vv214: Row Rank=Column Rank.

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Isomorphism

Definition: Let V, W be linear spaces.
A *linear* operator $T\colon V \to W$ is called an isomorphism if T is bijective, that is T^{-1} exists.
Examples:

1. $T\colon M_{2\times 2} \to M_{2\times 2},\ T(A) = S^{-1}AS$ with $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2. $L : M_{2\times 2} \to \mathbb{R}^4,\ L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. To generalize 2, consider a linear space with a *finite* basis $V = span(f_1, f_2 \ldots, f_n)$ and define the coordinate transformation $L_{\mathfrak{B}}\colon V \to \mathbb{R}^n$ is $L(f) = f_{\mathfrak{B}}$

Isomorphism

Theorem: Any n-dimensional linear space V is isomorphic to $\mathbb{R}^n$.
Rank-Nullity Theorem: $\dim V = \dim KerT + \dim ImT$
Properties of isomorphisms:

1. A linear operator $T\colon V \to W$ is an isomorphism if and only if

$$KerT = \{0\} \quad \text{and} \quad ImT = W$$

- If $KerT = \{0\}$, $ImT = W$, apply the rank-nullity theorem $\Rightarrow \dim V = \dim W$
- Let $\dim V = \dim W = n \Rightarrow \exists\, v_1, \ldots, v_n$ (basis of V) and $\exists\, w_1, \ldots, w_n$ (basis of W).
  Define an operator $T\colon V \to W$ by

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$$

  $\Rightarrow T$ is linear and one-to-one and onto (isomorphism).

2. If V is isomorphic to W then $\dim V = \dim W$.
3. A linear operator $T\colon V \to W$ with $KerT = \{0\}$ is an isomorphism if $\dim V = \dim W$.
4. A linear operator $T\colon V \to W$ with $ImT = V$ is an isomorphism if $\dim V = \dim W$.

More Examples

1. Let $V = span(\cos x, \sin x) = \{a \cos x + b \sin x,\ a, b \in \mathbb{R}\} \subset C^\infty$

$$T\colon V \to V,\ T(f) = 3f + 2f' - f''$$

T is an isomorphism

2. $T\colon V \to V,\ T(x_1, x_2, \ldots, x_n, \ldots) = (x_1,\ x_3,\ x_5, \ldots,)$

T is not an isomorphism

3. Let $Z_n = \{p(t) \in P_n(\mathbb{R})\colon\ p(0) = 0\}$ and

$$T\colon P_{n-1} \to Z_n,\ Tp(t) = \int_0^t p(x)\,dx$$

T is an isomorphism

More Examples

4. Define the operations

$$x \oplus y = xy, \quad k \odot x = x^k \quad \forall x \in \mathbb{R}_+$$

Let $T\colon \mathbb{R}_+ \to \mathbb{R}$, $Tx = \ln x$
T is an isomorphism
5. Can one define binary operations on $\mathbb{R}$ and make $\dim(\mathbb{R}^2) = 1$?

$$\bar{x} \oplus \bar{y} = T^{-1}(T\bar{x} + T\bar{y}), \quad k \odot \bar{x} = T^{-1}(kT\bar{x})$$

for any invertible $T\colon \mathbb{R}^2 \to \mathbb{R}$
6. If S is the set of all students in your linear algebra class. Can one define operations on S that make S into a real linear space?
No.

Row-Rank=Column-Rank

Goal: to prove that the number of linearly independent columns of a matrix A is the same as the number of linearly independent rows $\Rightarrow$ the rank of a matrix is the number of linearly independent rows or columns!!!

$$\boxed{\text{the row-rank } = \text{ the column rank}}$$

How to prove:

1. Linear functionals
2. The dual space V' (of all linear functionals defined on V)
3. Dual basis
4. Dual map
5. Annihilator $U^o = \{\varphi \in V' \colon \varphi(u) = 0 \, \forall u \in U\}$ of a linear subspace
6. $U \subset V \Rightarrow \dim U + \dim U^o = \dim V$

Linear Functionals and Dual Basis

Definition: A linear operator $f\colon V \to \mathbb{R}$ is called a linear functional.

Examples:

1. $f\colon \mathbb{R}^3 \to \mathbb{R} \quad f(x,y,z) = 4x - 5y + 2z$
2. $x\colon C[a,b] \to \mathbb{R} \quad x(t) = \int_a^b x(t)\,dt$

We shall write $L(V,W)$ to denote the linear space of all linear operators from V to W.

Definition: Dual Space: $V' = L(V,\mathbb{R})$

$$\dim V' = \dim V \dim \mathbb{R} \Rightarrow \dim V' = \dim V$$

Let $v_1,\ldots,v_n$ be a basis for V. The dual basis of $v_1,\ldots,v_n$ is

$$\{\varphi_1,\ldots,\varphi_n\} \in V'\colon \varphi_j(v_k) = \left\{\begin{array}{ll} 1 & k = j \\ 0 & k \neq j \end{array}\right.$$

Example: $e_1,\ldots,e_n \in \mathbb{R}^n \Rightarrow$ let $\varphi_i(x_1,\ldots,x_n) = x_j$. Then $\varphi_j(e_k) = \left\{\begin{array}{ll} 1 & k = j \\ 0 & k \neq j \end{array}\right. \Rightarrow \{\varphi_i\}$ is the dual basis.

Dual Map

Remark: Dual basis is indeed a basis. Let $\dim V = m \Rightarrow \dim V' = m$. Then $\varphi_1, \ldots, \varphi_m$ are linearly independent:

$$\alpha_1\varphi_1 + \ldots \alpha_m\varphi_m = 0 \Rightarrow (\alpha_1\varphi_1 + \ldots \alpha_m\varphi_m)(v_k) = 0$$

$$\Rightarrow \alpha_k\varphi_k(v_k) = 0 \Rightarrow \alpha_k = 0 \quad \forall k = 1, \ldots, m$$

Definition: Dual map: $T' \colon W' \to V' \quad T'(\varphi) = \varphi \circ T \quad \forall \varphi \in W'$
Dual map is well-defined:

$$T'(\varphi)(\underbrace{v}_{\in V}) = \varphi \circ T(v) = \varphi(\underbrace{Tv}_{\in W})$$

Pick a functional φ defined on W and then $T'(\varphi)$ is a functional defined on V.
Example: Consider $T \colon P_n \to P_n$, $Tp(t) = p'(t)$.
Let $\varphi \colon P_n \to \mathbb{R}$, $\varphi(p) = p(3)$

$$T'(\varphi) = \varphi \circ T \Rightarrow T'(\varphi)(p) = \varphi \circ T(p) = \varphi(Tp) = \varphi(p') = p'(3)$$

Matrix of the Dual Map

Let $T\colon V \to W$, $\dim V = m$, $\dim W = n$ with the bases $v_1, \dots, v_m$, $w_1, \dots, w_n \Rightarrow T$ is defined by a matrix $A_{n\times m} = (a_{ij})$.
Then $T'\colon W' \to V'$, $\dim W' = n$, $\dim V' = m \Rightarrow T$ is given by a matrix $B_{m\times n} = (b_{ij})$.
Let $\psi_1, \dots, \psi_n$ be a basis for W', and $\varphi_1, \dots, \varphi_m$ be a basis for V'.

$$\forall j = 1, \dots, n \quad T'(\psi_j) \in V' \Rightarrow T'(\psi_j) = \sum_{r=1}^{m} b_{rj}\varphi_r$$

But $T'(\psi_j) = \psi_j \circ T$

$$\Rightarrow \psi_j \circ T(v_k) = T'(\psi_j)(v_k) = \sum_{r=1}^{m} b_{rj}\varphi_r(v_k) = b_{kj}$$

On another hand,

$$\psi_j \circ T(v_k) = \psi_j(Tv_k) = \psi_j\left(\sum_{r=1}^{n} a_{rk}w_r\right) = a_{jk}$$

$$\Rightarrow b_{kj} = a_{jk} \quad \forall j = 1, .., n,\ k = 1..m \Rightarrow \boxed{B = A^T}$$

Annihilators

Definition: An annihilator of a linear subspace $U \subset V$ is

$$U^\circ = \{\varphi \in V' \colon \varphi(u) = 0 \, \forall u \in U\}$$

Example: Let $U = \{p(t) \in P_n(\mathbb{R}) \colon p(t) = t^2 g(t) \, \forall g(t) \in P_n(\mathbb{R})\}$

$$\Rightarrow U^\circ = \{\varphi \in P_n'(\mathbb{R}) \colon \varphi(p) = 0\}$$

If $\varphi(p) = p'(0)$, then $\varphi(p) = 0 \, \forall p \Rightarrow \varphi \in U^\circ$

Lemma: $\dim U + \dim U^\circ = \dim V$
Proof: Let $i(u) = u \quad \forall u \in U \Rightarrow i' \colon V' \to U'$

$$\dim V = \dim V' = \dim \underbrace{Ker\, i'}_{U^\circ} + \underbrace{\dim Im\, i'}_{\dim U}$$

$\dim Im\ T = \dim Im\ T'$

Lemma: $Ker\ T' = (Im\ T)^\circ$

Proof:

1. $\varphi \in Ker\ T' \Rightarrow 0 = T'(\varphi) = \varphi \circ T = 0$

$\Rightarrow 0 = \varphi \circ T(v) = \varphi(Tv)\, \forall v \in V \Rightarrow \varphi \in (Im\ T)^\circ \Rightarrow Ker\ T \subset (Im\ T)^\circ$

2. Let $\varphi \in (Im\ T)^\circ$

$\Rightarrow \varphi(Tv)\, \forall v \in V \Rightarrow 0 = T'(\varphi) \Rightarrow \varphi \in Ker\ T' \Rightarrow (Im\ T)^\circ \subset Ker\ T$

Theorem: $\dim Im\ T' = \dim Im\ T$

$\dim Im\ T' = \dim W' - \dim Ker\ T' = \dim W - \dim (Im\ T)^\circ = \dim Im\ T$

$$\boxed{rank(A^T) = rank(A)}$$

$$\boxed{\text{N of linearly independent columns} = \text{N of linearly independent rows}}$$

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix} \Rightarrow \text{row rank} = 2 \quad \text{column rank} = 2$$