Columbia University
MATHG4074: Programming for Quant and Comp Finance

Case Study 1 (tentative due date: Thursday Feb 11, 2016)

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The characteristic function of the log of stock price in Black-Scholes framework is given by:

$$\phi(u) = \mathbb{E}(e^{iu\ln S_T}) = \mathbb{E}(e^{ius_T}) = \exp\left(i(\ln S_0 + (r - q - \frac{\sigma^2}{2})T)u - \frac{1}{2}\sigma^2 u^2 T\right)$$

For the following parameters: spot price, $S_0 = \$1800$; maturity, $T = 0.5$ year; volatility, $\sigma = 0.30$; risk-free interest rate, $r = 0.50\%$, continuous dividend rate, $q = 1.75\%$ and strike price of $K = \$2100$ evaluate the premium price of a European call option via fast Fourier transform by first forming the vector $\mathbf{X}$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{\eta}{2}\frac{\mathrm{C}}{(\alpha+i\nu_1)(\alpha+i\nu_1+1)}e^{-i\beta\nu_1}\phi\left(\nu_1-(\alpha+1)i\right) \\ \eta\frac{\mathrm{C}}{(\alpha+i\nu_2)(\alpha+i\nu_2+1)}e^{-i\beta\nu_2}\phi\left(\nu_2-(\alpha+1)i\right) \\ \vdots \\ \eta\frac{\mathrm{C}}{(\alpha+i\nu_N)(\alpha+i\nu_N+1)}e^{-i\beta\nu_N}\phi\left(\nu_N-(\alpha+1)i\right) \end{pmatrix}$$

with $\eta = 0.25$, $\alpha = 1.5$, $N = 2^n$ for $n = 12$, $\beta = \ln K$, $\lambda = \frac{2\pi}{N\eta}$, $\nu_j = (j-1)\eta$, and $\mathrm{C} = e^{-rT}$. Now, vector $\mathbf{X}$ becomes the input to the FFT routine, and its output is vector $\mathbf{Y}$ of the same size, $\mathbf{Y} = \mathrm{fft}(\mathbf{X})$; then call prices at strikes $K_m = \exp(\beta+(m-1)\lambda)$ for $m = 1, \ldots, N$ are given by

$$\begin{pmatrix} C_T(K_1) \\ C_T(K_2) \\ \vdots \\ C_T(K_N) \end{pmatrix} = \begin{pmatrix} \frac{e^{-\alpha\beta}}{\pi}\mathrm{Re}(y_1) \\ \frac{e^{-\alpha(\beta+\lambda)}}{\pi}\mathrm{Re}(y_2) \\ \vdots \\ \frac{e^{-\alpha(\beta+(N-1)\lambda)}}{\pi}\mathrm{Re}(y_N) \end{pmatrix}$$

where $\mathrm{Re}(y_j)$ is the real part of y_j. Note that the first entry i.e. $C_T(K_1)$ corresponds to call price at strike K.