Lab 5 Solutions

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Instructions

Make sure that you upload an RMarkdown file to the canvas page (this should have a .Rmd extension) as well as the PDF output after you have knitted the file (this will have a .pdf extension). The files you upload to the Canvas page should be updated with commands you provide to answer each of the questions below. You can edit this file directly to produce your final solutions. The lab is due 11:59pm on Saturday, November 9th.

Goal

The goal of this lab is to investigate the empirical behavior of a common hypothesis testing procedure through simulation using R. We consider the traditional two-sample t-test.

Two-Sample T-Test

Consider an experiment testing if a 35 year old male's heart rate statistically differs between a control group and a dosage group. Let X denote the control group and let Y denote the drug group. One common method used to solve this problem is the two-sample t-test. The null hypothesis for this study is:

$$H_0 : \mu_1 - \mu_2 = \Delta_0,$$

where Δ_0 is the hypothesized value. The assumptions of the two sample pooled t-test follow below:

Assumptions

1. $X_1, X_2, \ldots, X_m$ is a random sample from a normal distribution with mean μ_1 and variance σ_1^2.
2. $Y_1, Y_2, \ldots, Y_n$ is a random sample from a normal distribution with mean μ_2 and variance σ_2^2.
3. The X and Y samples are independent of one another.

Procedure

The test statistic is

$$t_{calc} = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}},$$

where $\bar{x}, \bar{y}$ are the respective sample means and s_1^2, s_2^2 are the respective sample standard deviations.

The approximate degrees of freedom is

$$df = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}$$

Under the null hypothesis, t_{calc} has a student's t-distribution with df degrees of freedom.

Rejection rules

Alternative Hypothesis	P-value calculation
$H_A : \mu_1 - \mu_2 > \Delta_0$ (upper-tailed)	$P(t_{calc} > T)$
$H_A : \mu_1 - \mu_2 < \Delta_0$ (lower-tailed)	$P(t_{calc} < T)$
$H_A : \mu_1 - \mu_2 \neq \Delta_0$ (two-tailed)	$2 * P(\lvert t_{calc} \rvert > T)$

Reject H_0 when:

$$Pvalue \leq \alpha$$

Tasks

1) Using the **R** function **t.test**, run the two sample t-test on the following simulated dataset. Note that the **t.test** function defaults a two-tailed alternative. Also briefly interpret the output.

```r
set.seed(5)
sigma=5
Control <- rnorm(30,mean=10,sd=sigma)
Dosage <- rnorm(35,mean=12,sd=sigma)
t.test(Control,Dosage)
```

```
##
##  Welch Two Sample t-test
##
## data:  Control and Dosage
## t = -1.9684, df = 62.014, p-value = 0.05349
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -4.96460632  0.03821408
## sample estimates:
## mean of x mean of y
##  10.05649  12.51969
```

2) Write a function called **emperical.size** that simulates **R** different samples of X for control and **R** different samples of Y for the drug group and computes the proportion of test statistics that fall in the rejection region. The function should include the following:

- Inputs:
  - **R** is the number of simulated data sets (simulated test statistics). Let **R** have default 10,000.
  - Parameters **mu1**, **mu2**, **sigma1** and **sigma2** which are the respective true means and true standard deviations of X & Y. Let the parameters have respective defaults **mu1=0**, **mu1=0**, **sigma1=1** and **sigma2=1**.
  - Sample sizes **n** and **m** defaulted at **m=n=30**.
  - **level** is the significance level as a decimal with default at $\alpha = .05$.
  - **value** is the hypothesized value defaulted at 0.
- The output should be a **list** with the following labeled elements:
  - **statistic.list** vector of simulated t-statistics (this should have length **R**).

- **pvalue.list** vector of empirical p-values (this should have length **R**).
- **empirical.size** is a single number that represents the proportion of simulated test statistics that fell in the rejection region.

The function is below:

```
emperical.size <- function(R=10000,
                           mu1=0,mu2=0,
                           sigma1=1,sigma2=1,
                           m=30,n=30,
                           level=.05,
                           value=0,
                           direction="two.sided") {

  #Define empty lists
  statistic.list <- rep(0,R)
  pvalue.list <- rep(0,R)

  for (i in 1:R) {

    # Sample realized data
    Control <- rnorm(m,mean=mu1,sd=sigma1)
    Dosage <- rnorm(n,mean=mu2,sd=sigma2)

    # Testing values
    testing.procedure <- t.test(Control,Dosage,mu=value)
    statistic.list[i] <- testing.procedure$statistic
    pvalue.list[i] <-  testing.procedure$p.value

    }
    size <- mean(pvalue.list<=level)
    # return values
    return(list(statistic.list=statistic.list,
                pvalue.list=pvalue.list,
                emperical.size=size)
           )

}
```

Evaluate your function with the following inputs **R=10**,**mu1=10**,**mu1=12**,**sigma1=5** and **sigma2=5**.

```
emperical.size(R=10,
               mu1=10,mu2=12,
                sigma1=5,sigma2=5
               )
```

```
## $statistic.list
##  [1] -1.5594821 -1.6265940 -0.3916181 -3.0267377 -0.6315979  0.5023321
##  [7]  0.1796087 -2.7168991 -2.1448224 -0.7961500
##
## $pvalue.list
##  [1] 0.124574455 0.109418121 0.697191662 0.003707140 0.530363425
##  [6] 0.617452414 0.858106351 0.008684406 0.036512133 0.429195763
##
## $emperical.size
## [1] 0.3
```

3) Assuming the null hypothesis

$$H_0 : \mu_1 - \mu_2 = 0$$

is true, compute the empirical size using 10,000 simulated data sets. Use the function **emperical.size** to accomplish this task and store the object as **sim**. Output the empirical size quantity **sim$size**. Comment on this value. What is it close to?

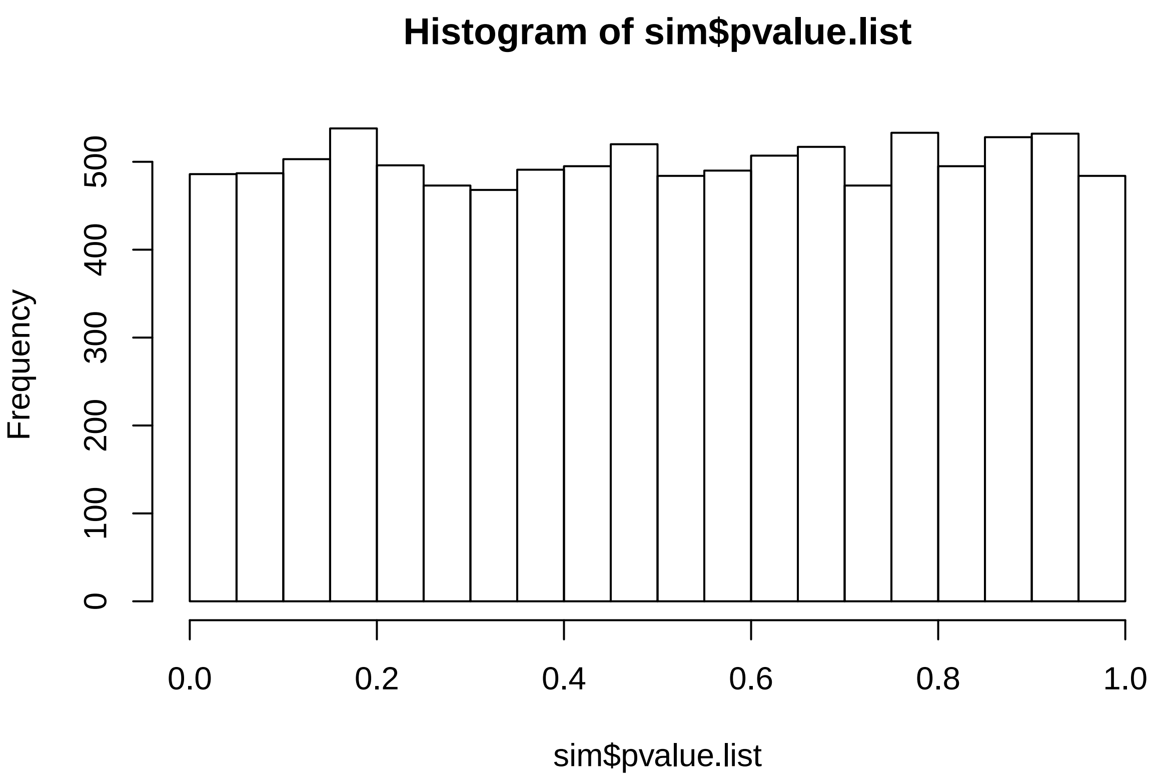
Note: use **mu1=mu1=10** (i.e., the null is true). Also set **sigma1=5**,**sigma2=5** and **n=m=30**.

```
sim <- emperical.size(R=10000,
               mu1=10,mu2=10,
                sigma1=5,sigma2=5
               )
sim$emperical.size
```

```
## [1] 0.0486
```

4) Plot a histogram of the simulated P-values, i.e., **hist(sim$pvalue.list)**. What is the probability distribution shown from this histogram? Does this surprise you?

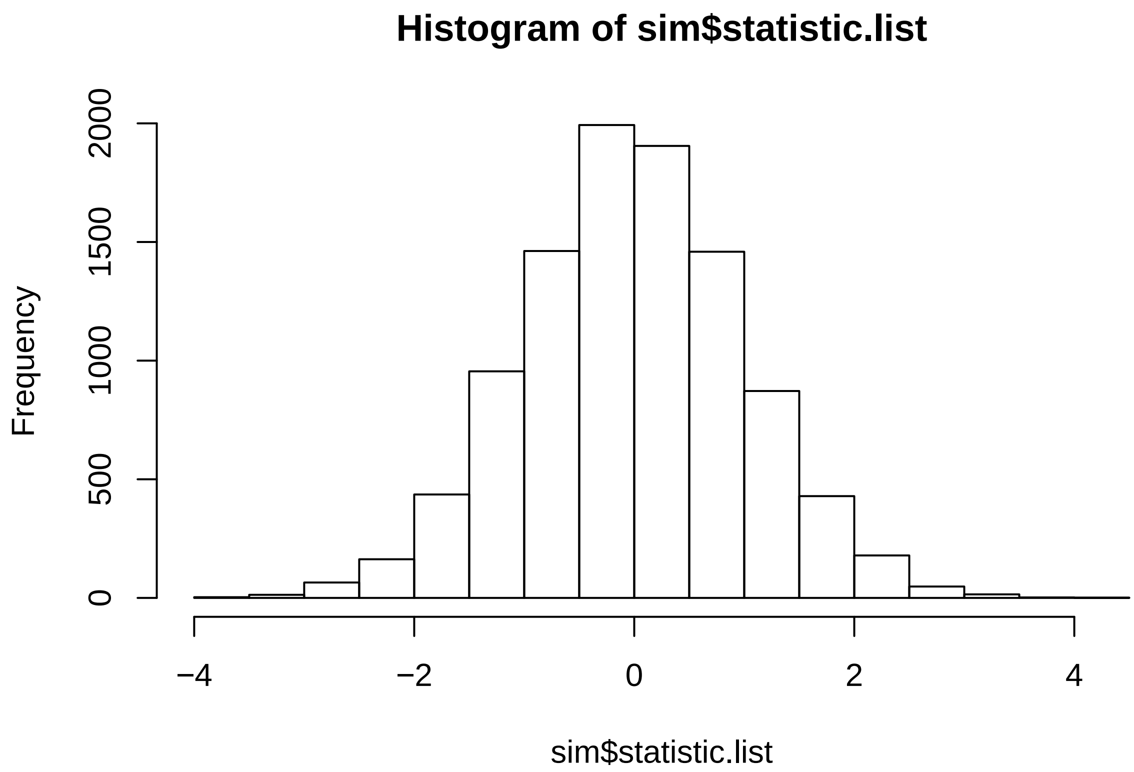
```
hist(sim$pvalue.list)
```



Under the null hypothesis $H_0 : \mu_1 - \mu_2 = 0$, the empirical probability distribution of p-values is uniform. That is so cool!

5) Plot a histogram illustrating the empirical sampling sampling of the t-statistic, i.e., **hist(sim.pooled$statistic.list,prob =TRUE)**. What is the probability distribution shown from this histogram?

```
hist(sim$statistic.list)
```



6) Run the following four lines of code:

```
emperical.size(R=10000,mu1=10,mu2=10,sigma1=5,sigma2=5)$emperical.size
```

```
## [1] 0.0486
```

```
emperical.size(R=10000,mu1=10,mu2=12,sigma1=5,sigma2=5)$emperical.size
```

```
## [1] 0.341
```

```
emperical.size(R=10000,mu1=10,mu2=14,sigma1=5,sigma2=5)$emperical.size
```

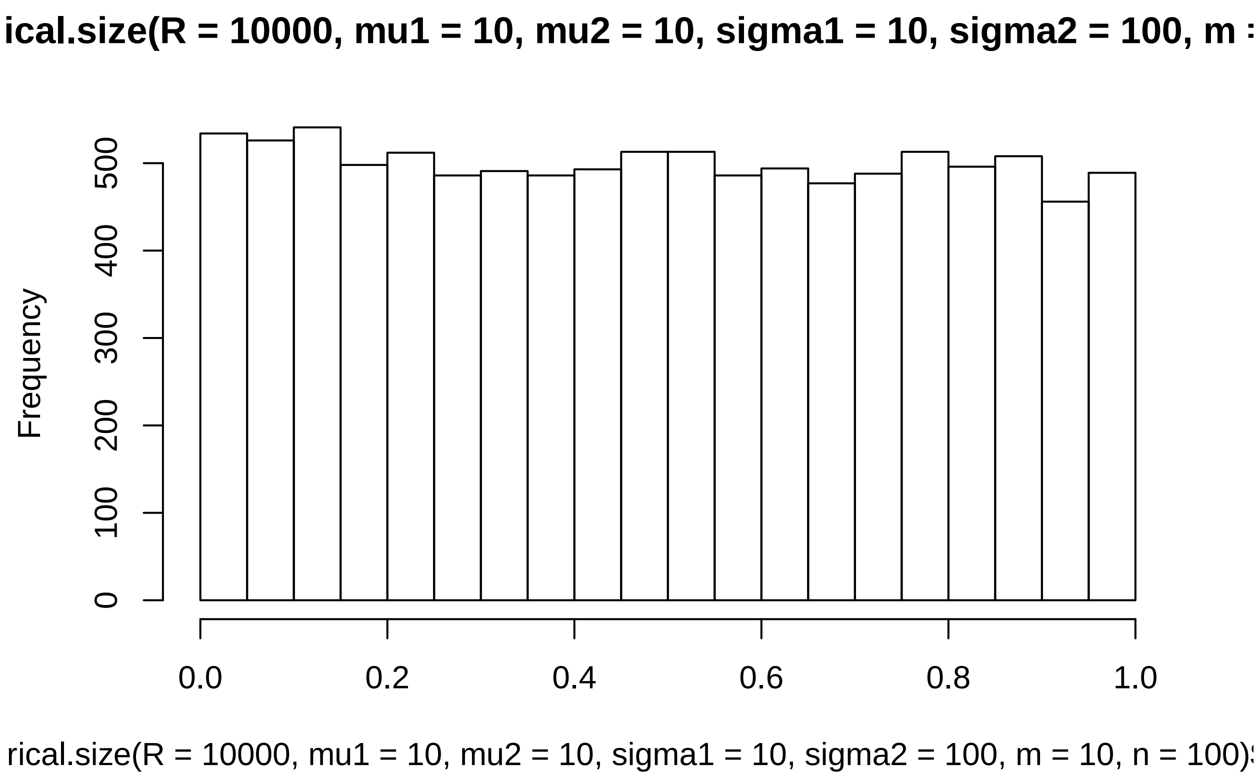
```
## [1] 0.862
```

```
emperical.size(R=10000,mu1=10,mu2=16,sigma1=5,sigma2=5)$emperical.size
```

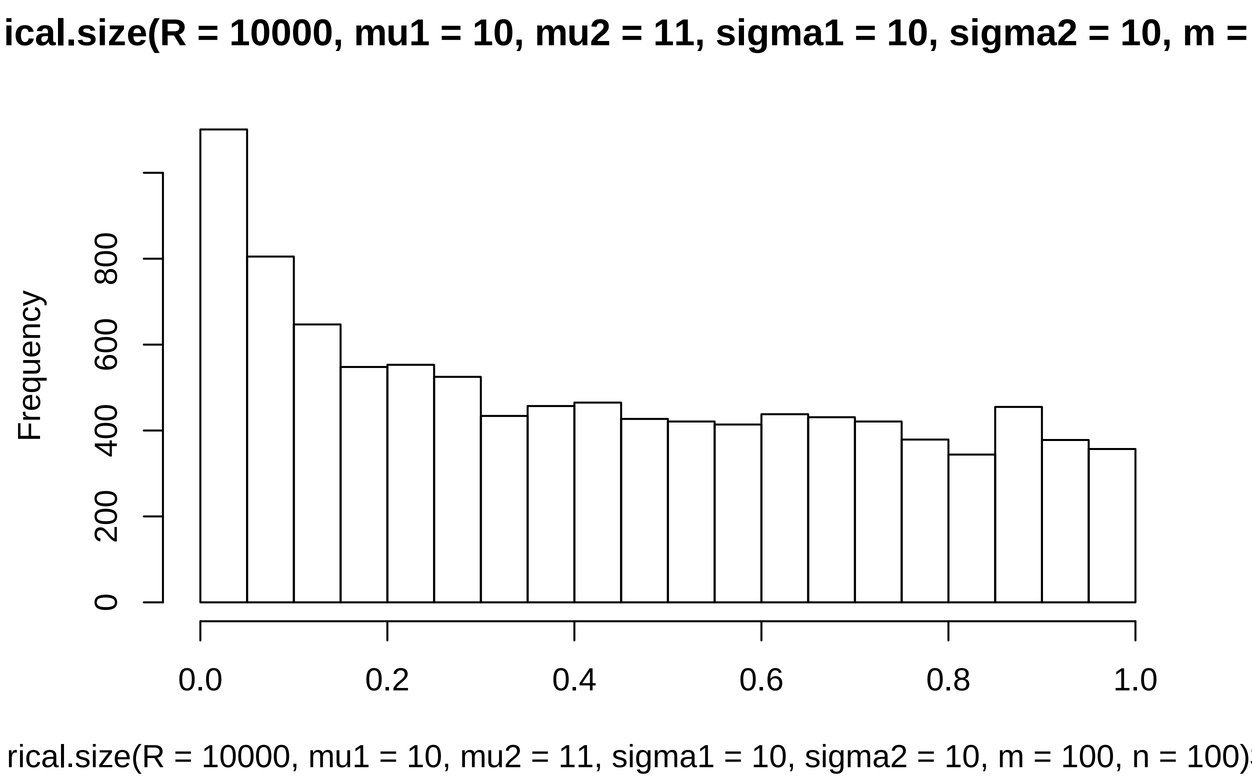
```
## [1] 0.9957
```

```
  Comment on the results.
```

```
hist(emperical.size(R=10000,mu1=10,mu2=10,sigma1=10,sigma2=100,m=10,n=100)$pvalue.list)
```



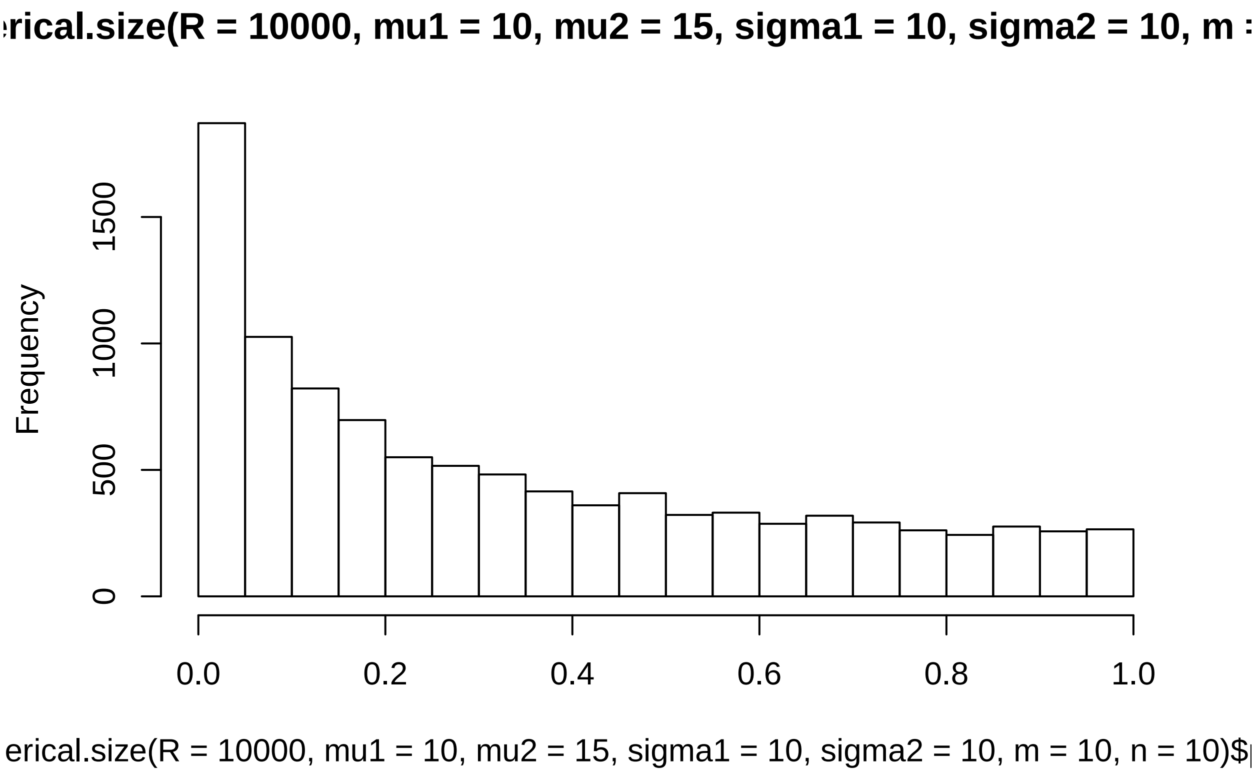

```r
#hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=30,n=30)$pvalue.list)
#hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=50,n=50)$pvalue.list)
hist(emperical.size(R=10000,mu1=10,mu2=11,sigma1=10,sigma2=10,m=100,n=100)$pvalue.list)
```



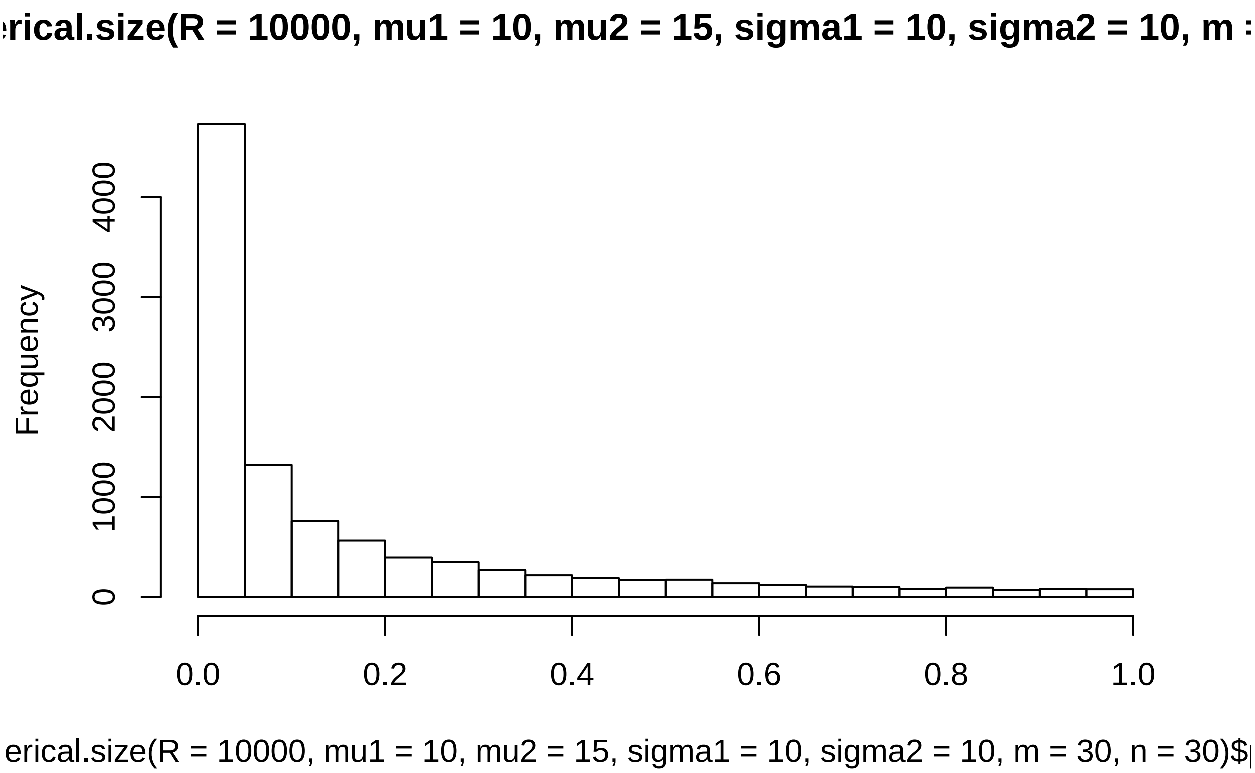
For fixed sample sizes m and n, the empirical the probability of rejecting the null increases for larger departures from the null hypothesis. More formally, as the effect size increases, the empirical power of the test increases.

7) Run the following four lines of code:

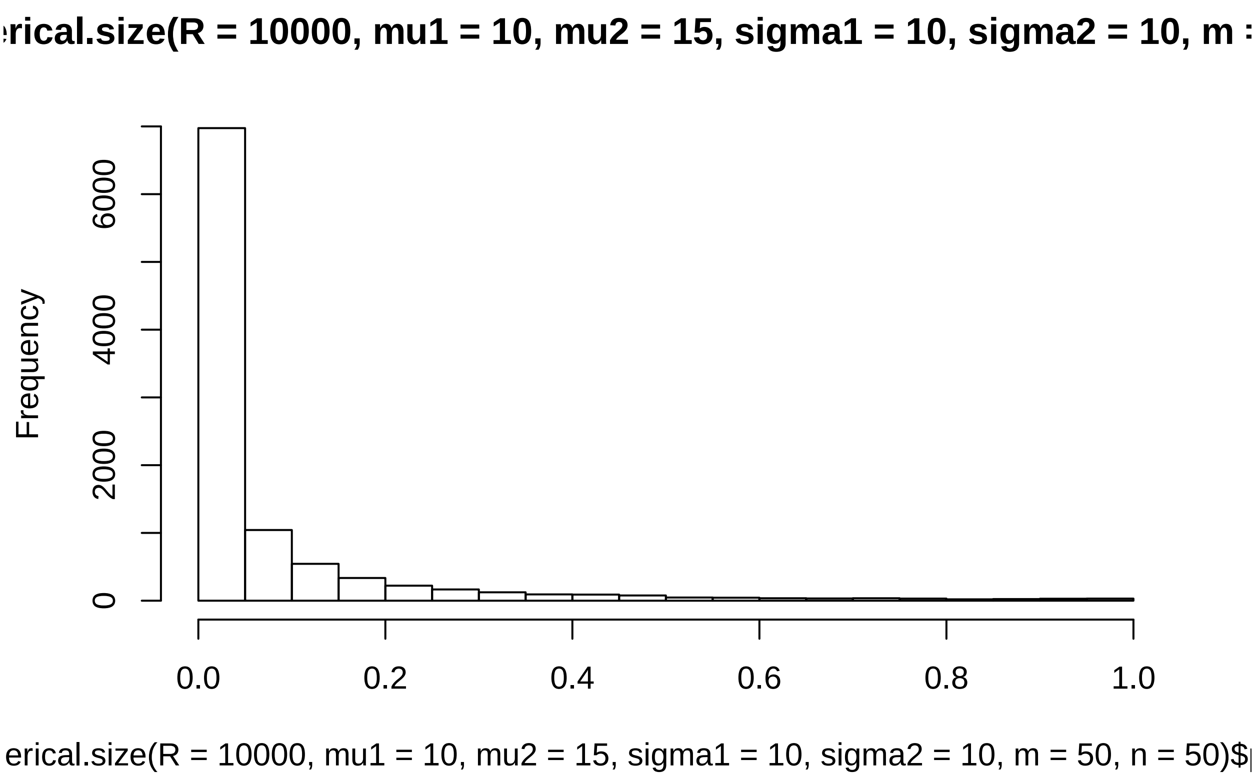
```
hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=10,n=10)$pvalue.list)
```



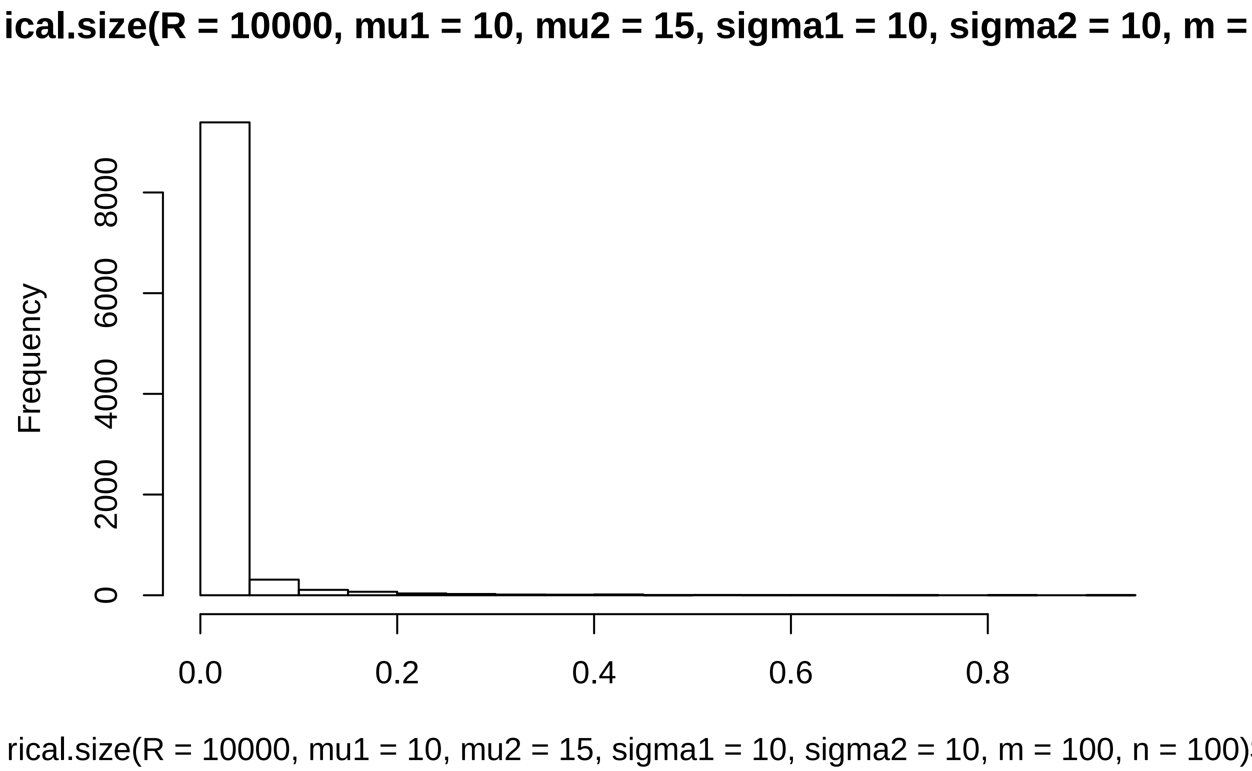

```
hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=30,n=30)$pvalue.list)
```




```r
hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=50,n=50)$pvalue.list)
```




```r
hist(emperical.size(R=10000,mu1=10,mu2=15,sigma1=10,sigma2=10,m=100,n=100)$pvalue.list)
```



Comment on the results.

For a fixed difference in the means, the empirical the probability of rejecting the null increases as the sample size increases. More formally, for a fixed effect size, the empirical power of the test increases as the sample size increases.