hw5_yw3204

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1.

Assume $X \sim MVN(\mu, \Sigma)$.

Then, we have $\mathbb{E}((X-\mu)(X-\mu)^T) = \Sigma$.

For AX, we know its mean is $\mathbb{E}(AX) = A\mu$.

By definition,

$Cov(AX)$

$= \mathbb{E}((AX - A\mu)(AX - A\mu)^T)$

$= A * \mathbb{E}((X-\mu)(X-\mu)^T) * A^T$

$= A\Sigma A^T$

2.

Denote the response vector as Y and the original design matrix as $X = (1, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_p)$.

Assume X's column vectors are lineraly independent. Then by Gram-Schmidt orthogonalization, we can turn them into a set of orthogonal vectors with length 1.

Denote the new design matrix as $\tilde{X} = (\tilde{\mathbf{x}}_\mathbf{0}, \tilde{\mathbf{x}}_\mathbf{1}, ..., \tilde{\mathbf{x}}_\mathbf{p})$.

Then, we rewrite the model as $y = \gamma_0 * \tilde{x}_0 + \gamma_1 * \tilde{x}_1 + ... + \gamma_p * \tilde{x}_p$.

The hypothesis is thus equivalent to

$H_0 : \gamma_1 = 0$ and $H_1 : \gamma_1 \neq 0$.

The estimator of γ is

$\hat{\gamma} = \tilde{X}^T Y$ or $\hat{\gamma}_i = \tilde{x}_i^T * Y$.

Due to the orthogonal property of $\tilde{X}^T$, the covariance matrix of $\hat{\gamma}$ is $\sigma^2 I_{p+1}$, which means $\hat{\gamma}_i$ are mutually independent.

The estimator of $\hat{\sigma^2}$ is

$\frac{(Y-\tilde{X}\tilde{X}^T Y)^T (Y-\tilde{X}\tilde{X}^T Y)}{n-p-1}$, which has a chi-square istribution with $df = n - p - 1$.

We can prove it by expanding the column space of $\tilde{X}$. But here, we just take it for granted.

The t-statistic is thus

$t = \frac{\hat{\gamma}_1}{\sqrt{\hat{\sigma^2}}}$

And the F-statistic is

$F = \frac{SSE(\tilde{x}_0, \tilde{x}_2, ..., \tilde{x}_p) - SSE(\tilde{x}_0, \tilde{x}_1, \tilde{x}_p)}{SSE(\tilde{x}_0, \tilde{x}_1, \tilde{x}_p)/(n-p-1)}$

The denominator is exactly $\hat{\sigma^2}$.

For the numerator, it can be written as

$SSE(\tilde{x}_0, \tilde{x}_2, \ldots, \tilde{x}_p) - SSE(\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_p)$

$= z^T z - (z - \hat{\gamma}_1 * \tilde{\mathbf{x}}_1)^T (z - \hat{\gamma}_1 * \tilde{\mathbf{x}}_1)$, in which $z = Y - \Sigma_{i \neq 1} \hat{\gamma}_i \tilde{\mathbf{x}}_\mathbf{i}$.

$= 2 * \hat{\gamma}_1 * z^T * \tilde{\mathbf{x}}_1 - \hat{\gamma}_1^2$

$= 2 * \hat{\gamma}_1 * Y^T * \tilde{\mathbf{x}}_1 - \hat{\gamma}_1^2$ (because of the orthogonality)

$= 2 * \hat{\gamma}_1 * \hat{\gamma}_1 - \hat{\gamma}_1^2$

$= \hat{\gamma}_1^2$

Hence, the F-statistic can be written as

$F = \frac{\hat{\gamma}_1^2}{\hat{\sigma^2}}$

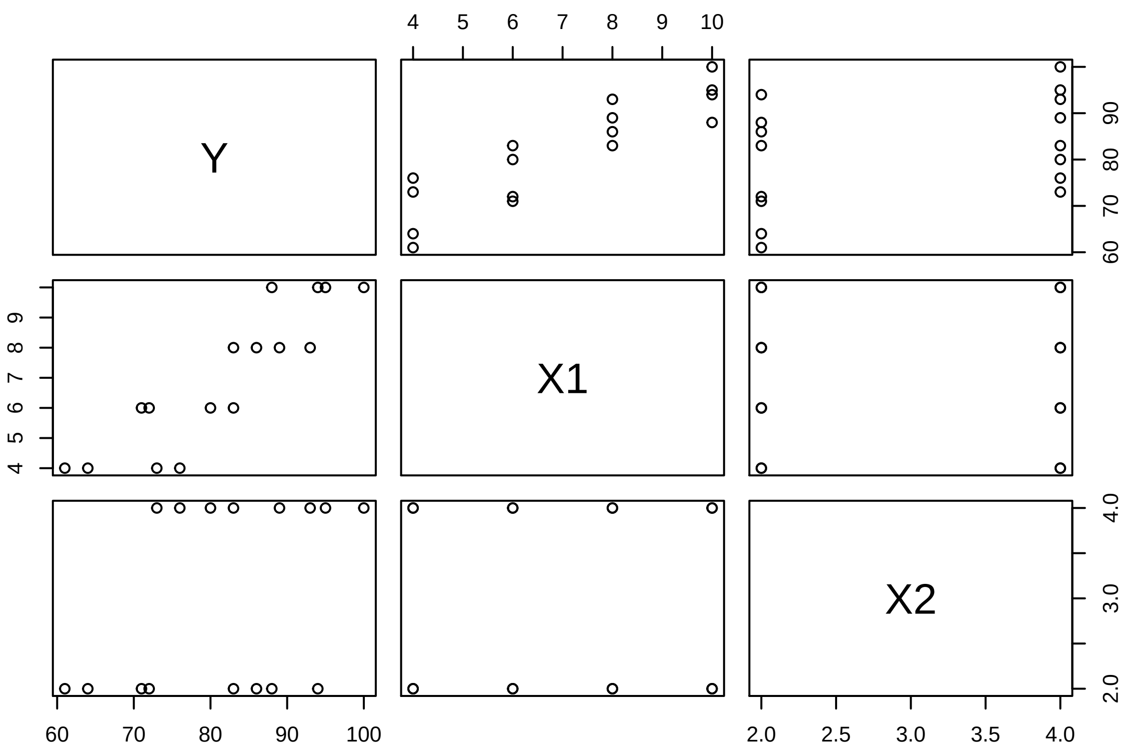
It is also the square of the t-statistc. Thus, we proved the equivalence between t-test and F-test.

6.5

a.

```r
brand <- read.table("CH06PR05.txt")
names(brand) <- c("Y", "X1", "X2")

pairs(brand)
```




```r
cor(brand)
```

```
##            Y        X1        X2
## Y  1.0000000 0.8923929 0.3945807
## X1 0.8923929 1.0000000 0.0000000
## X2 0.3945807 0.0000000 1.0000000
```

Observe the scatter plot matrix and the correlation matrix, we find there is a strong linear relationship between Y and X_1. And X_2 can be regarded as a categorical variable.

b.

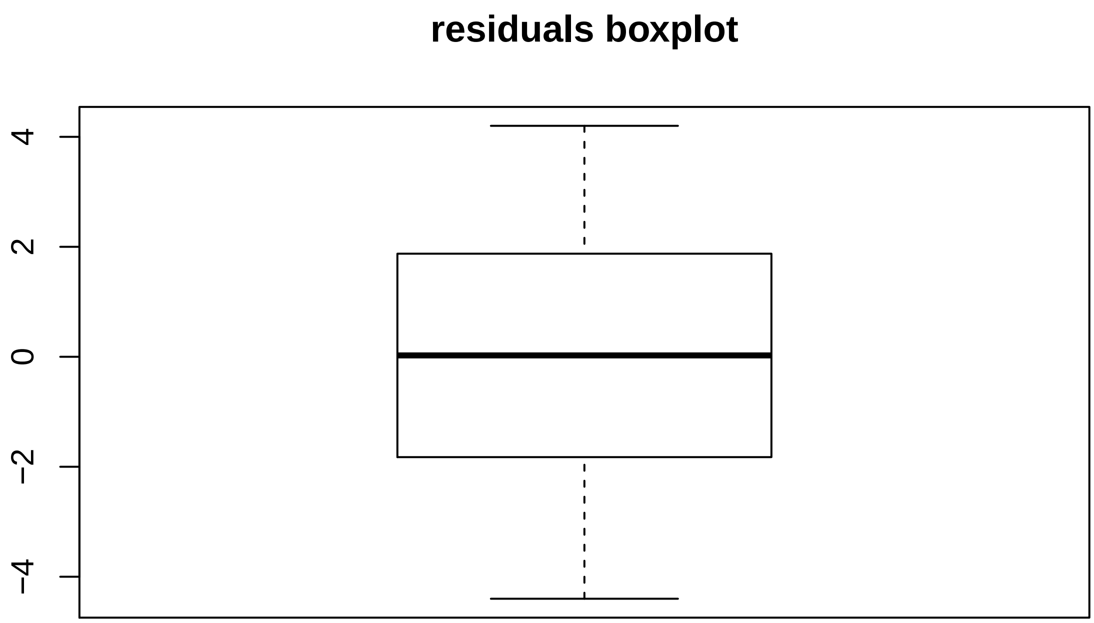
```
lm1 <- lm(Y ~ X1+X2, brand)
lm1
```

```
##
## Call:
## lm(formula = Y ~ X1 + X2, data = brand)
##
## Coefficients:
## (Intercept)           X1           X2
##      37.650        4.425        4.375
```

The fitted line is $Y = 37.650 + 4.425X_1 + 4.375 * X_2$. Here, b_1 means holding other variables fixed, the response Y changes 4.425 units when X_1 changes 1 unit.

c.

```
boxplot(resid(lm1), main = "residuals boxplot")
```



The plot seems quite symmetric with mean around 0 which matches with the equal varaince and zero mean assumption in the model.

6.7

a.

```
summary(lm1)$r.squared
```

```
## [1] 0.952059
```

The coefficient of multiple determination is 0.95 here which is incredibly high.

b.

```r
Y <- brand$Y
Y_hat <- predict(lm1, brand[, c(2, 3)])
summary(lm(Y~Y_hat))$r.squared
```

```
## [1] 0.952059
```

Yes, they are equal.

6.8

a.

```r
predict(lm1, newdata = data.frame(X1 = 5, X2 = 4), interval = "confidence", level = 0.99)
```

```
##      fit      lwr      upr
## 1 77.275 73.88111 80.66889
```

The interval is $[73.88, 80.67]$ and it means the true mean at $X_1 = 5$ and $X_2 = 4$ falls into it with probability 99%.

b.

```r
predict(lm1, newdata = data.frame(X1 = 5, X2 = 4), interval = "prediction", level = 0.99)
```

```
##      fit      lwr      upr
## 1 77.275 68.48077 86.06923
```

The interval is $[68.48, 86.07]$ and it means the new observation at $X_1 = 5$ and $X_2 = 4$ falls into it with probability 99%.

6.25

We can subtract $\beta_2 * X_{i2}$ from Y_i. That's to say, we regress $(Y_i - \beta_2 * X_{i2})$ on the remaining variables.