ECE2150J Intro to Circuits

Chapter 4. Circuit Theorems

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4.1 Introduction

- **A major advantage** of analyzing circuits using skills in Chapter 2 and 3 is that we can analyze a circuit without tampering with its original configuration. **A major disadvantage** of this approach is that for a large, complex circuit, tedious computation is involved.
- Therefore, it is useful if we can **simplify circuits**. In this chapter, we will study some theorems, such as, **Thevenin and Norton** (for linear circuit).

4.2 Linearity

- A linear circuit is one whose output/response is **linearly** related to its input/excitation.

- **The linearity property** is a combination of both the **homogeneity (scaling)** property and the **additivity** property.

- **A circuit is linear if it is both additive and homogeneous**.

- **Homogeneity** requires that if the input is multiplied by a constant, then the output is multiplied by the same constant.
- e.g. If $y = ax$, then $a\cdot(kx) = ky$

  because $a\cdot(kx)=k(ax)$

- In a circuit, homogeneity can be seen, for example, in Ohm's law.
- $V=iR$, and if the current increased by a constant k, then $\mathbf{ki\cdot R = k\cdot V}$

- Inhomogeneity example

$y = ax^2$, then $a(kx)^2 = k^2y$, because $ak^2x^2=k^2(ax^2)$
The output is **NOT** multiplied by the same constant.

In a circuit, the relationship between **power and voltage/current is nonlinear,** $P = i^2R = V^2/R$.

- **Additivity** property requires that the response to a sum of inputs is the sum of the responses to each input applied separately.

- Assuming input x_1 produces output y_1; input x_2 produces output y_2 (y_1=ax_1, y_2=ax_2)
- e.g. $y = ax$ where input $x = x_1+x_2$

$$\rightarrow a(x_1+x_2) = ax_1+ax_2 = y_1 + y_2$$

- A function with a square law, e.g. $y = ax^2$, does not show additivity.

Example 4.1 For the circuit in Fig. 4.2, find I_o when $v_s = 12$ V and $v_s = 24$ V. This example illustrate the homogeneity property.

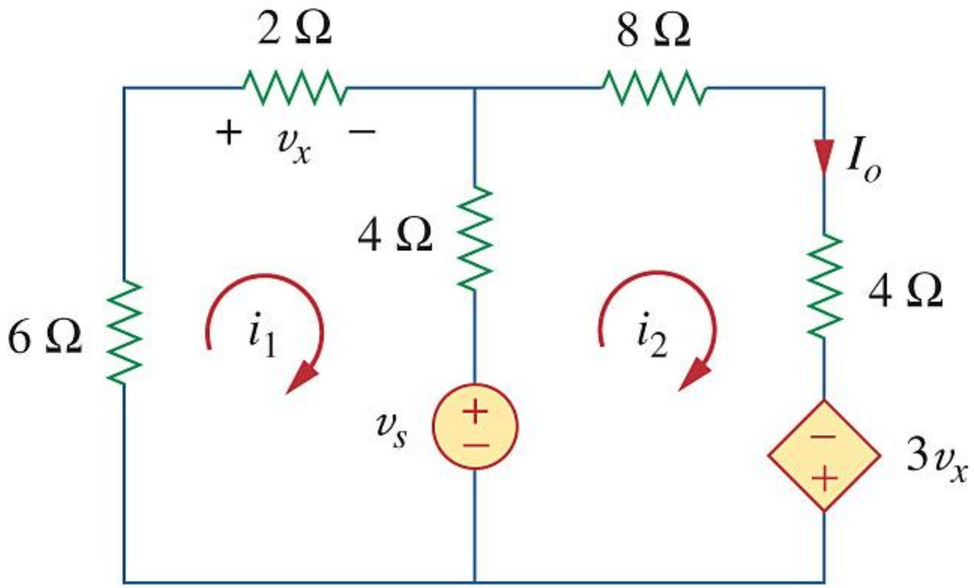


Figure 4.2

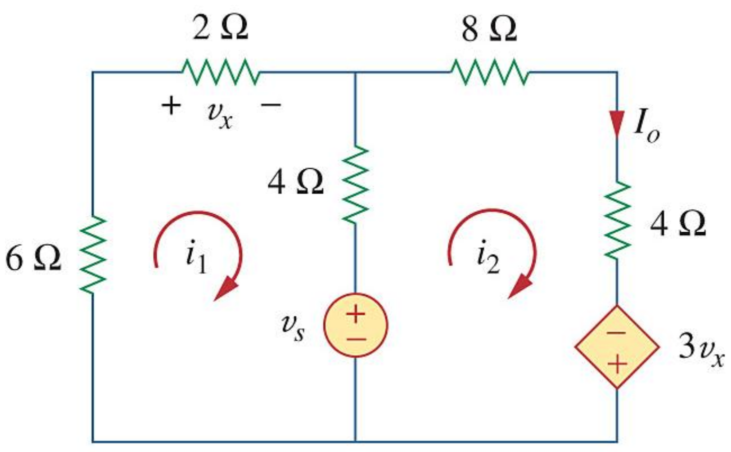


Figure 4.2

By using the inspection method

$$\begin{bmatrix} 6+2+4 & -4 \\ -4 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ 3v_x + v_s \end{bmatrix}$$

Because $V_x = 2i_1$

$$\begin{bmatrix} 6+2+4 & -4 \\ -4 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ 6i_1 + v_s \end{bmatrix}$$

$$\begin{bmatrix} 6+2+4 & -4 \\ -4-6 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ v_s \end{bmatrix}$$

$$\begin{bmatrix} 12 & -4 \\ -10 & 16 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ v_s \end{bmatrix}$$

I_o = i_2 and thus get a solution for i_2 using Cramer's rule

$$\Delta = \begin{vmatrix} 12 & -4 \\ -10 & 16 \end{vmatrix} = 152 \quad \Delta_2 = \begin{vmatrix} 12 & -v_s \\ -10 & v_s \end{vmatrix} = 2v_s \quad I_o = i_2 = \frac{\Delta_2}{\Delta} = \frac{v_s}{76}$$

When $v_s = 12$ V,

$$I_o = \frac{12}{76} = \frac{3}{19} \text{ (A)}$$

When $v_s = 24$ V,

$$I_o = \frac{24}{76} = \frac{6}{19} \text{ (A)}$$

Homogeneous circuit

- **$V_s \rightarrow I_o$**
- **$2V_s \rightarrow 2I_o$**

4.3 Superposition

- Superposition principle is based on **additivity**.
- A linear circuit with **more than one independent source**: the total response is the sum of the individual responses.
- Calculate the contribution of each independent source **separately** and **add all the contributions** to find the total contribution.

- To apply the superposition principle, we must keep three points in mind:

(1) Only one independent source at a time. All other independent sources are **turned off**

Voltage source: a short circuit (0 V)

Current source: an open circuit (0 A)

(2) Dependent sources are left intact because they are controlled by circuit variables.

(3) Superposition is based on linearity: Not applicable to Power

Linear circuit: homogeneous + additive

- Linear circuit

  $x_1 \rightarrow y_1$; $x_2 \rightarrow y_2$

  then, $ax_1+bx_2 \rightarrow ay_1+by_2$

- Superposition

  (1) contribution from input x_1: $ax_1 \rightarrow ay_1$

  (2) contribution from input x_2: $bx_2 \rightarrow by_2$

  adding all the contributions: ay_1+by_2

Example 4.3

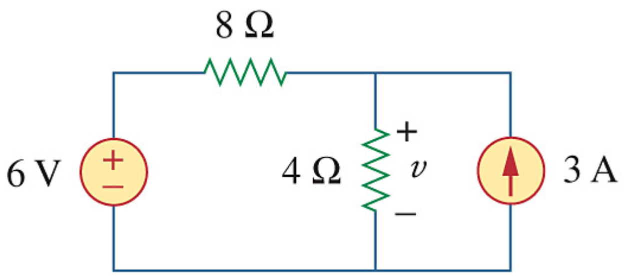
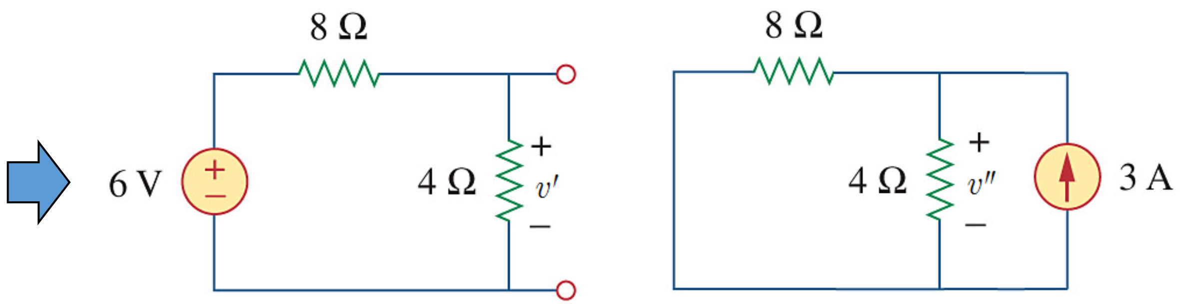


Figure 4.6



(a) (b)

Figure 4.7

By superposition, output $v = v' + v''$ where v' and v'' are the contributions by the voltage and current source, respectively.

$$v' = \frac{4}{8+4} \times 6 = 2 \text{ (V)}$$

$$v'' = 3 \times (4 \parallel 8) = 3 \times \frac{4 \times 8}{4+8} = 8 \text{ (V)}$$

$$v = v' + v'' = 2 + 8 = 10 \text{ [V]}$$

Practice Problem 4.4 Use superposition to find v_x in the circuit of Fig. 4.11.

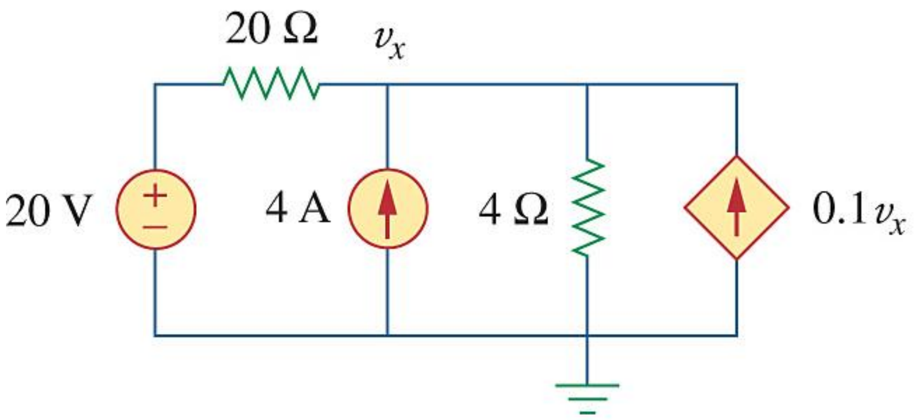
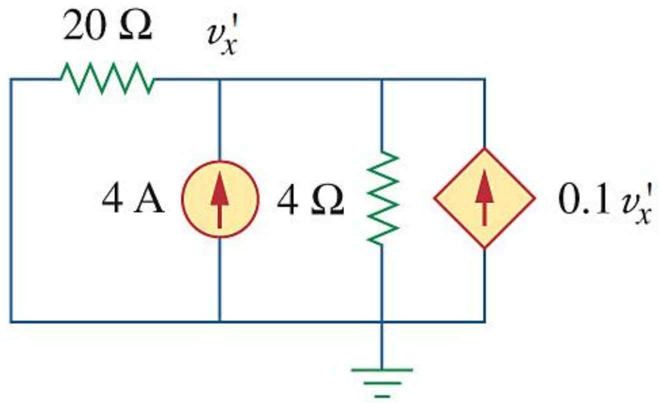


Figure 4.11

Voltage source: a short circuit (0 V)
Current source: an open circuit (0 A)

(i) Contribution from the current source

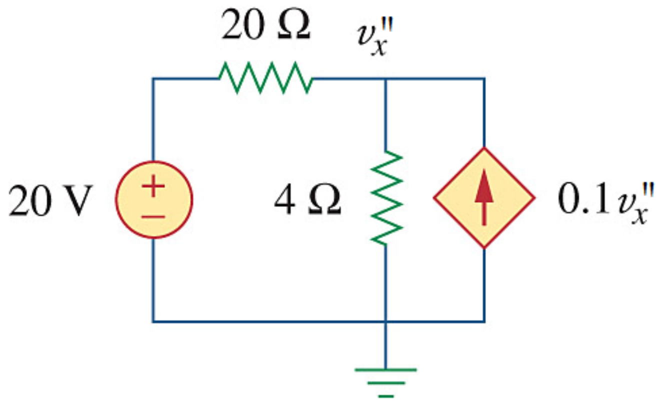


Let $v_x = v_x' + v_x''$

Set the voltage source to zero,

$$\frac{v_x'}{20} - 4 + \frac{v_x'}{4} - 0.1v_x' = 0 \Rightarrow v_x' = 20 \text{ (V)}$$

(ii) Contribution from the potential source



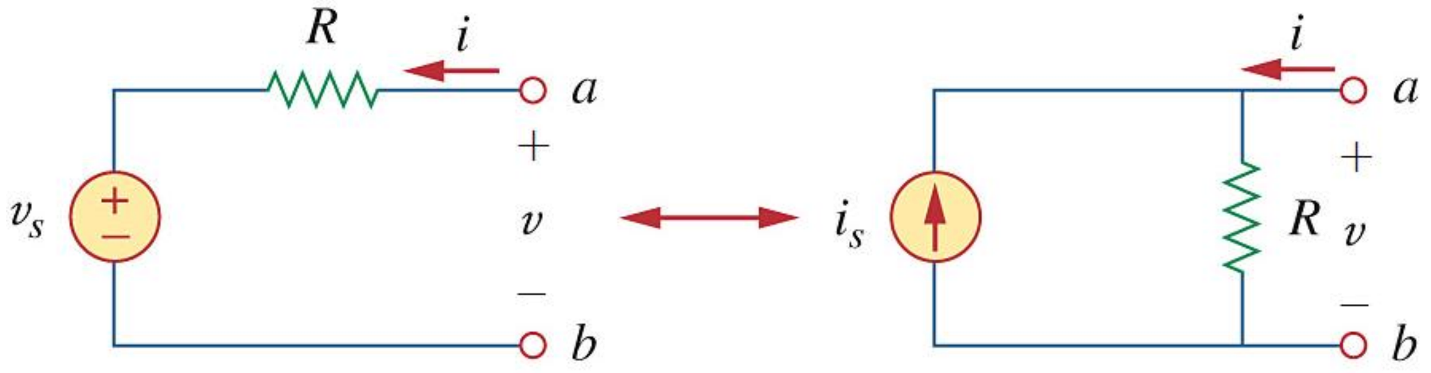
Set the current source to zero,

$$\frac{v_x'' - 20}{20} + \frac{v_x''}{4} = 0.1v_x'' \Rightarrow v_x'' = 5 \text{ (V)}$$

$$v_x = v_x' + v_x'' = 25 \text{ (V)}$$

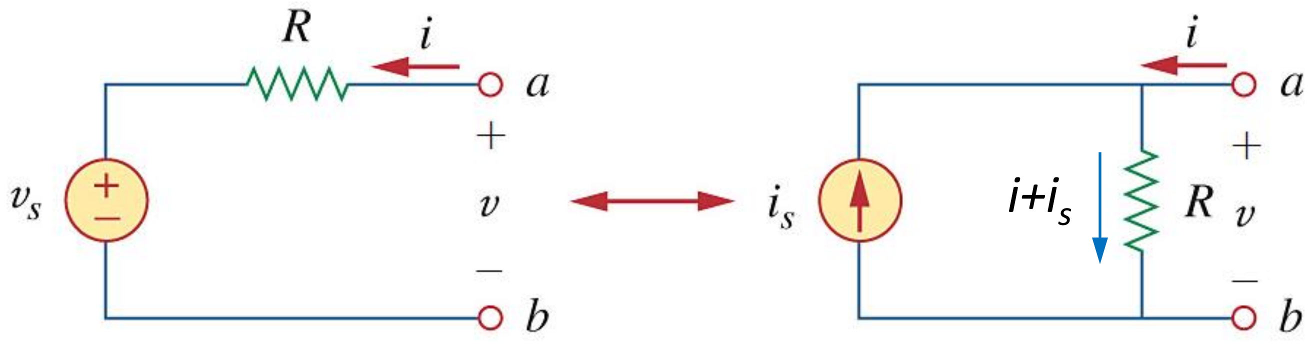
4.4 Source Transformation

- A source transformation is a tool for **simplifying circuits**.
- The source transformation is the process of replacing a **voltage source in series with a resistor** by **a current source in parallel with a resistor**, or vice versa.



$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$

Proof



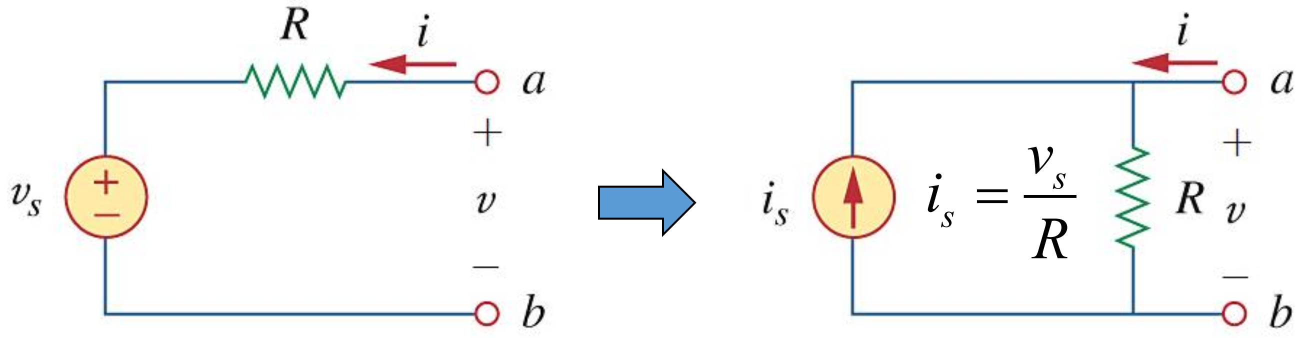
Two circuits are said to be equivalent if they have the same i-v relation.

- Circuit left: v = iR + $\mathbf{v_s}$
- Circuit right: v = iR + $\mathbf{i_s R}$

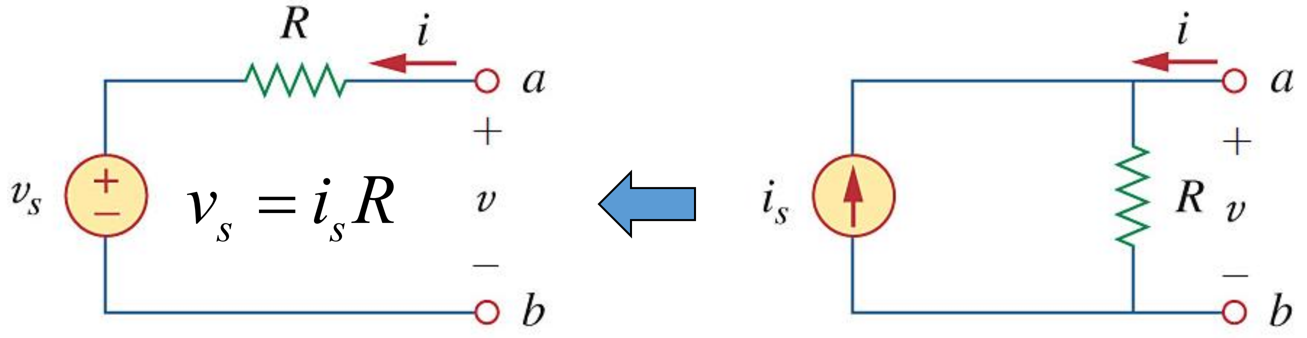
Two circuits (two iv relations) are identical provided that

$$v_s = i_s R \text{ or } i_s = {}^{v_s}/_{R}$$

(i) Voltage source to Current source

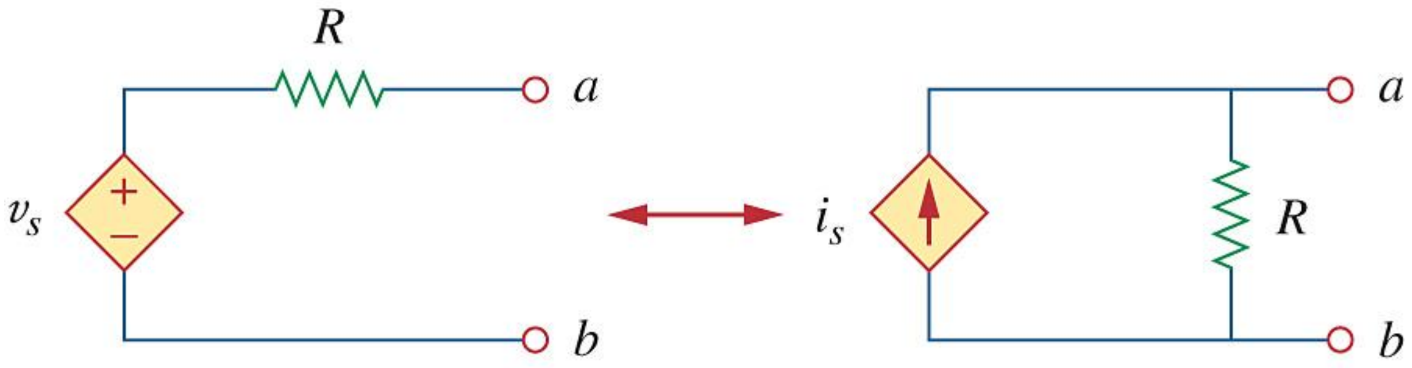


(ii) Current source to Voltage source



Be careful about the direction of current source

- Source transformation also applies to dependent sources, provided we carefully handle the dependent variable.



Example 4.6 Use source transformation to find v_o in the circuit of Fig. 4.17.

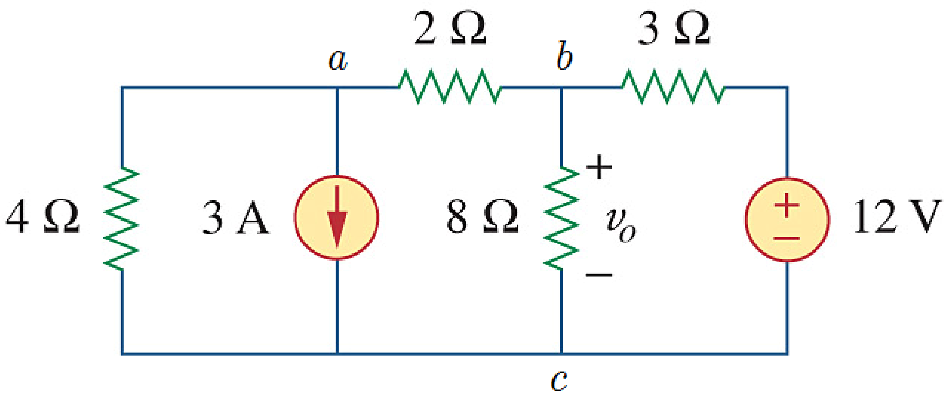
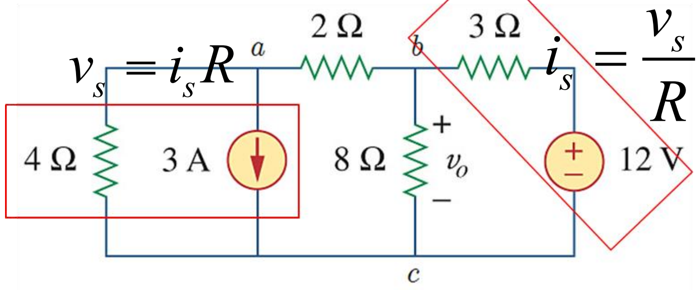


Figure 4.17

$$v_o = 2\times(8 \parallel 2) = 2\times\frac{8\times 2}{8+2} = 3.2 \text{ (V)}$$

Step 1

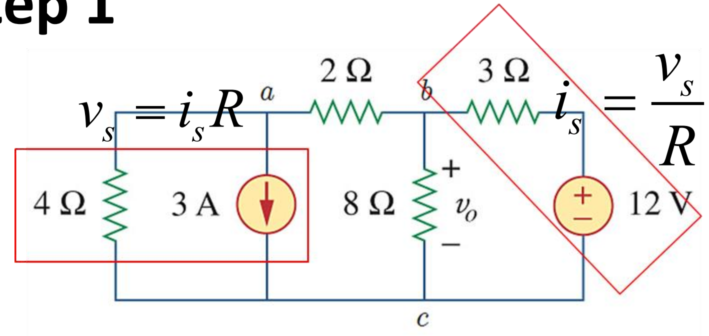


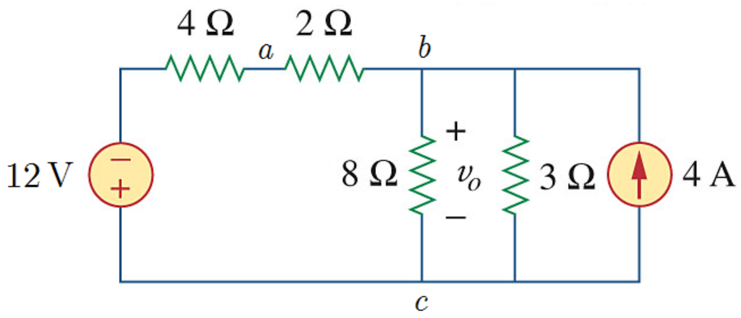
$v_s = i_s R$

$i_s = \frac{v_s}{R}$

Step 1

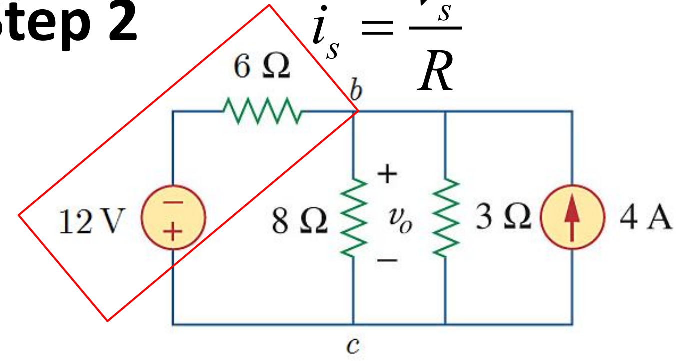
$$v_s = i_s R \qquad i_s = \frac{v_s}{R}$$





Step 2

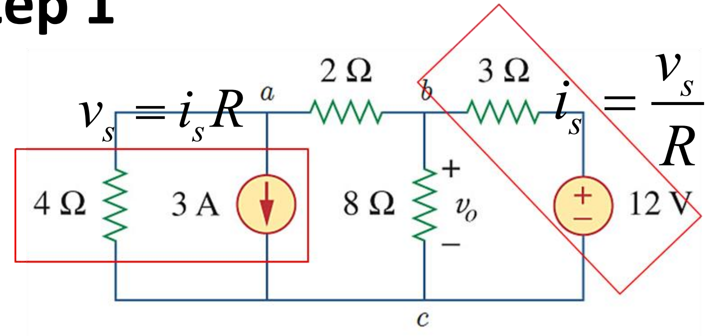
$$i_s = \frac{v_s}{R}$$

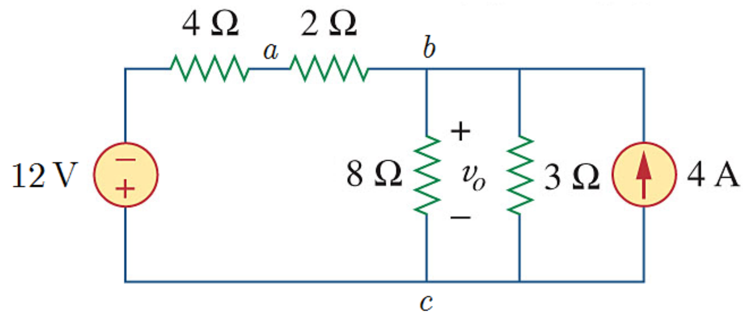


Step 1

$v_s = i_s R$

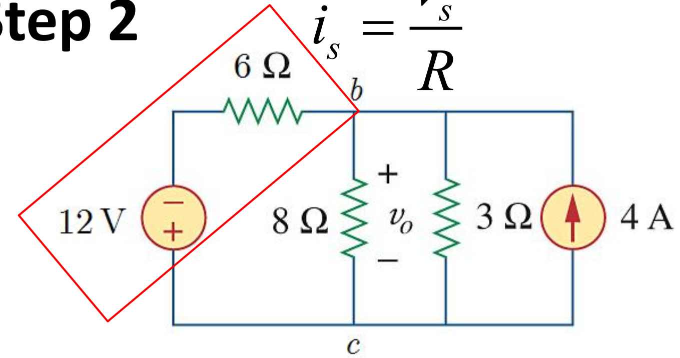
$i_s = \dfrac{v_s}{R}$

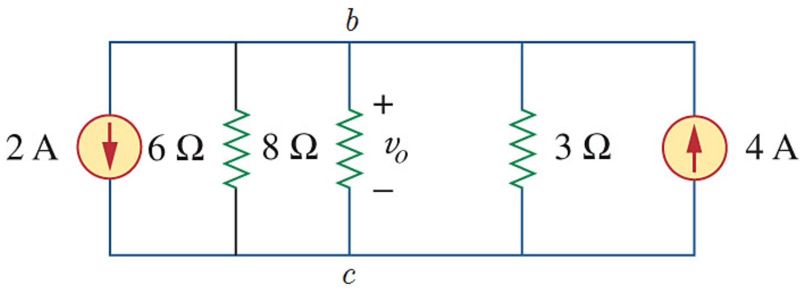




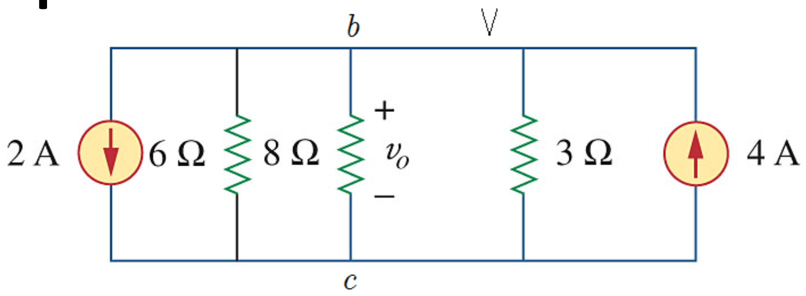
Step 2

$i_s = \dfrac{v_s}{R}$

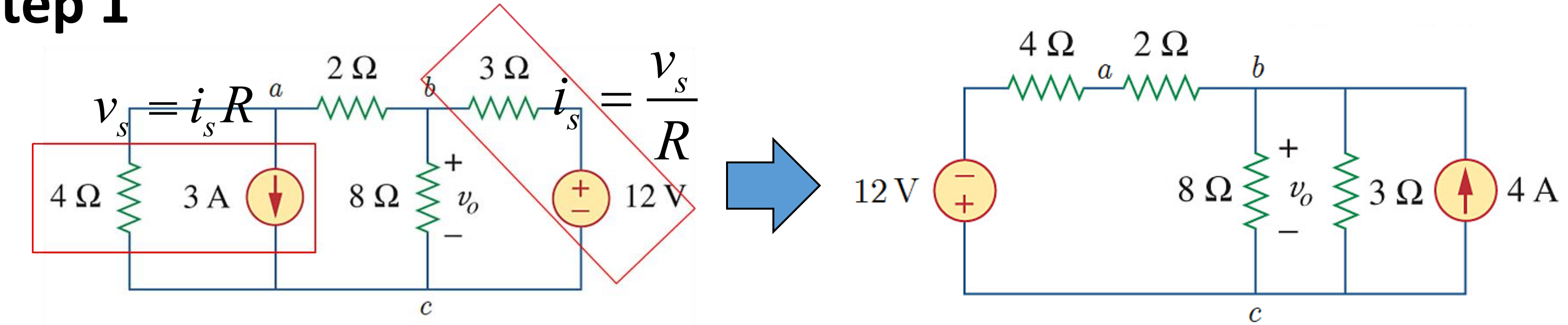




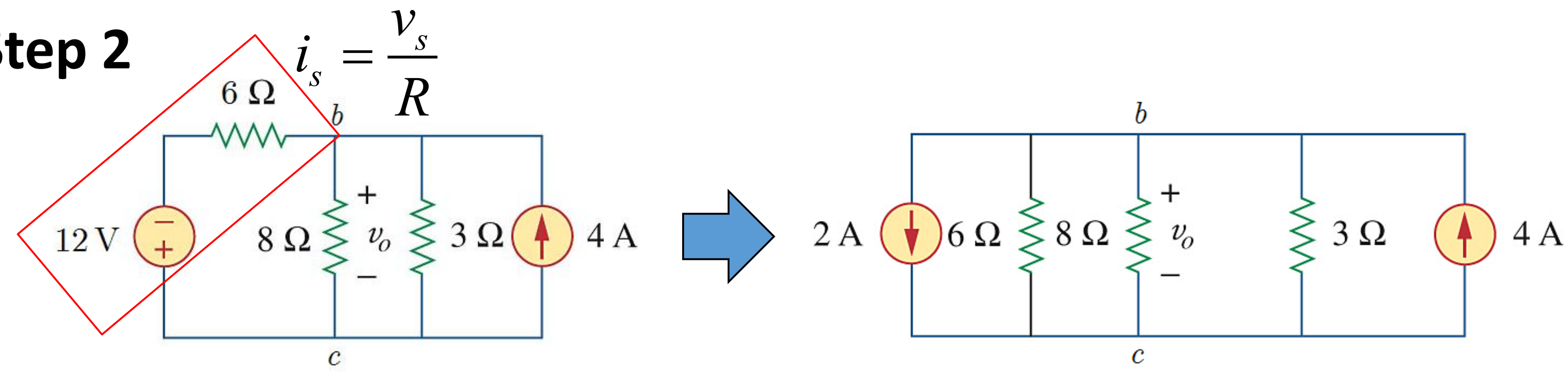
Step 3



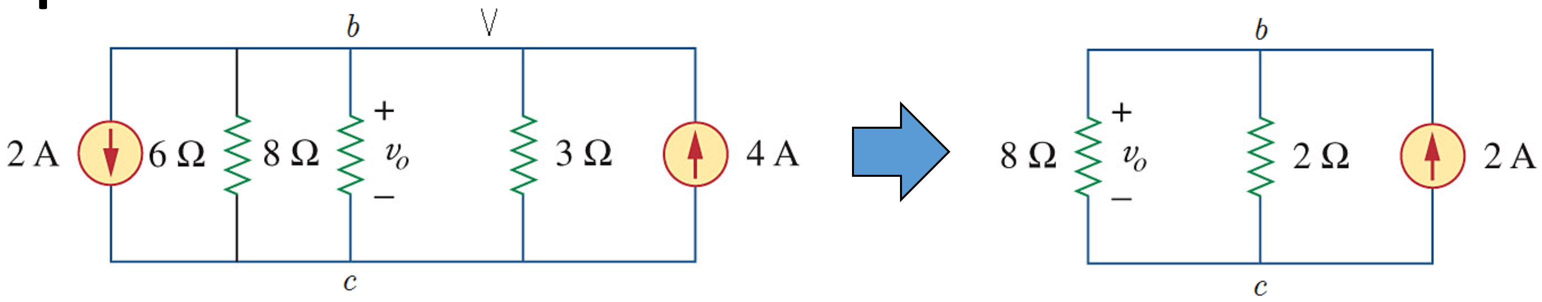
Step 1



Step 2



Step 3



Practice Problem Use source transformation to find i_x in the circuit shown in Fig. 4.22.

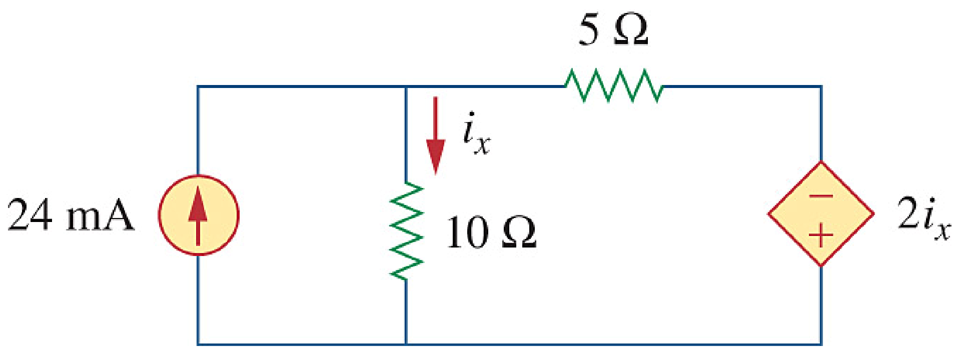
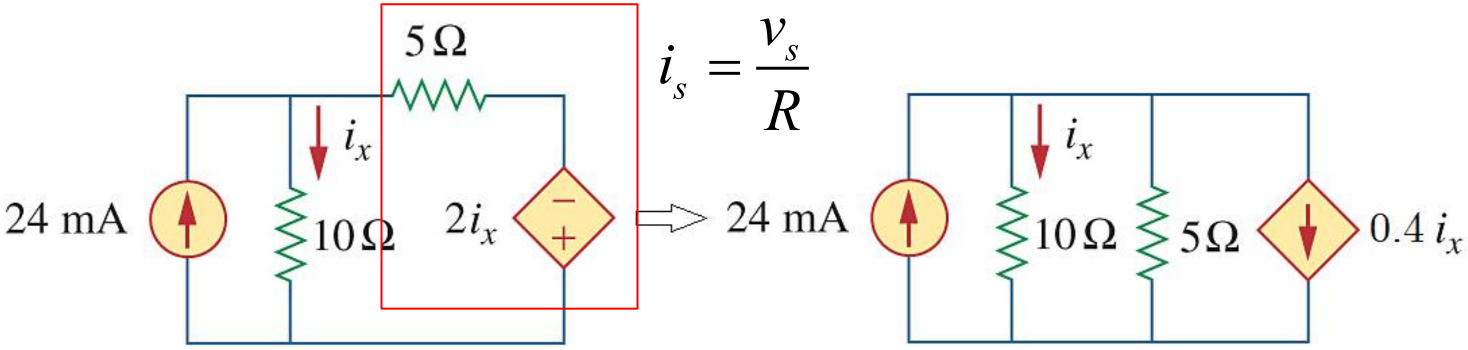


Figure 4.22



$$i_s = \frac{v_s}{R}$$

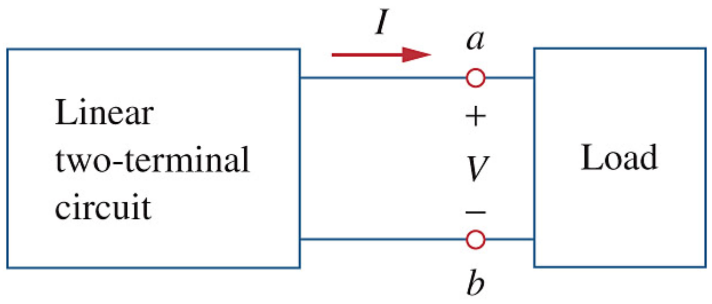
By KCL Current division

$$-24 + i_x + \frac{10 i_x}{5} + 0.4 i_x = 0$$

$$i_x = \frac{24}{3.4} \approx 7.06\ (\text{mA})$$

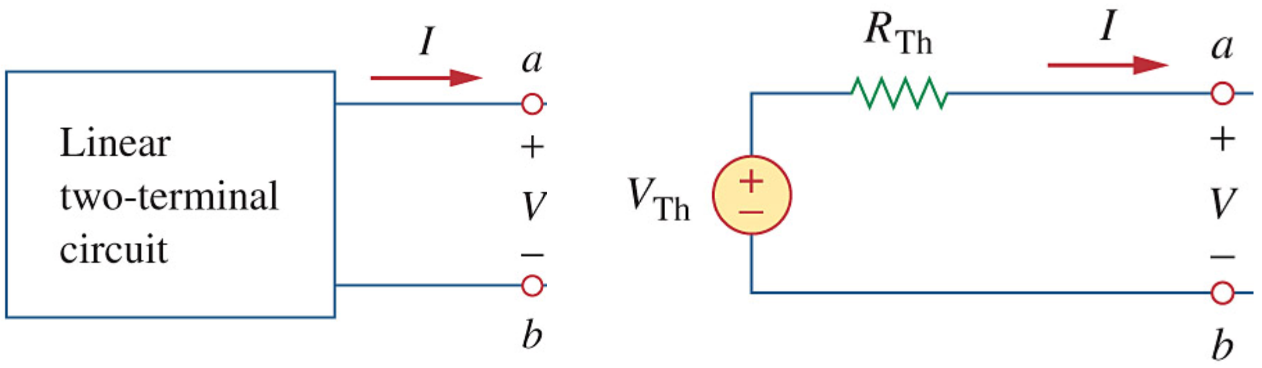
4.5 Thevenin's Theorem

- In practice, a particular element in a circuit is variable while other elements are fixed.



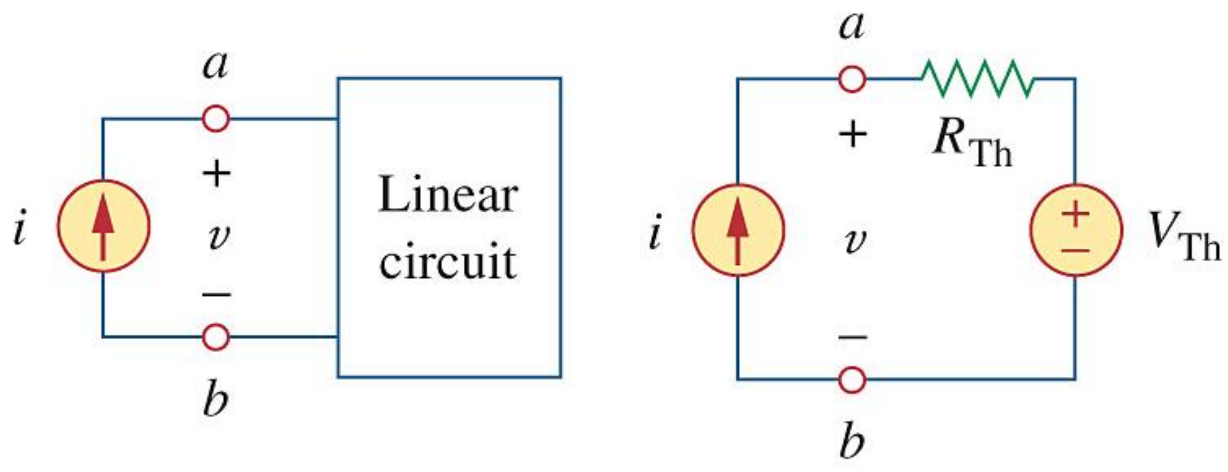
- e.g. When we connect a load to a linear two-terminal circuit, we are primarily interested in the **voltage and current at the terminals** of the load.

- **Thevenin's theorem**: A linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th} → a tool of simplifying circuit analysis.



- **V_{Th}**: open-circuit voltage at the terminals
- **R_{Th}**: equivalent resistance at the terminals when the <u>independent sources are turned off</u>.

Proof of Thevenin Equivalent Circuit

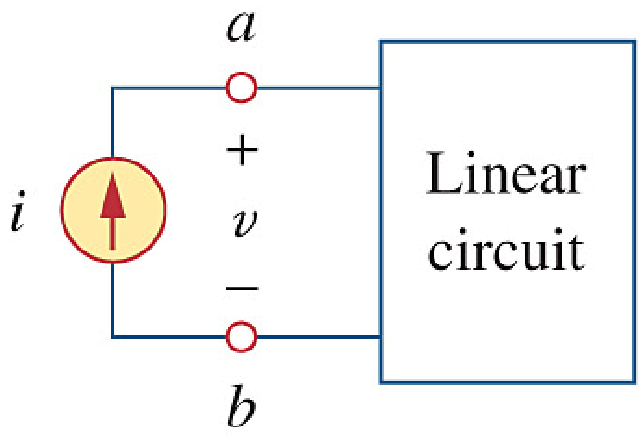


Consider the linear circuit that contains resistors and dependent and independent sources through which **current from an external source** is applied.

Our objective is to ensure that the **voltage-current relation** at terminals *a* and *b* is **identical to that of the Thevenin equivalent**.

We suppose the linear circuit contains two independent voltage source v_{s1} and v_{s2} and two independent current sources i_{s1} and i_{s2}.

By the **superposition**, the terminal voltage v is



External and internal contributions

$$v = \boxed{A_0 i} + \boxed{A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}}$$

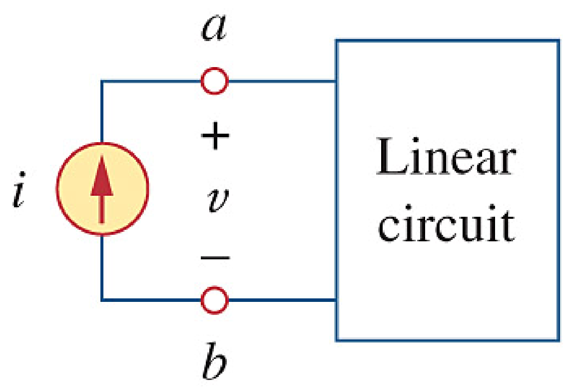
where A_0, A_1, A_2, A_3, and A_4 are constants.

$$\text{Let } B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$$

$$v = A_0 i + B_0$$

$$\text{Let } B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$$

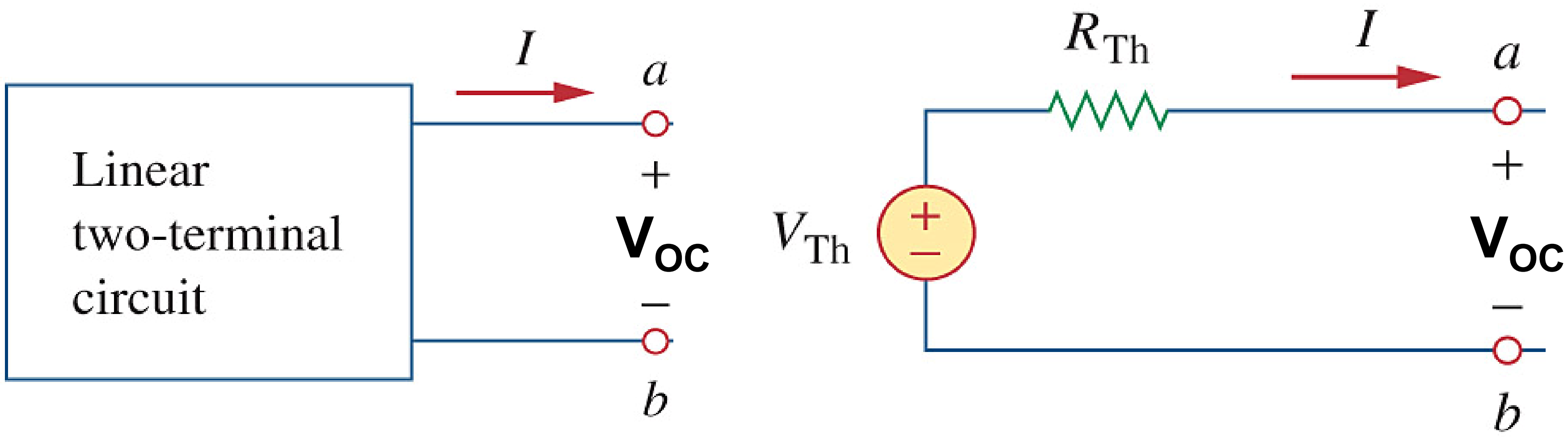
$$v = A_0 i + B_0$$



Evaluate the values of constants A_0 and B_0.

(i) Open Circuit: i = 0 and v = $\mathbf{v_{oc}}$ **=** B_0, **Contribution from internal sources.**

(ii) All **internal sources are turned off**, $B_0 = 0 \rightarrow$ the circuit can be replaced by an equivalent resistance $\mathbf{R_{eq}}$ **=** v/i = A_0.



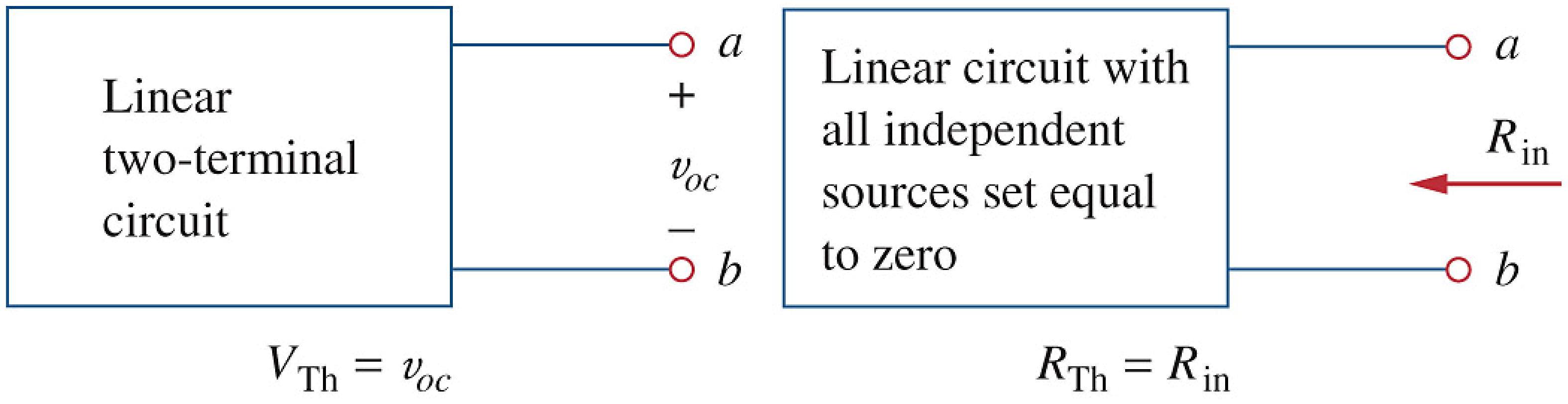
$B_0 = V_{oc}$, and $V_{oc} = V_{Th}$

$A_0 = V_{oc}/I = R_{eq}$, and $R_{eq} = R_{Th}$

Therefore, $v = \mathbf{A_0} i + \mathbf{B_0} = \mathbf{R_{Th}} i + \mathbf{V_{Th}}$

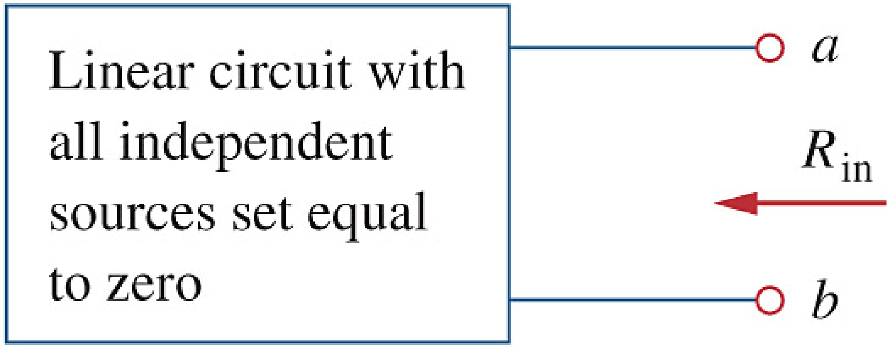
How to find the Thevenin equivalent circuit

Two circuits are said to be **equivalent** if they have **the same voltage-current relation** at their terminals.



- Make two circuits **open-circuits** → No current flows → **terminal voltage must be equal, i.e. $V_{Th} = V_{oc}$**
- **Turn off** all the independent sources → the input resistance must be equal, i.e. $\mathbf{R_{Th} = R_{in}}$

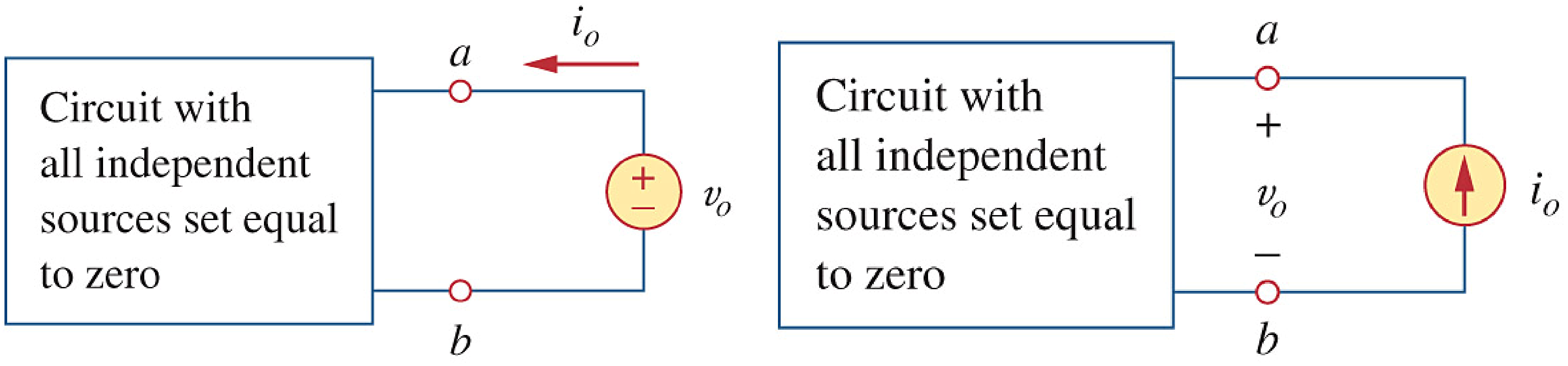
To apply this idea in finding the **Thevenin resistance R_{Th}**, we need to consider two cases.



$$R_{Th} = R_{in}$$

- **Case 1**: If the network has **no dependent sources**, we **turn off all independent sources**. R_{Th} is the input resistance of the network looking between terminals *a* and *b*.

- **Case 2**: The network has **dependent sources.** Similar to case 1, we **turn off all independent sources**, but **dependent sources cannot be turned off**.
  (i) Apply a voltage source v_o at terminal a and b, and determine the resulting current i_o. Then, $R_{Th} = v_o/i_o$
  (ii) Alternatively, apply a current source i_o at terminal a-b, and find the terminal voltage v_o. Again $R_{Th} = v_o/i_o$



Example 4.8 Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals a - b. then find the current through $R_L = 6, 16,$ and $36\ \Omega$.

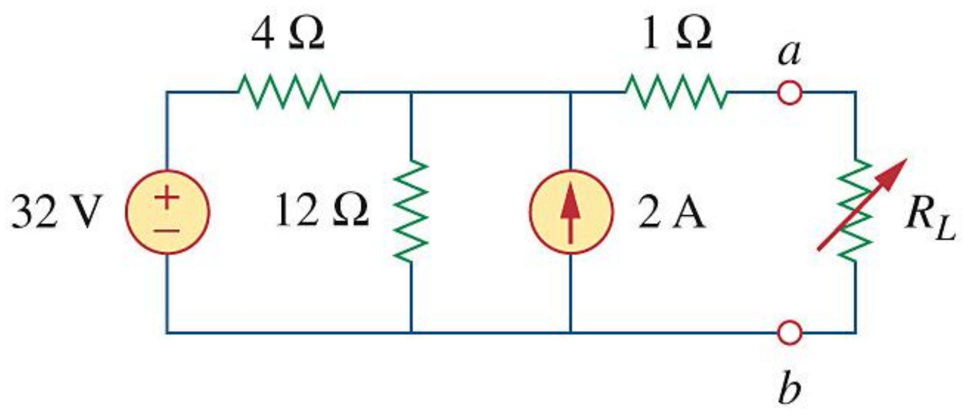


Figure 4.27

(i) R_{TH}

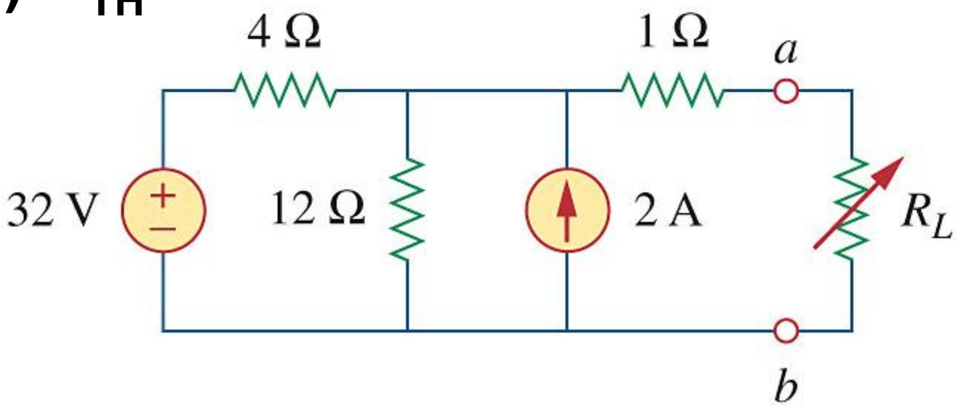


Figure 4.27

(ii) V_{TH}

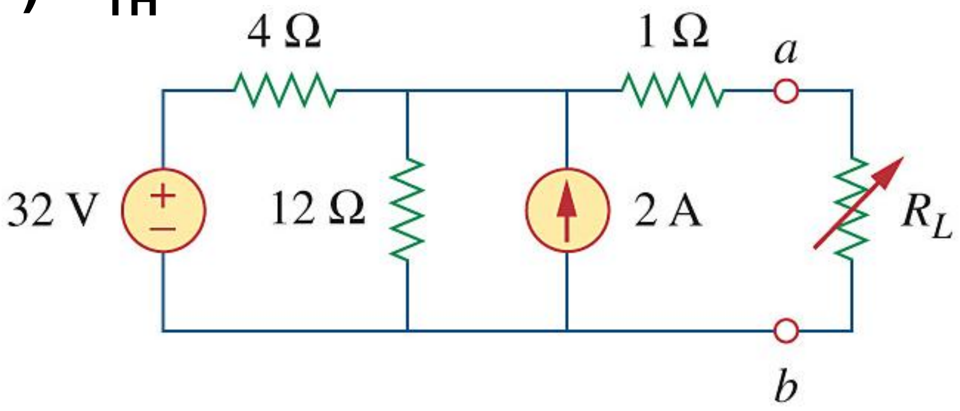
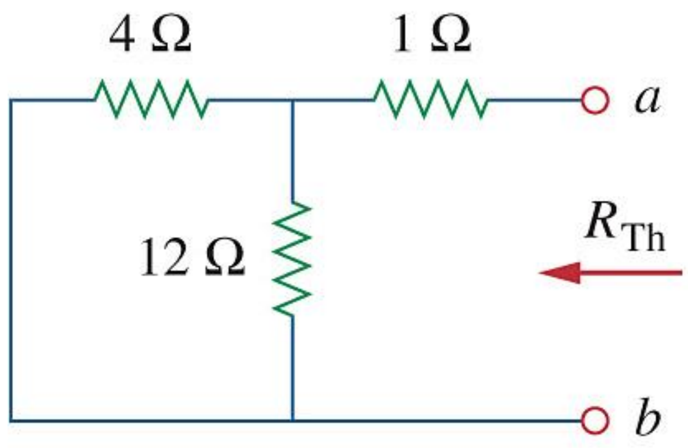


Figure 4.27

(i) R_{TH}

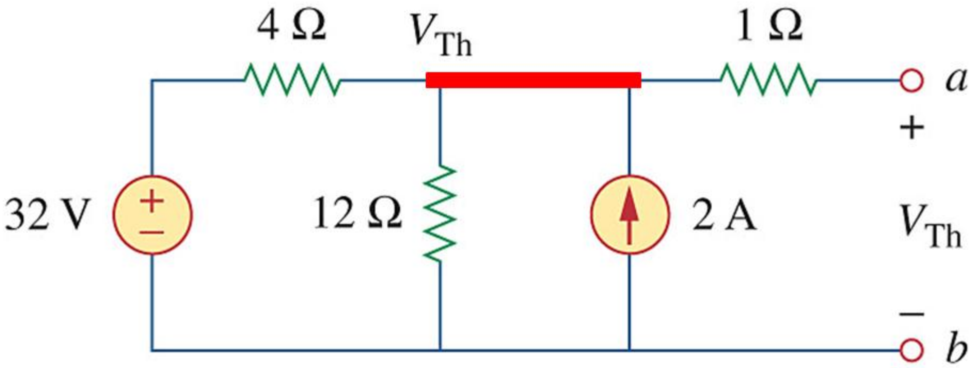


Voltage source → short
Current source → open

Turn off all independent sources, the circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{4+12} + 1 = 4\ (\Omega)$$

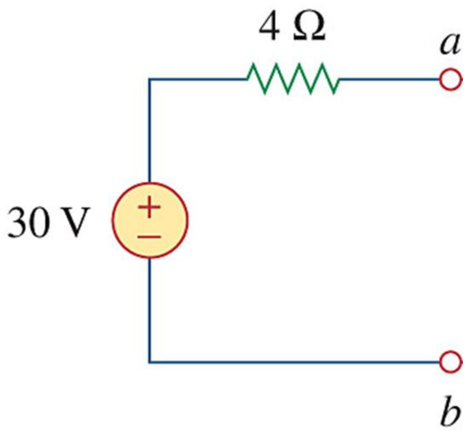
(ii) V_{TH}

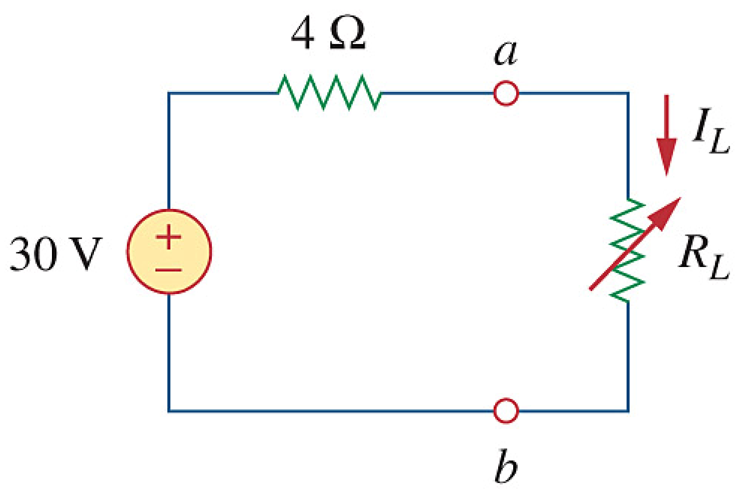


Get V_{TH} in an open circuit

$$\frac{V_{Th}-32}{4}+\frac{V_{Th}}{12}-2=0 \Rightarrow V_{Th}=30\ (\mathrm{V})$$

Thevenin Equivalent Circuit



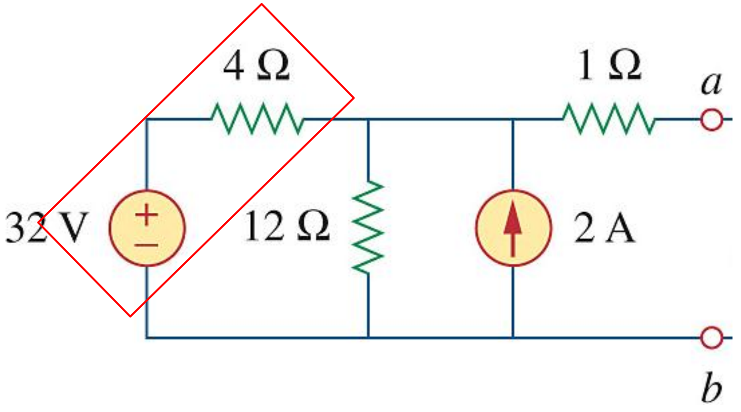


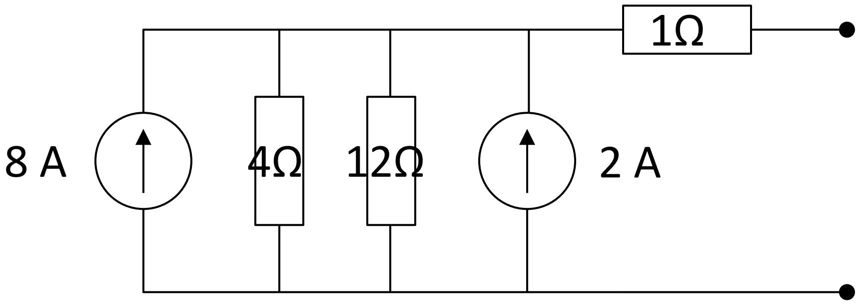
Finally, we can get I_L by using the Thevenin equivalent circuit

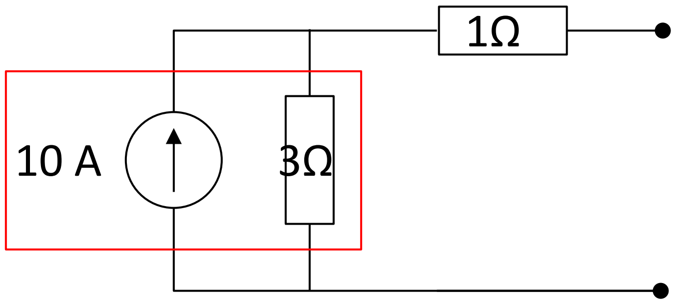
$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L} = 3, 1.5, 0.75 \text{ (A)}$$

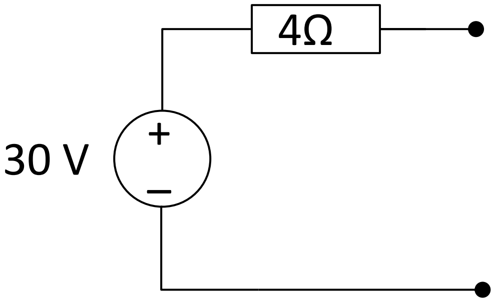
when $R_L = 6, 16, 26\ \Omega.$

Interestingly.. Example 4.8 can be solved by source transformation









$V_{TH} = 30$ V

$R_{TH} = 4\ \Omega$

Practice Problem 4.9 Find the Thevenin equivalent circuit of the circuit in Fig. 4.34 to the left of the terminals.

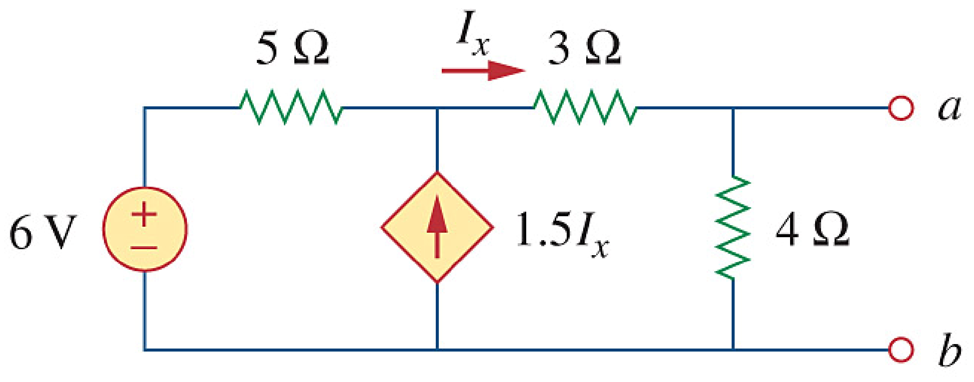
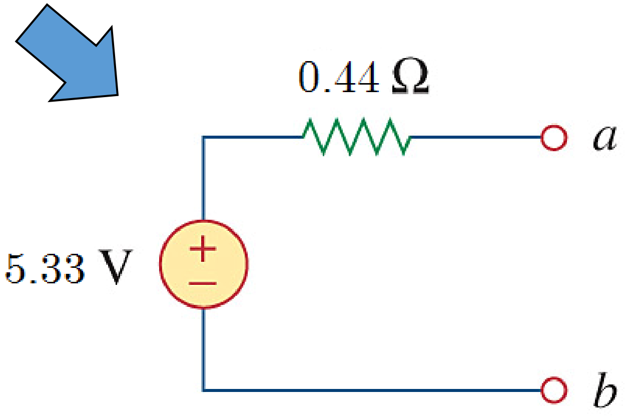
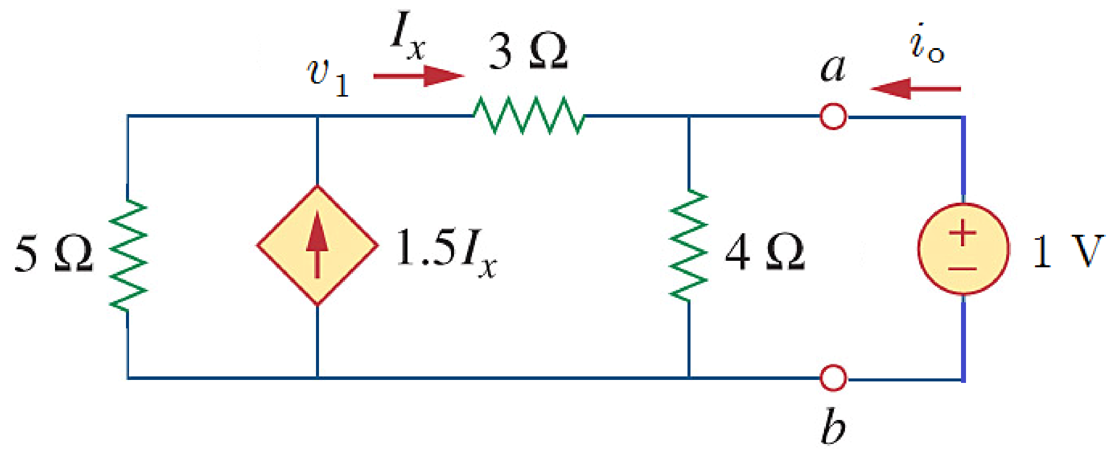


Figure 4.34



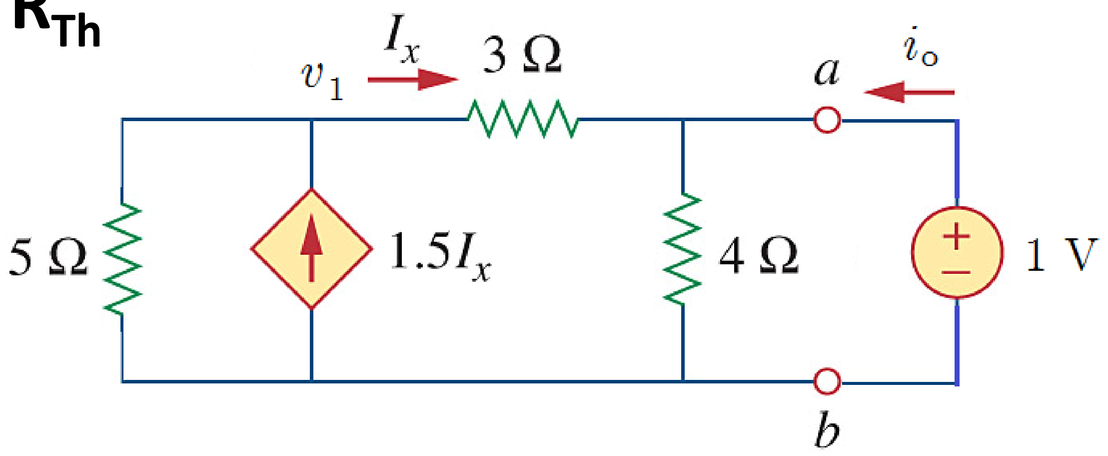
(i) R_{Th}

To find R_{Th}, first we turn off the independent sources. Because there is a dependent source we apply a test voltage at the terminals a-b.

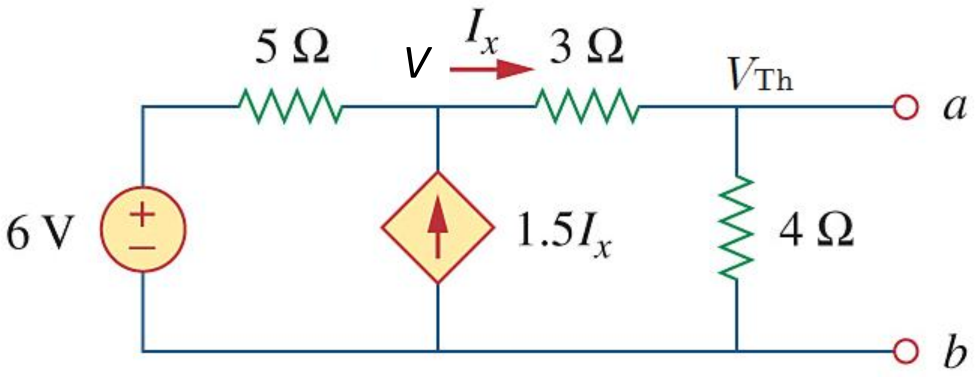


***Note: A test voltage can be any value**

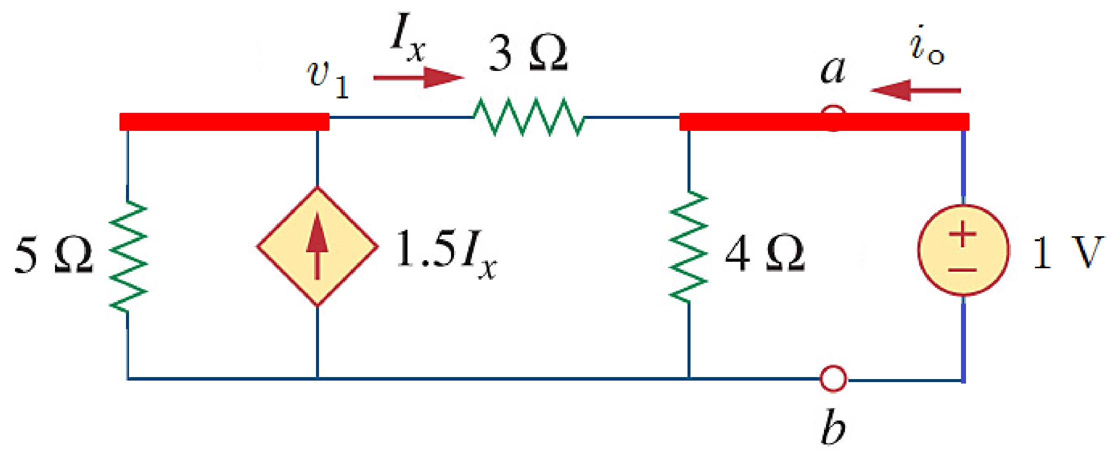
(i) R_{Th}



(ii) V_{Th}



(i) R_{Th}

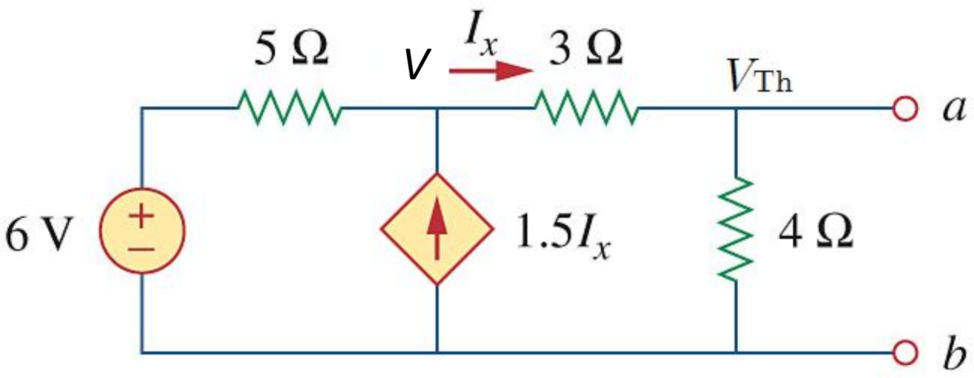


$$\begin{cases} I_x = \dfrac{v_1 - 1}{3} \\ \dfrac{v_1}{5} + I_x = 1.5 I_x \end{cases} \Rightarrow I_x = -2\ (A)$$

$$i_o = -I_x + \frac{1}{4} = \frac{9}{4}\ (A)$$

$$R_{Th} = \frac{1}{i_o} = \frac{4}{9} \approx 0.44\ (\Omega)$$

(ii) V_{Th}



$$(1)\ \frac{V-6}{5} - 1.5I_x + \frac{V-V_{Th}}{3} = 0$$

$$(2)\ \frac{V_{Th}-V}{3} + \frac{V_{Th}}{4} = 0 \rightarrow 7V_{Th} = 4V$$

$$(3)\ I_x = \frac{V-V_{Th}}{3}$$

Put (3) into (1), and then solve the equation

V_{Th} = 5.33 [V]

V = 9.33 [V]

I_x = 1.33 [A]

Try 10 V test voltage source

Practice Problem 4.9 Find the Thevenin equivalent circuit of the circuit in Fig. 4.34 to the left of the terminals.

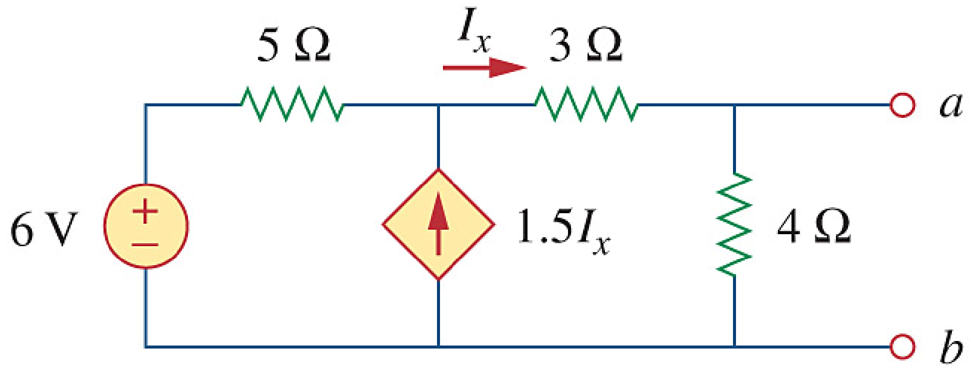
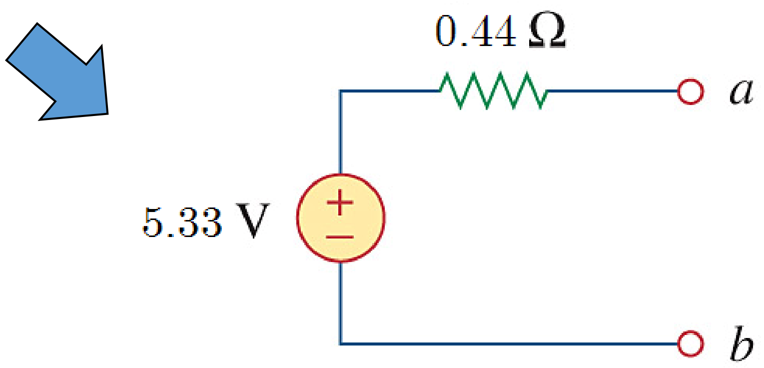
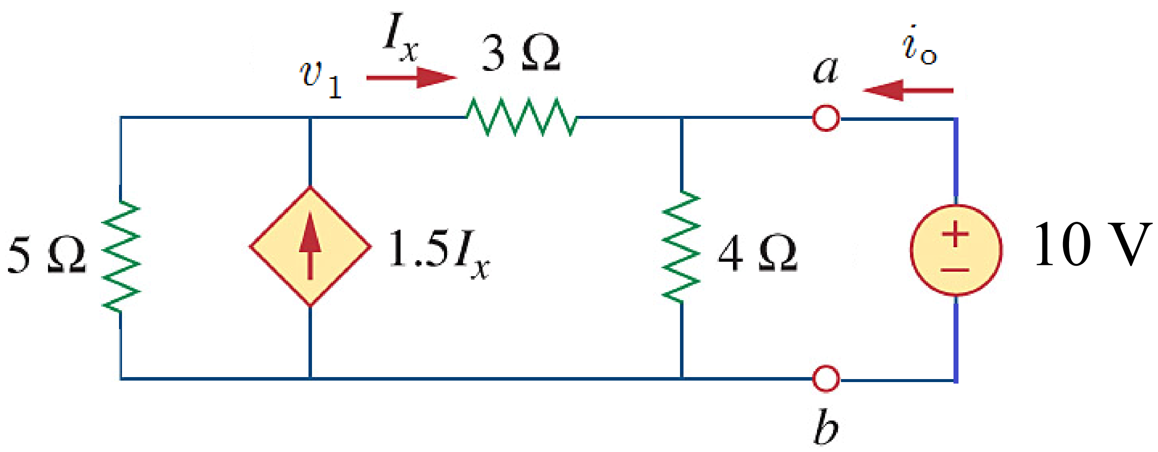


Figure 4.34



(i) R_{Th}



$$\begin{cases} I_x = \dfrac{v_1 - 10}{3} \\ \dfrac{v_1}{5} + I_x = 1.5 I_x \end{cases} \Rightarrow I_x = \text{-20 [A]}$$

$$i_o = -I_x + \frac{10}{4} = \text{ 22.5 [V]}$$

$$R_{Th} = \frac{10}{22.5} \quad \approx 0.44\ (\Omega)$$

Example 4.10 Determine the Thevenin equivalent of the circuit in Fig. 4.35(a) at terminals a - b.

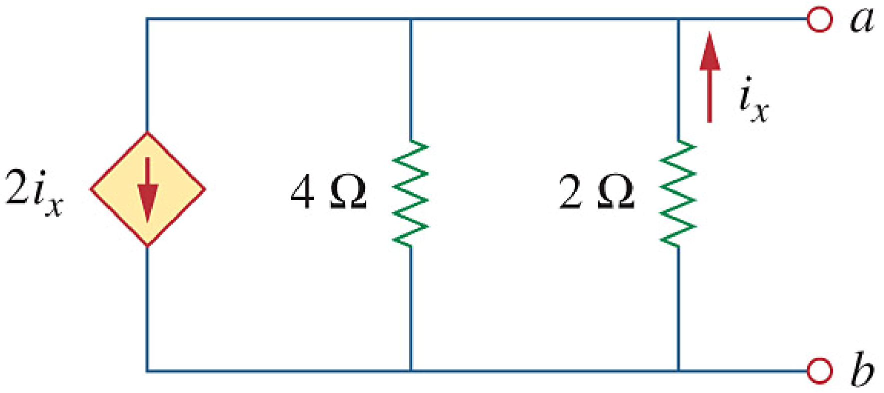


Figure 4.35(a)

(i) V_{Th}

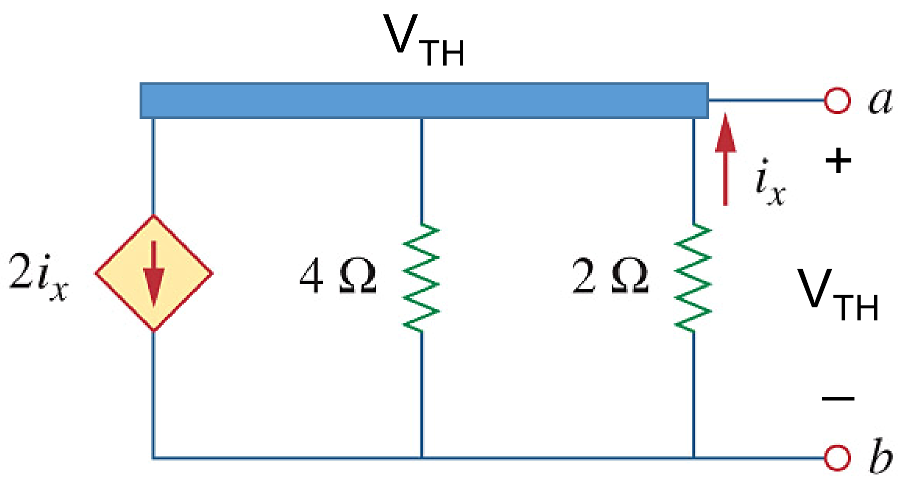


Figure 4.35(a)

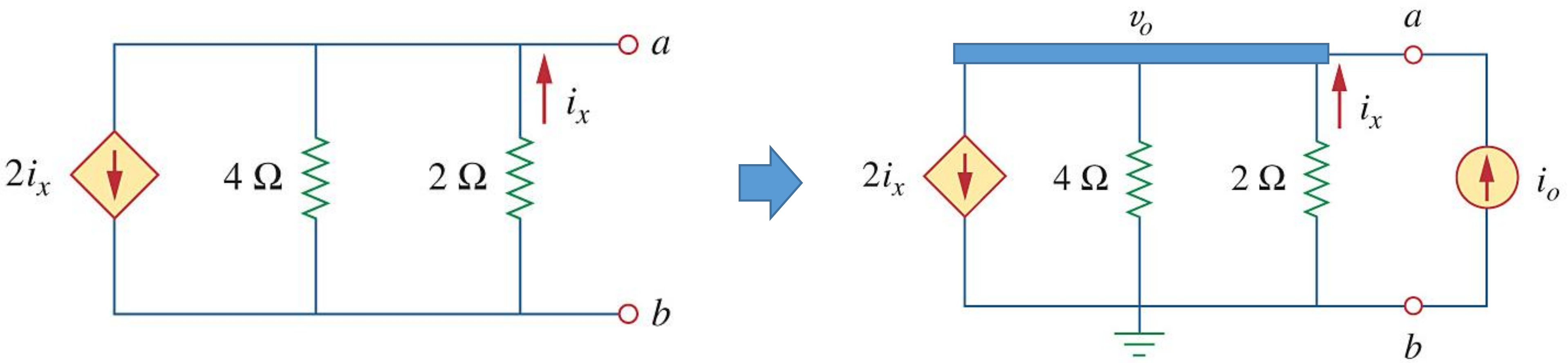
Using KCL

(1) $\frac{V_{TH}}{4} + \frac{V_{TH}}{2} - 2i_x = 0$

(2) $i_x = \frac{V_{TH}}{2}$

$\rightarrow V_{Th} = 0$

(ii) R_{Th}



(1) Use a test current source i_o

(2) By KCL at the node v_o

$$2i_x + \frac{v_o}{4} = i_x + i_o$$

$$\Rightarrow v_o = -4i_o$$

(3) By Ohm's law at 2Ω Resistor

$$v_o = -2i_x$$

$$R_{Th} = \frac{v_o}{i_o} = -4\ (\Omega)$$

R_{Th} with negative value

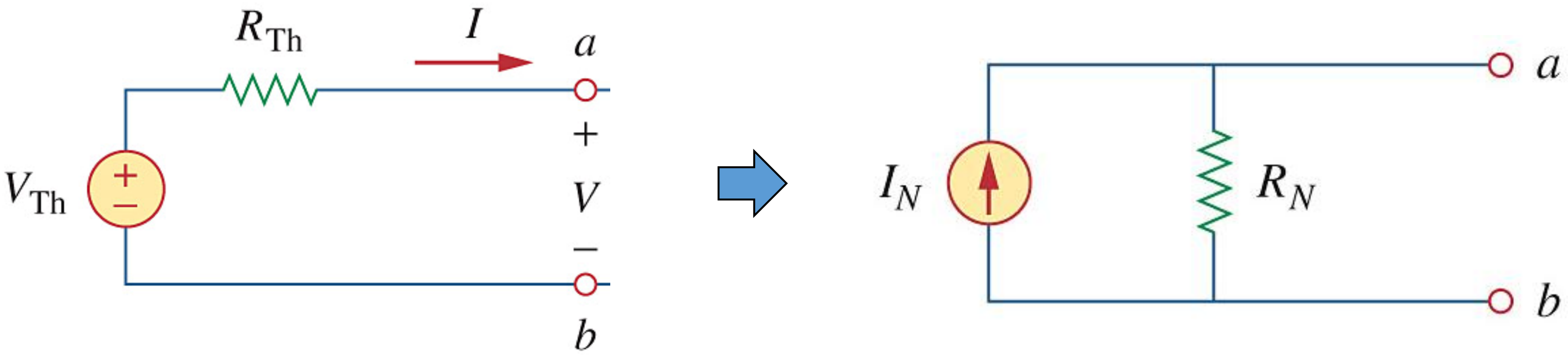
It often occurs that R_{Th} takes a negative value. In this case, the negative resistance implies that **the circuit is supplying power**. This is possible in a circuit with dependent sources.

*It is just an equivalent model and does not mean that there is a resistor (passive) with negative value.

4.6 Norton's Theorem

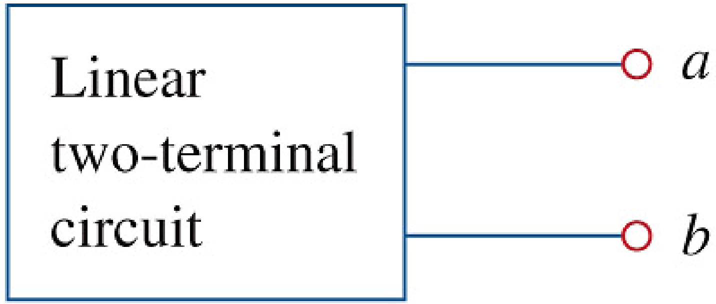
Norton's theorem is similar to Thevenin's theorem.

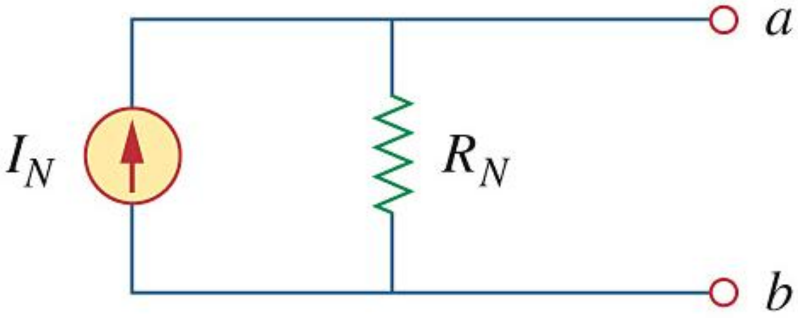
→ Source transformation of Thevenin's theorem.



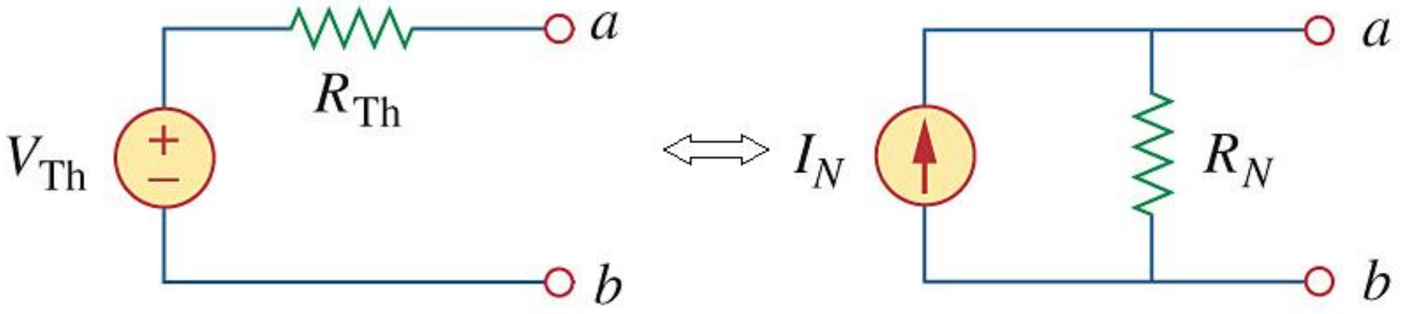
Edward Lawry Norton (28 July 1898, Rockland, Maine–28 January 1983, Chatham, New Jersey) was an accomplished Bell Labs engineer and scientist famous for developing the concept of the Norton equivalent circuit.

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit **consisting of a current source I_N in parallel with a resistor R_N**, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.





We can derive a Norton equivalent circuit from a Thevenin equivalent circuit simply by making a source transformation.



$$\frac{V_{Th}}{R_{Th}} = I_N \ \ or \ V_{Th} = I_N R_{Th} \qquad\qquad I_N R_N = V_{Th}$$

From the Source Transformation, **R_{Th} = R_N** and the Norton current equals the Thevenin voltage divided by the Thevenin resistance $I_N = \frac{V_{Th}}{R_{Th}}$

How to find a Norton equivalent circuit

(i) R_N

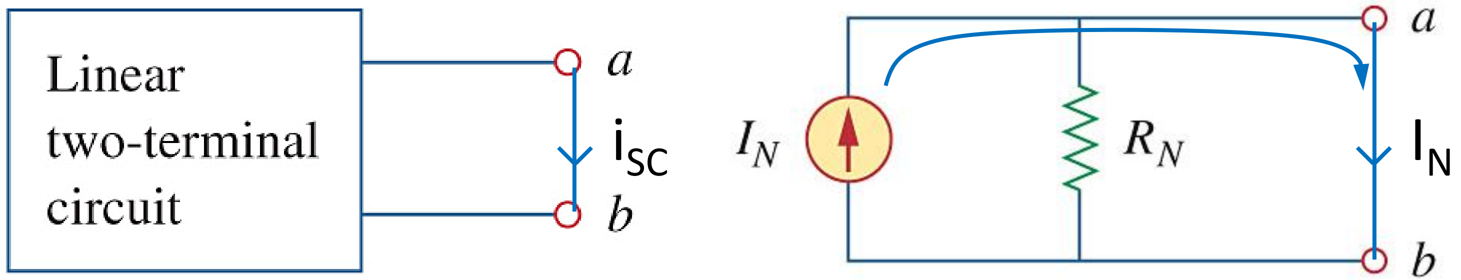
- We find R_N in the same way we find R_{Th}
- In fact, we have just seen that $R_N = R_{Th}$ by the source transformation.

Case 1: If the network has **no dependent sources**, we **turn off all independent sources**. R_N is the input resistance of the network looking between terminals *a* and *b*.

Case 2: The network has **dependent sources.** Similar to case 1, we **turn off all independent sources**, but **dependent sources cannot be turned off**.

(ii) I_N

To find the **Norton current** we determine **the short-circuit current** flowing from terminal *a* to *b* in both circuits

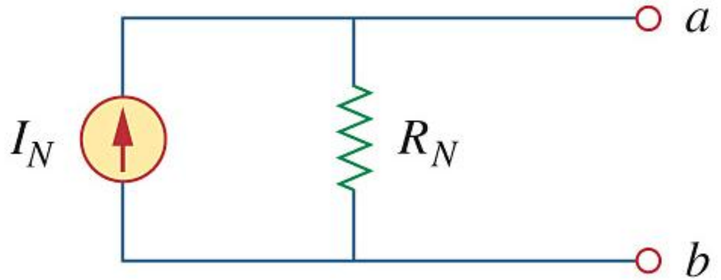


Because these two circuits are equivalent, the two currents ***i_{SC}* and *I_N* must be the same**.

Therefore, ***i_{SC} = I_N***

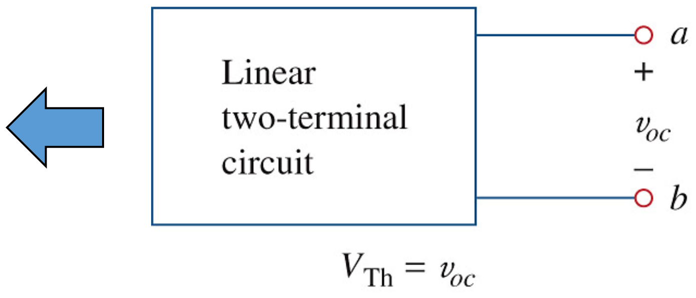
Now we found $\mathbf{R_N}$ **and** $\mathbf{I_N}$ **(**$\mathbf{i_{sc}}$**)**

Norton Equivalent Circuit becomes

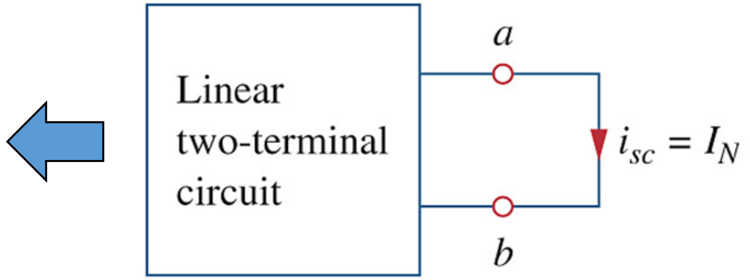


Combining Thevenin and Norton theorems, we have seen that $I_N = \frac{V_{Th}}{R_{Th}} \leftrightarrow R_{Th} = \frac{V_{Th}}{I_N}$

because $\mathbf{V_{Th} = v_{oc}}$



and $\mathbf{I_N = I_{SC}}$



The Thevenin or Norton resistance is the ratio of the open-circuit voltage to the short-circuit current: $\boldsymbol{R_{Th} = R_N = \frac{v_{oc}}{i_{sc}}}$

Practice Problem 4.11 Find the Norton equivalent circuit for the circuit in Fig. 4.42, at terminals *a* - *b*.

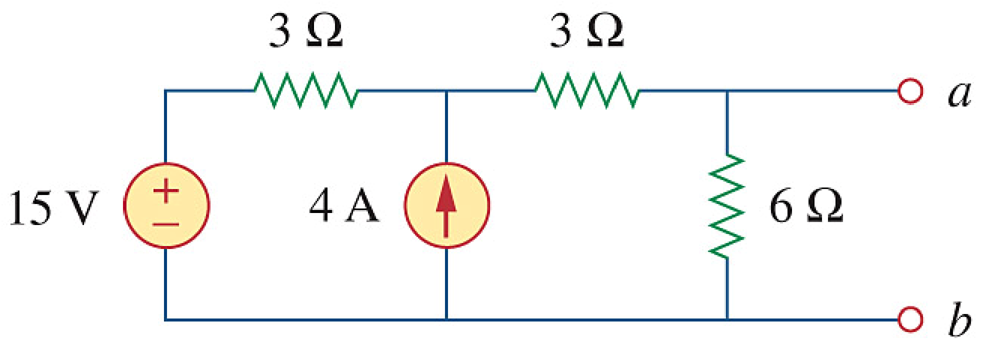
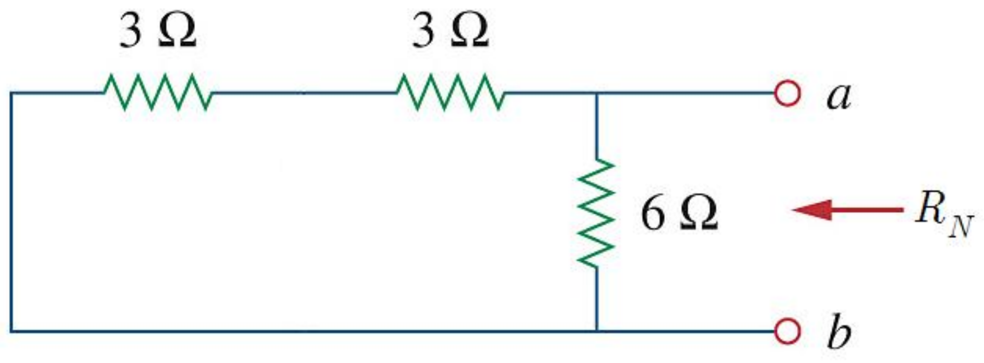


Figure 4.42

(i) R_N



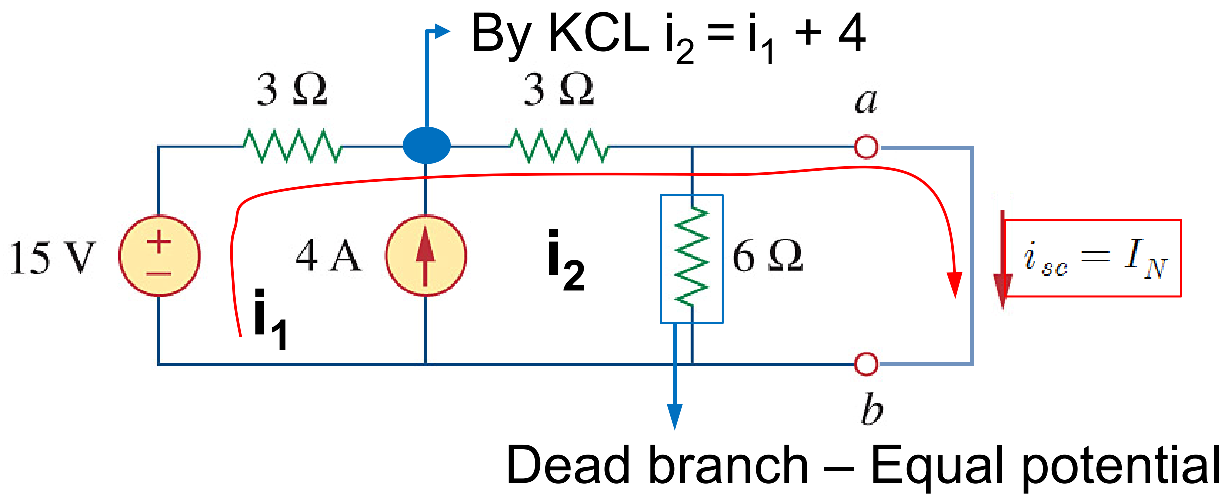
Turn off the voltage and current sources,

the Norton resistance is

$$R_N = (3+3) \parallel 6 = 6 \parallel 6 = 3\ (\Omega)$$

(ii) I_N

We short the terminal a-b. Then, we get a circuit



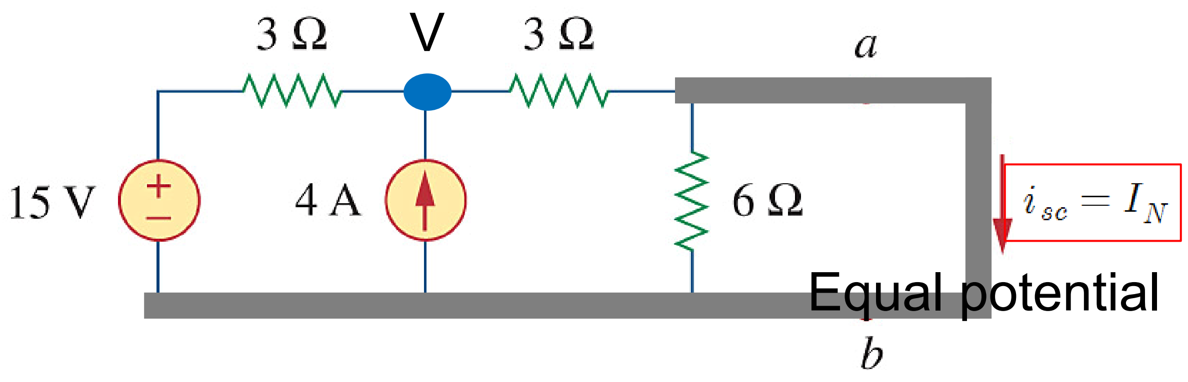
A Supermesh:

$$-15 + 3i_1 + 3(i_1 + 4) = 0 \;\rightarrow i_1 = 0.5\;[A]$$

$i_2 = i_1 + 4 =$ **4.5 [A] =** $\mathbf{I_N}$

(ii) I_N

We short the terminal a-b. Then, we get a circuit



By KCL

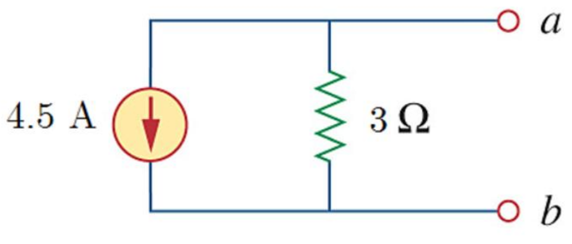
$$\frac{V-15}{3} - 4 + \frac{V}{3} = 0$$

$$V = \frac{27}{2}, i_{sc} = \frac{27}{6} = 4.5\ A$$

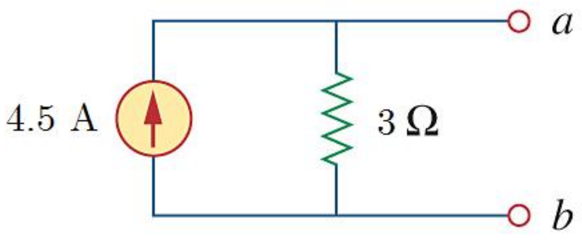
We got the parameters for a Norton equivalent circuit

$R_N = 3$ [Ω]
$I_N = 4.5$ [A]

Norton equivalent circuit is..? (a) Or (b)?



(a)



(b)

4.8 Maximum Power Transfer

- In many practical situations, a circuit is designed to provide power to a load.
- How can we deliver the **maximum power** to the load?
- **The Thevenin equivalent** is useful in finding the maximum power a linear circuit can deliver to the load.
- Max. Power Transfer = Max. Power Efficiency?

(i) Maximum power efficiency

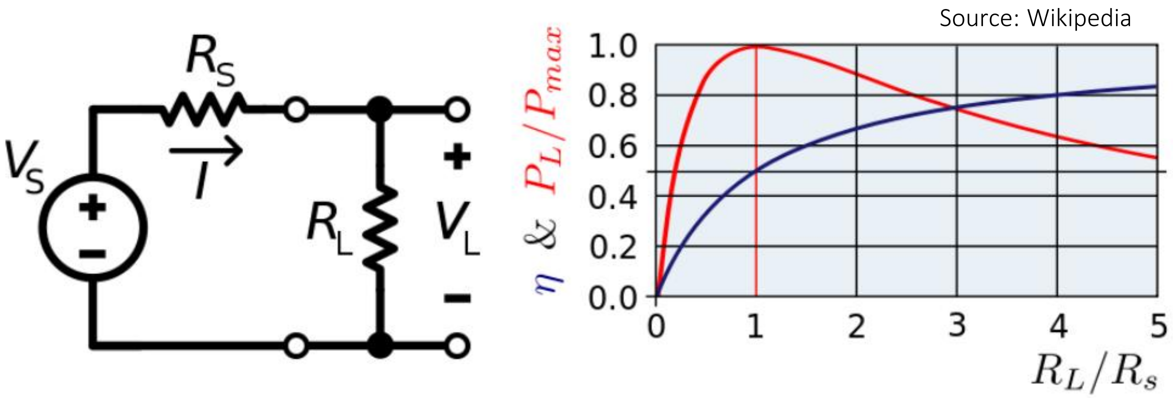
Power utility systems are concerned with the generation, transmission, and distribution of large quantities of electric power. This type of systems emphasizes **the efficiency of the power transfer**.

$$\text{Efficiency} = \frac{Useful\ Power\ Output}{Total\ Power\ Input}$$

(ii) Maximum power transfer

In some cases, for example, communication and electronic systems, the efficiency is not a primary concern. It is often desirable to **transmit as much of power as possible** to the load.

Maximum power efficiency ≠ Maximum power transfer



Efficiency *η (eta):* the ratio of power dissipated by the load, R_L, to power developed by the source, V_S

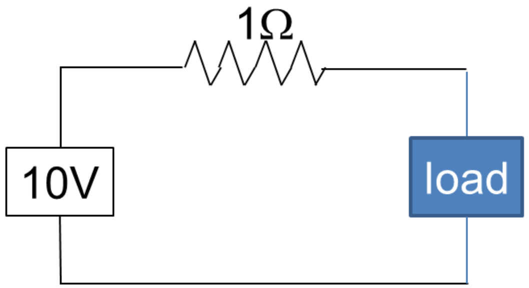
$$\eta = \frac{R_\mathrm{L}}{R_\mathrm{L} + R_\mathrm{S}} = \frac{1}{1 + R_\mathrm{S}/R_\mathrm{L}}$$

- $R_L = R_S$, then $\eta = 0.5$
- $R_L \rightarrow \infty$ or $R_S = 0$, then $\eta = 1$
- $R_L = 0$, then $\eta = 0$

The condition of maximum power transfer does not result in maximum efficiency.

e.g.

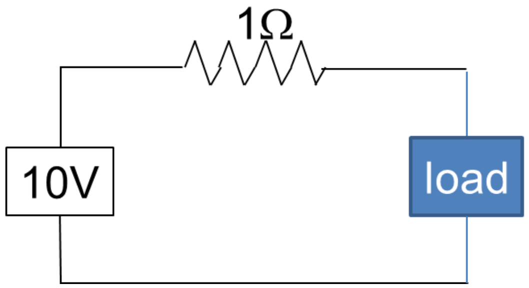
(i) Power utility systems



When R_L= 9 Ω
Power generated from the source = 10W
Power delivered to the load = 9W
Efficiency of power transfer **90%** (loss=10%)

Care electricity bills

(ii) Communication & Electronic systems

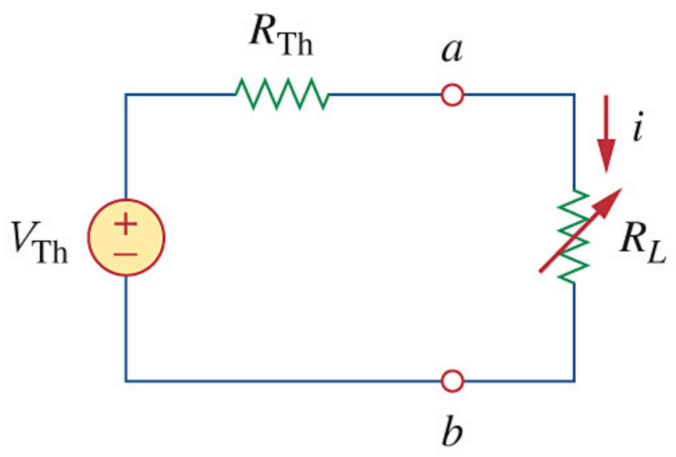


R_L= 1 Ω
Power generated from the source = 50W
Power delivered to the load = **25 W**
Efficiency of power transfer 50% (loss=50%)

Care supplying sufficient power

Circuit types	I. Power utility systems	II. Communication & Electronic systems
Power scale	**Large quantities** of electric power (generation, etc.)	**Small amount** of power is being transferred
Primary concern	**Efficiency** of the power transfer	Transmit **as much of power as** possible to the load

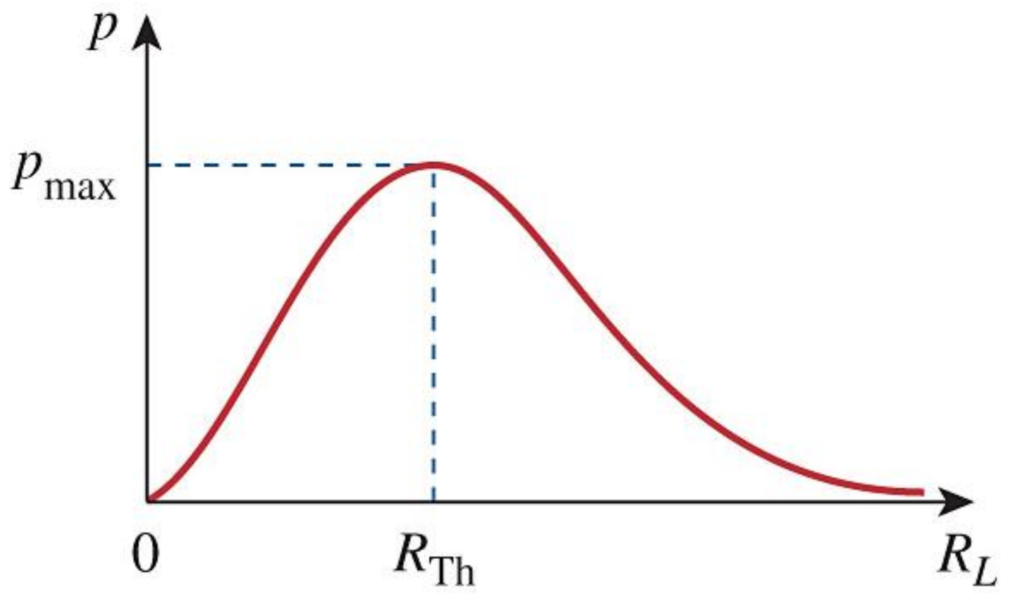
We now consider maximum power transfer in systems.



If the entire circuit is replaced by its Thevenin equivalent except for the load, the power delivered to the load is

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

The maximum power theorem states that the maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).



To prove the maximum power transfer theorem, we differentiate p with respect to R_L and set the result equal to zero

$$\frac{dp}{dR_L} = V_{Th}^2 \frac{R_{Th} - R_L}{(R_{Th} + R_L)^3} = 0$$

We have $R_L = R_{Th}$.

$$\left.\frac{d^2 p}{dR_L^2}\right|_{R_L = R_{Th}} = V_{Th}^2 \left.\frac{2R_L - 4R_{Th}}{(R_{Th} + R_L)^4}\right|_{R_L = R_{Th}} = -\frac{V_{Th}^2}{8R_{Th}^3}$$

$\left.\frac{d^2 p}{dR_L^2}\right|_{R_L = R_{Th}} < 0$ implies that at $R_L = R_{Th}$, p takes the maximum value.

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$ where $R_{Th} = R_L$

The maximum power transferred is

$$\boxed{p_{\max} = \frac{V_{Th}^2}{4R_{Th}}}$$

It should be noted that delivering the maximum power to the load results in significant internal losses.

Example 4.13 Find the value of R_L for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

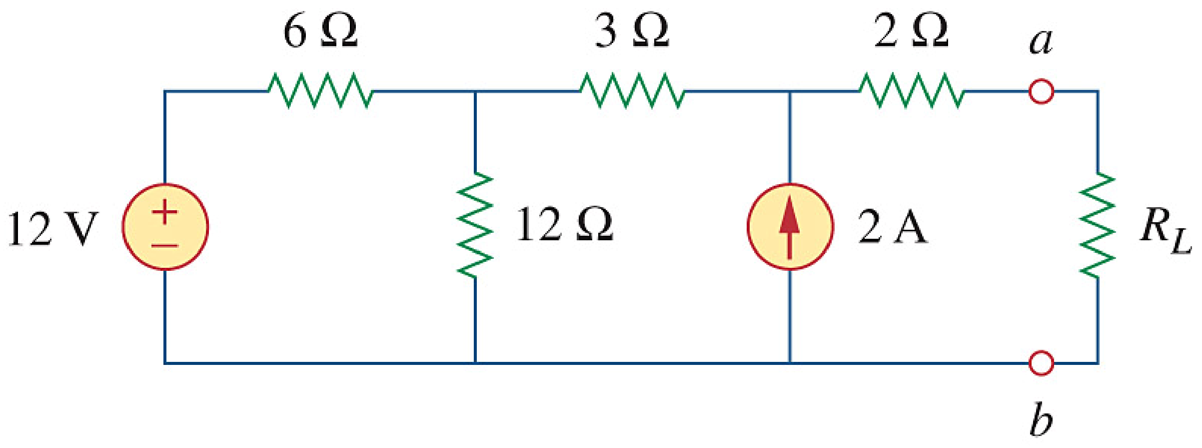


Figure 4.50

First, we will need to find a **Thevenin equivalent circuit**.

(i) R_{Th}

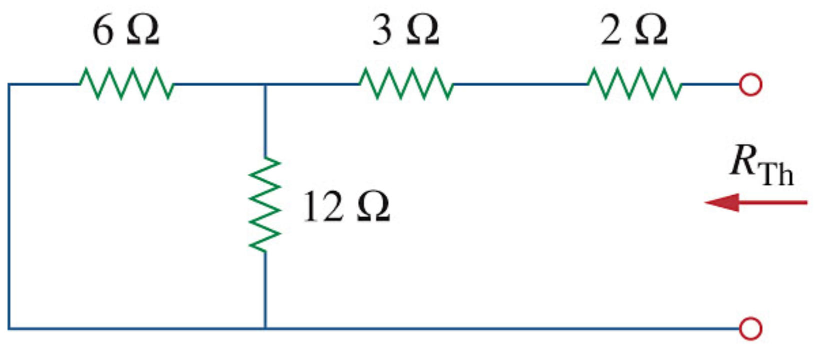


Figure 4.51(a)

First, we will need to find a **Thevenin equivalent circuit**.

(i) R_{Th}

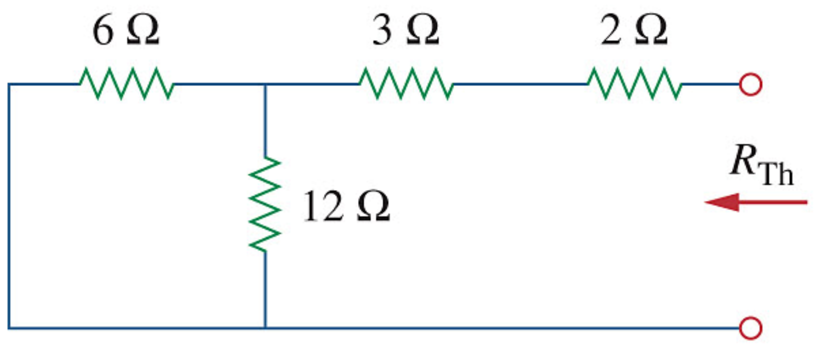
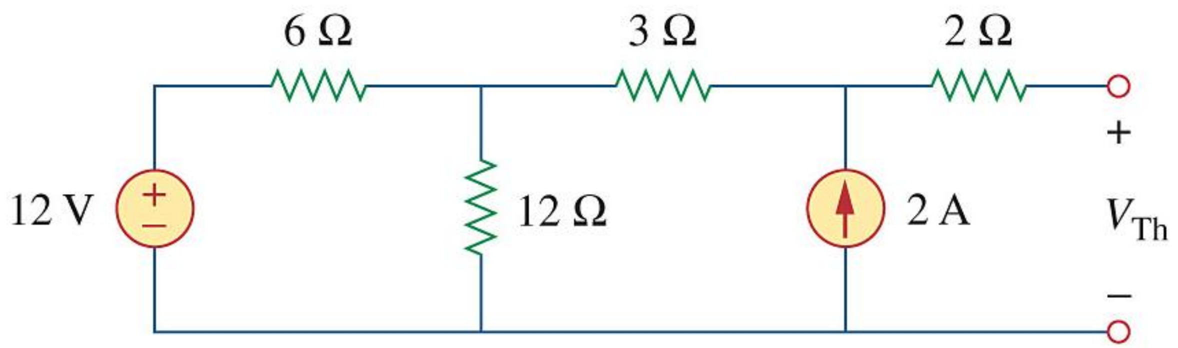


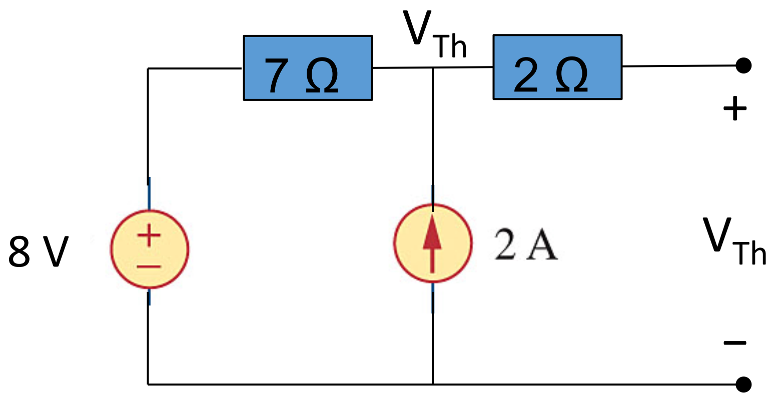
Figure 4.51(a)

$$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{6 + 12} = 9\ (\Omega)$$

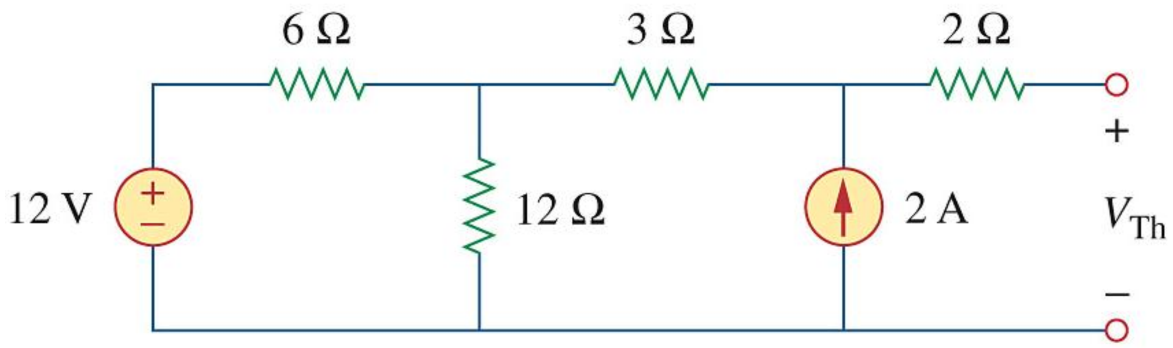
(ii) V_{Th}



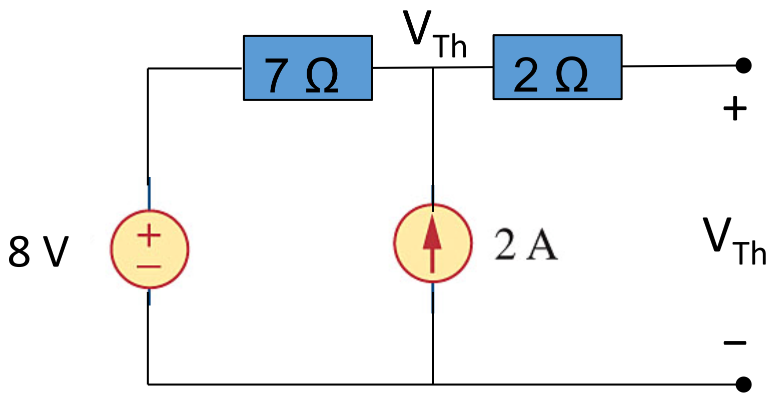
By Source Transformation



(ii) V_{Th}



By Source Transformation

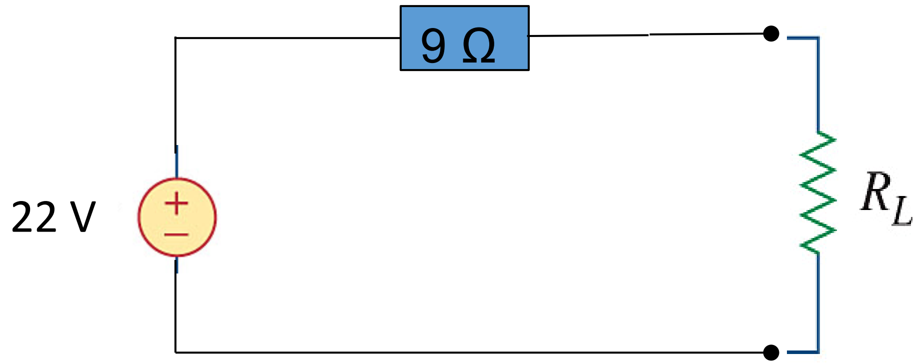


By KCL

$$\frac{V_{Th} - 8}{7} - 2 = 0 \rightarrow V_{Th} = 22\ [V]$$

Second, Set $R_L = R_{Th}$

And, finally, calculate the maximum power delivered to R_L



If R_L = 9 Ω, maximum power will be transferred

$$p_{\max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{22^2}{4 \times 9} = \frac{121}{9} \approx 13.44 \text{ (W)}$$