For office use only	Team Control Number	For office use only
T1 ________	**55869**	F1 ________
T2 ________		F2 ________
T3 ________	Problem Chosen	F3 ________
T4 ________	**A**	F4 ________

2018
MCM/ICM
Summary Sheet

Summary

In this paper, We build a model of the temperature of the bathtub water in space and time, analysis the impact of the bathtub's volume, shape and factors of people, then make a non-technical statement.

First, to calculate the value of dissipated heat, we use the heat conduction formula, the heat convection formula and the formula of water surface evaporation, to build a basic heat dissipation model, including the dissipation of the surface of water and interface between water and bathtub. Through simulation, under the condition of $18°C$ degrees at room temperature, the initial water quantity of 150L, the water temperature is reduced by $3.5°C$ for half an hour, which is basically consistent with the data provided by the merchant.

In order to keep the temperature and save the water, we design two ways of water adding. The first type is to add water by manual adjustment, while the second type is to adjust the water automatically by PID. Under the same environmental conditions as the heat dissipation model, the conditions of the temperature measuring accuracy of 0.01 degrees and the upper limit of the water amount are added, and the simulation results are obtained as follows. Both two methods can make temperature fluctuations remain within the range ($0.1°C$) when parameters are property. And the amount of water used by PID reduces by 8-13% compared with manual adjustment. The second order central moment of two methods are 44.874 and 0.0296 (on condition of round bathtub, no bubbles, 30 minutes of bath time, no human body, 37.3ml/s water inflow speed, and PID adjustment)

The influence of human lies in the absorption of heat, the change of water level, and action. For calculating the effect of human surface area, we use Newton's law of cooling to calculate the heat absorbed by body. We add a water overflow which volume equals to 0.1% volume of person to simulate human action (per 30 seconds on average). The results show that the average water consumption when person in bathtub is about 4.76% more than the condition of no person. (on condition of oval bathtub, no bubble, with action, PID adjustment)

Bubble is considered to mainly influence heat loss model. We calculate the reflection coefficient to modify the value of radiation in heat loss model and ignore the evaporation in the heat loss model. The loss of heat caused by radiation is 0.94 times that of the original. Water consumption has been saved by 4.832L from 68.838L to 64.006L, on condition of oval bathtub, no person, PID adjustment.

Finally, we give the strengths and weaknesses of our models.

Keywords: **Heat dissipation model, Continuous water addition model, Continues system simulation, Simulate Anneal Arithmetic(SAA), PID**

Contents

1 Introduction

The traditional bathtub is a simple waterproof container. The bathtub temperature is decreasing and people need to constantly add hot water to maintain the temperature.

We want to establish such a viable model which can automatically add hot water to keep the temperature of the bathtub at the comfort temperature. At the same time saving water resources needs to be considered.

To solve this problem, we need to consider the impact of the bathtub shape and size on this model. Person's shape, size and temperature are also under consideration.

First, if we want to establish the water temperature model of the bathtub, we need to consider two aspects of heat about dissipation and heating.

For the heat dissipation model of the bathtub, we should determine which part of the bathtub heat dissipation and the dissipation mode of these parts. For the heating model of the bathtub, we need to consider whether the heating mode is manually heated or automatically heated. We have taken into account the upper limit of the tub volume. So we should constraint the upper limit of the water volume and establish a drainage model.

After establishing the model, we have to consider how to solve the model. And we want to use the simulation software to solve the model by finding the appropriate value to make the temperature constant. Then we will think about how to optimize the model and make the error of the value smaller.

At the same time we also need to consider the effect of the shape and volume of the bathtub on the model. We want to abstract the geometric properties of the bathtub as a function expression. We will try to minimize the impact of geometric properties of the bathtub on the model. We also want to design specific bathtub shapes to analysis.

Considering the effect of the human body on the model of the bathtub, we mainly determine how the temperature and volume of people effect the water temperature model. We will also explore how the behavior of people changes which attributes of the bathtub model. Consider the impact of bubble. We will mainly through the thermodynamic equations to explore how the bubble will change the thermodynamic parameters.

2 Assumptions

(1) The temperature distribution in the bathtub is evenly distributed. Because the temperature difference is small and the temperature distribution has little influence on the heat dissipation model, we can assume temperature distribution is evenly distributed to simplify the model.

(2) Evaporation will result in no loss of water. The loss of water from evaporation is too little to consider, so we ignore the change of quantity of water.

(3) The flux of inflow can be set precisely. Both two water adding methods can accurately control the flux of water.

(4) The temperature of overflowing water equals to the average temperature in bathtub. Because of the assumption that the temperature is uniform, the temperature of the overflow water equals to the average temperature in bathtub.

(5) The temperature of the room does not change. The hot steam have little influence on the temperature of the room, so the temperature of the room almost remain unchanged.

3 Symbol Description

Table 1:Constants

Symbol	uint	Meaning	value
P_v''	hP_a	vapor pressure of thin saturated layer on water surface	12
P_v	hP_a	water vapor partial pressure in wet air	1000
P_a	hP_a	atmospheric pressure	6.2
L	kj/kg	heat of water vaporization	2500
ε		emissivity	0.97
C_p	$KJ/(kg \cdot K)$	specific heat capacity of dry air at 300k	1.005
C_{water}	$KJ/(kg \cdot K)$	specific heat capacity of water at 300k	4.196

Table 2:Notation

Symbol	uint	Meaning
$V(h)$	m^3	the relationship between volume of water and liquid height
$S_1(h)$	m^2	the top surface area of the liquid
$S_2(h)$	m^2	the lower surface area of the bathtub
$S_3(h)$	m^2	the side contact area of the bathtub and liquid
S	m^2	the surface area of the body
h_p	cm	height of a person
w	kg	weight of a person
α	$W/(m^2 \cdot ^\circ C)$	heat dissipation coefficient
t	$^\circ C$	surface temperature of water
θ	$^\circ C$	dry bulb temperature of air
β	$W/(m^2 \cdot hP_a)$	evaporation coefficient
Q_{upper}	J	bathtub top surface unit time heat dissipation
Q_{side}	J	bathtub side surface unit time heat dissipation
t_{in}	$^\circ C$	the temperature of the water poured in
Q_{in}	J	the energy brought into the bathtub in period of time

4 Heat Dissipation Model

In order to analyze the situation of heat dissipation. We need to start from the bathtub. The effect of a bathtub on heat loss is mainly in material and shape.

The surface area of the liquid is variable because of the different shapes and attributes about bathtubs. Likewise the contact area of liquid and bathtub. Therefore we use functions to describe the relationship between the surface area of liquid, the liquid surface height and the contact area of liquid and bathtub.

We use these abstract expression to avoid the effect of exact shapes and attributes on the model.

$$\begin{aligned} h &= H(v) \\ S_1 &= S_1(h) \\ S_2 &= S_2(h) \end{aligned} \tag{4.1}$$

The water in the bathtub dissipates heat by contacting the air on the upper surface and the bathtub wall on the side surface. So we So we decompose the model into two parts:

- Liquid surface heat loss model.
- contact area of liquid and bathtub heat loss model.

4.1 Liquid surface heat loss model

The water dissipation mode including three parts: evaporation heat, convection heat and radiative heat dissipation.We use the coefficient of heat dissipation and dissipation general formula to get results.

The evaporation heat is the amount of heat transmitted to the air by the water at unit time Q_a. It can be represented by following formula:

$$dQ_a = \alpha(t - \theta)dS \tag{4.2}$$

The convection heat is the amount of heat that evaporates through the water at unit time Q_b. It can be represented by following formula:

$$Q_b = \beta(p''_v - p_v)dS \tag{4.3}$$

Radiative heat dissipation is a heat transfer method that emits light in a straight line in all directions centered on a heat source Q_c. It can be represented by following formula:

$$dQ_c = \varepsilon\sigma(t + 273)^4 dS \tag{4.4}$$

In unit time, the heat dissipation of the water surface is[1]

$$dQ = dQ_a + dQ_b + dQ_c = [\alpha(t - \theta) + \beta(p''_v - p_v) + \varepsilon\sigma(t + 273)^4]dS \tag{4.5}$$

ε is the emissivity. The value is 0.97
σ is a constant. The value is 5.6×10^{-8}
In the actual process, β and α can be obtained by General formula[2]:

$$\beta = \sqrt{[22.0 + 12.5W^2 + 2.0(t - \theta)} \tag{4.6}$$

$$\frac{\alpha}{\beta} = b = \frac{P_a C_p}{0.623L} \tag{4.7}$$

C_p is the specific heat capacity of dry air. As the temperature changes, there is little change in the value. So we take the value as the temperature equals to 300k.

$$C_p = 1.005KJ/(kg \cdot K) \tag{4.8}$$

According to assumption (3), the temperature of the water has the same value. So during unit time, the total heat dissipating capacity in the upper surface is:

$$\begin{aligned} Q_{upper} &= \int_{upper} dQ \\ &= \int_{upper} \alpha(t-\theta) + \beta(p_v'' - p_v) + \varepsilon\sigma(t+273)^4 dS \\ &= [\alpha(t-\theta) + \beta(p_v'' - p_v) + \varepsilon\sigma(t+273)^4] \cdot S_1 \end{aligned} \tag{4.9}$$

4.2 Side area of bathtub heat loss model

The heat dissipated area is the contact part between bathtub and liquid.So we just need to think about the effect of contact area.

For the water in the side surface of the bathtub is not in direct contact with the air, there will be no evaporation. Therefore there will not be dQ_b in the equation (4.2). Considering of the reflex of thermal radiation between water and the side surface, dQ_c will change.

The material of the side surface of the bathtub is acrylic. The index of refraction of acrylic is $n_1 \approx 1.4$, the index of refraction of water is $n_0 \approx 1.33$.

So the reflection coefficient is

$$R = \frac{(n_0 - n_1)^2}{(n_0 + n_1)^2} = \frac{(1.4 - 1.33)^2}{(1.4 + 1.33)^2} = 0.06 \tag{4.10}$$

Then in the equation(4.2), the dQ_c will be multiplied by $(1 - R = 0.94)$. Therefore, the heat dissipating capacity during unit time in the side surface is :

$$dQ = dQ_a + 0.94dQ_c = [\alpha(t-\theta) + 0.94\varepsilon\sigma(t+273)^4]dS \tag{4.11}$$

According to assumption (3), the temperature of the water has the same value. So during unit time, the total heat dissipating capacity in the side surface is :

$$\begin{aligned} Q_{side} &= \int_{side} dQ \\ &= \int_{side} [\alpha(t-\theta) + 0.94\varepsilon\sigma(t+273)^4]dS \\ &= [\alpha(t-\theta) + 0.94\varepsilon\sigma(t+273)^4]S_3 \end{aligned} \tag{4.12}$$

4.3 The bathtub heat loss model

Just as we mentioned above, the water in the bathtub dissipates heat by contacting the air on the upper surface and the bathtub wall on the side surface.

So the total heat dissipating capacity during unit time is:

$$Q_0 = Q_{upper} + Q_{side} \tag{4.13}$$

The specific forms of Q_{upper} and Q_{side} can be obtained from the above.

5 Model of Hot Water Addition

5.1 Influence of inflowing water on quantity of heat and volume

First, we should establish an abstract model to describe the heating process.

The water in the bathtub is heated by the hot water flowing into it. So we should consider the energy and the volume of water in unit time.

Set the unit time flow of the faucet as dV, and the temperature of the inflow water is t_{in}. Because the heat capacity of water changes in a very small degree, we can use $0°C$ as the reference point which means the relative quantity of heat is 0.

Compared to the energy reference, during unit time the energy of water poured in is:

$$q_{in} = C\rho dV t_{in} \tag{5.1}$$

Adding hot water is a continuous process. Then in the constant T_1 seconds of adding hot water, the total energy brought into the bathtub is:

$$Q_{in} = q_{in}T_1 = C\rho dV t_{in} T_1 \tag{5.2}$$

Meanwhile, the volume of water brought into the bathtub is:

$$\triangle V = VT_1 \tag{5.3}$$

Here, two different heating methods are given, one is adding water by manual adjustment, and the other one is adjusting the water automatically by PID. Taking into account the user's actual situation, such as whether there is a temperature sensor, combined with the specific parameters of the indoor environment, we will quantitatively compare the two models and recommend a model that makes the temperature more stable and more water saving.

5.2 Add a constant flow of hot water

In order to facilitate user operation, we give up the discrete model using complicated intermittent inflow water.

We discuss the process of adding water in two cases.

- When the water level does not reach the upper limit of the maximum water level,we can use (5.5) to calculate the temperature change rate

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{C\rho dV T_{high} + C\rho V(h)T}{V(h) + dV} \tag{5.4}$$

- When the water level reaches the overflow, the volume of water is almost invariable. Because we assume that the water in the bathtub is quickly mixed and uniform,we can get the temperature change rate function

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{C\rho dV T_{high} + C\rho V_{max}T}{V_{max} + dV} \tag{5.5}$$

  where V_{max} means the maximum amount of water contained in the bathtub.

5.3 Use automatic control algorithm to control flow velocity

The PID algorithm is a classic automatic control algorithm. Suppose we can get relatively accurate temperatures, accurately to $0.01°C$. And suppose the maximum flow rate of hot water is 100ml/s.

We can establish an automatic control equation, which includes three items of P, I and D.

$$T_{error} = T_{target} - T_{real} \tag{5.6}$$

But considering the accuracy of temperature measurement

$$T_{error} = \begin{cases} T_{target} - T_{real} & \text{if } |T_{target} - T_{real}| \geq 0.01 \\ 0 & \text{if } |T_{target} - T_{real}| < 0.01 \end{cases} \tag{5.7}$$

P is a proportion term

$$P = Kp \times T_{error} \tag{5.8}$$

I is an integral term

$$I = Ki \times \sum T_{error} \tag{5.9}$$

and the function of I is to Eliminate static error.

D is a differential term.

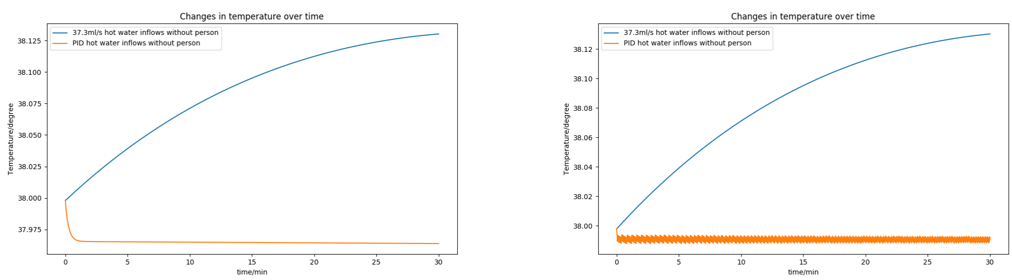
$$D = Kd \times (T_{error} - T_{last_error}) \tag{5.10}$$

The function of D is to Forecast the change trend.

The output water is dV :

$$dV = P + I + D \tag{5.11}$$

If the amount of output water is larger than $100ml/s$, calculate the dV as $100ml/s$. And if the amount is less than $0ml/s$, alculate the dV as $0ml/s$.



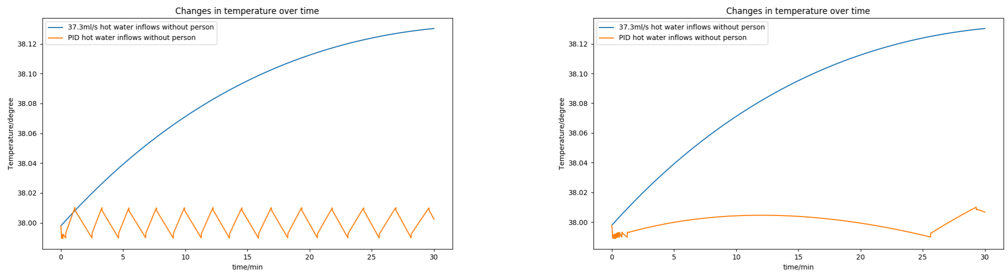
(a) When Kp is too small (b) When Kp is too large

Figure 1: Only use Kp

We simulated the PID algorithm in the environment of under the condition of $18.0°C$ degrees at room temperature, the initial water quantity of 150L.

We found that only the use of P is not a good way to adjust the temperature of the water. For example, the subgraph (a) in figure 1 shows the situation when Kp is small. There is a static difference between the expected temperature and the actual temperature. So we need to increase the value of Kp.

The subgraph (b) in figure 1 shows when the Kp is increased, the temperature produces a high frequency shock



(a) When parameters are unsuitable

(b) When parameters are suitable

Figure 2: Use Kp, Ki, Kd

We introduce the I and D items to reduce the frequency of the shock and make the water temperature as stable as possible. The subgraph (a) in figure 2 shows when we introduce I, the static difference can be eliminated.

The subgraph (b) in figure 2 shows when we also introduce the D item at the same time. The temperature trend becomes more stable.

6 Solutions for Tasks

6.1 Model Simulation with Computer

In the above paper, we've got a general model of heating and heat loss. Take advantage of the two general models we have already obtained, we established a simulation program built with Python. It can simulate the process of heating and heat loss. We can use the simulation program to solve the water temperature model of bathtub.

The figure below shows the process of simulation:

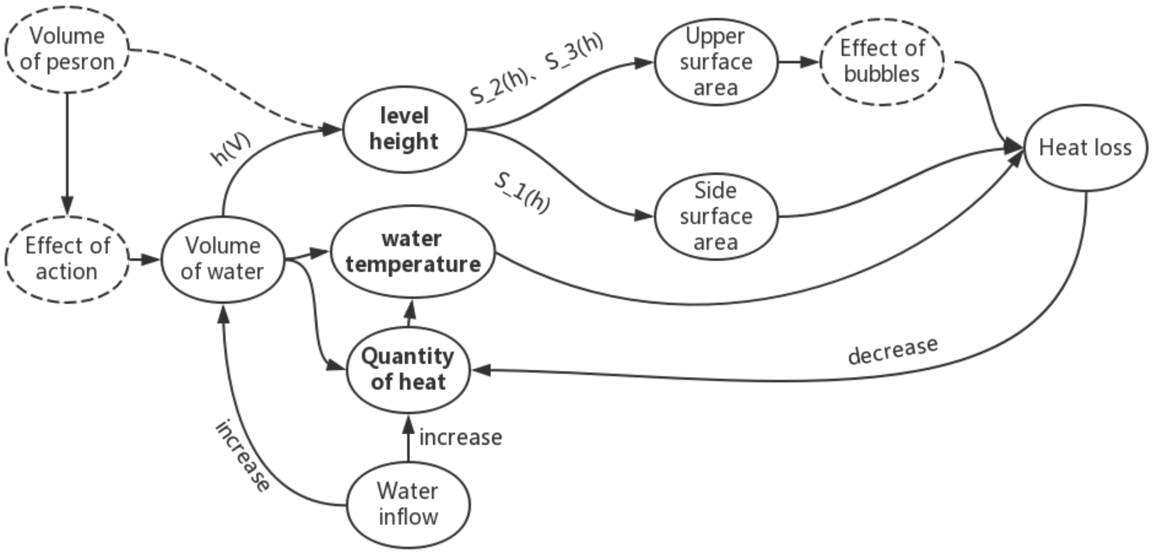


Figure 3: Interdependence of variables

A circle of dotted lines in Figure 3 may or may not exist. Arrows indicate the impact. $h(V), S_1(h), S_2(h), S_3(h)$ depend on the shape of bathtub. We can find that a change in a variable can lead to a series of changes. So we use **Object Oriented** design program met and take the bathtub as a **Class**. By using the **attribute mechanism** of Python, we solve the interdependence between variables. Different bathtubs are objects of different initial conditions for the class. The following picture shows the framework of the simulation.

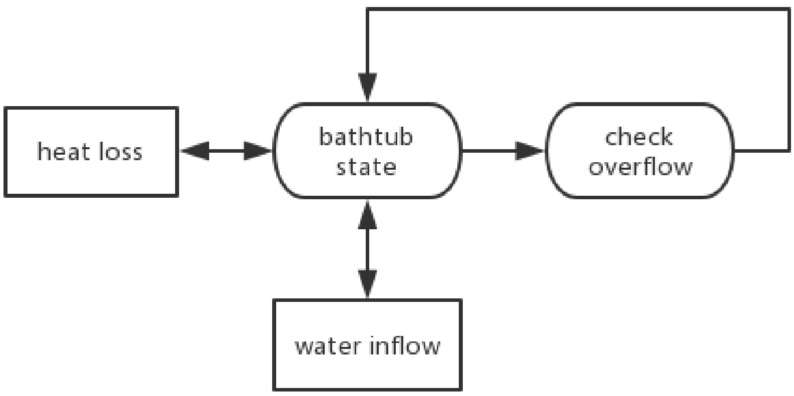


Figure 4: Single step simulation

Figure 4 shows the single step simulation. The simulation results of the last step will be used as the initial conditions for the next step. All of our simulations uses a step length of 1 second.

The initial temperature of water is set at $38.0°C$, the room temperature is set at $18.0°C$. And the amount of inflowing water is regarded as variable. We can change the value of it to control the amount of inflowing water.

6.2 Simulation Conditions

If there is no special explanation, the basic simulation conditions are the same as above. The default bathtub shape is based on circle.

- Indoor temperature: $18.0°C$
- The temperature of water inflow: $50.0°C$
- The initial temperature of water in bathtub: $38.0°C$
- The initial volume of water in bathtub: 200L(Circle bathtub), 150L(Oval bathtub).
- The upper limit of water volume: 300L(circle bathtub), 260L(oval bathtub).
- Atmospheric pressure $1.015 \times 10^3 hPa$
- Partial pressure of water vapor: 0.5 times of the current saturation pressure at indoor temperature
- Partial pressure of water vapor: 0.5 times of the current saturation pressure at indoor temperature
- The maximum water flux is 100ml/s. The temperature measurement precision of PID adjustment is $0.01°C$, and the control frequency is 1 hertz
- Indoor wind speed is set to 0
- PID parameters: Kp = 0.002,Ki = 0.0001,Kd = 0.001

6.3 Model Testing

First, we need to verify the accuracy of the heat dissipation model. We want to verify whether the model of the temperature change over time is in accordance with the actual situation.

We set the bath time as half an hour. And we don't add hot water or bubble bath additive in the bathtub.

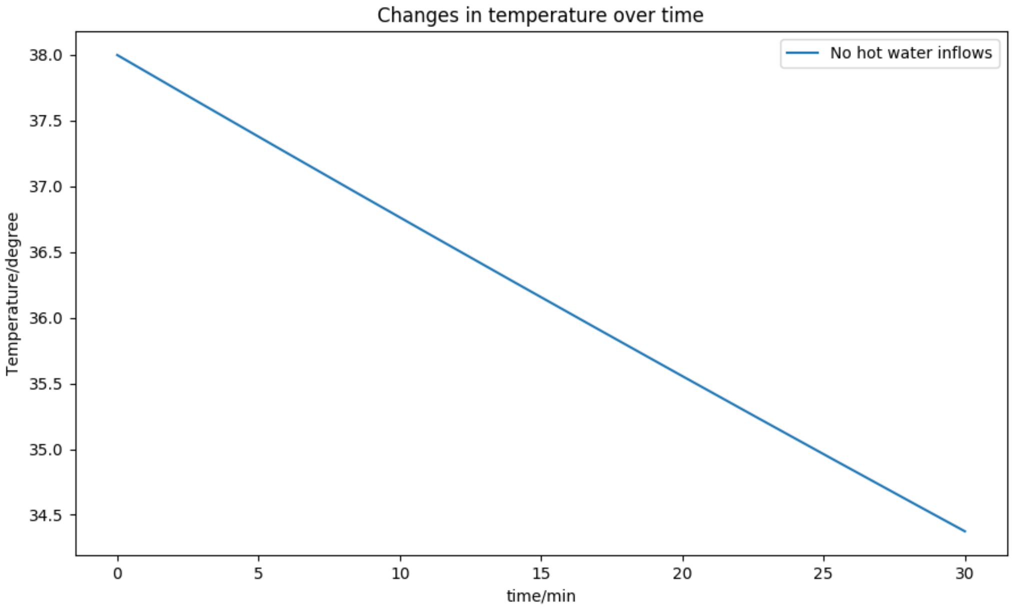


Figure 5: changes in temperature over time

The model implementation with computer is shown in the figure. We can get that the temperature in the bathtub has fallen as time goes on.It drops $4.5°C$ as half an hour in the bathtub. And in the indoor environment, we find the temperature of most bathtub without constant temperature system drops from 3 to $5°C$ in half an hour.

Our results is within this range. So the model fits the reality.

6.4 Solve the Model

As we mentioned in the model of hot water addition, when the inflow of water per unit time changes, the heat that the bathtub get is different. So we can change the amount of inflowing water during unit time to make the temperature of bathtub different.

We want to find a water inflowing per unit time which make the temperature is kept near the initial temperature of the bathtub. So we use the minimum dichotomy by constantly changing the amount of inflowing water during unit time.

And finally we found the water inflowing per unit time through more than a dozen iterations as $V_{property}$.

$$V_{property} = 40ml/s \tag{6.1}$$

The figure below shows the image of temperature varying with time when hot water is added.

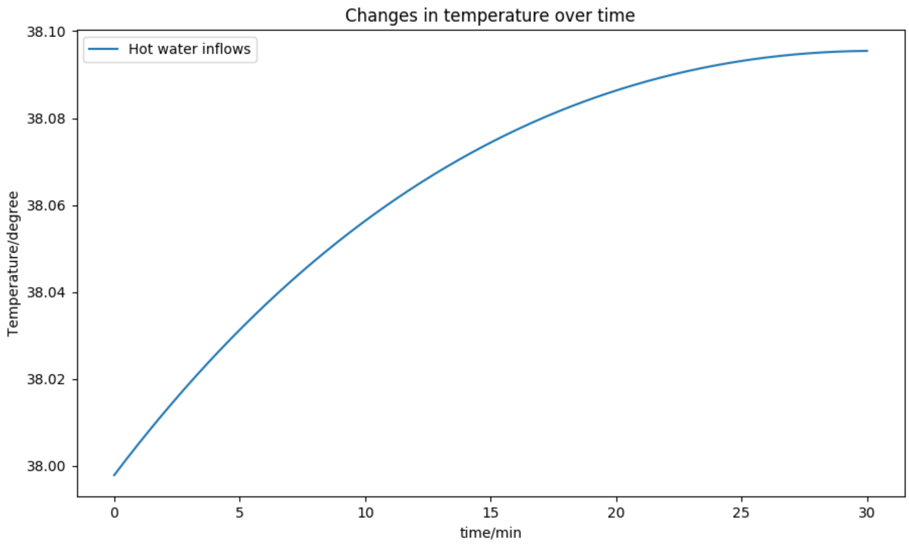


Figure 6: temperature varying with time when hot water is added

When hot water is added at in $V_{property}$, the temperature will only change in a very small range from 38 to 38.09. So we get a result that the fluctuation of the temperature is always kept in a range from 0 to 0.1.

It means the temperature will nearly not change.

Next, we want to find what is the result of adding hot water as $V_{property}$ to the model. So we made a comparison between temperature varying with time when hot water is added and not added. As shown in the picture:

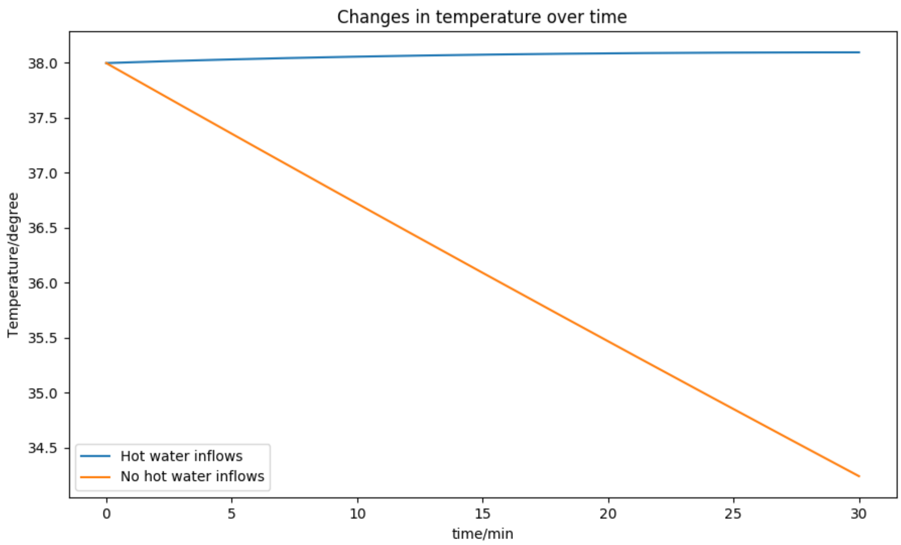


Figure 7: changes in temperature over time

The curves tell that our hot water is set perfectly, the temperature will neither get too high nor get too low.

When the water inflowing per unit time is $V_{property}$. We got a result that the fluctuation of the temperature is always kept in a range from 0 to 0.1. And the goal of keeping the temperature as close as possible to the initial temperature is achieved.

6.5 Model Optimization

It can be obtained from the above that when we use the minimum dichotomy to find the V_{best}, there always have a range from 0 to 0.1. It may bring some error. And we want to minimize the error of the model as we can. So we try to use the simulated annealing algorithm to optimize our model.

6.5.1 Simulated Annealing Algorithm

Starting from a higher initial temperature, the simulated annealing algorithm randomly searches for the global optimal solution of the objective function in the solution space with the sudden drop of the temperature parameter. In other words, the local optimal solution can jump out of probability Eventually tends to the global optimum.

6.5.2 Optimization Results

We use Python to implement simulated annealing algorithm.We call library functions of simulated annealing algorithm optimize our model. Finally, we successfully optimize our model and made the range of the V_{best} get less. As shown in the picture:

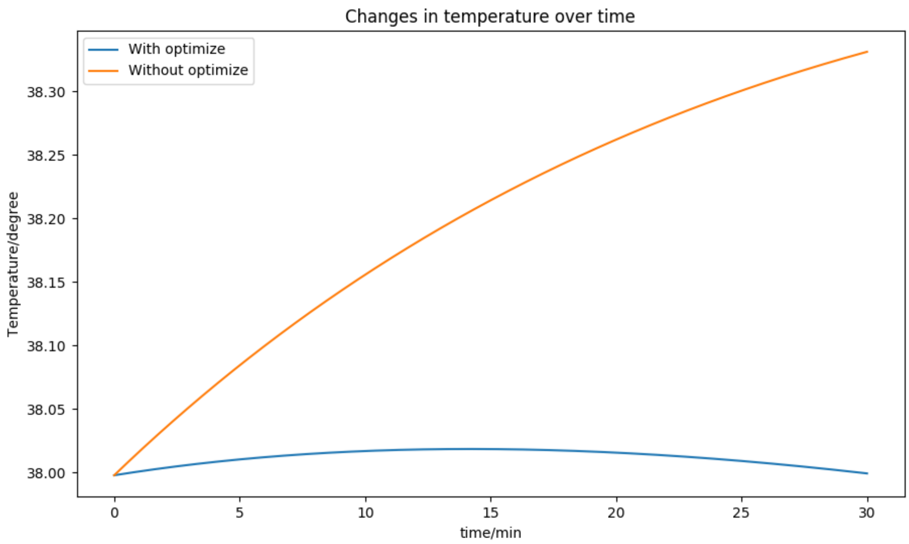


Figure 8: temperature varying with time when optimized

We can find that the curve of temperature with optimization approximately coincides with another curve without optimization. We got two curves with less variance. So The fluctuation of the temperature gets smaller. We successfully optimize our model.

6.6 Verifying the Stability of the Model

In real life, everyone's bath time is different. We should consider that some persons want to bath for longer time. So we should prolong the abscissa with more bath time to verify the stability of the model.

We prolong the bath time to 150 minutes. The figure below shows the change when time prolongs.

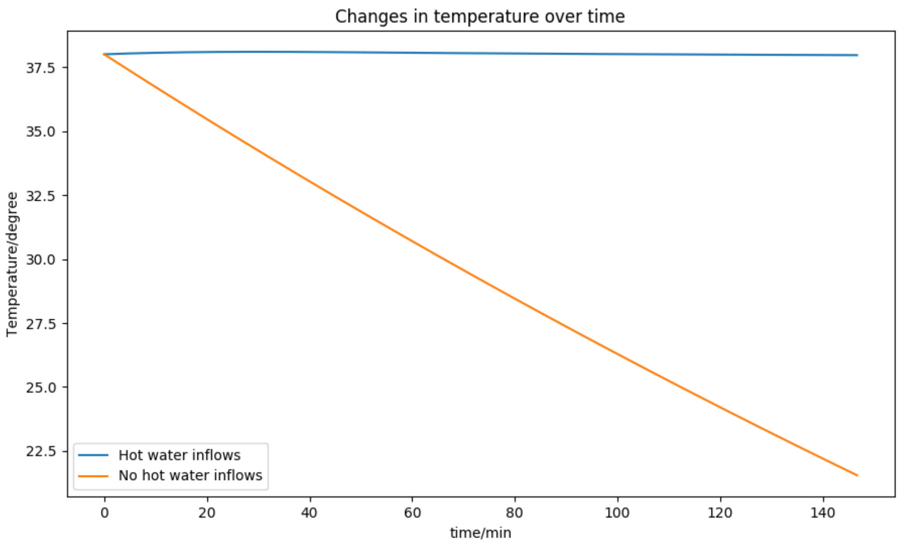


Figure 9: temperature varying with time when hot water is added

Therefore, when hot water is not added, the temperature will decrease with time constantly, even the temperature will be lower than the room temperature set previously. But when hot water is added, the temperature can also be kept well. So our model is **stable** and **effective**.

7 Factors that Affect the Model

7.1 The Influence of the Bubble

In our model, when bubble exists in the surface of water, the heat dissipation will change.

In the discussion above, we know that heat dissipation consists mainly of thermal evaporation, thermal radiation, and thermal convection. Thermal evaporation occurs mainly in the air when the humidity is not yet saturated. When bubble exists, the little air between the surface of water and bubble will soon reach saturation. So there will be no thermal evaporation.

Thermal radiation is an radiation of an electromagnetic wave. When the wave passes through a different medium(bubble and air), it will reflect back to the water. The amount of reflection is measured by reflection coefficient R.

And the index of refraction of air is :

$$n_1 \approx 1.0 \tag{7.1}$$

the index of refraction of bubble is:

$$n_0 \approx 1.33 \tag{7.2}$$

So the reflection coefficient is:

$$R = \frac{(n_0 - n_1)^2}{(n_0 + n_1)^2} = \frac{(1.0 - 1.33)^2}{(1.0 + 1.33)^2} = 0.02 \tag{7.3}$$

Then in the equation (10), the dQ_c will be multiplied by $(1 - R = 0.98)$.

In heat Dissipation Model the equation (10) will change into :

$$\begin{aligned} Q_{upper} &= \int_{upper} dQ \\ &= \int_{upper} [\alpha(t-\theta) + 0.98\varepsilon\sigma(t+273)^4]dS \\ &= [\alpha(t-\theta) + 0.98\varepsilon\sigma(t+273)^4]S_3 \end{aligned} \tag{7.4}$$

After the equation(10) has been changed, we simulate the process of bath again.

The figure below shows the image of temperature varying with time when bubble is added(with proper amount of hot water added).

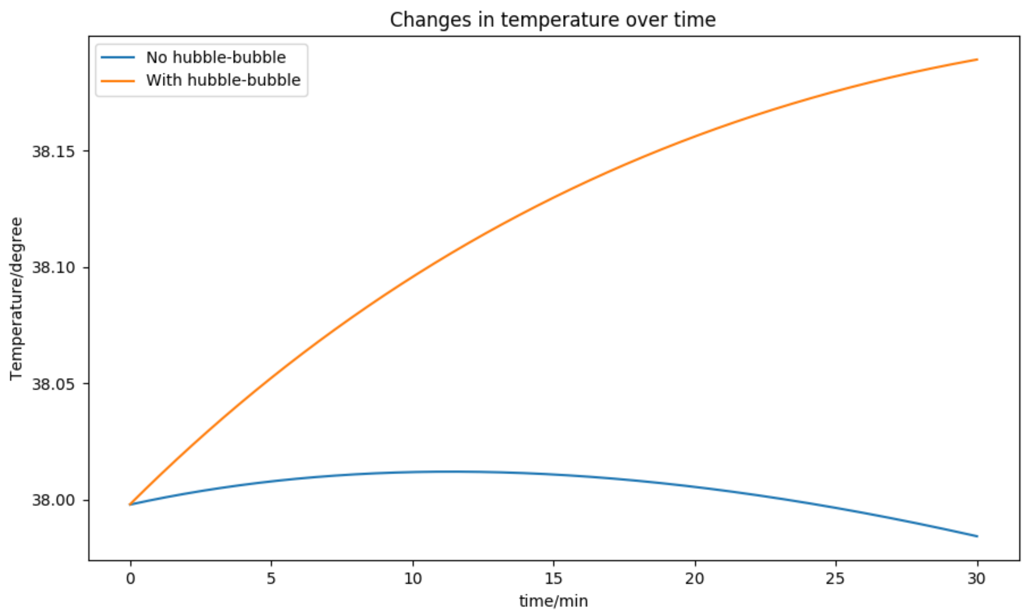


Figure 10: temperature varying with time when bubble is added

Through the analysis of Figure 10, we can find that bubbles can increase a small amount of thermal insulation.

7.2 Different Shapes of the Bathtub

We model and analysis several different shapes of bathtubs.

7.2.1 Bathtub based on Ovals

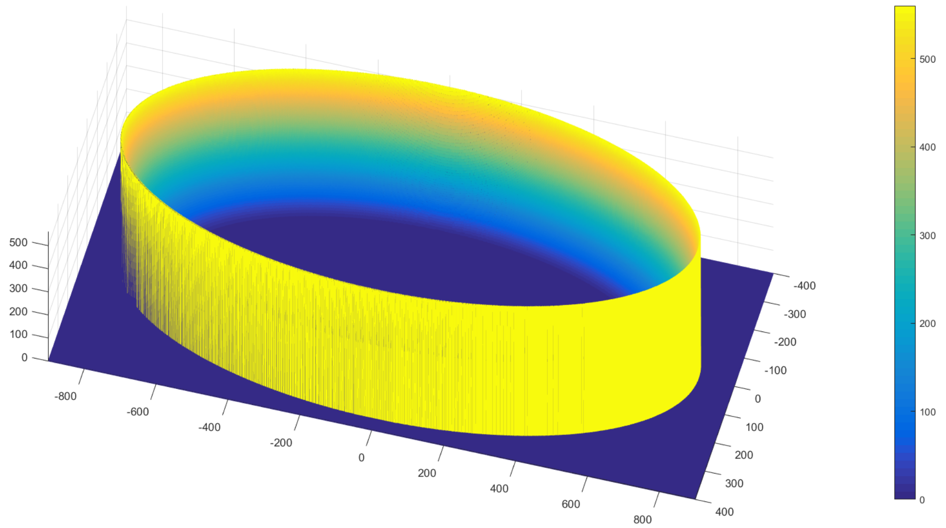


Figure 11: The shape of bathtub based on ovals

The elliptical long axis on the bottom of the bathtub is A, the short axis is B, the height of the whole bathtub is h, and the lower bottom long axis is $A - 2d$.

We divide the bathtub into an infinite number of elliptical columns with a height of dz, with a long semi-axis of each elliptical element

$$a(z) = \frac{A}{2} - d + \frac{d}{H^2}z^2 \tag{7.5}$$

Short semi-axis of each elliptical element is

$$b(z) = \frac{B}{2} - d\frac{B}{A} + \frac{Bd}{AH^2}z^2 \tag{7.6}$$

Calculation of the volume of water

The volume of the small cylinder is

$$dV = Sdz = \pi a(z)b(z) \tag{7.7}$$

The relationship between the volume of water in the bathtub and the height of the liquid surface is

$$\begin{aligned} V(h) &= \int dV = \int Sdz = \int_0^h \pi a(z)b(z)dz \\ &= \int_0^h \pi(\frac{A}{2} - d + \frac{dz^2}{H^2})(\frac{B}{2} - d\frac{B}{A} + \frac{dBz^2}{AH^2})dz \end{aligned} \tag{7.8}$$

We can get relationship between the volume of water and the height of liquid surface by replacing the bathtub size into the formula (7.8).

Calculation of the contact area of water with the bathtub and air

Without considering the effect of person, the upper and lower surface of the liquid contact are standard ovals.

The area of upper surface is

$$S_1 = \frac{\pi AB}{4} \tag{7.9}$$

And area of the lower surface is

$$S_2 = \pi(\frac{A}{2} + (\frac{h^2}{H^2} - 1)d)(\frac{B}{2} + (\frac{h^2}{H^2} - 1)d\frac{B}{A}) \tag{7.10}$$

In order to calculate the contact area between the liquid and side of the bathtub, we use the calculus method.

The area of the small cylinder is

$$ds = \sqrt{1 + (\frac{\partial a(z)}{\partial z}^2)}L(z) \tag{7.11}$$

where $a(z)$ is the long axis of a cross section ellipse with a height of h and $L(x)$ is the circumference of the ellipse of the cross section. And $L(x)$ is

$$L(z) = \pi(\frac{3}{2}(a(z) + b(z)) - \sqrt{a(z)b(z)}) \tag{7.12}$$

So we can get the lateral area through the integral (7.11)

$$S_3 = \int ds = \int_0^h \pi\sqrt{1 + \frac{4d^2z^2}{h^4}}(\frac{3}{2}(a(z) + b(z)) - \sqrt{a(z)b(z)})dz \tag{7.13}$$

Simulation Result

We simulate the optimization of water inflow using the annealing algorithm and water inflow adjusted by PID. The result is as follows

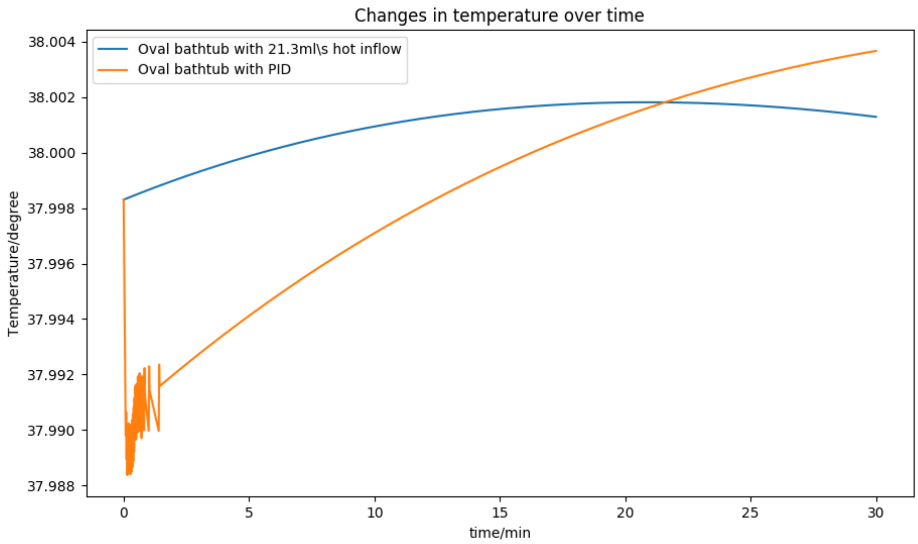


Figure 12: Comparison of two kinds of water adding methods

We can find that the optimal manual adjustment is a little better than the PID control. The reason is that the PID automatic regulation depends on the temperature measurement accuracy. But the temperature stability of two methods is also very good.

7.2.2 Bathtub based on circles

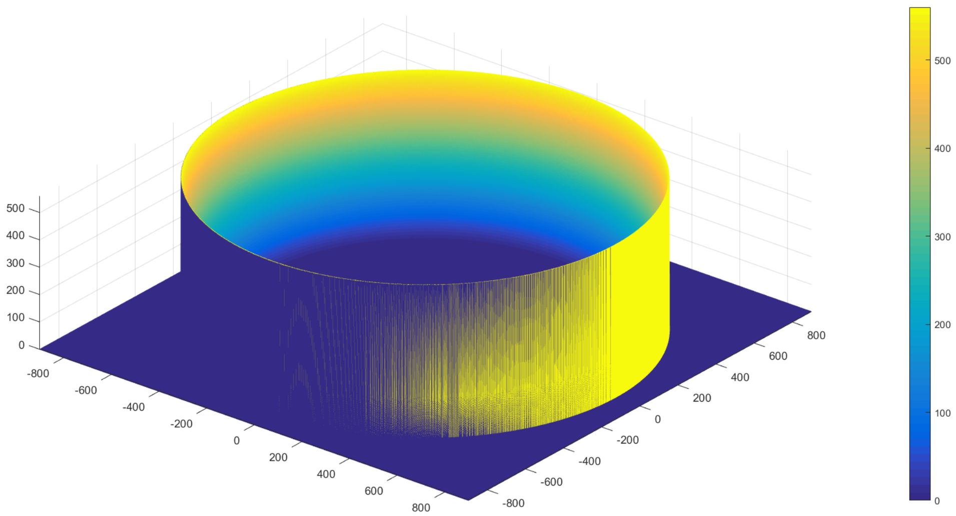


Figure 13: The shape of bathtub based on circles

Also without considering the effect of person, the upper and lower surface of the liquid contact are standard ovals.

The area of upper surface is

$$S_1 = \pi(R + (\frac{h^2}{H^2} - 1)d)^2 \quad (7.14)$$

And area of the lower surface is

$$S_2 = \pi(R - d)^2 \quad (7.15)$$

where R is the upper surface area of the bathtub, H is the height of bathtub, d is the the difference between the radius of the upper surface and the lower surface, and h is the height of the liquid surface.

In order to calculate the contact area between the liquid and side of the bathtub, we use the calculus method like the oval-based bathtub.
And we can get the area of side surface S_3 of the bathtub.

$$S_3 = \int ds = \int_0^h \pi\sqrt{1 + \frac{4d^2z^2}{h^4}}(3r(z)) - r(z))dz \quad (7.16)$$

where $r(z)$ is

$$R + (\frac{h^2}{H^2} - 1)d \quad (7.17)$$

Simulation Result

We compare two ways of adding hot water.

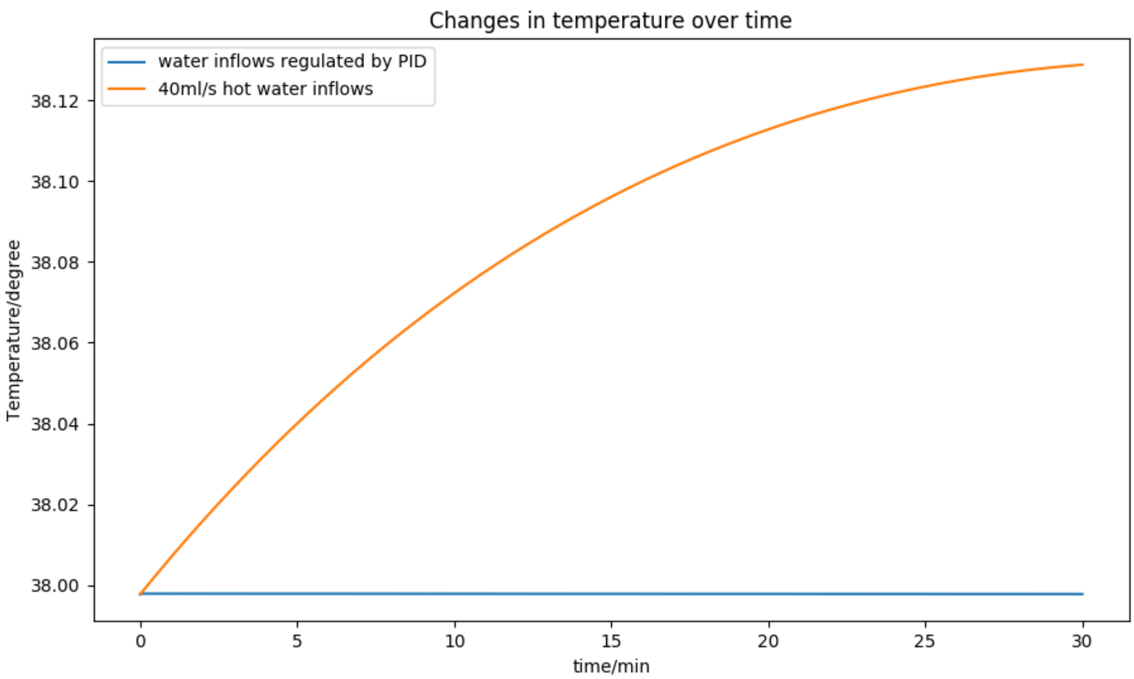


Figure 14: Comparison of two kinds of water adding methods

The results show that the PID regulation is more stable.

Comparison of Simulation Results

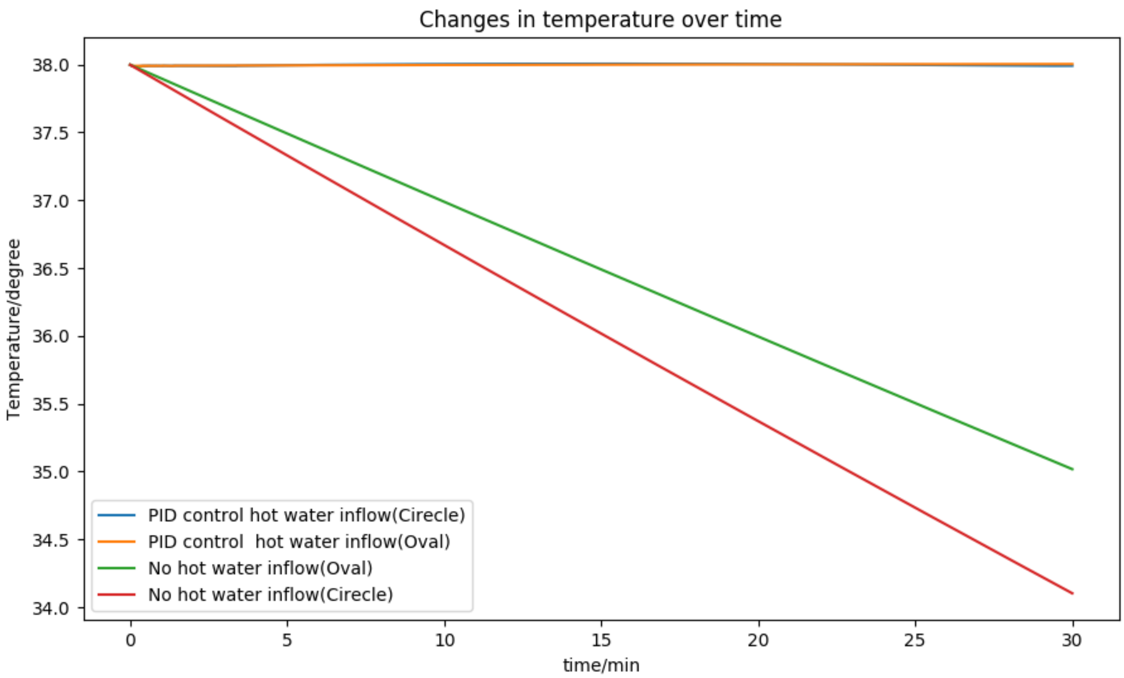


Figure 15: Comparison of bathtub shapes and water adding methods

We compare the heat dissipation and the automatic water injection model of two kinds of bathtub models. The results show that the PID algorithm has a good adaptability. And because the surface area of the elliptical bathtub is small and the heat dissipation is slow, the result is consistent with the reality.

7.3 The effect of human body and body temperature on the model

We improved our model in view of the human body influence. Our improvement is mainly in the following aspects.

7.3.1 The influence of the heat conduction

The interior of the body is basically maintained at a constant temperature, and the temperature of the skin is close to the water in bathtub. So we can simplify the question.The simplified question is illustrated as follows.

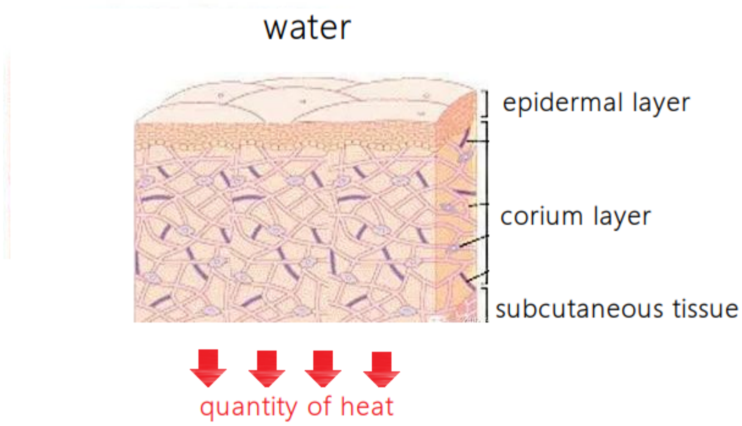


Figure 16: Heat conduction model of human body

We can assume that the temperature of the skin is the same as the water. We select the temperature of the rectum about 37.5°C as the body temperature. The problem can be simplified as the skin heat dissipating to the interior of the body.

So we can use **Newton's law of cooling** to calculate the speed of heat dissipation.

$$\begin{aligned} \Delta T &= |T_w - T_f| \\ q &= h\Delta T \\ \phi &= qA = Ah\Delta T \end{aligned} \tag{7.18}$$

Where ***q*** is the heat flux density. ***h*** is the convection heat transfer coefficient of matter. ϕ is heat transfer power (or heat transfer per unit time). ***A*** is the heat transfer area.

The surface area of the human body can be calculated by the following formula.

$$S = \begin{cases} 0.0057 * h + 0.0121 * w + 0.0820(\text{ man }) \\ 0.0073 * h + 0.0127 * w - 0.2106(\text{woman}) \end{cases} \tag{7.19}$$

According to formula 16, the quantity of heat dissipated(dQ) in unit time is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \phi = qA = Sh\Delta T \tag{7.20}$$

where h=1.48,$T_f = 37.5°C$.

We add this part of heat loss to our heat dissipation model. The change of temperature with time is obtained by the simulation program. And the result is drawn into a graph as follows

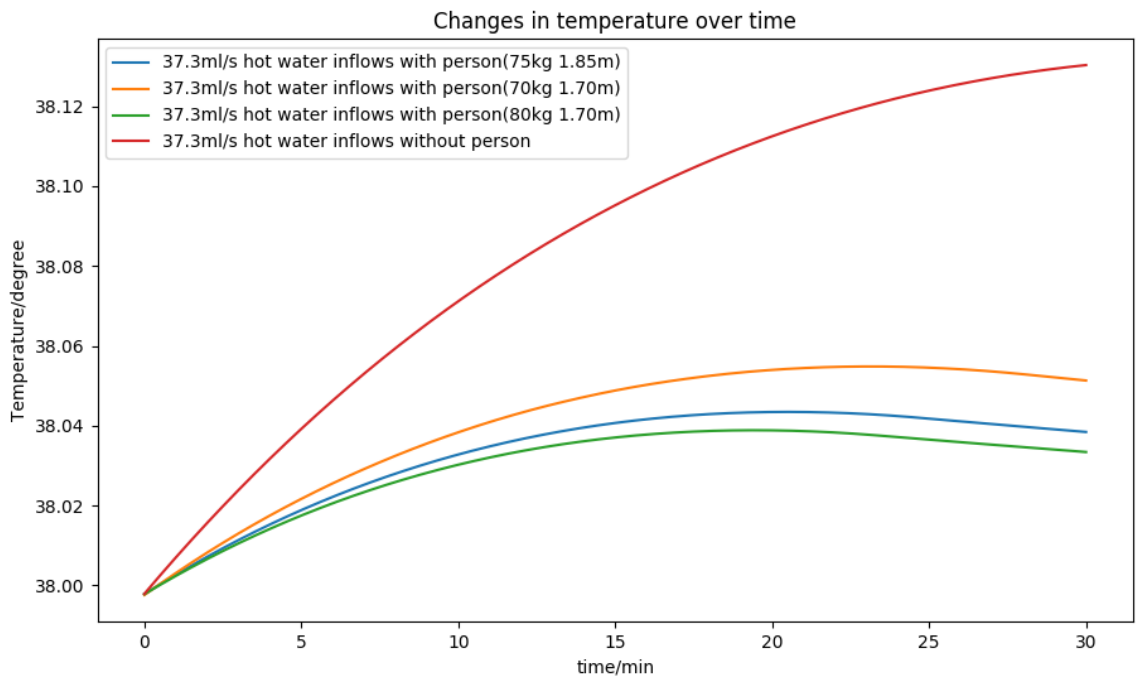


Figure 17: The change of temperature influenced by human body over time

7.3.2 The influence of the body shape and action

The volume of the person in the water affects the height of the liquid. In order to discuss the influence on the liquid surface conveniently. We use the weight to estimate the volume of the body. We use $\rho = 1.06 \times 10^3 kg/m^3$ as the density of the human body, so the volume can be calculated:

$$v_{body} = \frac{m}{\rho} = \frac{m}{1.06 \times 10^3}(m^3) \tag{7.21}$$

If we consider the effect of the body volume, we need to correct of the height of the liquid surface. We assume 65% volume of the body is in the water. So the height of the liquid surface should be corrected to

$$h = H(v_{water} + 0.65 v_{body}) \tag{7.22}$$

Because the effect of body action on convection and radiation heat dissipation is very small, we simplify the effect of body action on our model.

We assume that the movements of persons can only lead to a water spill that is equivalent to a certain proportion of the body's volume. We set the volume of liquid for each overflow is dv

$$dv = 5 \times 10^{-3} v_{body} \tag{7.23}$$

We set the probability of water overflow per second is 0.02, which means there is an overflow of water on average in half a minute.

Simulation Result

We built the model with the added action impact and compare it with the simulation results that has no action impact, and we get the following figure

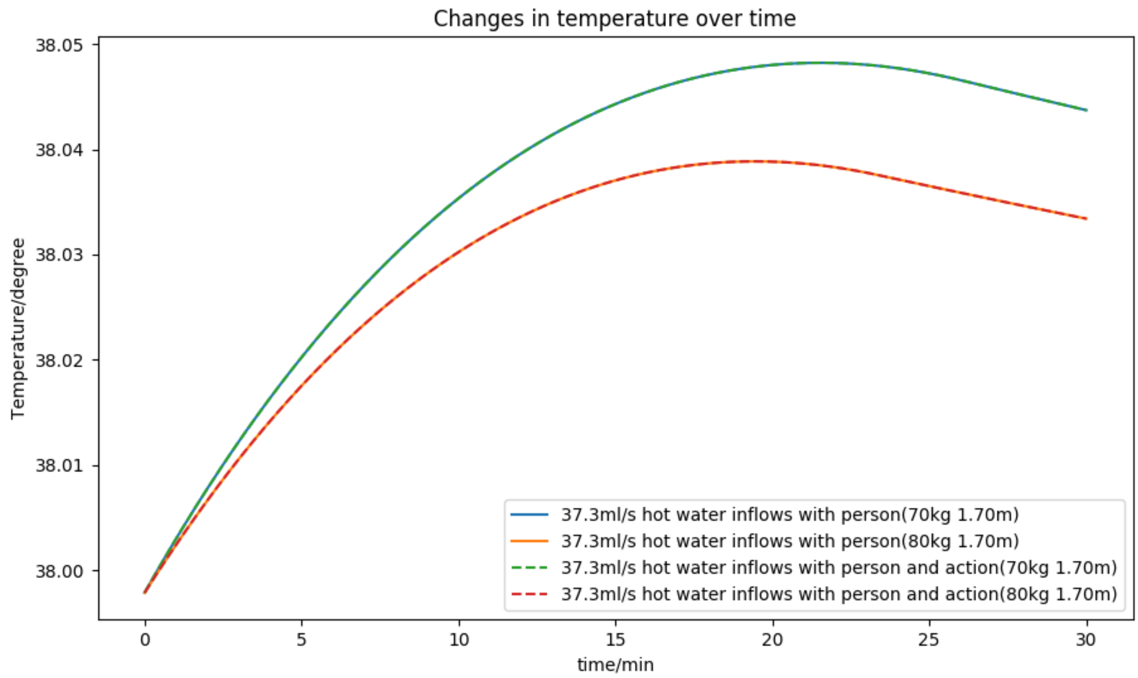


Figure 18: The effect of body action

When we **manually** adjust the amount of water, by analyzing the Figure 18, we can find that when action and no action curve basically coincide. Therefore, the action has little effect on the trend of temperature change.

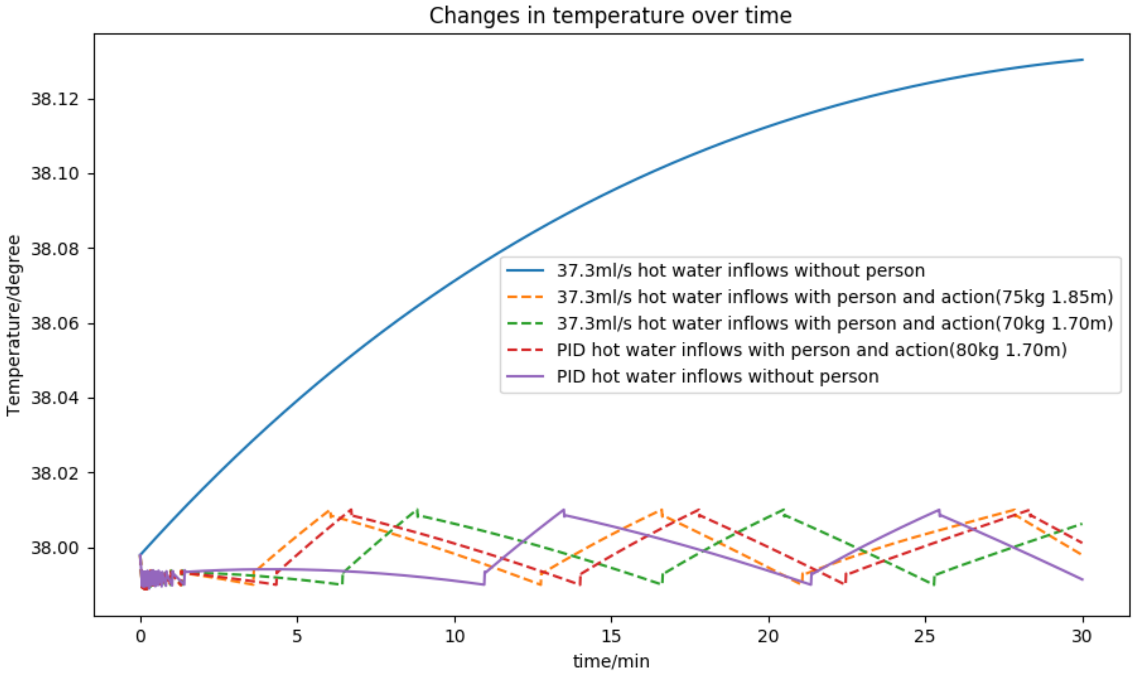


Figure 19: The effect of body action on PID model

When we use **PID** to adjust the amount of water, by analyzing the Figure 19, we can find that there are some difference of action and no action curves. But the stability of the temperature is basically the same

We generally use the single variable principle to sum up the simulation data. The data obtained under different heating methods and influencing factors are shown in the table below

Table 3:The simulation result
(Environment:Initial water volume 200L,Room temperature 18°C)

water flow	Water consumption	bubble	body shape	action	temperature variance
PID	68.838L	No	No	No	0.0661
40.0ml/s	72.000L	No	No	No	15.66907
PID	64.006L	Yes	No	No	0.0621
40.0ml/s	72.000L	Yes	No	No	86.95530
37.3ml/s	67.140L	Yes	No	No	15.61321
37.3ml/s	67.140L	Yes	75kg 1.85m	Yes	2.242
37.3ml/s	67.140L	Yes	70kg 1.70m	Yes	3.612
37.3ml/s	67.140L	Yes	80kg 1.70m	Yes	3.279
PID	63.856L	Yes	No	No	0.06619
PID	66.367L	Yes	75kg 1.85m	Yes	0.05963
PID	66.222L	Yes	70kg 1.70m	Yes	0.06282
PID	66.170L	Yes	80kg 1.70m	Yes	0.062188

Table 4:Oval shaped bathtub simulation result
(Environment:Initial water volume 150L,Room temperature 18°C)

water flow	Water consumption	bubble	body shape	action	Two order center moment
37.3ml/s	38.339L	Yes	No	No	44.8740
37.3ml/s	38.339L	Yes	75kg 1.85m	Yes	40.6900
37.3ml/s	38.339L	Yes	70kg 1.70m	Yes	40.1090
37.3ml/s	38.339L	Yes	80kg 1.70m	Yes	40.3992
PID	34.246L	Yes	No	No	0.02956
PID	34.331L	Yes	75kg 1.85m	Yes	0.07450
PID	34.357L	Yes	70kg 1.70m	Yes	0.05084
PID	34.362L	Yes	80kg 1.70m	Yes	0.08233

Finally, the results show that the average water consumption when person in bathtub is about 4.76% more than the condition of no person. And the PID regulation saves about 9-12% of water. The second order central moment of two methods are 44.874 and 0.0296. (On condition of round bathtub, no bubbles, 30 minutes of bath time, no human body, 37.3ml/s water inow speed, and PID adjustment)

7.4 Conclusion

Through the analysis of Table 3 and 4, We can get the following conclusions

- When the long half axis of the oval bathtub is equal to the radius of a round bathtub, the hot water consumption of the oval bathtub is obviously small(about 40%).
- PID regulation saves about 9-12% of water.
- Bubbles can slightly enhance the thermal insulation performance. The final water temperature is about 0.2°C higher than that without bubbles under the same conditions.
- The amount of water consumption increases with the increase of human body shape, human movements have no special effect on the model. But to some extent, the more water overflow, less water consumption by PID regulation.

8 Strengths and Weaknesses

8.1 Strengths

- **The simulation result of our model is appealing.** As we have shown before, almost all the simulation of our model is according with the real world situation. And this means our model is effective to describe the process of bath.
- **Our model is steady.** When the different influential factors are considered, our model is still effective. Our model is steady to describe the process in a varied situation.

- **Our model is easy to implement.** We give the specific amount of inflowing water. When the real environmental parameters are given, the users can easily adjust the amount of inflowing water to keep the initial temperature.

8.2 Weaknesses

- **The motions of people is simplified.**The motions of people are too complicated and variable to consider. Some complicated motions may bring more influence on our model.

9 Explanation for users

The bathtub we use in our daily life is a simple water containment, in order to take a shower comfortably, we need constantly add hot water in the bathtub.

Here, we give the explanation of the strategy to add water researched in this paper.

If the volume of your bathtub is a little bigger than normal size, you should increase the amount of inflowing water. You'd better buy the bathtub with smaller top surface.

If you are fat, first you should increase the amount of inflowing water. But if the water in the bathtub reaches the upper limit, you can slightly reduce your water inflowing. Bubble additive can provide you with a more comfortable bathing.

And if you want to save water, you should use the automatic control.

While, keeping the temperature as close as possible to the initial temperature is difficult.

There are all kinds of factors about heat loss like wind speed, saturated pressure and bathtub material. And environmental conditions are constantly changing. The different temperature, volume and motions of people also influence the temperature in the bathtub. So it's difficult to keep the temperature as close as possible to the initial temperature.

10 References

References

[1] Zhenguo Zhao. Formula of enthalpy difference for water surface heat dissipation and its application[J]. Journal of Hydraulic Engineering, 2004(02):34-38.

[2] Peimin Pu. Study on the formula of evaporation and heat dissipation of water surface(1)[J]. Journal of Lake Sciences, 1994(01):1-12.

[3] Peimin Pu. Study on the formula of evaporation and heat dissipation of water surface(2)[J]. Journal of Lake Sciences, 1994(03):201-210

[4] Xinyong Chen, Jing Zhu, Longxi Han. Determination method and sensitivity analysis of water surface comprehensive heat dissipation coefficient [J]. Water Sciences and Engineering Technology, 2007(05):9-13.

[5] Bowen L S. The ratio of heat losses by conduction and by evaporation from any water surface [J].phts.Rev.,1926,27:779-787.

[6] Baowei Xiang. Research and application of simulated annealing algorithm in structural optimization[D]. Dalian University of Technology,2004.