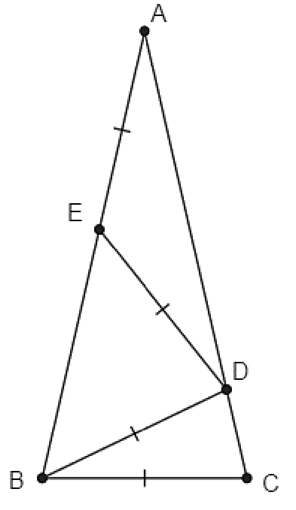
Problem 1

In ΔABC with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that

$$AE = ED = DB = BC.$$

The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



设 $\angle A = 2\alpha$，$\angle B = \angle C = 90^\circ - \alpha$，则 $\angle ADE = 2\alpha$，$\angle BDC = 90^\circ - \alpha$

同时有外角 $\angle BED = 4\alpha = \angle EBD$，于是

$8\alpha = 2\alpha + 90^\circ - \alpha \Rightarrow \alpha = \frac{90^\circ}{7}$，$\angle B = \frac{540^\circ}{7}$，答案 $\boxed{547}$

Problem 2

There is a unique positive real number x such that the three numbers

$$\log_8(2x), \log_4 x, \text{and } \log_2 x,$$

in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

设 $x=2^a$ （$a \in \mathcal{R}$），则 $\frac{a+1}{3} \cdot a = (\frac{a}{2})^2 \Rightarrow a=-4$

$x=\frac{1}{16}$ 验算后也是对的，答案 $\boxed{017}$

Problem 3

A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation

$$\underline{1}\,\underline{b}\,\underline{c}\,\underline{a},$$

where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

首先 $121a+11b+c=512+64b+8c+a$，即 $120a-53b-7c=512$，要求 (a,b,c) 最小

来凑一下，$a_{\min}=5$，此时 $53b+7c=88$，故 $b=1, c=5$

最后验算 $(515)_{11}=(1155)_8$，答案为 $\boxed{621}$

Problem 4

Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, $42,020$ is in S because 4 is a divisor of $42,020$. Find the sum of all the digits of all the numbers in S. For example, the number $42,020$ contributes $4+2+0+2+0=8$ to this total.

设 $n=\overline{d2020}$，要求 $d|n \Rightarrow d|2020$，故
$d=1,2,4,5,10,20,101.202,404,505,1010,2020$

手动计算数字之和，记得算上 12 个 2020 的贡献，答案 $\boxed{093}$

Problem 5

Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

考虑去掉了某个数 i 后剩下数单调递增（递降的相当于全部翻转，最后 ×2 即可）

我们从 1 到 6 依次枚举 i，可以发现，当 $i=1$ 时有 6 种方案，而当 $i>1$ 时有 4 种方案（另 2 种出现过）

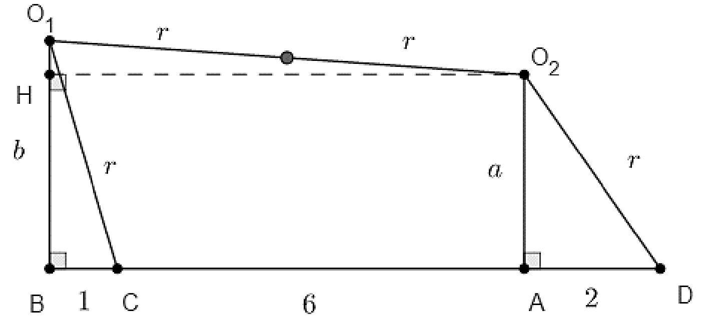
故答案为 $\boxed{052}$，其中的证明部分可以自己尝试一下（或者全部列出来看看）

Problem 6

A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

题面的 sit 用得太精妙了，我看了三遍才明白，其实是说两个球卡在了两个洞里，半径相同且外切

设球半径为 r，简单画个竖过来的截面：



方便起见设 $a=O_1A, b=O_2B$，有 $a^2=r^2-4, b^2=r^2-1$

$$
\begin{aligned}
O_1H^2+O_2H^2 &= O_1O_2^2 \\
(b-a)^2+7^2 &= (2r)^2 \\
(\sqrt{r^2-1}-\sqrt{r^2-4})^2 &= 4r^2-49 \\
-2\sqrt{r^4-5r^2+4} &= 2r^2-44 \\
r^4-5r^2+4 &= r^4-44r^2+484 \\
39r^2 &= 480 \\
\Rightarrow r^2 &= \frac{160}{13}
\end{aligned}
$$

最后答案是 $\boxed{173}$

Problem 7

A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

组合恒等变形以后，判断每个质因数是否出现在 N 里，答案为

$2+7+13+17+19+23=\boxed{081}$

$$\sum_{i=0}^{11}\binom{11}{i}\binom{12}{i+1}=\sum_{i=0}^{11}\binom{11}{11-i}\binom{12}{i}=\sum_{i=0}^{11}\binom{23}{11}=\frac{23!}{11!12!}$$

Problem 8

A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then

$$OP^2=\frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

小虫的最终位置相当于每一天行走的向量之和，稍微画个图就能发现，只需要计算出前 6 天的结果+等比数列求和即可。剩下的事情应该不必多说了，答案是

$\frac{100}{3}\Rightarrow\boxed{103}$（算错的老老实实检查...）

Problem 9

Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random from the set S and labeled $a_1, a_2,$ and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

$20^9=2^{18}\cdot 5^9$，可看成分别随机 2 和 5 的幂次，要求 $a\leq b\leq c$，枚举下来概率是:

$$\frac{19\cdot 70}{19\cdot 19\cdot 19}\cdot\frac{11\cdot 20}{10\cdot 10\cdot 10}$$

约分后分子是 $\boxed{077}$

Problem 10

Let m and n be two positive integers such that

$$\gcd(210, m+n) = 1,$$

where m^m is a multiple of n^n and m is not a multiple of n. Find the smallest possible value of $m + n$.

210 是什么东西？$2 \times 3 \times 5 \times 7$，也就是说，$n$ 不可以有这些质因数，否则显然 m 也会有（不过 $\gcd(210, m) \neq 1$ 没关系）

根据条件，必须有质因数 p 使得 $\nu_p(n) > \nu_p(m)$，由于 p 的地位几乎是等价的，不妨先取 $p = 11$

另一方面，根据比赛规则，$n, m < 1000$，则在 $p \geq 11$ 时 $\nu_p(n) < 3$，即 $\nu_p(n) = 2, \nu_p(m) = 1$

$*$ 如果你觉得这还不够严谨，可以先凑出一组不优解缩小范围，也能很快得到类似的结论

我们基本可以确定 $n = 11^2$ 了，再来看条件 $n^n \mid m^m$，这要求 $n\nu_p(n) \leq m\nu_p(m) \Leftrightarrow 2n \leq m$

因此，只要 $m = 11 \cdot k$（$k \geq 23$），那我们取 $k = 23$ 就好了......

在写上答案之前，幸运的你又看了一眼条件——$\gcd(210, m+n)$ 好像不是 1？

你一边怀疑自己是不是做错了，一边悄悄把 k 改成了 $26 \Rightarrow \boxed{407}$ 完美！

Problem 11

For integers a, b, c, and d, let

$$f(x) = x^2 + ax + b \text{ and } g(x) = x^2 + cx + d.$$

Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that

$$g(f(2)) = g(f(4)) = 0.$$

$f(2) = 4 + 2a + b$ 和 $f(4) = 16 + 4a + b$ 是 $g(x) = 0$ 的根

韦达定理：$c = -6a - 2b - 20$，显然 d 一定存在我们不用管，只要求满足
$-10 \le a, b, -6a - 2b - 20 \le 10$ 的 (a, b) 个数，整理一下
$-15 - 3a \le b \le -5 - 3a$

a	b	个数
-8	$[9, 10]:$	2
-7	$[6, 10]:$	5
-6	$[3, 10]:$	8
-5	$[0, 10]:$	11
-4	$[-3, 7]:$	11
-3	$[-6, 4]:$	11
-2	$[-9, 1]:$	11
-1	$[-10, -2]:$	9
0	$[-10, -5]:$	6
1	$[-10, -8]:$	3

总共 77 个——怎么和第9题一模一样？好像不应该啊?

问题出在了 $f(2) = f(4)$ 没有考虑，此时 $a = -6$ 而 b, c 无限制

答案为 $77 - 8 + 21 \cdot 21 = \boxed{510}$

Problem 12

Let n be the smallest positive integer such that

$$149^n - 2^n \text{is a multiple of } 3^3 \cdot 5^5 \cdot 7^7.$$

Determine the number of positive integer factors of n.

第一眼看，149 的意义大概就是 $149 - 2 = 3 \cdot 7^2$ 了，我们升幂一下

$$\begin{aligned} 3 \le \nu_3(149^n - 2^n) = \nu_3(147) + \nu_3(n) = 1 + \nu_3(n) \\ 7 \le \nu_7(149^n - 2^n) = \nu_7(147) + \nu_7(n) = 2 + \nu_7(n) \end{aligned}$$

所以 n 至少要有 $3^2 \cdot 7^5$，再来看 5^5

对模 25 进行枚举，不难发现 $20 \mid n$，这提示我们 $4 \mid n$，$5 \mid n$

于是同样模 5 再做一次升幂（因为 $5 || 149^4 - 2^4$）可知 $\nu_5(n) \ge 4$

因此，n 最小是 $4 \cdot 3^2 \cdot 5^4 \cdot 7^5$，这也是足够的，因数个数为 $3 \cdot 3 \cdot 5 \cdot 6 = \boxed{270}$

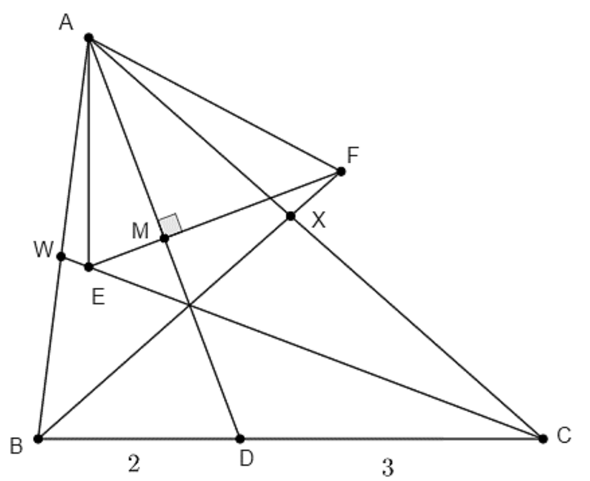
Problem 13

Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB=4$, $BC=5$, $CA=6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

精准作图很简单，甚至能快速猜出一些结论，不过我们还是建系验证一下

$*$ 我在做的时候（以及下面的过程）E, F 反掉了，不过这无关紧要

设 CW, BX 分别是 $\angle C, \angle B$ 的角平分线



准备工作

设 $B(0,0),C(5,0)$，角平分线定理 $D(2,0)$，海伦公式或余弦定理 $A(\frac{1}{2},\frac{3}{2}\sqrt{7})$，故 $M(\frac{5}{4},\frac{3}{4}\sqrt{7})$

直线 AD：$y=-\sqrt{7}(x-2)$，AD 中垂线 EF：$y=\frac{1}{\sqrt{7}}x+\frac{4\sqrt{7}}{7}$

E 的坐标

角平分线定理+定比分点：$W(\frac{5}{22},\frac{15\sqrt{7}}{22})$，直线 CW：$y=-\frac{\sqrt{7}}{7}(x-5)$

故 CW 和 EF 的交点 E 满足 $x_E=\frac{1}{2}$

F 的坐标

和上面类似地，$X(\frac{5}{2},\frac{5}{6}\sqrt{7})$，直线 BX：$y=\frac{\sqrt{7}}{3}x$

BX 和 EF 的交点 F 满足 $x_F=3$

结合 EF 的斜率可知 $EF=\frac{5\sqrt{14}}{7}$，因此 $S_{\triangle AEF}=\frac{1}{2}EF\cdot AM=\frac{15\sqrt{7}}{14}$，答案 $\boxed{036}$

$*$ AD 千万不能算错，余弦定理或坐标距离都行

Problem 14

Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation

$$P(P(x))=0$$

has four distinct solutions, $x=3,4,a,b$. Find the sum of all possible values of $(a+b)^2$.

* 尝试消元的过程我好像经历了一个世纪之久，做着做着就又把自己绕进去了

称题目中的两根 a,b 为 A,B，设 $P(x)=x^2+ax+b$，满足 $P(x)=0$ 的根是 x_1,x_2

我们需要将 $3,4,A,B$ 分两组，分别对应 $P(x)=x_1,P(x)=x_2$ 的根

① $3,4$ 和 A,B 一组

因为 $3,4$ 是 $x^2+ax+b=x_1$ 的两根，所以 $a=-7$，$A+B=-a=7$

第一个解：49，这太轻松了，一定还有其他解

* 我们只关心所有可能的 $(A+B)^2$，而并不关心此时 a,b 的取值（也没法关心）

② $3,A$ 和 $4,B$ 一组

假设 $3,A$ 是 $x^2+ax+b-x_1=0$ 的根，$4,B$ 是 $x^2+ax+b-x_2=0$ 的根

韦达定理立刻告诉了我们很多事情，列出来看看：

$$\begin{aligned} x_1+x_2=-a,\ x_1x_2=b \\ 3+A=-a,\ 4+B=-a \\ 3A=b-x_1,\ 4B=b-x_2 \end{aligned}$$

考虑用 a 表示 b，把 b 消掉：

$$\begin{aligned} 3A+4B&=b+a \\ 3(-a-3)+4(-a-4)&=2b+a \\ 2b+8a&=-25 \\ \Rightarrow b&=\frac{-25-8a}{2} \end{aligned}$$

回到之前的式子，我们有两种方法来表示 AB：

$$\begin{aligned} (a+3)(a+4)&=\frac{1}{12}(b-x_1)(b-x_2) \\ 12(a+3)(a+4)&=b^2+ab+b \\ 48(a+3)(a+4)&=(25+8a)^2-(a+1)(50+16a) \\ \Rightarrow 48\times 12&=25^2-2a-50 \\ a&=-\frac{1}{2} \end{aligned}$$

不放心的话再验算一下，$b=-\frac{21}{2}$，代回去可得 $(A+B)^2=36$

综上，答案是 $49+36=\boxed{085}$

Problem 15

Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with

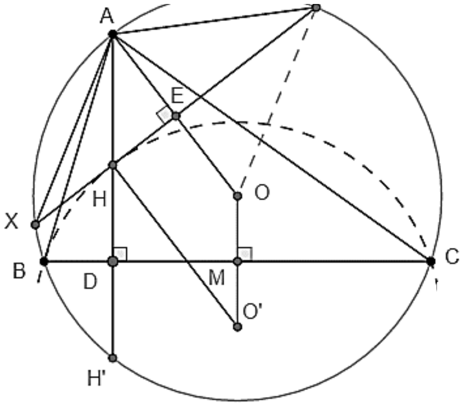
$$HA = 3, HX = 2, HY = 6.$$

The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

看到 AH 顺手延长，这样 H 和 H'、O 和 O' 都关于 BC 对称

小结论：$AH = 2OM$，证明只需要简单的三角计算，于是我们有平行四边形 $AHO'O$

相切：$O'H \perp XY \Rightarrow AO \perp XY$，又一个垂径，$XE = EY = 4$，再对 $\triangle AHE$ 用勾股知 $AE = \sqrt{5}$



相交弦：$HD = DH' = 2$，即 $AD = 5$，求 $S_{\triangle ABC} \Leftrightarrow$ 求 BC，而 $OM = \frac{3}{2}$，只要求半径 r 就好了

稍微观察一下，对 $\triangle OEY$ 用勾股，$(r-\sqrt{5})^2 + 4^2 = r^2 \Rightarrow r = \frac{21\sqrt{5}}{10}$

因此 $S_{\triangle ABC} = 3\sqrt{55}$，答案为 $\boxed{058}$