Solutions to Homework 3

Chapter 4

10. Since α has the uniform distribution on the interval $[-\pi/2, \pi/2]$, the p.d.f. of α is

$$f(\alpha) = \begin{cases} \dfrac{1}{\pi} & \text{for } -\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

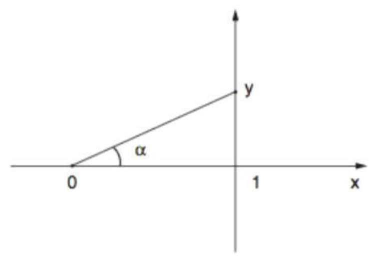

Figure S.4.1: Figure for Exercise 10 of Sec. 4.1.

Also, $Y = \tan(\alpha)$. Therefore, the inverse transformation is $\alpha = \tan^{-1} Y$ and $d\alpha/dy = 1/(1+y^2)$. As α varies over the interval $(-\pi/2, \pi/2)$, Y varies over the entire real line. Therefore, for $-\infty < y < \infty$, the p.d.f. of Y is

$$g(y) = f(\tan^{-1} y)\frac{1}{1+y^2} = \frac{1}{\pi(1+y^2)}.$$

6. Let $X_i = 1$ if the ith jump of the particle is one unit to the right and let $X_i = -1$ if the ith jump is one unit to the left. Then, for $i = 1, \ldots, n$,

$$E(X_i) = (-1)p + (1)(1-p) = 1 - 2p.$$

The position of the particle after n jumps is $X_1 + \cdots + X_n$, and

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n(1-2p).$$

10. The m.g.f. of Z is

$$\begin{aligned} \psi_1(t) &= E(\exp(tZ)) = E[\exp(t(2X - 3Y + 4))] \\ &= \exp(4t)E(\exp(2tX)\exp(-3ty)) \\ &= \exp(4t)E(\exp(2tX))E(\exp(-3tY)) \qquad \text{(since } X \text{ and } Y \text{ are independent)} \\ &= \exp(4t)\psi(2t)\psi(-3t) \\ &= \exp(4t)\exp(4t^2 + 6t)\exp(9t^2 - 9t) \\ &= \exp(13t^2 + t). \end{aligned}$$

12.

$$\begin{aligned}
E(X) &= \int_0^1 \int_0^2 x \cdot \frac{1}{3}(x+y)dy\,dx = \frac{5}{9}, \\
E(Y) &= \int_0^1 \int_0^2 y \cdot \frac{1}{3}(x+y)dy\,dx = \frac{11}{9}, \\
E(X^2) &= \int_0^1 \int_0^2 x^2 \cdot \frac{1}{3}(x+y)dy\,dx = \frac{7}{18}, \\
E(Y^2) &= \int_0^1 \int_0^2 y^2 \cdot \frac{1}{3}(x+y)dy\,dx = \frac{16}{9}, \\
E(XY) &= \int_0^1 \int_0^2 xy \cdot \frac{1}{3}(x+y)dy\,dx = \frac{2}{3}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}, \\
\mathrm{Var}(Y) &= \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}, \\
\mathrm{Cov}(XY) &= \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = -\frac{1}{81}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathrm{Var}(2X - 3Y + 8) &= 4\mathrm{Var}(X) + 9\,\mathrm{Var}(Y) - (2)(2)(3)\,\mathrm{Cov}(X, Y) \\
&= \frac{245}{81}.
\end{aligned}$$

7. The marginal p.d.f. of X is

$$f_1(x) = \int_0^1 (x+y)dy = x + \frac{1}{2} \qquad \text{for} \quad 0 \le x \le 1.$$

Therefore, for $0 \le x \le 1$, the conditional p.d.f. of Y given that $X = x$ is

$$g(y \mid x) = \frac{f(x,y)}{f_1(x)} = \frac{2(x+y)}{2x+1} \qquad \text{for } 0 \le y \le 1.$$

Hence,

$$\begin{aligned}
E(Y \mid x) &= \int_0^1 \frac{2(xy+y^2)}{2x+1}dy = \frac{3x+2}{3(2x+1)}, \\
E(Y^2 \mid x) &= \int_0^1 \frac{2(xy^2+y^3)}{2x+1}dy = \frac{4x+3}{6(2x+1)},
\end{aligned}$$

and

$$\mathrm{Var}(Y \mid x) = \frac{4x+3}{6(2x+1)} - \left[\frac{3x+2}{3(2x+1)}\right]^2 = \frac{1}{36}\left[3 - \frac{1}{(2x+1)^2}\right].$$

<u>Chapter 5</u>

10. The number of children X in the family who will inherit the disease has the binomial distribution with parameters n and p. Let $f(x|n,p)$ denote the p.f. of this distribution. Then

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - f(0|n,p) = 1 - (1-p)^n.$$

For $x = 1, 2, \ldots, n$,

$$\Pr(X = x|X \geq 1) = \frac{\Pr(X = x)}{\Pr(X \geq 1)} = \frac{f(x|n,p)}{1-(1-p)^n}.$$

Therefore, the conditional p.f. of X given that $X \geq 1$ is $f(x|n,p)/(1-[1-p]^n)$ for $x = 1, 2, \ldots, n$. The required expectation $\mathrm{E}(X \mid X \geq 1)$ is the mean of this conditional distribution. Therefore,

$$E(X|X \geq 1) = \sum_{x=1}^{n} x \frac{f(x \mid n,p)}{1-(1-p)^n} = \frac{1}{1-(1-p)^n} \sum_{x=1}^{n} x f(x|n,p).$$

However, we know that the mean of the binomial distribution is np; i.e.,

$$E(X) = \sum_{x=0}^{n} x f(x \mid n,p) = np.$$

Furthermore, we can drop the term corresponding to $x = 0$ from this summation without affecting the value of the summation, because the value of that term is 0. Hence, $\sum_{x=1}^{n} x f(x|n,p) = np$. It now follows that $E(X|X \geq 1) = np/(1-[1-p]^n)$.

9. Let N denote the total number of items produced by the machine and let X denote the number of defective items produced by the machine. Then, for $x = 0, 1, \ldots,$

$$\Pr(X = x) = \sum_{n=0}^{\infty} \Pr(X = x \mid N = n) \Pr(N = n).$$

Clearly, it must be true that $X \leq N$. Therefore, the terms in this summation for $n < x$ will be 0, and we may write

$$\Pr(X = x) = \sum_{n=x} \Pr(X = x \mid N = n) \Pr(N = n).$$

Clearly, $\Pr(X = 0 \mid N = 0) = 1$. Also, given that $N = n > 0$, the conditional distribution of X will be a binomial distribution with parameters n and p. Therefore,

$$\Pr(X = x \mid N = n) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}.$$

Also, since N has the Poisson distribution with mean λ,

$$\Pr(N = n) = \frac{\exp(-\lambda)\lambda^n}{n!}.$$

Hence,

$$\Pr(X = x) = \sum_{n=x}^{\infty} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \frac{\exp(-\lambda)\lambda^n}{n!} = \frac{1}{x!} p^x \exp(-\lambda) \sum_{n=x}^{\infty} \frac{1}{(n-x)!} (1-p)^{n-x} \lambda^n.$$

If we let $t = n - x$, then

$$\begin{aligned}
\Pr(X = x) &= \frac{1}{x!} p^x \exp(-\lambda) \sum_{t=0}^{\infty} \frac{1}{t!} (1-p)^t \lambda^{t+x} \\
&= \frac{1}{x!} (\lambda p)^x \exp(-\lambda) \sum_{t=0}^{\infty} \frac{[\lambda(1-p)]^t}{t!} \\
&= \frac{1}{x!} (\lambda p)^x \exp(-\lambda) \exp(\lambda(1-p)) = \frac{\exp(-\lambda p)(\lambda p)^x}{x!}.
\end{aligned}$$

It can be seen that this final term is the value of the p.f. of the Poisson distribution with mean λp.

5. By Eq. (5.5.6), the m.g.f. of X_i is

$$\psi_i(t) = \left(\frac{p}{1-(1-p)\exp(t)}\right)^{r_i} \quad \text{for} \quad t < \log\left(\frac{1}{1-p}\right).$$

Therefore, the m.g.f. of $X_1 + \cdots + X_k$ is

$$\psi(t) = \prod_{i=1}^{k} \psi_i(t) = \left(\frac{p}{1-(1-p)\exp(t)}\right)^{r_1+\cdots+r_k} \quad \text{for} \quad t < \log\left(\frac{1}{1-p}\right).$$

Since $\psi(t)$ is the m.g.f. of the negative binomial distribution with parameters $r_1 + \cdots + r_k$ and p, that must be the distribution of $X_1 + \cdots + X_k$.

20. If X has the Weibull distribution with parameters a and b, then the c.d.f. of X is

$$F(x) = \int_0^x \frac{b}{a^b} t^{b-1} \exp(-(t/a)^b) dt = [-\exp(-(t/a)^b)]_0^x = 1 - \exp(-(x/a)^b).$$

Therefore,

$$h(x) = \frac{b}{a^b} x^{b-1}.$$

If $b > 1$, then $h(x)$ is an increasing function of x for $x > 0$, and if $b < 1$, then $h(x)$ is an decreasing function of x for $x > 0$.

6. The joint p.d.f. of X and Y will be the product of their marginal p.d.f.'s Therefore, for $x > 0$ and $y > 0$,

$$\begin{aligned} f(x,y) &= \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \exp(-\beta x) \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} \exp(-\beta y) \\ &= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1} y^{\alpha_2 - 1} \exp(-\beta(x+y)). \end{aligned}$$

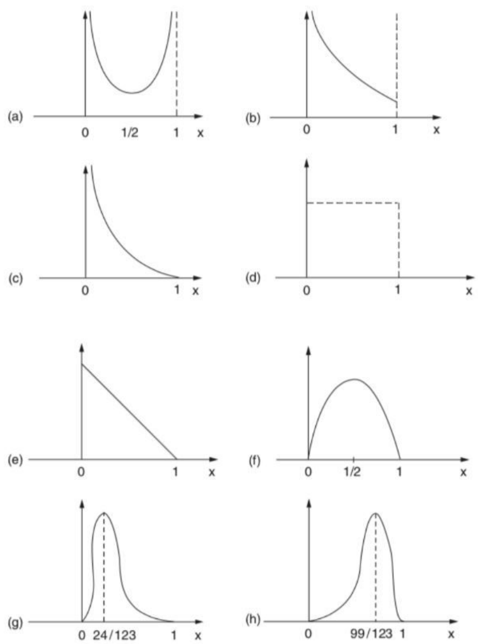

Figure S.5.4: Figure for Exercise 3 of Sec. 5.8.

Also, $X = UV$ and $Y = (1-U)V$. Therefore, the Jacobian is

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} v & u \\ -v & 1-u \end{bmatrix} = v.$$

As x and y vary over all positive values, u will vary over the interval (0, 1) and v will vary over all possible values. Hence, for $0 < u < 1$ and $v > 0$, the joint p.d.f. of U and V will be

$$g(u,v) = f[uv, (1-u)v]v = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} v^{\alpha_1+\alpha_2-1} \exp(-\beta v).$$

It can be seen that this joint p.d.f. has been factored into the product of the p.d.f. of a beta distribution with parameters α_1 and α_2 and the p.d.f. of a gamma distribution with parameters $\alpha_1 + \alpha_2$ and β. Therefore, U and V are independent, the distribution of U is the specified beta distribution, and the distribution of V is the specified gamma distribution.

7. For any nonnegative integers $x_1, \dots, x_k$ such that $\sum_{i=1}^k x_i = n$,

$$\Pr\left(X_1 = x_1, \dots, X_k = x_k \left| \sum_{i=1}^k X_i = n \right.\right) = \frac{\Pr(X_1 = x_1, \dots, X_k = x_k)}{\Pr\left(\sum_{i=1}^k X_i = n\right)}$$

Since $X_1, \dots, X_k$ are independent,

$$\Pr(X_1 = x_1, \dots, X_k = x_k) = \Pr(X_1 = x_1) \dots \Pr(X_k = x_k).$$

Since X_i has the Poisson distribution with mean λ_i,

$$\Pr(X_i = x_i) = \frac{\exp(-\lambda_i)\lambda_i^{x_i}}{x_i!}.$$

Also, by Theorem 5.4.4, the distribution of $\sum_{i=1}^k X_i$ will be a Poisson distribution with mean $\lambda = \sum_{i=1}^k \lambda_i$. Therefore,

$$\Pr\left(\sum_{i=1}^k X_i = n\right) = \frac{\exp(-\lambda)\lambda^n}{n!}.$$

It follows that

$$\Pr\left(X_1 = x_1, \dots, X_k = x_k \left| \sum_{i=1}^k X_i = n \right.\right) = \frac{n!}{x_i! \dots x_k!} \prod_{i=1}^k \left(\frac{\lambda_i}{\lambda}\right)^{x_i}$$

2. Let X_1 denote the student's score on test A and let X_2 denote his score on test B. The conditional distribution of X_2 given that $X_1 = 80$ is a normal distribution with mean $90 + (0.8)(16)\left(\frac{80 - 85}{10}\right) = 83.6$ and variance $(1 - 0.64)(256) = 92.16$. Therefore, given that $X_1 = 80$, the random variable $Z = (X_2 - 83.6)/9.6$ will have the standard normal distribution. It follows that

$$\Pr(X_2 > 90 \,|\, X_1 = 80) = \Pr\left(Z > \frac{2}{3}\right) = 1 - \Phi\left(\frac{2}{3}\right) = 0.2524.$$