Recitation Class 4
Linear Algebra

Yuzhou Li

UM-SJTU Joint Institute

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Definition

Coordinates in a subspace of $\mathbb{R}^n$

Consider a basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ of a subspace V of $\mathbb{R}^n$. By Theorem 3.2.10, any vector $\vec{x}$ in V can be written uniquely as

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m.$$

The scalars $c_1, c_2, \ldots, c_m$ are called the $\mathfrak{B}$-*coordinates* of $\vec{x}$, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

is the $\mathfrak{B}$-*coordinate vector* of $\vec{x}$, denoted by $[\vec{x}]_{\mathfrak{B}}$. Thus

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{means that } \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m.$$

Note that

$$\vec{x} = S\,[\vec{x}]_{\mathfrak{B}}, \quad \text{where } S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}, \text{ an } n \times m \text{ matrix.}$$

Linearity of Coordinates

If $\mathfrak{B}$ is a basis of a subspace V of $\mathbb{R}^n$, then

a. $\left[\vec{x}+\vec{y}\right]_{\mathfrak{B}} = \left[\vec{x}\right]_{\mathfrak{B}} + \left[\vec{y}\right]_{\mathfrak{B}}$, for all vectors $\vec{x}$ and $\vec{y}$ in V, and

b. $\left[k\vec{x}\right]_{\mathfrak{B}} = k\left[\vec{x}\right]_{\mathfrak{B}}$, for all $\vec{x}$ in V and for all scalars k.

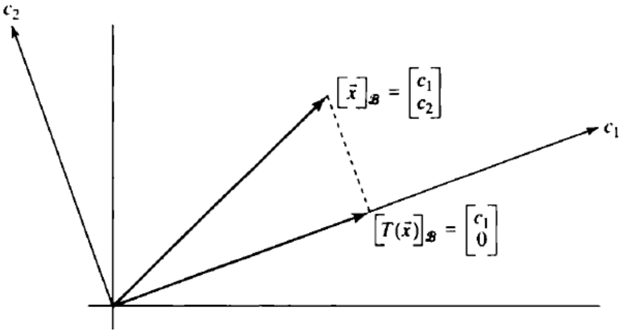
Exercise

Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

a. If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathfrak{B}}$.

b. If $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{y}$.

Example


c_2
$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
c_1
$[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

Example

The matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ that transforms $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ into $[T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ is called the $\mathfrak{B}$-matrix of T:

$$[T(\vec{x})]_{\mathfrak{B}} = B\,[\vec{x}]_{\mathfrak{B}}.$$

We can organize our work in a diagram as follows:

$$\begin{tikzcd}[column sep=huge]
\vec{x} = \overbrace{c_1\vec{v}_1}^{\text{in } L} + \overbrace{c_2\vec{v}_2}^{\text{in } L^\perp} \arrow[r, "T"'] \arrow[d] & T(\vec{x}) = c_1\vec{v}_1 \arrow[d] \\
[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \arrow[r, "B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"'] & [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}.
\end{tikzcd}$$

Easiest way for finding matrix B

Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $\mathfrak{B}$ of $\mathbb{R}^n$. The $n \times n$ matrix B that transforms $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of T:

$$[T(\vec{x})]_{\mathfrak{B}} = B\,[\vec{x}]_{\mathfrak{B}},$$

for all $\vec{x}$ in $\mathbb{R}^n$. We can construct B column by column as follows: If $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$, then

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix}.$$

We need to verify that the columns of B are $[T(\vec{v}_1)]_{\mathfrak{B}}, \ldots, [T(\vec{v}_n)]_{\mathfrak{B}}$. Let $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Using first the linearity of T and then the linearity of coordinates (Theorem 3.4.2), we find that

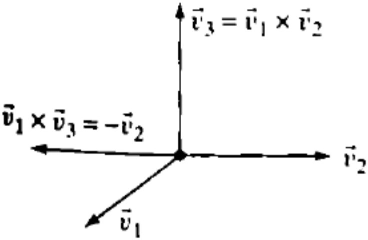
$$T(\vec{x}) = c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)$$

and

$$[T(\vec{x})]_{\mathfrak{B}} = c_1 [T(\vec{v}_1)]_{\mathfrak{B}} + \cdots + c_n [T(\vec{v}_n)]_{\mathfrak{B}}$$

$$= \underbrace{\begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix}}_{B} [\vec{x}]_{\mathfrak{B}}.$$

Exercise (You can try both ways)



Find the $\mathfrak{B}$-matrix B of the linear transformation $T(\vec{x}) = \vec{v}_1 \times \vec{x}$.

Answers: Way 1

$$
\begin{tikzcd}[column sep=huge]
\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \arrow[r, "T"] \arrow[d] & \begin{aligned} T(\vec{x}) &= \vec{v}_1 \times (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \\ &= c_1(\vec{v}_1 \times \vec{v}_1) + c_2(\vec{v}_1 \times \vec{v}_2) + c_3(\vec{v}_1 \times \vec{v}_3) \\ &= c_2\vec{v}_3 - c_3\vec{v}_2 \end{aligned} \arrow[d] \\
[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \arrow[r, swap, "B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}"] & [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}
\end{tikzcd}
$$

Alternatively, we can construct B column by column,

$$
B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & [T(\vec{v}_3)]_{\mathfrak{B}} \end{bmatrix}.
$$

Answers: Way 2

$$[T(\vec{v}_1)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(\vec{v}_2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [T(\vec{v}_3)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & [T(\vec{v}_3)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Use B-matrix to find standard matrix A of T

$$
\begin{tikzcd}
\vec{x} \arrow[r, "A"] & T(\vec{x}) \\
{[\vec{x}]_{\mathfrak{B}}} \arrow[u, "S"'] \arrow[r, "B"] & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, "S"']
\end{tikzcd}
$$

Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$. Let B be the $\mathfrak{B}$-matrix of T, and let A be the standard matrix of T (such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$). Then

$$AS = SB, \quad B = S^{-1}AS, \quad \text{and} \quad A = SBS^{-1}, \quad \text{where} \quad S = \begin{bmatrix} \vec{v}_1 & & \vec{v}_n \end{bmatrix}. \blacksquare$$

Back to the first exercise, we can get A as we know the value of S and B.

Similar matrices

Consider two $n \times n$ matrices A and B. We say that A is similar to B if there exists an invertible matrix S such that

$$AS = SB, \quad \text{or} \quad B = S^{-1}AS.$$

Thus two matrices are similar if they represent the same linear transformation with respect to different bases.

a. An $n \times n$ matrix A is similar to A itself (*reflexivity*).

b. If A is similar to B, then B is similar to A (*symmetry*).

c. If A is similar to B and B is similar to C, then A is similar to C (*transitivity*).

More general case: functions, matrices...

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "L_{\mathfrak{B}}"'] & V \arrow[d, "L_{\mathfrak{B}}"] \\
\mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
f \arrow[r, "T"] \arrow[d, "L_{\mathfrak{B}}"'] & T(f) \arrow[d, "L_{\mathfrak{B}}"] \\
[f]_{\mathfrak{B}} \arrow[r, "B"] & [T(f)]_{\mathfrak{B}}
\end{tikzcd}
$$

We can write B in terms of its columns. Suppose that $\mathfrak{B} = (f_1, \ldots, f_i, \ldots, f_n)$. Then

$$\left[T(f_i)\right]_{\mathfrak{B}} = B\left[f_i\right]_{\mathfrak{B}} = B\vec{e}_i = (i\text{th column of } B)$$

Exercise

Consider the linear transformation

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{from } \mathbb{R}^{2\times 2} \text{ to } \mathbb{R}^{2\times 2}.$$

a. Find the matrix B of T with respect to the standard basis $\mathfrak{B}$ of $\mathbb{R}^{2\times 2}$.

b. Find bases of the image and kernel of B.

c. Find bases of the image and kernel of T, and thus determine rank and nullity of transformation T.

Answer

a. For the sake of variety, let us find B by means of a diagram.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}$$

$$\begin{tikzcd} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \arrow[r, "T"] \arrow[d, "L_{\mathfrak{B}}"'] & \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} \arrow[d, "L_{\mathfrak{B}}"] \\ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \arrow[r, "B"'] & \begin{bmatrix} c \\ d-a \\ 0 \\ -c \end{bmatrix} \end{tikzcd}$$

We see that

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Answer

b. Note that columns $\vec{v}_2$ and $\vec{v}_4$ of B are redundant, with $\vec{v}_2 = \vec{0}$ and $\vec{v}_4 = -\vec{v}_1$, or $\vec{v}_1 + \vec{v}_4 = \vec{0}$. Thus the nonredundant columns

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{form a basis of } \operatorname{im}(B),$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{is a basis of } \ker(B).$$

Answer

c. We apply $L_{\mathfrak{B}}^{-1}$ to transform the vectors we found in part (b) back into $\mathbb{R}^{2\times 2}$, the domain and target space of transformation T:

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{is a basis of } \operatorname{im}(T),$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a basis of } \ker(T).$$

Thus $\operatorname{rank}(T) = \dim(\operatorname{im} T) = 2$ and $\operatorname{nullity}(T) = \dim(\ker T) = 2$.

Change of basis

Consider two bases $\mathfrak{A}$ and $\mathfrak{B}$ of an n-dimensional linear space V. Consider the linear transformation $L_{\mathfrak{A}} \circ L_{\mathfrak{B}}^{-1}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, with standard matrix S, meaning that $S\vec{x} = L_{\mathfrak{A}}\left(L_{\mathfrak{B}}^{-1}(\vec{x})\right)$ for all $\vec{x}$ in $\mathbb{R}^n$. This invertible matrix S is called the *change of basis matrix* from $\mathfrak{B}$ to $\mathfrak{A}$, sometimes denoted by $S_{\mathfrak{B}\to\mathfrak{A}}$. See the accompanying diagrams. Letting $f = L_{\mathfrak{B}}^{-1}(\vec{x})$ and $\vec{x} = [f]_{\mathfrak{B}}$, we find that

$$[f]_{\mathfrak{A}} = S\,[f]_{\mathfrak{B}}, \quad \text{for all } f \text{ in } V.$$

If $\mathfrak{B} = (b_1, \ldots, b_i, \ldots, b_n)$, then

$$[b_i]_{\mathfrak{A}} = S\,[b_i]_{\mathfrak{B}} = S\vec{e}_i = (i\text{th column of } S),$$

so that

$$S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix} [b_1]_{\mathfrak{A}} & & [b_n]_{\mathfrak{A}} \end{bmatrix}$$

$$\begin{tikzcd} & \mathbb{R}^n \\ V \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\ & \mathbb{R}^n \arrow[uu, "S"'] \end{tikzcd} \qquad \begin{tikzcd} & {[f]_{\mathfrak{A}}} \\ f \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\ & {[f]_{\mathfrak{B}}} \arrow[uu, "S"'] \end{tikzcd}$$

Exercise

Let V be the subspace of C^∞ spanned by the functions e^x and e^{-x}, with the bases $\mathfrak{A} = (e^x, e^{-x})$ and $\mathfrak{B} = (e^x + e^{-x}, e^x - e^{-x})$. Find the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{A}}$.

Answer

$$S = \begin{bmatrix} [e^x + e^{-x}]_{\mathfrak{A}} & [e^x - e^{-x}]_{\mathfrak{A}} \end{bmatrix}.$$

$$e^x + e^{-x} = 1 \cdot e^x + 1 \cdot e^{-x} \qquad e^x - e^{-x} = 1 \cdot e^x + (-1) \cdot e^{-x}$$

$$[e^x + e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [e^x - e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S_{\mathfrak{B} \to \mathfrak{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Full version

$$
\begin{tikzcd}
{[f]_{\mathfrak{A}}} \arrow[rrr, "A"] & & & {[T(f)]_{\mathfrak{A}}} \\
& f \arrow[ul, "L_{\mathfrak{A}}"'] \arrow[dl, "L_{\mathfrak{B}}"] \arrow[r, "T"] & T(f) \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\
{[f]_{\mathfrak{B}}} \arrow[uu, "S"] \arrow[rrr, "B"'] & & & {[T(f)]_{\mathfrak{B}}} \arrow[uu, "S"']
\end{tikzcd}
$$

Let V be a linear space with two given bases $\mathfrak{A}$ and $\mathfrak{B}$. Consider a linear transformation T from V to V, and let A and B be the $\mathfrak{A}$- and the $\mathfrak{B}$-matrix of T, respectively. Let S be the change of basis matrix from $\mathfrak{B}$ to $\mathfrak{A}$. Then A is similar to B, and

$$AS = SB \quad \text{or} \quad A = SBS^{-1} \quad \text{or} \quad B = S^{-1}AS. \qquad \blacksquare$$

Exercise

As in Example 5, let V be the linear space spanned by the functions e^x and e^{-x}, with the bases $\mathfrak{A} = (e^x, e^{-x})$ and $\mathfrak{B} = (e^x + e^{-x}, e^x - e^{-x})$. Consider the linear transformation $D(f) = f'$ from V to V.

a. Find the $\mathfrak{A}$-matrix A of D.

b. Use part (a), Theorem 4.3.5, and Example 5 to find the $\mathfrak{B}$-matrix B of D.

c. Use Theorem 4.3.2 to find the $\mathfrak{B}$-matrix B of D in terms of its columns.

Answer

a. Let's use a diagram. Recall that $(e^{-x})' = -e^{-x}$, by the chain rule.

$$\begin{array}{ccc} ae^x + be^{-x} & \xrightarrow{D} & ae^x - be^{-x} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \end{bmatrix} & \xrightarrow[A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}]{} & \begin{bmatrix} a \\ -b \end{bmatrix} \end{array}$$

b. In Example 5 we found the change of basis matrix $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ from $\mathfrak{B}$ to $\mathfrak{A}$. Now

$$B = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

c. $B = \left[\begin{array}{cc} [D(e^x + e^{-x})]_{\mathfrak{B}} & [D(e^x - e^{-x})]_{\mathfrak{B}} \end{array} \right]$

$$= \left[\begin{array}{cc} [e^x - e^{-x}]_{\mathfrak{B}} & [e^x + e^{-x}]_{\mathfrak{B}} \end{array} \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Isomorphism

An invertible linear transformation T is called an *isomorphism*. We say that the linear space V is isomorphic to the linear space W if there exists an isomorphism T from V to W.

Properties

a. A linear transformation T from V to W is an isomorphism if (and only if) $\ker(T) = \{0\}$ and $\mathrm{im}(T) = W$.

In parts (b) through (d), the linear spaces V and W are assumed to be finite dimensional.

b. If V is isomorphic to W, then $\dim(V) = \dim(W)$.

c. Suppose T is a linear transformation from V to W with $\ker(T) = \{0\}$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

d. Suppose T is a linear transformation from V to W with $\mathrm{im}(T) = W$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

To prove a transformation is a isomorphism, first prove it is linear!

Exercise

Show that the transformation

$$T(A) = S^{-1}AS \quad \text{from } \mathbb{R}^{2\times 2} \text{ to } \mathbb{R}^{2\times 2}$$

is an isomorphism, where $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Answer

We need to show that T is a linear transformation, and that T is invertible.

Let's check the linearity first:

$$T(A_1 + A_2) = S^{-1}(A_1 + A_2)S = S^{-1}(A_1S + A_2S) = S^{-1}A_1S + S^{-1}A_2S$$

equals

$$T(A_1) + T(A_2) = S^{-1}A_1S + S^{-1}A_2S,$$

and

$$T(kA) = S^{-1}(kA)S = k(S^{-1}AS) \quad \text{equals} \quad kT(A) = k(S^{-1}AS).$$

The most direct way to show that a function is invertible is to exhibit the inverse. Here we need to solve the equation $B = S^{-1}AS$ for input A. We find that $A = SBS^{-1}$, so that T is indeed invertible. The inverse transformation is

$$T^{-1}(B) = SBS^{-1}. \qquad \blacksquare$$

Orthogonality

a. Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are called *perpendicular* or *orthogonal*[1] if $\vec{v} \cdot \vec{w} = 0$.

b. The *length* (or magnitude or norm) of a vector $\vec{v}$ in $\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

c. A vector $\vec{u}$ in $\mathbb{R}^n$ is called a *unit vector* if its length is 1, (i.e., $\|\vec{u}\| = 1$, or $\vec{u} \cdot \vec{u} = 1$).

The vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ in $\mathbb{R}^n$ are called *orthonormal* if they are all unit vectors and orthogonal to one another:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

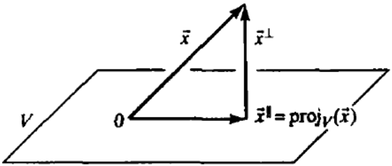
a. Orthonormal vectors are linearly independent.

b. Orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.

Orthogonal projection

The vector $\vec{x}^{\parallel}$ is called the *orthogonal projection* of $\vec{x}$ onto V, denoted by $\operatorname{proj}_V(\vec{x})$. See Figure 4.

The transformation $T(\vec{x}) = \operatorname{proj}_V(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is linear.



If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$, then

$$\operatorname{proj}_V(\vec{x}) = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m.$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Consider an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_n$ of $\mathbb{R}^n$. Then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n.$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Orthogonal complement

Consider a subspace V of $\mathbb{R}^n$. The *orthogonal complement* $V^\perp$ of V is the set of those vectors $\vec{x}$ in $\mathbb{R}^n$ that are orthogonal to all vectors in V:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\}.$$

Note that $V^\perp$ is the kernel of the orthogonal projection onto V.

Consider a subspace V of $\mathbb{R}^n$.

a. The orthogonal complement $V^\perp$ of V is a subspace of $\mathbb{R}^n$.

b. The intersection of V and $V^\perp$ consists of the zero vector alone: $V \cap V^\perp = \{\vec{0}\}$.

c. $\dim(V) + \dim(V^\perp) = n$

d. $(V^\perp)^\perp = V$