Final

2019年8月4日 12:28

- [x] Inner product spaces and their structure.
- [ ] The Cayley-Hamilton theorem and its applications:
  - [x] order reduction,
  - [x] matrix exponential,
  - [x] analytic functions
  - [ ] matrix arguments.
- [x] Eigenvalues and eigenvectors of a matrix.
- [x] Diagonalization and orthogonal diagonalization of a matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad |A-\lambda I| = 0 \Rightarrow \lambda^2(3-\lambda) = 0 \Rightarrow \lambda_{1,2} = 0,\ \lambda_3 = 3$$

可直接代入

$$\lambda_{1,2} = 0:\ \bar{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \bar{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ \lambda_3 = 3:\ \bar{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

orthonormal

$$\Rightarrow \bar{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},\ \bar{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 2 \end{pmatrix},\ \bar{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 2 & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix},\ D = S^{-1}AS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

A is orthogonally diagonalizable

- [x] Applications of matrix diagonalization in solving discrete dynamical systems (recall that $A^n = SD^nS^{-1}$
- [x] Adjoint, self-adjoint and normal operators: properties and derivation of adjoint operators.
- [x] Singular Value Decomposition and its properties.

Some questions:

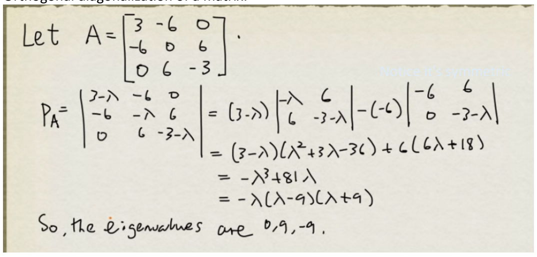
7.4.3

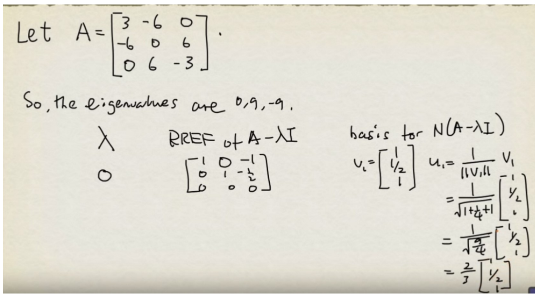
- [x] Condition for diagonalizable matrix
- [x] algebraic multiplicities.

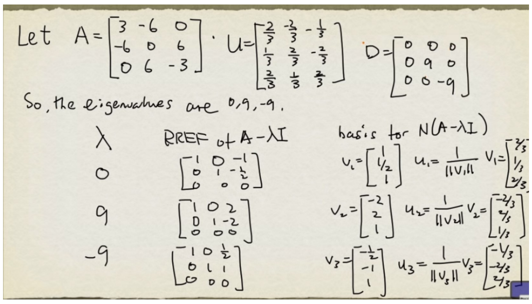
Theorem (Spectral Theorem): A matrix A is orthogonally diagonalizable iff A is symmetric ($A^T = A$).

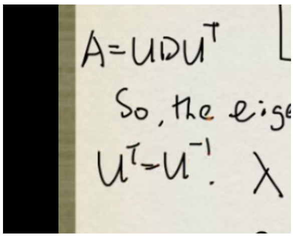
- [ ]

★ Orthogonal diagonalization of a matrix:









Eigenvalues and Eigenvectors

- λ is an eigenvalue of A.
  $\Updownarrow$
  There exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I_n)\vec{v} = \vec{0}$.
  $\Updownarrow$
  $\ker(A - \lambda I_n) \neq \{\vec{0}\}$.
  $\Updownarrow$
  Matrix $A - \lambda I_n$ fails to be invertible.
  $\Updownarrow$
  $\det(A - \lambda I_n) = 0$.
- $$f_A(\lambda) = \det(A - \lambda I_n) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} - \lambda \end{bmatrix}.$$
- Algebraic multiplicity

5 Algebraic multiplicity of an eigenvalue

We say that an eigenvalue λ_0 of a square matrix A has *algebraic multiplicity* k if λ_0 is a root of multiplicity k of the characteristic polynomial $f_A(\lambda)$, meaning that we can write

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

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for some polynomial $g(\lambda)$ with $g(\lambda_0) \neq 0$.

- In Example 4, the algebraic multiplicity of the eigenvalue $\lambda_0 = 1$ is $k = 3$, since

$$f_A(\lambda) = (\underset{\substack{\uparrow \\ \lambda_0}}{1} - \lambda)^{\overset{\substack{k \\ \downarrow}}{3}} \underbrace{(2-\lambda)^2}_{g(\lambda)}.$$

- $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$, the product of the eigenvalues

- $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, the sum of the eigenvalues.

Eigenspaces

- Consider an eigenvalue λ of an $n \times n$ matrix A. Then the kernel of the matrix $A - \lambda I_n$ is called the *eigenspace* associated with λ, denoted by E_λ:

$$E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}.$$

Note that the eigenvectors with eigenvalue λ are the *nonzero* vectors in the eigenspace E_λ.

Geometric multiplicity

- Consider an eigenvalue λ of an $n \times n$ matrix A. The dimension of eigenspace $E_\lambda = \ker(A - \lambda I_n)$ is called the *geometric multiplicity* of eigenvalue λ. Thus, the geometric multiplicity is the nullity of matrix $A - \lambda I_n$, or $n - \operatorname{rank}(A - \lambda I_n)$.

Eigenbasis

- Consider an $n \times n$ matrix A. A basis of $\mathbb{R}^n$ consisting of eigenvectors of A is called an *eigenbasis* for A.

7.3.5 An $n \times n$ matrix with n distinct eigenvalues

- If an $n \times n$ matrix A has n distinct eigenvalues, then there exists an eigenbasis for A. We can construct an eigenbasis by finding an eigenvector for each eigenvalue.

7.3.6 The eigenvalues of similar matrices

- Suppose matrix A is similar to B. Then

  a. Matrices A and B have the same characteristic polynomial, that is, $f_A(\lambda) = f_B(\lambda)$.

  b. $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $\operatorname{nullity}(A) = \operatorname{nullity}(B)$.

  c. Matrices A and B have the same eigenvalues, with the same algebraic and geometric multiplicities. (However, the eigenvectors need not be the same.)

  d. Matrices A and B have the same determinant and the same trace: $\det A = \det B$ and $\operatorname{tr} A = \operatorname{tr} B$.

Eigenbases and geometric multiplicities

- **a.** Consider an $n \times n$ matrix A. If we find a basis of each eigenspace of A and concatenate all these bases, then the resulting eigenvectors $\vec{v}_1, \ldots, \vec{v}_s$ will be linearly independent. (Note that s is the sum of the geometric multiplicities of the eigenvalues of A.)

- **b.** There exists an eigenbasis for an $n \times n$ matrix A if (and only if) the geometric multiplicities of the eigenvalues add up to n (meaning that $s = n$ in part a).

The eigenvalues of similar matrices

- Suppose matrix A is similar to B. Then

  a. Matrices A and B have the same characteristic polynomial, that is, $f_A(\lambda) = f_B(\lambda)$.

  b. $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $\operatorname{nullity}(A) = \operatorname{nullity}(B)$.

  c. Matrices A and B have the same eigenvalues, with the same algebraic and geometric multiplicities. (However, the eigenvectors need not be the same.)

  d. Matrices A and B have the same determinant and the same trace: $\det A = \det B$ and $\operatorname{tr} A = \operatorname{tr} B$.

1. A matrix A is diagonalizable iff there exists an eigenbasis for A.

2. If $A_{n\times n}$ has n distinct eigenvalues then A is diagonalizable.

Diagonalization

7.4.1 The matrix of a linear transformation with respect to an eigenbasis

Consider a linear transformation $T(\vec{x}) = A\vec{x}$, where A is a square matrix. Suppose $\mathfrak{D} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ is an eigenbasis for T, with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then the $\mathfrak{D}$-matrix D of T is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}, \quad \text{where} \quad S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}.$$

Matrix D is diagonal, and its diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of T. ■

7.4.2 Diagonalizable matrices

An $n \times n$ matrix A is called *diagonalizable* if A is similar to some diagonal matrix D, that is, if there exists an invertible $n \times n$ matrix S such that $S^{-1}AS$ is diagonal.

As we just observed, matrix $S^{-1}AS$ is diagonal if (and only if) the column vectors of S form an eigenbasis for A. This implies the following result.

Eigenbases and diagonalization

a. Matrix A is diagonalizable if (and only if) there exists an eigenbasis for A.

b. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable. ■

If an n x n matrix A has fewer than n distinct eigenvalues, then A may or may not be diagonalizable.

$$\begin{tikzcd}
{[f]_{\mathfrak{A}}} \arrow[rrr, "A"] & & & {[T(f)]_{\mathfrak{A}}} \\
& f \arrow[ul, "L_{\mathfrak{A}}"'] \arrow[dl, "L_{\mathfrak{D}}"] \arrow[r, "T"] & T(f) \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{D}}"'] & \\
{[f]_{\mathfrak{D}}} \arrow[uu, "S"] \arrow[rrr, "D"'] & & & {[T(f)]_{\mathfrak{D}}} \arrow[uu, "S"']
\end{tikzcd}$$

7.4.4 Diagonalization

Suppose we are asked to determine whether a given $n \times n$ matrix A is diagonalizable. If so, we wish to find an invertible matrix S such that $S^{-1}AS$ is diagonal.

We can proceed as follows.

a. Find the eigenvalues of A, that is, solve the characteristic equation $f_A(\lambda) = \det(A - \lambda I_n) = 0$.

b. For each eigenvalue λ, find a basis of the eigenspace $E_\lambda = \ker(A - \lambda I_n)$.

c. Matrix A is diagonalizable if (and only if) the dimensions of the eigenspaces add up to n. In this case, we find an eigenbasis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ by concatenating the bases of the eigenspaces we found in step (b). Let $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$. Then matrix $S^{-1}AS = D$ is diagonal, and the ith diagonal entry of D is the eigenvalue λ_i associated with $\vec{v}_i$. ■

7.4.5 Powers of a diagonalizable matrix

To compute the powers A^t of a diagonalizable matrix A (where t is a positive integer), proceed as follows:

Use Theorem 7.4.4 to diagonalize A, that is, find an invertible S and a diagonal D such that $S^{-1}AS = D$.

Then

$$A = SDS^{-1} \quad \text{and} \quad A^t = SD^tS^{-1}.$$

To compute D^t, raise the diagonal entries of D to the tth power. ■

7.4.6 The eigenvalues of a linear transformation

Consider a linear transformation T from V to V, where V is a linear space. A scalar λ is called an *eigenvalue* of T if there exists a nonzero element f of V such that

$$T(f) = \lambda f.$$

Such an f is called an *eigenfunction* if V consists of functions, an *eigenmatrix* if V consists of matrices, and so on. In theoretical work, the inclusive term *eigenvector* is often used for f.

Now suppose that V is finite dimensional. Then a basis $\mathfrak{D}$ of V consisting of eigenvectors of T is called an *eigenbasis* for T. We say that transformation T is *diagonalizable* if the matrix of T with respect to some basis is diagonal. Transformation T is diagonalizable if (and only if) there exists an eigenbasis for T (see Theorem 7.4.3a).

Adjoint

For an invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we find

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Properties of Adjoint Operators

-
  1. $T \in L(V, W) \Rightarrow T^\star \in L(W, V)$

  $$(v, T^\star(w_1 + w_2)) = (Tv, w_1 + w_2) = (Tv, w_1) + (Tv, w_2)$$
  $$= (v, T^\star w_1) + (v, T^\star w_2) = (v, T^\star w_1 + T^\star w_2)$$
  $$(v, T^\star(\lambda w)) = (Tv, \lambda w) = \bar{\lambda}(Tv, w) = \bar{\lambda}(v, T^\star w) = (v, \lambda T^\star w)$$

  2. $(T_1 + T_2)^\star = T_1^\star + T_2^\star \quad \forall S, T \in L(V, W)$
  3. $(\lambda T)^\star = \bar{\lambda} T^\star \quad \forall \lambda \in \mathbb{K} \forall T \in L(V, W)$
  4. $(T^\star)^\star = T \quad \forall T \in L(V, W)$

  $$(w, (T^\star)^\star v) = (T^\star w, v) = \overline{(v, T^\star w)} = \overline{(Tv, w)} = (w, Tv) \forall v \in V$$

  5. $I^\star = I$
  6. $(T_1 T_2)^\star = T_2^\star T_1^\star \quad \forall T_1 \in L(W, U),\ T_2 \in L(V, W)$
  7. $Ker\ T^\star = (Im\ T)^\perp$

  $$w \in Ker\ T^\star \Leftrightarrow T^\star w = 0 \Leftrightarrow (v, T^\star w) = 0 \quad \forall v \in V$$
  $$\Leftrightarrow (Tv, w) = 0 \quad \forall v \in V \Leftrightarrow w \in (Im\ T)^\perp$$

  8. The matrix of the adjoint $T^\star$ w.r.t orthonormal bases $e_1, \dots, e_m \in V$; $f_1, \dots f_n \in W$ is the conjugate transpose of the matrix of T.

  $$A_T = (Te_1\ Te_2 \dots Te_m) \quad Te_k \in W,\ f_1, \dots, f_n \text{ is orthonormal}$$
  $$\Rightarrow Te_k = (Te_k, f_1)f_1 + \dots + (Te_k, f_n)f_n \Rightarrow (A_T)_{jk} = (Te_k, f_j)$$
  $$A_{T^\star} = (T^\star f_1\ T^\star f_2 \dots T^\star f_m) \quad T^\star f_k \in V,\ e_1, \dots, e_m \text{ is orthonormal}$$
  $$\Rightarrow T^\star f_k = (T^\star f_k, e_1)e_1 + \dots + (T^\star f_k, e_m)e_m$$
  $$\Rightarrow (A_{T^\star})_{jk} = (T^\star f_k, e_j) = \overline{(e_j, T^\star f_k)} = \overline{(Te_j, f_k)}$$

Self-Adjoint Operators

- **Definition:** The operator $T \in L(V, V)$ is called self-adjoint if $T^\star = T$:

  $$(Tv, w) = (v, Tw) \quad \forall v, w \in V$$

  Hermitian=self-adjoint

Normal Operators

- **Definition:** An operator on an inner product space is called normal if it commutes with its adjoint, i.e $TT^\star = T^\star T$.

  Remarks:
  1. Every self-adjoint operator is normal.
  2. T is normal iff $||Tv|| = ||T^\star v|| \forall v$

  $$T \text{ is normal} \Leftrightarrow T^\star T - TT^\star = 0 \Leftrightarrow ((T^\star T - TT^\star)v, v) = 0$$
  $$\Leftrightarrow (T^\star Tv, v) = (TT^\star v, v) \Leftrightarrow ||Tv||^2 = ||T^\star v||^2$$

  3. Let $Tv = \lambda v$. T is normal $\Leftrightarrow T - \lambda I$ is also normal.
  4. Let $Tv = \lambda v$. Then $T^\star v = \bar{\lambda} v$

  $$0 = ||(T - \lambda I)v|| = ||(T - \lambda I)^\star v|| = ||(T^\star - \bar{\lambda} I)v|| \Rightarrow T^\star v = \bar{\lambda} v$$

  5. Suppose $T \in L(V, V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

  $$Tu = \alpha u,\ Tv = \beta v \Rightarrow T^\star v = \bar{\beta} v$$
  $$(\alpha - \beta)(u, v) = \alpha(u, v) - \beta(u, v) = (Tu, v) - (u, T^\star v) = 0$$

Singular Value

- **Definition:** The singular values of a matrix $A_{n\times m}$ are the square roots of the eigenvalues of the symmetric matrix $(A^T A)_{m\times m}$ listed with their algebraic multiplicities:

  $$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$$

  Theorem: Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$, $L\bar{x} = A\bar{x}$ be invertible. The image of the unit circle under the map L is an ellipse E. Singular values of A are the length of semi-axes of E.

SVD

Lemma: If $rank\ A_{n\times m} = r$, then its singular values

$$\sigma_1, \ldots, \sigma_r \neq 0 \quad \text{and} \quad \sigma_{r+1}, \ldots, \sigma_m = 0$$

- **Theorem (SVD):** Any matrix $A_{n\times m}$ can be represented in the form

$$A = U\Sigma V^T,$$

U is an orthogonal $n \times n$ matrix

V is an orthogonal $m \times m$ matrix

Σ is an $n \times n$ matrix whose first r diagonal entries are nonzero singular values of A, $r = rank\ A$, and all other entries vanish

- Example 1

$$A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$$

$$\bar{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \bar{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow V = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{\sigma_1}A\bar{v}_1 = \frac{1}{10\sqrt{5}}\begin{pmatrix} 10 \\ -20 \end{pmatrix}, \bar{u}_2 = \frac{1}{\sigma_2}A\bar{v}_2 = \frac{1}{5\sqrt{5}}\begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

- Example 2

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\bar{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \bar{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \bar{v}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{\sigma_1}A\bar{v}_1 = \frac{1}{\sqrt{3}\sqrt{6}}\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \bar{u}_2 = \frac{1}{\sigma_2}A\bar{v}_2 = \frac{1}{1\cdot\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

Symmetric Matrices

8.1.1 Spectral theorem

- A matrix A is *orthogonally diagonalizable* (i.e., there exists an orthogonal S such that $S^{-1}AS = S^TAS$ is diagonal) if and only if A is *symmetric* (i.e., $A^T = A$). ■

- **8.1.2** Consider a symmetric matrix A. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with *distinct* eigenvalues λ_1 and λ_2, then $\vec{v}_1 \cdot \vec{v}_2 = 0$; that is, $\vec{v}_2$ is orthogonal to $\vec{v}_1$.

- **8.1.3** A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicities.

8.1.4 Orthogonal diagonalization of a symmetric matrix A

- **a.** Find the eigenvalues of A, and find a basis of each eigenspace.
  b. Using the Gram–Schmidt process, find an *orthonormal* basis of each eigenspace.
  c. Form an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for A by concatenating the orthonormal bases you found in part (b), and let

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & & \vec{v}_n \\ | & | & & | \end{bmatrix}.$$

S is orthogonal (by Theorem 8.1.2), and $S^{-1}AS$ will be diagonal. ■

Quardatic forms

8.2.1 Quadratic forms

- A function $q(x_1, x_2, \ldots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called a *quadratic form* if it is a $\boxed{\text{linear combination of functions of the form } x_i x_j}$ (where i and j may be equal). A quadratic form can be written as

$$\boxed{q(\vec{x}) = \vec{x} \cdot A\vec{x} = \vec{x}^T A\vec{x},}$$

for a unique symmetric $n \times n$ matrix A, called the matrix of q.

8.2.2 Diagonalizing a quadratic form

- Consider a quadratic form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$, where A is a symmetric $n \times n$ matrix. Let $\mathfrak{B}$ be an orthonormal eigenbasis for A, with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2,$$

where the c_i are the coordinates of $\vec{x}$ with respect to $\mathfrak{B}$.[3] ■