GR5241 HW1

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In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import gamma
%matplotlib inline
```

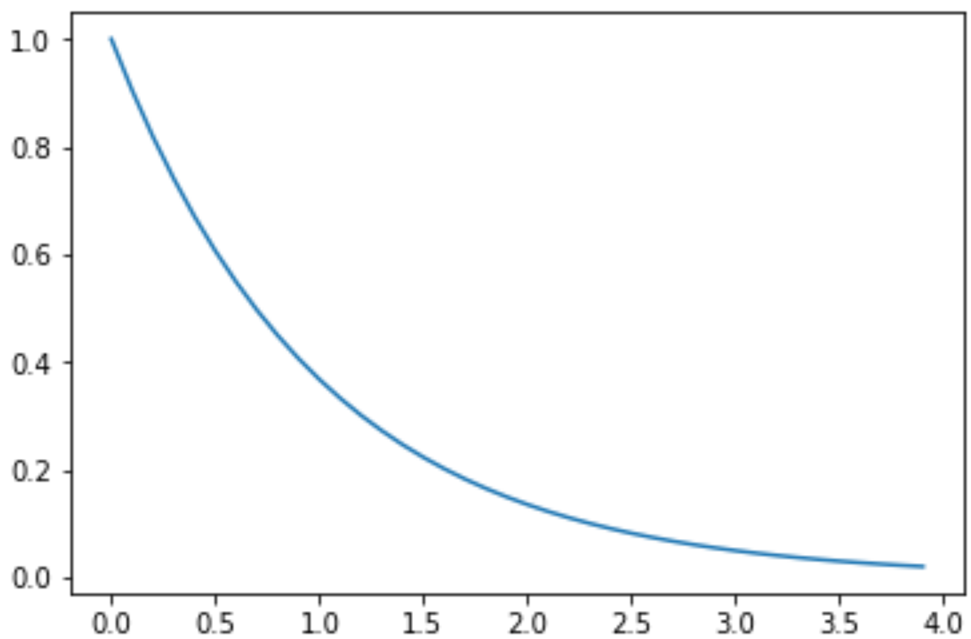
Problem 1

In [2]:

```
x = np.arange(0.0, 4.0, 0.1)
y = np.exp(-x)

plt.plot(x, y)
```

Out[2]:

```
[<matplotlib.lines.Line2D at 0x11f63c588>]
```

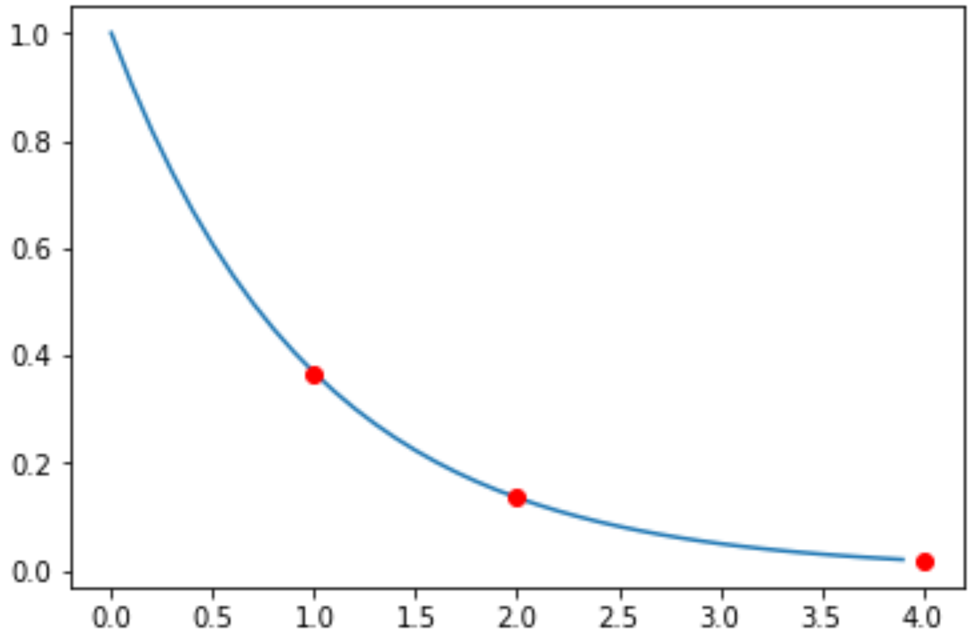


In [3]:

```
plt.plot(x, y)
plt.plot(1, np.exp(-1), 'ro')
plt.plot(2, np.exp(-2), 'ro')
plt.plot(4, np.exp(-4), 'ro')
```

Out[3]:

```
[<matplotlib.lines.Line2D at 0x1a219e9be0>]
```

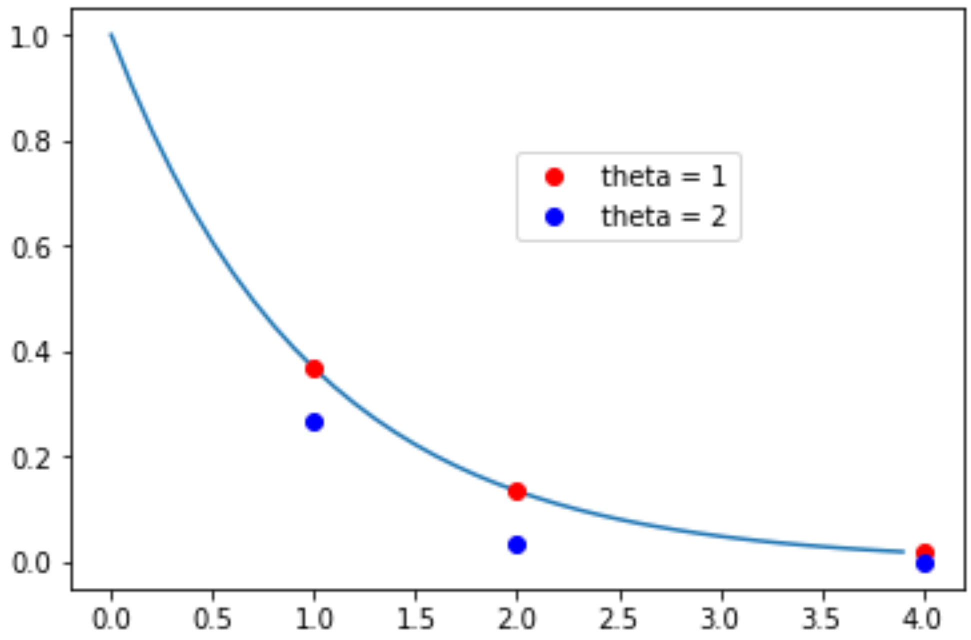


In [4]:

```
plt.plot(x, y)
plt.plot(1, np.exp(-1), 'ro', label = "theta = 1")
plt.plot(2, np.exp(-2), 'ro')
plt.plot(4, np.exp(-4), 'ro')

plt.plot(1, 2*np.exp(-1*2), 'bo', label = "theta = 2")
plt.plot(2, 2*np.exp(-2*2), 'bo')
plt.plot(4, 2*np.exp(-4*2), 'bo')

plt.legend(loc = (0.5, 0.6))
```

Out[4]:

```
<matplotlib.legend.Legend at 0x1a21b35320>
```



According to the graph, it is obvious that when $\theta = 2$, the likelihood of each example in the toy data set decreases. And more generally, when $x \geq ln(2)$, $p(x; \theta = 2) \leq p(x; \theta = 1)$ always holds.

Question 1.

Given the n observations and that $q(\theta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} * \theta^{\alpha_0 - 1} * e^{-\beta_0 \theta}$, we have

$q(\theta | x_1, \ldots, x_n)$

$\propto q(\theta) * \Pi_{i=1}^{n} p(x_i | \theta)$

$= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} * \theta^{\alpha_0 - 1} * e^{-\beta_0 \theta} * \theta^n * e^{-\theta * \Sigma_{i=1}^{n} x_i}$

$\propto \theta^{n + \alpha_0 - 1} * e^{-\theta(\beta_0 + \Sigma_{i=1}^{n} x_i)}$

Therefore, the posterior corresponds to a Gamma density function with parameter $\alpha = \alpha_0 + n$ and $\beta = \beta_0 + \Sigma_{i=1}^{n} x_i$

Question 2.

a.

Based on question 1, we have $\alpha_{n-1} = \alpha_0 + n - 1$ and $\beta_{n-1} = \beta_0 + \Sigma_{i=1}^{n-1} x_i$.

We then calculate the posterior based on the n^{th} sample and prior $\tilde{q}(\theta) = \Pi(\theta | x_{1:n-1})$, which is proportional to:

$\tilde{q}(\theta) * \theta * e^{-\theta x_n}$

$= \frac{\beta_{n-1}^{\alpha_{n-1}}}{\Gamma(\alpha_{n-1})} * \theta^{\alpha_{n-1} - 1} * e^{-\beta_{n-1} \theta} * \theta * e^{-\theta x_n}$

$\propto \theta^{\alpha_{n-1} + 1 - 1} * e^{-\theta(\beta_{n-1} + x_n)}$

Thus, it is a Gamma distribution with parameter $\alpha_{n-1} + 1$ and $\beta_{n-1} + x_n$, which are exactly α_n and β_n. That's to say, $\alpha_n = \alpha_{n-1} + 1$ and $\beta_n = \beta_{n-1} + x_n$. Therefore, we can calculate the posterior $\Pi(\theta | x_{1:n})$ as the posterior given the n^{th} observation using the prior $\tilde{q}(\theta)$.

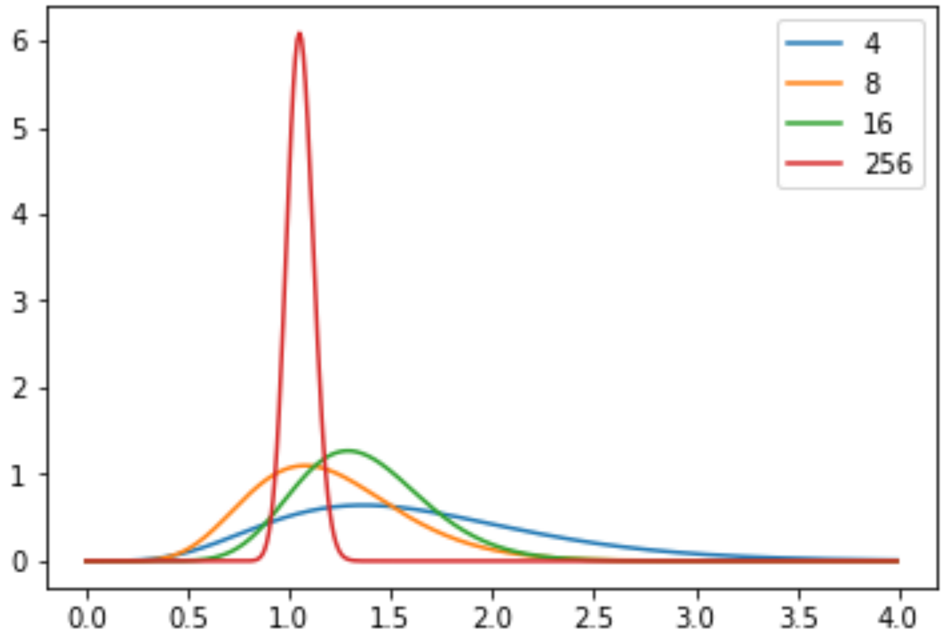
b.

In [7]:

```
smps = np.random.exponential(1, 256)

a0 = 2
b0 = 0.2
theta = np.arange(0.0, 4.0, 0.01)

for i in [4, 8, 16, 256]:
    at = a0 + i
    bt = b0 + sum(smps[:i])
    plt.plot(theta, gamma.pdf(theta, a=at, scale=1/bt), label = i)

plt.legend()
```

Out[7]:

```
<matplotlib.legend.Legend at 0x1a21d67f28>
```



As n increases, the posterior density function tends to center at 1 and becomes more and moer steep.

Problem 2.

By Bayesian Theorem, the posterior density can be written as:

$f(\pi^1, \pi^2 | Y_1^{T_1}, \ldots, Y_n^{T_n}, T_1, \ldots, T_n)$

$= \frac{f(\pi^1,\pi^2) * \mathbb{P}(Y_1^{T_1},\ldots,Y_n^{T_n},T_1,\ldots,T_n|\pi^1,\pi^2)}{\int f(\pi^1,\pi^2) * \mathbb{P}(Y_1^{T_1},\ldots,Y_n^{T_n},T_1,\ldots,T_n|\pi^1,\pi^2) d\pi_1 d\pi_2}$

Because of the independence of $(Y_i^{T_i}, T_i), i = 1, \ldots, n$, we have:

$\mathbb{P}(Y_1^{T_1}, \ldots, Y_n^{T_n}, T_1, \ldots, T_n | \pi^1, \pi^2)$

$= \Pi_{i=1}^n \mathbb{P}(Y_i^{T_i}, T_i | \pi^1, \pi^2)$

$= \Pi_{i=1}^n \frac{1}{2} * (\pi^{T_i})^{Y_i^{T_i}} * (1 - \pi^{T_i})^{1-Y_i^{T_i}}$

Therefore, the posterior density is:

$f(\pi^1, \pi^2 | Y_1^{T_1}, \ldots, Y_n^{T_n}, T_1, \ldots, T_n)$

$= \frac{\Pi_{i=1}^n \frac{1}{2} * (\pi^{T_i})^{Y_i^{T_i}} * (1-\pi^{T_i})^{1-Y_i^{T_i}}}{\int \Pi_{i=1}^n \frac{1}{2} * (\pi^{T_i})^{Y_i^{T_i}} * (1-\pi^{T_i})^{1-Y_i^{T_i}} d\pi^1 d\pi^2}$

$= \frac{\Pi_{i=1}^n (\pi^{T_i})^{Y_i^{T_i}} * (1-\pi^{T_i})^{1-Y_i^{T_i}}}{\int \Pi_{i=1}^n (\pi^{T_i})^{Y_i^{T_i}} * (1-\pi^{T_i})^{1-Y_i^{T_i}} d\pi^1 d\pi^2}$

Problem 3.

(a)

Beacuse the expectation of a Poisson(λ) random variable is λ and the linearity of expectation, we have:

$\mathbb{E}(\bar{X})$

$= \frac{\Sigma_{i=1}^n X_i}{n}$

$= \frac{\Sigma_{i=1}^n \mathbb{E}(X_i)}{n}$

$= \frac{n\lambda}{n}$

$= \lambda$

(b)

Note that $\mathbb{E}_\lambda(\bar{X} - \lambda)^2 = Var(\bar{X}) = \frac{\lambda}{n}$.

We then try to prove $\mathbb{E}_\lambda(T_n - \lambda)^2 \geq \frac{\lambda}{n}$.

Denote $f(X_1, \ldots, X_n) = \Pi_{i=1}^n \frac{\lambda^{X_i}}{X_i!} e^{-\lambda}$,

$l(X_1, \ldots, X_n) = log(f(X_1, \ldots, X_n)) = (\Sigma_{i=1}^n X_i) log\lambda - \Sigma_{i=1}^n log(X_i!) - n\lambda$ and

$\dot{l}(X_1, \ldots, X_n) = \frac{\partial}{\partial\lambda} l(X_1, \ldots, X_n) = \frac{\Sigma_{i=1}^n X_i}{\lambda} - n.$

Then, we have $\mathbb{E}(\dot{l}(X_1, \ldots, X_n)) = 0.$

We now calculate

$Cov(\dot{l}(X_1, \ldots, X_n), T_n)$

$= \mathbb{E}(\dot{l}(X_1, \ldots, X_n) * T_n)$

$= \Sigma_{\mathbf{x}} f(x_1, \ldots, x_n) * \frac{\partial}{\partial\lambda} l(x_1, \ldots, x_n) * T_n$

$= \Sigma_{\mathbf{x}} f(x_1, \ldots, x_n) * \frac{1}{f(x_1, \ldots, x_n)} * \frac{\partial}{\partial\lambda} f(x_1, \ldots, x_n) * T_n$

$= \Sigma_{\mathbf{x}} \frac{\partial}{\partial\lambda} f(x_1, \ldots, x_n) * T_n$

$= \frac{\partial}{\partial\lambda} \Sigma_{\mathbf{x}} f(x_1, \ldots, x_n) * T_n$

$= \frac{\partial}{\partial\lambda} \mathbb{E}(T_n)$

$= 1$

Also, the variance of $\dot{l}(X_1, \ldots, X_n)$ is

$Var(\dot{l}(X_1, \ldots, X_n))$

$= Var(\frac{\Sigma_{i=1}^n X_i}{\lambda} - n)$

$= Var(\frac{\Sigma_{i=1}^n X_i}{\lambda})$

$= \frac{Var(\Sigma_{i=1}^n X_i)}{\lambda^2}$

$= \frac{n}{\lambda}$

Now, according to the Cauchy-Schwarz inequality, we have:

$Cov(\dot{l}(X_1, \ldots, X_n), T_n)^2 \leq Var(\dot{l}(X_1, \ldots, X_n)) * Var(T_n)$

$<=> 1 \leq \frac{n}{\lambda} * \mathbb{E}_\lambda(T_n - \lambda)^2$

which is exacly what we want: $\mathbb{E}_\lambda(T_n - \lambda)^2 \geq \frac{\lambda}{n}$.