18.]

3.46 Given two metric spaces (M, d) and (N, ρ), we can define a metric on the product $M \times N$ in a variety of ways. Our only requirement is that a sequence of pairs (a_n, x_n) in $M \times N$ should converge precisely when both coordinate sequences (a_n) and (x_n) (in (M, d) and (N, ρ) respectively). Show that each of the following define metrics on $M \times N$ that enjoy this property and hat all three are equivalent:

$$\begin{aligned} d_1\left((a,x),(b,y)\right) &= d(a,b) + \rho(x,y), \\ d_2\left((a,x),(b,y)\right) &= \left(d(a,b)^2 + \rho(x,y)^2\right)^{1/2}, \\ d_\infty\left((a,x),(b,y)\right) &= \max\{d(a,b), \rho(x,y)\}. \end{aligned}$$

Chapter 4: Open Sets and Closed Sets

4.3 Some authors say that two metrics d and ρ on a set M are equivalent if they generate the same open sets. Prove this. (Recall that we have defined equivalence to mean that d and ρ generate the same convergent sequences. See Exercise 3.42.)

4.5 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

4.8 Show that every open interval (and hence every open set) in $\mathbb{R}$ is a countable intersection of open intervals.

4.11 Let $e^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, where the kth entry is 1 and the rest are 0s. Show that $\{e^{(k)} : k \geq 1\}$ is closed as a subset of l_1.

4.17 Show that A is open if and only if $A^\circ = A$ and that A is closed if and only if $\overline{A} = A$.

4.18 Given a nonempty bounded subset E of $\mathbb{R}$, show hat $\sup E$ and $\inf E$ are elements of $\overline{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is *closed*.

4.19 Show that $\operatorname{diam}(A) = \operatorname{diam}\left(\overline{A}\right)$

4.33 Let A be a subset of M. A point $x \in M$ is called a **limit point** of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \varnothing$ for every $\varepsilon > 0$. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

4.34 Show that x is a limit point of A if and only if there is a sequence (x_n) in A such that $x_n \to x$ and $x_n \neq x$ for all n.

4.46 A set A is said to be **dense** in M (or, as some authors say, *everywhere dense*) if $\overline{A} = M$. For example, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$. Show that A is dense in M if and only if any of the following hold:

(a) Every point in M is the limit of a sequence from A.

(b) $B_\varepsilon(x) \cap A \neq \varnothing$ for every $x \in M$ and every $\varepsilon > 0$.

(c) $U \cap A \neq \varnothing$ for every nonempty open set U.

(d) A^c has an empty interior.

4.48 A metric space is called **separable** if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}$, in $\mathbb{R}^2$, and in $\mathbb{R}^n$.

4.61 Complete the proof of Proposition 4.13.

4.62 Suppose that A is open in (M, d) and that $G \subset A$. Show that G is open in A if and only if G is open in M. Is the result still true if "open" is replaced everywhere by "closed"? Explain.

Chapter 5: Continuity

5.1 Given a function $f : S \to T$ and sets $A, B \subset S$ and $C, D \subset T$, establish the following:

(i) $A \subset f^{-1}(f(A))$, with equality for all A if and only if f is one-to-one.

(ii) $f(f^{-1}(C)) \subset C$, with equality for all C if and only if f is onto.

(iii) $f(A \cup B) = f(A) \cup f(B)$.

(iv) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(v) $f(A \cap B) \subset f(A) \cap f(B)$, with equality for all A and B if and only if f is one-to-one.

(vi) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(vii) $f(A) \setminus f(B) \subset f(A \setminus B)$.

(viii) $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$.

5.2 Given a subset A of some "universal" set S, we define $\chi_A : S \to \mathbb{R}$, the **characteristic function** of A, by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Prove or disprove the following formulas: $\chi_{A \cup B} = \chi_A + \chi_B$, $\chi_{A \cap B} = \chi_A \cdot \chi_B$, $\chi_{A \setminus B} = \chi_A - \chi_B$. What corrections are necessary?

5.8 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some interval $(-a, a)$.

(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x. Will this result hold with "≥ 0" replaced by "> 0"? Explain.

5.9 Let $A \subset M$. Show that $f : (A, d) \to (N, \rho)$ is continuous at $a \in A$ if and only if, given $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(f(x), f(a)) < \varepsilon$ whenever $d(x, a) < \delta$ *and* $x \in A$. We paraphrase this statement by saying "f has a point of continuity relative to A."

5.17 Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

5.19 A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a **Lipschitz condition** if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. More economically, we may say that f is Lipschitz (or Lipschitz with constant K if a particular constant seems to matter). Show that $\sin x$ is Lipschitz with constant $K = 1$. Prove that a Lipschitz function is (uniformly) continuous.

5.20 If d is a metric on M, show that $|d(x, z) - d(y, z)| \leq d(x, y)$ and conclude that the function $f(x) = d(x, z)$ is continuous on M for any fixed $x \in M$. This says that $d(x, y)$ is *separately continuous* – continuous in each variable separately.

5.25 A function $f : (M, d) \to (N, \rho)$ is called **Lipschitz** if there is a constant $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

5.30 Let $f : (M, d) \to (N, \rho)$. Prove that f is continuous if and only if $f\left(\overline{A}\right) \subset \overline{f(A)}$ for every $A \subset M$ if and only if $f^{-1}\left(B^{\circ}\right) \subset \left(f^{-1}(B)\right)^{\circ}$ for every $B \subset N$. Give an example of a continuous f such that $f\left(\overline{A}\right) \neq \overline{f(A)}$ for some $A \subset M$.

5.34 Show that d is continuous on $M \times M$, where $M \times M$ is supplied with "the" product metric (see Exercise 3.46). This says that d is *jointly* continuous, that is, continuous as a function of two variables. [Hint: If $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$.]

5.36 Suppose that we are given a point x and a sequence (x_n) in a metric space M, and suppose that $f(x_n) \to f(x)$ for every continuous, real-valued function f on M. Does it follow that $x_n \to x$ in M? Explain.

5.46 Show that every metric space is homeomorphic to one of finite diameter. [Hint: Every metric is equivalent to a bounded metric.]

5.48 Prove that $\mathbb{R}$ is homeomorphic to $(0, 1)$ and that $(0, 1)$ is homeomorphic to $(0, \infty)$. Is $\mathbb{R}$ *isometric* to $(0, 1)$? to $(0, \infty)$? Explain.

5.52 Prove Theorem 5.5.

5.53 Suppose that we are given a point x and a sequence (x_n) in a metric space M, and suppose that $f(x_n) \to f(x)$ for every continuous real-valued function f on M. Prove that $x_n \to x$ in M.

5.54 Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Prove that the following are equivalent: (i) f is homeomorphism and (ii) $g : N \to \mathbb{R}$ is continuous if and only if $g \circ f : M \to \mathbb{R}$ is continuous. [Hint: Use the characterization given in Theorem 5.5(ii).]

5.56 Let $f : (M, d) \to (N, \rho)$.

- **(i)** We say that f is an **open** map if $f(U)$ is open in N whenever U is open in M; that is, f maps open sets to open sets. Give examples of a continuous map that is not open and an open map that is not continuous. [Hint: Please note that the definition depends on the target space N.]
- **(ii)** Similarly, f is called **closed** if it maps closed sets to closed sets. Give examples of a continuous map that is not closed and a closed map that is not continuous.

5.57 Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto. Show that the following are equivalent: (i) f is open; (ii) f is closed; and (iii) f^{-1} is continuous. Consequently, f is a homeomorphism if and only if both f and f^{-1} are open (closed).

Chapter 6: Connectedness

6.5 If E and F are connected subsets of M with $E \cap F \neq \varnothing$, show that $E \cup F$ is connected.

6.6 More generally, if $\mathcal{C}$ is a collection of connected subsets of M, all having a point in common, prove that $\bigcup \mathcal{C}$ is connected. Use this to give another proof that $\mathbb{R}$ is connected.

6.7 If every pair of points in M is contained in some connected set, show that M is itself connected.

6.9 If $A \subset B \subset \overline{A} \subset M$, and if A is connected, show that B is connected. In particular, $\overline{A}$ is connected.

6.13 If $f : [a, b] \to [a, b]$ is continuous, show that f has a fixed point: that is, show that there is some point x in $[a, b]$ with $f(x) = x$.

Chapter 7: Completeness

7.1 If $A \subset B \subset M$, and if B is totally bounded, show that A is totally bounded.

7.2 Show that a subset A of $\mathbb{R}$ is totally bounded if and only if it is bounded. In particular, if I is a closed, bounded interval in $\mathbb{R}$ and $\varepsilon > 0$, show that I can be covered by finitely many closed subintervals $J_1, \cdots, J_n$, each of length at most ε.

7.5 Prove that A is totally bounded if and only if $\overline{A}$ is totally bounded.

7.9 Give an example of a closed bounded subset of l_∞ that is not totally bounded.

7.10 Prove that a totally bounded metric space M is separable. [Hint: for each n, let D_n be a finite $(1/n)$-net for M. Show that $D = \bigcup_{n-1}^{\infty} D_n$ is a countable dense set.]

7.12 Let A be a subset of an arbitrary metric space (M, d). If (A, d) is complete, show that A is closed in M.

7.16 Prove that $\mathbb{R}^n$ is complete under any of the norms $\|\cdot\|_1, \|\cdot\|_2$, or $\|\cdot\|_\infty$. [This is interesting because completeness is not usually preserved by the mere equivalence of *metrics*. Here we use the fact that all of the metrics involved are generated by *norms*. Specifically, we need the norms in question to be equivalent as functions: $\|\cdot\|_\infty \le \|\cdot\|_2 \le \|\cdot\|_1 \le n\|\cdot\|_\infty$. As we will see later, *any* two norms on $\mathbb{R}^n$ are comparable in this way.]

7.18 Fill in the details of the proofs that l_1 and l_∞ are complete.

7.26 Just as with the nested interval theorem, it is essential that the sets F_n used in the nested set theorem be closed and bounded. Why? Is the condition $\operatorname{diam}(F_n) \to 0$ really necessary? Explain.

7.27 Note that the version of the Bolzano-Weierstrass theorem given in Theorem 7.11 replaced boundedness with total boundedness. Is this really necessary? Explain.

Chapter 8: Compactness

8.1 If K is a nonempty compact subset of $\mathbb{R}$, show that $\sup K$ and $\inf K$ are elements of K.

8.2 Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$, considered as a subset of $\mathbb{Q}$ (with its usual metric). Show hat E is closed and bounded but *not* compact.

8.17 If M is compact, show that M is also separable.

8.20 Let E be a noncompact subset of $\mathbb{R}$. Find a continuous function $f : E \to \mathbb{R}$ that is (i) not bounded; (ii) bounded but has no maximum value.

8.23 Suppose that M is compact and that $f : M \to N$ is continuous, one-to-one, and onto. Prove that f is a homeomorphism.

8.30 Prove Lemma 8.8.

8.44 Show that any Lipschitz map $f : (M, d) \to (N, \rho)$ is uniformly continuous. In particular, any isometry is uniformly continuous.

8.48 Prove that a uniformly continuous map sends Cauchy sequences into Cauchy sequences.

8.54 Let E be a bounded, noncompact subset of $\mathbb{R}$. Show that there is a continuous function $f : E \to \mathbb{R}$ that is not uniformly continuous.

8.55 Give an example of a bounded continuous map $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous. Can an unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous? Explain.

8.57 A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a *Lipschitz condition of order* α, where $\alpha > 0$, if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all x, y. Prove that such a function is uniformly continuous.

8.58 Show that any function $f : \mathbb{R} \to \mathbb{R}$ having a bounded derivative is Lipschitz of order 1. [Hint: Use the mean value theorem.]