Stat GR 5205 Lecture 6

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Least squares estimate

- Least squares estimator

$$\hat{\beta} = \arg\min_{\beta} (Y - X\beta)^\top (Y - X\beta) = (X^\top X)^{-1} X^\top Y$$

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Frequentist distribution

- Unbiased distribution

$$E(\hat{\beta}) = \beta$$

- Variance and covariance

$$Var(\hat{\beta}) = \sigma^2 (X^\top X)^{-1}.$$

- Computation of covariance matrix
- Multivariate normal distribution

Frequentist distribution

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Variance estimation

- An unbiased estimator

$$\hat{\sigma}^2 = \frac{(Y - \hat{Y})^\top (Y - \hat{Y})}{n - p - 1}$$

- The distribution of $\hat{\sigma}^2$
- Prediction

$$\hat{Y} = X(X^\top X)^{-1} X^\top Y$$

- Hat matrix

$$H = X(X^\top X)^{-1} X^\top$$

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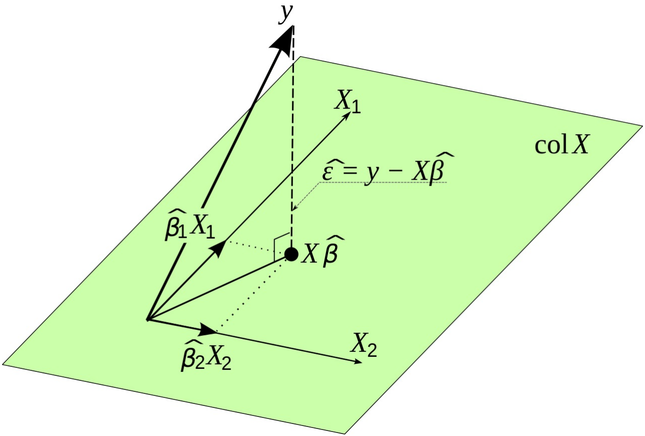
Projection


y
X_1
$\operatorname{col} X$
$\hat{\varepsilon} = y - X\hat{\beta}$
$\hat{\beta}_1 X_1$
$X\hat{\beta}$
$\hat{\beta}_2 X_2$
X_2

Joint distribution

- $\hat{\beta}$ and $\hat{\sigma}^2$ are independent.

Hypothesis testing

- Hypothesis testing

$$H_0 : \beta_i = \beta_i^0 \quad H_1 : \beta_i \neq \beta_i^0$$

- Z-statistic

$$Z - stat = \frac{\hat{\beta}_i - \beta_i^0}{SD(\hat{\beta}_i)}$$

- t-statistic

$$t - stat = \frac{\hat{\beta}_i - \beta_i^0}{SE(\hat{\beta}_i)}$$

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Confidence interval

- σ^2 known

$$\hat{\beta}_i \pm q(1-\alpha/2)SD(\hat{\beta}_i)$$

- σ^2 unknown

$$\hat{\beta}_i \pm t_{n-p-1}(1-\alpha/2)SE(\hat{\beta}_i)$$

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Prediction

- Prediction $x^\top \hat{\beta}$
- Prediction error

$$SD(x^\top \hat{\beta}) = x^\top SD(\hat{\beta})x = \sigma^2 x^\top (X^\top X)^{-1} x$$

- Prediction of future observations
- Simultaneous interval

$$\hat{Y} \pm \lambda SE(\hat{Y})$$

where

$$\lambda^2 = (p+1)F(1-\alpha; p+1, n-p-1).$$

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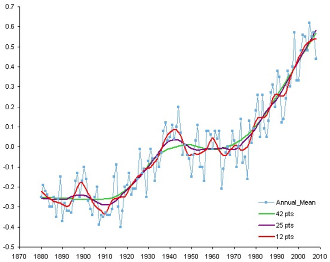
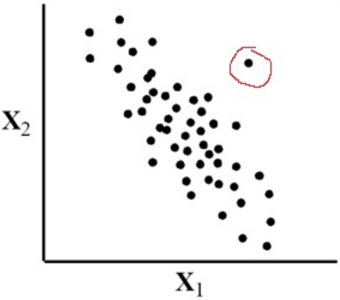
Extrapolation: one predictor and multiple predictors





Linear model is a good local approximation.

Analysis of variance

- ANOVA

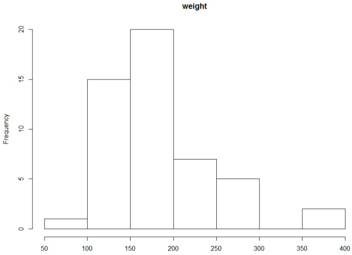
$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$$

- $R^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$

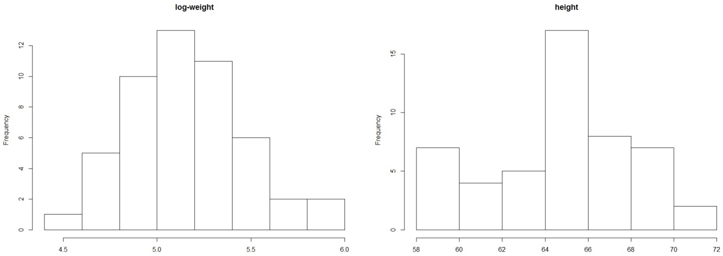
Example

- 50 samples
- 5 Asian, 15 African American, 30 Whites
- Weight and height
- Coding of the design matrix

Example


weight
Frequency
0
5
10
15
20
50
100
150
200
250
300
350
400
height
Frequency
0
5
10
15
58
60
62
64
66
68
70
72

Example


log-weight
Frequency
0
2
4
6
8
10
12
4.5
5.0
5.5
6.0
height
Frequency
0
5
10
15
58
60
62
64
66
68
70
72

Example

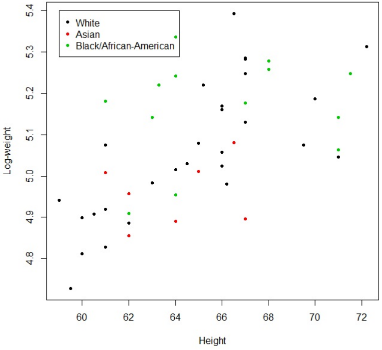


Figure: Height (inch) versus log-weight (log-lb)

Example

- $$\log(weight) = \beta_0 + \beta_1 height + \varepsilon$$

- $$\log(weight) = \beta_0 + \beta_{Asian} I_{Asian} + \beta_{Black} I_{Black} + \beta_1 height + \varepsilon$$

- $$\begin{aligned} \log(weight) &= \beta_0 + \beta_{Asian} I_{Asian} + \beta_{Black} I_{Black} + \beta_1 height \\ &\quad + \beta_{Asian,H} I_{Asian} height + \beta_{Black,H} I_{Black} height + \varepsilon \end{aligned}$$

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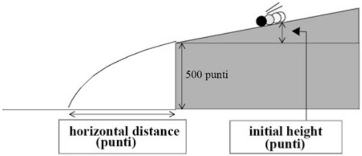
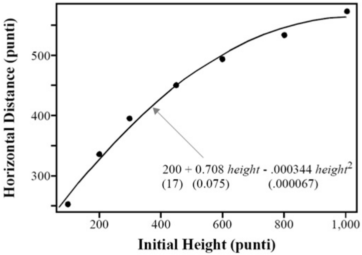
Galileo's experiment



horizontal distance (punti)	initial height (punti)
253	100
337	200
395	300
451	450
495	600
534	800
573	1000



$$distance = \beta_0 + \beta_1 height + \beta_2 height^2 + \varepsilon$$

Galileo's experiment

variable	coefficient	standard error	t-statistic	p-value
intercept	199.91	16.8	11.93	0.0003
height	0.71	0.075	9.5	0.0007
height^2	- 0.00034	0.000067	5.15	0.007

$R^2 = 0.99 \quad \hat{\sigma} = 13.6$

Galileo's experiment

$$distance = \beta_0 + \beta_1 height + \beta_2 height^2 + \beta_3 height^3 + \varepsilon$$

The extra sum-of-squares F test

- Question: is there any difference among the three groups aside from that due to height difference
- The formulation

$$\log(weight) = \beta_0 + \beta_{Asian} + \beta_{Black} + \beta_1 height + \varepsilon$$

- The hypotheses

$$H_0 : \beta_{Asian} = \beta_{Black} = 0 \quad H_1 : \ otherwise$$

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The full model versus the reduced model

- Full model (H_1)

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- Reduced model (H_0)

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- Comparing the full model against the reduce model

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Analysis of variance

- ANOVA of the full model

$$SST = SSR_{full} + SSE_{full}$$

- ANOVA of the reduced model

$$SST = SSR_{reduced} + SSE_{reduced}$$

- Inequality

$$SSE_{extra} = SSE_{reduced} - SSE_{full} > 0$$

- Reject H_0 if SSE_{extra} is large
- Distribution of SSR_{extra}

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Extra sums of squares test

- Test statistic

$$F - statistic = \frac{SSE_{extra} / (p_{full} - p_{reduced})}{SSE_{full} / (n - p_{full})}$$

Some notation

- Reduced model

$$SST = SSR(X_1) + SSE(X_1)$$

- Full model

$$SST = SSR(X_1, X_2) + SSE(X_1, X_2)$$

- Extra sums of squares

$$SSR(X_2|X_1) = SSE(X_1) - SSE(X_1, X_2)$$

- The coefficients of partial determination

$$R^2_{X_2|X_1} = \frac{SSR(X_2|X_1)}{SSE(X_1)}$$

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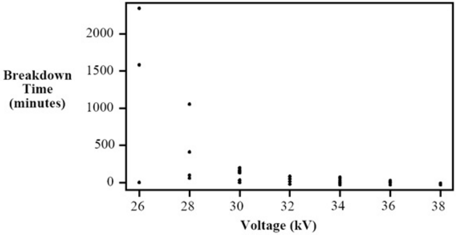
ANOVA table

$$SST = SSR + SSE$$

source	sums of sq	d.f.	mean sum of sq	F-stat	p-value
Regression	SST	p-1	SST/(p-1)		
Residual	SSE	n-p	SSE/(n-p)		
Total	SST				

Lack-of-fit test



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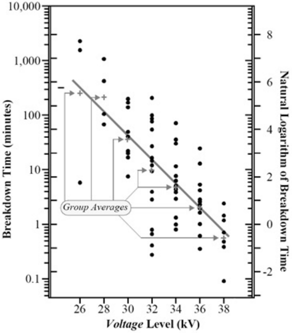

Lack-of-fit test

source	sum of sq	d.f.	mean sq	F-stat	p-value
regression	190	1	190	78	< 0.0001
residual	180	74	2.4		
total	370	75			

source	sum of sq	d.f.	mean sq	F-stat	p-value
between group	196	6	33	13	< 0.0001
residual	174	69	2.5		
total	370	75			

Lack-of-fit test

$$F - statisitc = \frac{(196 - 190)/5}{174/69} = 0.48$$