6.1. Intro to Determinants

Matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff. $\det A = ad-bc \neq 0$. (2.4.9a)

6.6.1 Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & \ddots & \\ \vdots & & a_{33} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$. A fails to be invertible if image of A isn't in $\mathbb{R}^3$

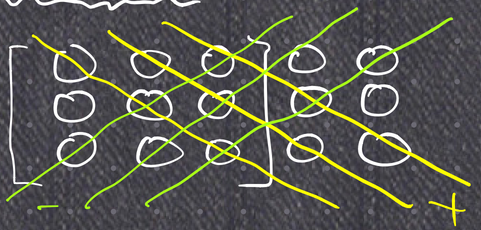
$\Rightarrow$ 3 column vectors are contained in some plane V.

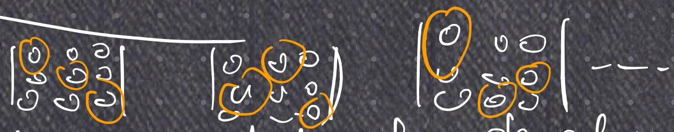
$\Rightarrow \vec{u}\cdot(\vec{v}\times\vec{w})=0$.

For A invertible

$\Rightarrow \vec{u}\cdot(\vec{v}\times\vec{w})\neq 0$.

6.1.2. Sarrus's Rule





Another exp: A choice of an element in each row and each column of a square matrix.

可取的独立个数 (无相交)

6.1.3. A pattern in an $n\times n$ matrix A is: a way to choose n entries of the matrix so that [there's one chosen entry in each row and in each column] of A. number $= n!$

prod P : product of all its entries.

Two entries in a pattern are inverted if one of them is located to the right and above the other in a matrix.

The signature of a pattern P is defined as

$\operatorname{sgn} P = (-1)^{(\text{number of inversions in } P)}$ swap number

$$\boxed{\det A = \sum (\operatorname{sgn} P)(\operatorname{prod} P)}$$

e.g. $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ no inversions $+$; $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ One inversion. $-$

$\det A = (-1)^0 ad + (-1)^1 bc = ad-bc$

6.1.4. Det of triangular matrix.

$\det\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \det\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = (\det A)(\det C)$.

"block"

$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det D) - \det(B)\det(C)$

6.2. Properties.

6.2.1. $\det(A^T) = \det A$

6.2.2. Linearity of determinant (in the ith row).

For fixed row vectors $\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots \vec{v}_n$ with n components, function

$$T(\vec{x}) = \det \begin{bmatrix} - \vec{v}_1 - \\ \cdots \\ - \vec{v}_{i-1} - \\ - \vec{v}_{i+1} - \\ \cdots \\ - \vec{v}_n - \end{bmatrix}$$

from $\mathbb{R}^{1\times n}$ to $\mathbb{R}$ is a linear trans...

6.2.3. Elementary row operations.

a. B: divide a row of A by k, $\det B = \frac{1}{k}\det A$

b. B: Get a row swap of A: $\det B = -\det A$ Alternating property.

c. B: Add a multiple of of a row of A to another row:

$$\det B = \det A.$$

6.2.4. A square mat A is invertible iff $\det A \neq 0$.

※ Algorithm 6.2.5

a. Swap row s times when computing $\operatorname{rref} A = I_n$.
Divide various rows by scalars $k_1, k_2, \cdots, k_r$

Then $\det A = (-1)^s k_1 k_2 \cdots k_r$.

b. ⋯ (P278)

6.2.6. $\det(AB) = \det(A)\det(B)$

Notice: $\det(A+B) \neq \det A + \det B$.
Non-linear.

6.2.8 For invertible A, $\det(A^{-1}) = \frac{1}{\det A}$

※ 6.2.10. Laplace/Cofactor expansion

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad \text{(down } j\text{th column)}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\underbrace{A_{ij}}_{\text{minor of } A\ (\text{余子式})}) \quad (i\text{th row})$$

6.2.11. $T: V \to V$, when $\mathfrak{B}$ is a basis of V, B is the $\mathfrak{B}$ matrix of T

$\Rightarrow \det T = \det B$

6.3. Geometric / Cramer's Rule

6.3.1 (Orthogonal Matrix: the columns form an orthogonal basis). $\|T(\vec{x})\| = \|\vec{x}\|$

The determinant of an orthogonal matrix is 1 or −1.

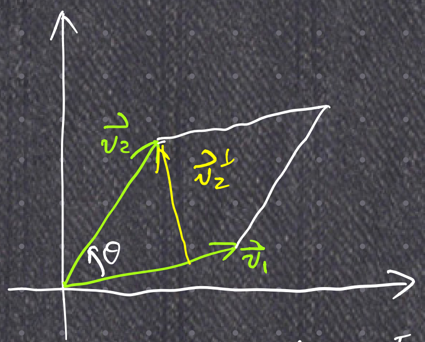
↳ like: $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$ ...

6.3.2. Rotation Matrices

Def: ① Orthogonal $n \times n$ matrix ② $\det A = 1$

Rotation: Linear trans... : $T(\vec{x}) = A\vec{x}$

※ The determinant as Area & Volume



$$\det A = \det[\vec{v}_1 \ \vec{v}_2] = \|\vec{v}_1\| \sin\theta \|\vec{v}_2\|.$$

$$|\det A| = \|\vec{v}_1\| \|\vec{v}_2^{\perp}\|$$

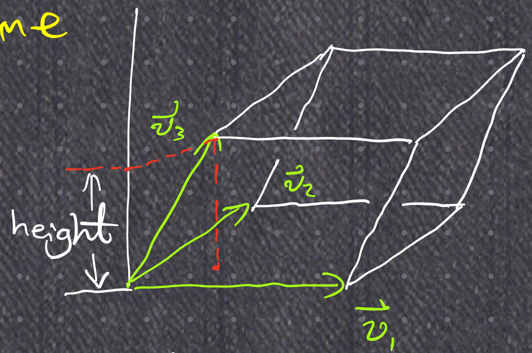
For $A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] = QR$ — Q: Orth. mat.

↳ R: upper trian... mat with: $r_{11} = \|\vec{v}_1\|$ ... $r_{jj} = \|\vec{v}_j^{\perp}\|$ for $j \geq 2$.

6.3.3. ⇒ $\det A = \|\vec{v}_1\| \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_n^{\perp}\|$.

where $\vec{v}_k^{\perp}$ is the component of $\vec{v}_k$ $\perp$ to the span $(\vec{v}_1, \ldots, \vec{v}_{k-1})$

↓ Volume



6.3.6. Volume of parallelepiped in $\mathbb{R}^n$.

① $V(\vec{v}_1, \ldots, \vec{v}_m) = V(\vec{v}_1, \ldots, \vec{v}_{m-1}) \|\vec{v}_m^{\perp}\|$.
$= \|\vec{v}_1\| \cdot \ldots \|\vec{v}_m^{\perp}\|$.

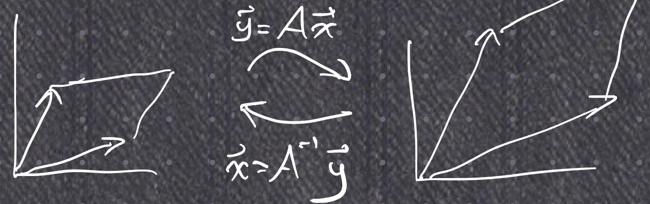
② Volume $= \sqrt{\det(A^T A)}$, with $A_{n\times n} = [\vec{v}_1 \ \cdots \ \vec{v}_m]$.

* If $m = n$, $V = |\det A|$.

※ The determinant as expansion factor.

expansion factor $= \dfrac{\text{Area}(T(\Omega))}{\text{Area}(\ldots)}$

$$V(A\vec v_1 \cdots A\vec v_n) = |\det A|\, V(\vec v_1 \cdots \vec v_n)$$



$*|\det(A^{-1})| = \frac{1}{|\det A|}$

Cramer's Rule.

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A^{-1} = \frac{1}{\det A}$

$T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$, $T(p(t)) = t\,p'(t)$, $B_1 = \{1, t, t^2, t^3\}$, $B_2 = \{1, t-1, t^2-1, t^3-1\}$.

Find: T

$$\underline{T(a_0 + a_1 t + a_2 t^2 + a_3 t^3)} = t(a_1 + 2a_2 t + 3a_3 t^2) = \underline{a_1 t + 2a_2 t^2 + 3a_3 t^3}$$

$\dim P_3(\mathbb{R}) = 4 \Rightarrow P_3(\mathbb{R}) \cong \mathbb{R}^4$

$$(Tp)_{B_1} = \begin{pmatrix} 0 \\ a_1 \\ 2a_2 \\ 3a_3 \end{pmatrix}, \quad (Tp)_{B_1} = A_{B_1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{A_{B_1}}$$

$$Tp = a_1 t + 2a_2 t^2 + 3a_3 t^3 = 1\cdot b_0 + b_1(t-1) + b_2(t^2-1) + b_3(t^3-t)$$
$$= (b_0 - b_1 - b_2) + t(b_1 - b_3) + b_2 t^2 + b_3 t^3$$

$$\begin{cases} b_0 = b_1 + b_2 \Rightarrow b_0 = a_1 + 2a_2 + 3a_3 \\ b_1 - b_3 = a_1 \Rightarrow b_1 = a_1 + 3a_3 \\ b_2 = 2a_2 \\ b_3 = 3a_3 \end{cases}$$

$$(Tp)_{B_2} = A_{B_2} \begin{pmatrix} a_0 \\ \vdots \\ a_3 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{A_{B_2}}$$

$$\det \begin{pmatrix} \sqrt2 & \sqrt3 & \sqrt5 & \sqrt3 \\ \sqrt6 & \sqrt{21} & \sqrt{10} & -2\sqrt3 \\ \sqrt{10} & 2\sqrt{15} & 5 & \sqrt6 \\ 2 & 2\sqrt6 & \sqrt{10} & \sqrt{15} \end{pmatrix} = \sqrt2\sqrt3\underline{\sqrt5}\sqrt3 \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sqrt3 & \sqrt7 & \sqrt2 & -2 \\ \sqrt5 & 2\sqrt5 & \sqrt5 & \sqrt2 \\ \sqrt2 & 2\sqrt2 & \sqrt2 & \sqrt5 \end{pmatrix} \begin{matrix} \to \underline{\sqrt5} \\ \to \sqrt2 \end{matrix}$$

$$= 30 \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ \sqrt3 & \sqrt7 & \sqrt2 & -2 \\ 1 & 2 & 1 & \sqrt2/\sqrt5 \\ 1 & 2 & 1 & \sqrt5/\sqrt2 \end{vmatrix}$$

$$0 \quad 0 \quad 0 \quad \frac{\sqrt{5}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}}$$

$$= 30 \begin{vmatrix} 1 & 1 & 1 & 1 \\ \sqrt{3} & \sqrt{7} & \sqrt{2} & -2 \\ 0 & 1 & 0 & \frac{\sqrt{2}}{\sqrt{3}} - 1 \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}} \end{vmatrix} \rightarrow 0 \quad \sqrt{7}-\sqrt{3} \quad \sqrt{2}-\sqrt{3} \quad -2-\sqrt{3}$$

$$= 30 \times \left(\frac{\sqrt{5}}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{3}}\right) \times (-1)^4 \begin{vmatrix} 1 & 1 & 1 \\ \sqrt{3} & \sqrt{7} & \sqrt{2} \\ 0 & 1 & 0 \end{vmatrix}$$

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$$= 30 \times \left(\frac{3}{\sqrt{10}}\right) \times (\sqrt{5} - \sqrt{2})$$

1. $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 3 & 1 & -3 \end{pmatrix} \times 2$

$\rightarrow 2 \times \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & -2 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 8 \end{pmatrix} \rightarrow 2 \times 2 \times 8 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\rightarrow 32 \times \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 128\, I_4$

2. Q: Consider map $\mathbb{R}^3 \rightarrow \mathbb{R}$, s.t.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \det \begin{pmatrix} x & 1 & 2 \\ y & 2 & 1 \\ z & -1 & 0 \end{pmatrix}$$

Show the map is linear, find its matrix.

A. $A = (A\vec{e}_1 \; A\vec{e}_2 \; A\vec{e}_3)$

$A_{1\times 3} \quad (\mathbb{R}^3 \rightarrow \mathbb{R})$

$A\vec{e}_1 = \det \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 1 \qquad A\vec{e}_2 = \det \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -2$

$A\vec{e}_3 = \ldots \qquad A = (1 \quad -2 \quad -3)$

3. Let $\det \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = 2$, evaluate $\det \begin{pmatrix} 3a+2c & c \\ 2d+3b & d \end{pmatrix}$

$$\begin{vmatrix} 3a+2c & c \\ 3b+2d & d \end{vmatrix} = \begin{vmatrix} 3a & c \\ 3b & d \end{vmatrix} + \begin{vmatrix} 2c & c \\ 2d & d \end{vmatrix}$$

$$3ad + \cancel{2cd} - \cancel{2cd} - 3bc$$

$$= -3(bc - ad) = -3 \times 2 = -6$$

4. Q: Let $\vec{v}_1, \dots, \vec{v}_{n-1} \in \mathbb{R}^n$ and

$$\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{x}) = 1,\quad \det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{y}) = 2,$$

$$\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{z}) = 3$$

Find: $\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ (-\vec{x} + 2\vec{y} + 3\vec{z} + 5\vec{v}_3))$

A: $\downarrow = -\underbrace{\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{x})}_{1\times(-1)} + 2\underbrace{\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{y})}_{2\times 2} + 3\underbrace{\det(\vec{v}_1 \cdots \vec{v}_{n-1}\ \vec{z})}_{3\times 3}$

$+ 5\underbrace{\det(\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \cdots \vec{v}_{n-1}\ \vec{v}_3)}_{0}$

Commutative Rings.

Def: A ring is a set R, with two operations

$\forall x, y \in R$, $x+y$, $x \cdot y \in R$.

Satisfying: a. R is an abelian group under addition.

b. $x(yz) = (xy)z$

c. $x(y+z) = xy + xz$, &

$(x+y)z = xz + yz$

If $xy = yx$, $\forall x, y \in R$, ring is commutative

If $1 \in R$ s.t. $1x = x1 = x$, 1 is identity for R

Property: Scalar $(-1)^{i+j}\det A_{ij}$ is the cofactor of the entree a_{ij}

$$C_{ij} = (-1)^{i+j}\det A_{ij} \Rightarrow \det A = \sum_{i=1}^{n} a_{ij} C_{ij}$$

$$\sum_{i=1}^{n} a_{ik} C_{ij} \neq 0 \quad \text{if } j \neq k.$$

Replace jth column by A