vv214: Linear transformations.

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This week

Today

1. Review: basis and dimension.
2. Linear transformations and linear operators. Linear operators in finite dimensional linear spaces.
3. Linear transformations in 2D and 3D: rotations, reflections, projections.

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Next class

1. Composition of linear transformations.
2. Inverse linear transformations.

Basis

- Any spanning set of vectors can be reduced to a basis of a linear space.
- Any set of linear independent set of elements can be extended to a basis of a linear space.

Example: $\mathbb{R}^3$: $(2,3,4)$, $(9,6,8)$ are linearly independent. Consider the linearly independent vectors with vectors that span $\mathbb{R}^3$:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eliminating linearly dependent vectors from this system, you obtain basis elements:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is a basis for } \mathbb{R}^3$$

Basis

- If V has a finite basis and U is a linear subspace of V, then there exists a linear subspace W of V such that $V = U \oplus W$
  1. U must have a finite basis $v_1, \ldots, v_m$ as well.
  2. Let $w_1, \ldots, w_n$ span V. Consider the basis of U and the span of V together:
  $$v_1, \ldots, v_m,\ w_1, \ldots, w_n$$
  3. Eliminating linearly dependent elements, we obtain a basis for V:
  $$v_1, \ldots, v_m,\ u_1, \ldots, u_k, \quad u_i = w_j$$
  4. Denote $W = span(u_1, \ldots, u_k) \Rightarrow V = U + W$
  5. It remains to show that $U \cap W = \{0\}$. Let $x \in U \cap W \Rightarrow x \in U$, $x \in W$
  $$x = \alpha_1 v_1 + \ldots + \alpha_m v_m = \beta_1 u_1 + \ldots + \beta_k u_k$$
  $$\Rightarrow \alpha_1 v_1 + \ldots + \alpha_m v_m - \beta_1 u_1 - \ldots - \beta_k u_k = 0$$
  6. $v_1, \ldots, v_m,\ u_1, \ldots, u_k$ is a basis, i.e. linearly independent
  $$\Rightarrow \alpha_1 = \ldots = \alpha_m = \beta_1 = \ldots = \beta_k = 0 \Rightarrow x = 0$$
  7. $V = U + W$, $U \cap W = \{0\} \Rightarrow V = U \oplus W$

Examples

1. $U = span\,(2,3,4),\ (9,6,8))$ is a linear subspace of $\mathbb{R}^3$, and $(2,3,4),\ (9,6,8),\ (0,1,0))$ is a basis for $\mathbb{R}^3$

$$\text{Let } W = span\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} \Rightarrow V = U \oplus W$$

2. Let $M = \{p(t) \in P_2(\mathbb{R}) :\ p(1) = 0\}$.

$$p(1) = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 0 \Rightarrow a_0 = -a_1 - a_2$$

$M = \{p(t) = a_1(t-1) + a_2(t^2-1)\} \Rightarrow \{t-1,\ t^2-1\}$ is a basis for M

Consider $t-1, t^2-1, 1, t, t^2$ and eliminate linearly dependent elements:

$$t-1, t^2-1, 1 \qquad \text{is a basis for } P_2(\mathbb{R})$$

$\Rightarrow W = span(1) \Rightarrow P_2(\mathbb{R}) = M \oplus W$

Dimension

Definition: The number of elements in the basis is called the dimension of a linear space.

Examples:

1. $\dim \mathbb{R}^n = n$
   a. The vectors $\bar{e}_1 = (1, 0, \ldots, 0), \ldots, \bar{e}_n = (0, \ldots, 1) \in \mathbb{R}^n$ are linearly independent:
   $$\alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \ldots + \alpha_n \bar{e}_n = \bar{0}$$
   $$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0) \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$$
   $$\dim \mathbb{R}^n \geq n$$
   b. Consider arbitrary $n+1$ vectors in $\mathbb{R}^n$: $\bar{x}^1 = (x_1^1, \ldots, x_n^1)$, $\ldots, \bar{x}^n = (x_1^n \ldots, x_n^n)$, $\bar{x}^{n+1} = (x_1^{n+1} \ldots, x_n^{n+1})$
   $$\alpha_1 \bar{x}^1 + \ldots + \alpha_n \bar{x}^n + \alpha_{n+1} \bar{x}^{n+1} = \bar{0}$$
   This is a homogeneous system of n linear equations in $n+1$ variables $\Rightarrow \exists \alpha_i \neq 0 \Rightarrow$ any $n+1$ vectors are linearly dependent in $\mathbb{R}^n \Rightarrow \dim \mathbb{R}^n < n+1$
   c. $\dim \mathbb{R}^n \geq n$, $\dim \mathbb{R}^n < n+1 \Rightarrow \dim \mathbb{R}^n = n$

Dimension

Examples:

2. dim $C[a, b] = \infty$
   a. Let $n \in \mathbb{N}$ be arbitrary. The functions $1, x, x^2, \ldots, x^n$ are continuous on any $[a, b] \Rightarrow 1, x, x^2, \ldots, x^n \in C[a, b]$
   b. Check linear dependence/independence of $1, x, x^2, \ldots, x^n$

$$\alpha_0 \cdot 1 + \alpha_1 x + \ldots + \alpha_n x^n = 0$$

   This equation has n roots $x_1, \ldots, x_n$ for any constants $\alpha_0, \ldots, \alpha_n$. If we want to keep this identity for any x, then $\alpha_0 = \ldots = \alpha_n = 0 \Rightarrow 1, x, x^2, \ldots, x^n$ are linearly independent.
   c. But $n \in \mathbb{N}$ can be any $\Rightarrow$ there is a system of linearly independent elements in $C[a, b]$ which is not finite

$$\Rightarrow \dim C[a, b] = \infty$$

3. dim $\mathbb{M}_{2\times 2} = 4$

Dimension

Examples:

4. $\dim P_n(\mathbb{R}) = n+1$
   a. $\forall p(t) \in P_n(\mathbb{R}) \quad p(t) = a_0 \cdot 1 + a_1 t + a_2 t^2 + \ldots a_n t^n$

   $$\Rightarrow P_n(\mathbb{R}) = span(1, t, \ldots, t^n)$$

   b. The system $1, t, \ldots, t^n$ is linearly independent $\Rightarrow \dim P_n(\mathbb{R}) = n+1$
5. $\dim U \oplus W = \dim U + \dim W$
   a. It is enough to prove that

   $$\dim (U+W) = \dim U + \dim W - \dim (U \cap W)$$

   b. Let $u_1, \ldots, u_m$ be a basis of $U \cap W \Rightarrow$ we can extend it up to the basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U and up to the basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of W.
   c. $\dim U = m + j$, $\dim W = m + k$
   d. Show that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is the basis for $U + W \Rightarrow \dim (U+W) = m + j + k =$ $(m+j) + (m+k) - m = \dim U + \dim W - \dim (U \cap W)$

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- Is the standard basis $e_i = (0, \ldots, \underbrace{1}_{i\,th}, \ldots, 0)$ a "good" basis in $\mathbb{R}^n$?
- It gives us only the coordinates of a point. Can we form bases that keep other information?
- Let each coordinate represent brightness of a pixel in an image $\Rightarrow$ the brightness of the whole image is $x_1 + \ldots + x_n$, $x_1 - x_2 + x_3 - \ldots + (-1)^n x_n$ is the "jaggedness" of the image.
- $\mathbb{R}^2$: the vectors $v_1 = (1,1)$, $v_2 = (1,-1)$ are linearly independent $\Rightarrow \{v_1,\ v_2\}$ is the basis.

$$x = \frac{x_1 + x_2}{2} v_1 + \frac{x_1 - x_2}{2} v_2$$

The coordinates of $x = (x_1, x_2)$ in the basis $\mathfrak{B} = \{v_1,\ v_2\}$ are

$$x_{\mathfrak{B}} = \frac{x_1 + x_2}{2}, \ \frac{x_1 - x_2}{2}$$

Lagrange Interpolation

- You know that p is a polynomial and $deg(p) \leq n-1$. Also $p(\alpha_i) = b_i$, $i = 1, \dots, n$. Find p.
- The n polynomials

$$g_j = \frac{\prod_{i=1}^{n}(x-\alpha_i)}{x-\alpha_j}$$

  are linearly independent.

  $\Rightarrow g_j,\ j = 1, \dots, n$ form a basis of $P_{n-1}(\mathbb{R})$.

  $\Rightarrow \forall p \in P_{n-1}(\mathbb{R}) \quad \exists c_j \colon p = \sum_j c_j g_j$
- The coefficients c_j equal

$$c_i = \frac{p(\alpha_i)}{(\alpha_i-\alpha_1)\dots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\dots(\alpha_i-\alpha_n)}$$

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- Give (a_i, b_i) to your ith friend.

Simplest Coding-Decoding Transformations

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- ⋆ A transformation of the form $\bar{y} = A\bar{x}$ is called a linear transformation.

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- ⋆ B is the coefficient matrix of the inverse transformation.

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- The inverse transformation does not exist.
- What do you notice about the coefficient matrix

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}?$$

Linear Transformations

Definition: A function $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is called a linear transformation if there exists an $n \times m$ matrix A such that

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Remark: A linear transformation is a special case of a linear operator: Let V, U be linear spaces over $\mathbb{K}$. A map $T\colon V \to U$ is a linear operator if

1. $T(v_1 + v_2) = Tv_1 + Tv_2 \quad \forall\, v_1, v_2 \in V$
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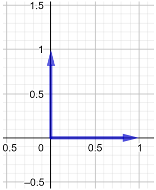
2. The identity transformation

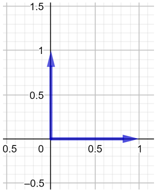
$$I\colon \mathbb{R}^n \to \mathbb{R}^n,\ I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Linear Transformations

3 Let $T\bar{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{x}$



Linear Transformations



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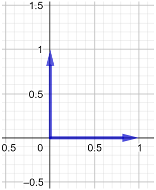

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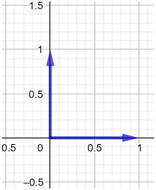

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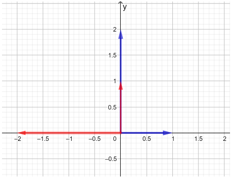

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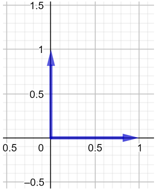

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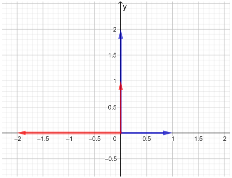
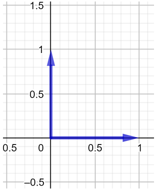


The rotation through $\frac{\pi}{2}$ in the counterclockwise direction.

Linear Transformations



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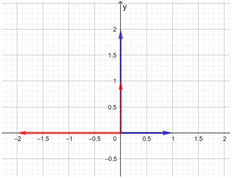

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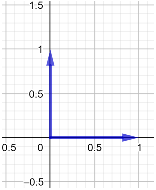

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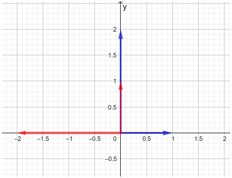

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$$\Rightarrow \sqrt{x_1^2 + x_2^2} = \sqrt{(-x_1)^2 + x_2^2}$$

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$$\begin{cases} ax_1 + bx_2 = & y_1 \\ cx_1 + dx_2 = & y_2 \end{cases}$$

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$$\bar{x} = B\bar{y}$$

Linear Transformations

Definition: For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the quantity $ad - bc$ is called the determinant of A:

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Lemma: Let $A = (\bar{a}_1 \quad \bar{a}_2)$ be a non-zero matrix. Then

1. $\det A = |\bar{a}_1| \sin\theta |\bar{a}_2|$ where θ is oriented from $\bar{a}_1$ to $\bar{a}_2$, $-\pi < \theta < \pi$
2. The area of the parallelogram spanned by $\bar{a}_1$, $\bar{a}_2$ is $\det A$.
3. $\det A = 0 \Rightarrow \bar{A}_1 || \bar{a}_2$

Problems

1. Let $A = \begin{pmatrix} 1 & \frac{1}{11} \\ 11 & 1 \end{pmatrix}$ shows that 1 Canadian dollar is worth 11 South African rand. After a trip you have C\$100 and ZAR 2200 in your pocket.

$$\bar{x} = \begin{pmatrix} 100 \\ 2200 \end{pmatrix}$$

What is the practical significance of the two components of the vector $A\bar{x}$? When does a system $A\bar{x} = \bar{b}$ have solutions?

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2. The cross product of two vectors

$$\bar{x} = (x_1, x_2, x_3),\ \bar{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$$

is given by

$$\bar{x} \times \bar{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

Let $\bar{v} = (v_1, v_2, v_3)$ be fixed and $T\bar{x} = \bar{v} \times \bar{x}$.
Is T a linear transformation?

Problems

3. Consider an arbitrary vector $\bar{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.
Is the transformation $T\bar{v} = (\bar{x},\, \bar{v})$ linear? $(\cdot, \cdot)$ denotes the dot product. If so, find the matrix of T. Show that the converse is also true: for a linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}$, there exists $\bar{v} \in \mathbb{R}^3$ such that $T\bar{v} = (\bar{x},\, \bar{v})$.

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4. Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

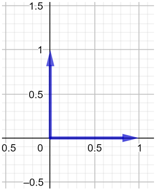
$$\bar{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \bar{y}_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\bar{x}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \to \bar{y}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

What is the matrix A^{-1} of the inverse transformation T^{-1}?
How can one use the representation $A^{-1} = (T^{-1}\bar{e}_1,\ T^{-1}\bar{e}_2)$ to find the matrix of T^{-1}?

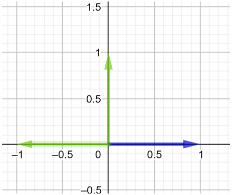
Examples of linear transformations in $\mathbb{R}^n$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



the identity transformation

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



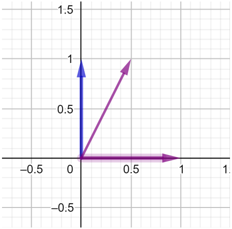
the reflexion about the vertical axis

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ the clockwise rotation through $\frac{\pi}{2}$

Examples of linear transformations in $\mathbb{R}^n$

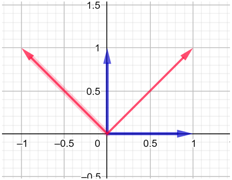
$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the orthogonal projection onto the horizontal axis

$A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$



the horizontal shear

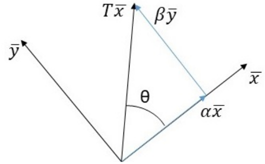
$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$



the scaling by the factor $\sqrt{2}$ and the rotation through $\frac{\pi}{4}$

Rotations

Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that rotates any vector $\bar{x}$ through a fixed angle θ in the counterclockwise direction.



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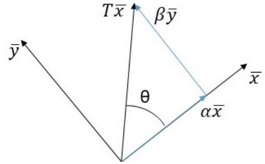

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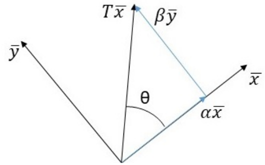

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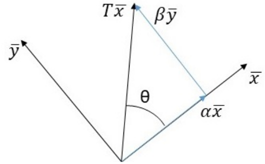

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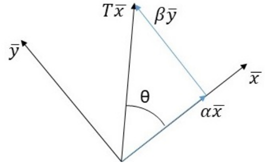

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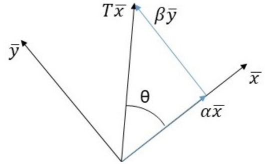

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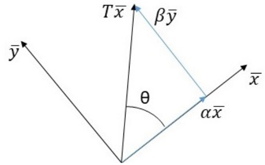

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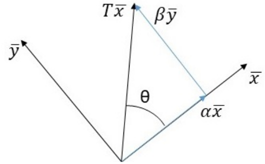

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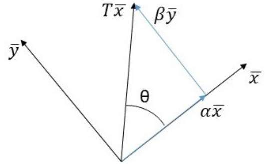

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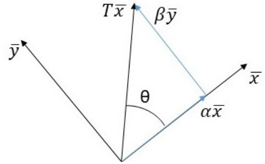

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$$T\bar{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow T \text{ is linear}$$

Rotations

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Example: The matrix of the counterclockwise rotation through $\frac{\pi}{4}$ is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Scaling

For any positive scalar k the matrix

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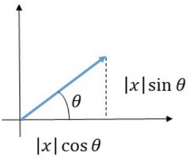
- $k > 1$ enlargement
- $0 < k < 1$ shrinking
- $k = -1$ the rotation through π
- $-1 < k < 0$ shrinking and rotation through π
- $k < -1$ enlargement and rotation through π

Rotation combined with scaling



$$\bar{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ||\bar{x}|| \cos\theta \\ ||\bar{x}|| \sin\theta \end{pmatrix}$$

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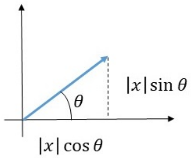


$$\bar{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ||\bar{x}|| \cos\theta \\ ||\bar{x}|| \sin\theta \end{pmatrix}$$

The matrix

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ||\bar{x}|| \cos\theta & -||\bar{x}|| \sin\theta \\ ||\bar{x}|| \sin\theta & ||\bar{x}|| \cos\theta \end{pmatrix}$$

$$= ||\bar{x}|| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

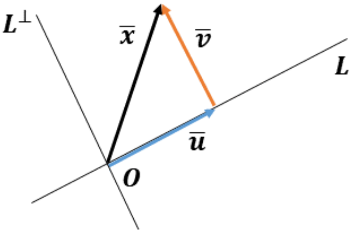
represents the rotation combined with scaling.

Orthogonal Projections in $\mathbb{R}^2$

Let L be a line running through the origin

$$\bar{x} = \bar{u} + \bar{v}, \quad \bar{u} || L, \quad \bar{v} \perp L$$

$$T\bar{x} = proj_L \bar{x} = \bar{u}, \quad \bar{v} = proj_{L^\perp} \bar{x}$$



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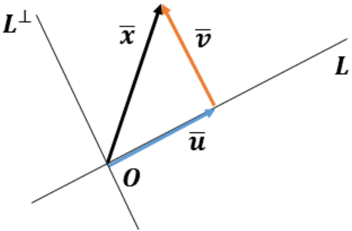

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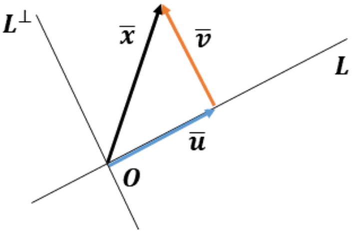

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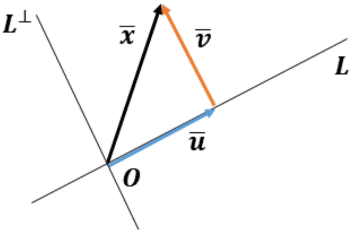

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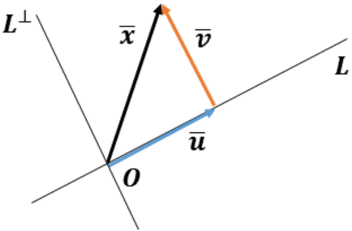

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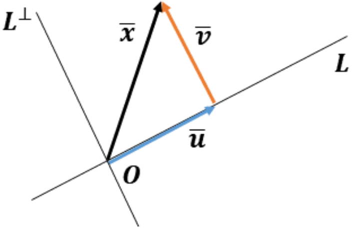

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If $||\bar{w}|| = 1$, then $(\bar{w}, \bar{w}) = 1$ and the matrix of the orthogonal projection becomes

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Example: The orthogonal projection onto the line $L = span\{(-1, 3)\}$ is given by the matrix

$$A = \frac{1}{(-1)^2 + 3^2} \begin{pmatrix} (-1)^2 & -1 \cdot 3 \\ -1 \cdot 3 & 3^2 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.3 \\ -0.3 & 0.9 \end{pmatrix}$$

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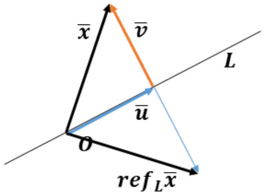

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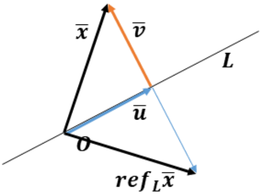

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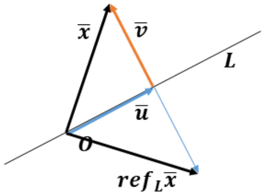

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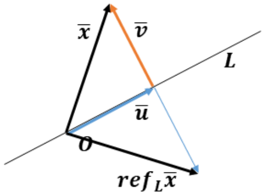

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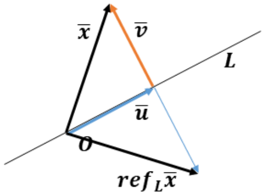

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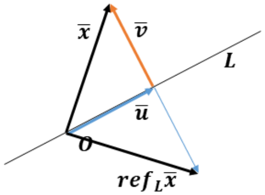

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$$A = \begin{pmatrix} 2w_1^2 - 1 & 2w_1w_2 \\ 2w_1w_2 & 2w_2^2 - 1 \end{pmatrix}, \; ||\bar{w}|| = 1, \; \bar{w}||L$$

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The reflection matrix is of the form

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \; a^2+b^2 = 1$$

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Let L be a line in $\mathbb{R}^3$ running through the origin.

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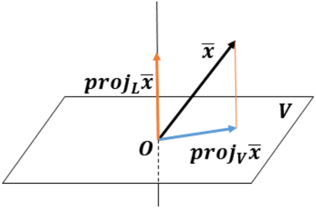
If $V = L^{\perp}$ is a plane through the origin perpendicular to L then



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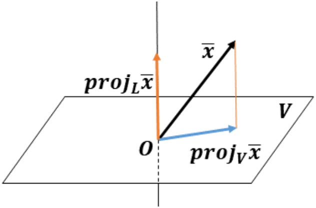

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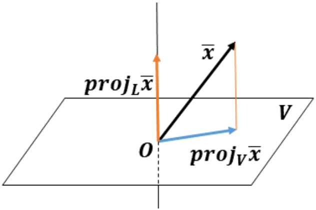

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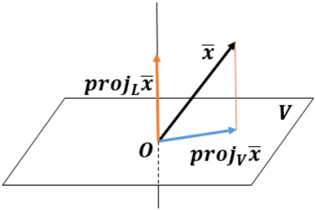

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Examples

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$$proj_L \bar{x} = \left(\frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3\right)\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

$$= \left(\frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3, \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{2}{9}x_3, \frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3\right)$$

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e. the rotation about the z-axis through $\pi/4$ turning the positive x-axis towards the positive y-axis

f. the orthogonal projection onto the line $y = x$ on the xy-plane

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The reflection about the line L with the direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ the rotation through} -\frac{\pi}{2}$$

Inverse Linear Transformations

Let X, Y be linear spaces. A linear operator $T\colon X \to Y$ is invertible if it is one-to-one and onto, i.e. the equation $Tx = y$ has a unique solution.
The inverse operator T^{-1} satisfies $T^{-1}y = x$, i.e there exists a unique x such that $Tx = y$.

$$T^{-1}(Tx) = y \quad \text{and} \quad T(T^{-1}y) = y \quad \forall x \in X \forall y \in Y$$

Remarks:

1. $(T^{-1})^{-1} = T$
2. Let $X, Y = \mathbb{R}^n$. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the system $A\bar{x} = \bar{y}$ has a unique solution

$$\iff \textit{rank}\, A = n \iff \textit{rref}\, A = I_n$$

Definition: A square matrix A is invertible if the linear transformation $T\bar{x} = A\bar{x}$ is invertible.

Inverse Linear Transformations

$$A = \frac{1}{5}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix} \Rightarrow A\bar{x} = I\bar{y}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$I\bar{y} = A^{-1}\bar{x} \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 2 \\ -7 & 3 & -1 \end{pmatrix}$$

Inverse Linear Transformations

1. Let $A_{n\times n}$. If $\exists A^{-1}$, the a system $A\bar{x} = \bar{b}$ has a unique solution.
2. $(AB)^{-1} = B^{-1}A^{-1}$