Second Recitation Class
Linear Algebra

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Compute A_G^2 and trace(A_G^2)

1. Let $A_G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ be the adjacency matrix of a graph G. Compute *trace* (A).

$$trace(A) = a_{11} + a_{22} + a_{33} = 0$$

2. Compute A_G^2 and *trace* (A_G^2). Find the good interpretation for *trace* (A_G^2)—the question is still open!

$$(A_G^2)_{ij} = \sum_{k=1}^{3} a_{ik} a_{kj} \Rightarrow \text{we count only terms with } a_{ik} a_{kj} \neq 0$$

$$\Rightarrow a_{ik} \neq 0,\ a_{kj} \neq 0 \iff \underbrace{a_{ik} = a_{kj} = 1}_{v_i \sim v_k \sim v_j} \text{ we count w.r.t } k$$

$$(A_G^2)_{ij} = \text{ the number of common neighbors of } v_i \text{ and } v_j$$

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Compute A_G^3

3 Compute A_G^3. What are the entries $(A_G^3)_{ij}$?

$$(A_G^3)_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} a_{ik}a_{km}a_{mj} \Rightarrow a_{ik}a_{km}a_{mj} \neq 0$$

$$\iff \underbrace{a_{ik} = a_{km} = a_{mj} = 1}_{v_i \sim v_k \sim v_m \sim v_j}$$

$(A_G^3)_{ij}$ is the number of walks of the length 3 from v_i to v_j

$\boxed{(A_G^m)_{ij} \text{ is the number of walks of the length } m \text{ from } v_i \text{ to } v_j}$

Remark: Recall, that a walk from v_i to v_j in a graph is a sequence of vertices

$$v_i - - v_p - - v_k - - v_r - - v_s - - \ldots - - v_j$$

The length of a walk is the number of edges in the walk.

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Remark

1. According to our definition of graphs, $trace(A_G) = 0$.
2. $(A_G^m)_{ij}$ is the number of walks of the length m from v_i to v_j.
3. In the definition of walks, one can "return" to the place where it comes from.
4. Between two adjacent points i and j, there always exists a walk of the length $(2n+1)$ for some $n \in N$ from v_i to v_j.
5. We can always directly write out A_G^n for any $n \in N$ since we know the definition of elements in such matrix.

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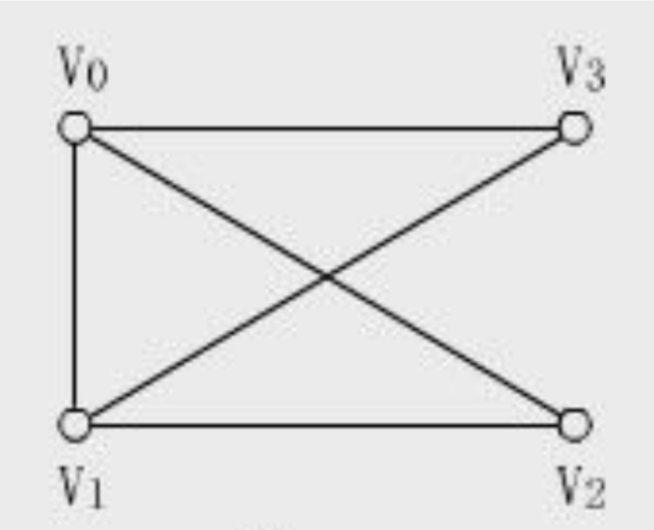
Exercise

Directly write out the adjacency matrix A_G, A_G^2 of this graph.



Linear Combination

Definition: If $\bar{y} = \alpha \bar{x}_1 + \ldots + \alpha_k \bar{x}_k$, then $\bar{y}$ is called a linear combination of $\bar{x}_1, \ldots, \bar{x}_k$ OR
we say that $\bar{y}$ is spanned by $\bar{x}_1, \ldots, \bar{x}_k$ and denote

$$\bar{y} = span(\bar{x_1}, \ldots, \bar{x_k})$$

Linear Independence

Definition: Elements $v_1, v_2, \ldots, v_n \in V$ are said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$$

Remark: An infinite set of vectors is said to be linearly independent if *every finite subset is linearly independent*.
Definition: If

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Rightarrow \exists \alpha_i \neq 0$$

then the elements $v_1, v_2, \ldots, v_n \in V$ are said to be linearly dependent.

Definition: A vector v_i is said to be redundant if it is represented as a linear combination of preceding vectors

$$v_i = \alpha_1 v_1 + \ldots + \alpha_{i-1} v_{i-1}$$

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Exercise

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$ is a linear combination of A and I_2.

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Answer

Solution

We have to find scalars c_1 and c_2 such that

$$A^2 = c_1 A + c_2 I_2,$$

or

$$\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$c_1 = 3, \; c_2 = 2.$

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Definition

Linear spaces (or vector spaces)

A *linear space*[3] V is a set endowed with a rule for addition (if f and g are in V, then so is $f+g$) and a rule for scalar multiplication (if f is in V and k in $\mathbb{R}$, then kf is in V) such that these operations satisfy the following eight rules[4] (for all f, g, h in V and all c, k in $\mathbb{R}$):

1. $(f+g)+h = f+(g+h)$.
2. $f+g = g+f$.
3. There exists a *neutral element* n in V such that $f+n = f$, for all f in V. This n is unique and denoted by 0.
4. For each f in V there exists a g in V such that $f+g=0$. This g is unique and denoted by $(-f)$.
5. $k(f+g) = kf+kg$.
6. $(c+k)f = cf+kf$.
7. $c(kf) = (ck)f$.
8. $1f = f$.

Figure: Definition of Linear Space

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Definition

Definition: Let X be a linear space over a scalar field $\mathbb{K}$. A real-valued function $||\cdot||: X \to \mathbb{R}$ defined on X is called a norm provided

1. $||x|| \geq 0 \quad \forall x \in X$ and $||x|| = 0$ iff $x = 0$
2. $||\alpha x|| = |\alpha|||x|| \quad \forall x \in X \, \forall \alpha \in \mathbb{K}$
3. $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in X$

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Definition

Definition: Let V be a linear space over $\mathbb{K}$.
If $U \subset V$ and U is also a linear space closed w.r.t binary operations defined for V, then we say that U is a linear subspace of V:

1. $0_U = 0_V \in U$
2. $u_1, u_2 \in U \to u_1 + u_2 \in U$
3. $\alpha \in \mathbb{K}$, $u \in U \to \alpha u \in U$

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Definition

Definition: Let $U_1, \ldots, U_m$ be linear subspaces of V.
The direct sum $U_1 \oplus \ldots \oplus U_m$ of $U_1, \ldots, U_m$ is a linear space s.t. any of its elements can be *uniquely* represented as

$$u_1 + \ldots + u_m, \quad u_i \in U_i,\ i = 1..m$$

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Definition

Definition: In a lin. space V, elements $v_1, \ldots, v_m$ form a basis if

1. $V = span(v_1, \ldots, v_m)$, and
2. $v_1, \ldots, v_m$ are linear independent.

Remark: If v_1, $v_2, \ldots, v_m \in V$ for a basis of V then $\forall x \in V$ there exists a unique representation

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_m v_m, \quad c1, \ldots, c_m \in \mathbb{K}.$$

Definition: The scalars $c_1, \ldots, c_m$ are called coordinates of $x \in V$ in the basis $v_1, \ldots,\ v_m$.
Examples: In class.

Definition: The number of elements in the basis is called the dimension of a linear space.

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Problem 1

Use the matrix A, along with its reduced row-echelon form, $rref(A)$, to answer the problems below:

$$A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 8 & 4 \end{bmatrix}, \qquad rref(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 points) What is the rank of A?

(b) (5 points) Give a formula that describes all solutions to $A\vec{x} = \vec{0}$.

(c) (4 points) Let V denote the set of all solutions from the previous part. Is V a subspace of $\mathbb{R}^4$? Explain.

(d) (5 points) Find a basis for the image of A. What is the dimension of $image(A)$?

[redacted]

(e) (4 points) Is $\begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix}$ in the image of A? Explain.

Answer

(a) (2 points) What is the rank of A?

Solution:
$\text{rank}(A) = 2$ because there are two leading ones in $rref(A)$.

(b) (5 points) Give a formula parametrizing all $\vec{x} \in \mathbb{R}^4$ satisfying $A\vec{x} = \vec{0}$.

Solution:

$$\vec{x} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Answer

(c) (4 points) Let V denote the set of all solutions from part (b). Is V a subspace of $\mathbb{R}^4$? Explain.

Solution:
Yes. The set of all solutions to $A\vec{x} = \vec{0}$ form the kernel, and the kernel of a matrix with four columns is a subspace of $\mathbb{R}^4$. Alternatively, the set of all solutions is the span of the vectors $(-1,-2,1,0)$ and $(-1,-1,0,1)$, and the span of a set of vectors is a subspace. Or, you could check the conditions defining a subspace directly: Let $\vec{v}_1 = (-1,-2,1,0)$ and $\vec{v}_2 = (-1,-1,0,1)$.

(i) If $s = t = 0$, then $\vec{0} = s\vec{v}_1 + t\vec{v}_2$, so $\vec{0}$ is an element of V.

(ii) If $\vec{x}_1$ and $\vec{x}_2$ are in V, there are scalars s_1, t_1, s_2, t_2 such that $\vec{x}_1 = s_1\vec{v}_1 + t_1\vec{v}_2$ and $\vec{x}_2 = s_2\vec{v}_1 + t_1\vec{v}_2$. Then $\vec{x}_1 + \vec{x}_2$ is equal to $(s_1+s_2)\vec{v}_1 + (t_1+t_2)\vec{v}_2$, and so by (b) is also in V.

(iii) If $\vec{x} = s\vec{v}_1 + t\vec{v}_2$ is any vector in V, and k is a scalar, then we have $k\vec{x} = ks\vec{v}_1 + kt\vec{v}_2$, and so $k\vec{x}$ is in V.

(d) (5 points) Find a basis for the image of A. What is the dimension of $image(A)$?

Solution:
$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$ is a basis for $image(A)$ because the columns of A span the image, but $rref(A)$ indicates the 3rd and 4th column are redundant. The dimension is 2 because the basis we have given has two elements.

Answer

(e) (4 points) Is $\begin{bmatrix}10\\1\\4\end{bmatrix}$ in the image of A? Explain.

Solution:

Yes: The vector $\begin{bmatrix}10\\1\\4\end{bmatrix}$ can be written as $4\begin{bmatrix}2\\0\\0\end{bmatrix}+\begin{bmatrix}2\\1\\4\end{bmatrix}$, and therefore is a linear combination of the two basis vectors of the image of A, which means that it is itself in the image of A.

Problem 2

(a) (5 points) Determine if the vectors below are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b) (6 points) Let $\vec{w}$ be the vector below, and let $\vec{v}_1$ and $\vec{v}_3$ be as in part (a). For which value(s) of b are the vectors $\vec{v}_1$, $\vec{w}$, and $\vec{v}_3$ linearly *dependent*?

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ b \\ 2 \end{bmatrix}$$

Answer

(a) (5 points) Determine if the vectors below are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution:

They are linearly independent because the reduced row echelon form of the matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in every column, none of $\vec{v}_1, \vec{v}_2$, or $\vec{v}_3$ are redundant.

Answer

(b) (6 points) Let $\vec{w}$ be the vector below, and let $\vec{v}_1$ and $\vec{v}_3$ be as in part (a). For which value(s) of b are the vectors $\vec{v}_1$, $\vec{w}$, and $\vec{v}_3$ linearly *dependent*?

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ b \\ 2 \end{bmatrix}$$

Solution:
We perform a few steps of row reduction:

$$[\vec{v}_1\ \vec{w}\ \vec{v}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & b & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & b-1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow[\frac{-1}{2}R_2]{R_4 + R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & b-1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From here we can see that the columns are linearly dependent exactly when rows 2 and 3 are scalar multiples of each other, and that this occurs only when $b = 0$.