vv214: Linear transformations II.

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This week

Today

1. Kerenel and image of a linear transpromation.
2. Rank-Nullity Theorem.
3. Inverse linear transformations.

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Next class

Coordinates.

Image and Kernel of a Linear Transformation

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3. $T: \mathbb{R}^3 \to \mathbb{R}^3,\ T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

$$Ker\ T = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ k = const,\ Im\ T = xy \text{ plane}$$

Image and Kernel of a Linear Transformation

4. $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T\bar{x} = A\bar{x},\ A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

$$Ker\,A = t\begin{pmatrix} -3/2 \\ 1 \end{pmatrix},\ Im\,A = span\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

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5. $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ Tp(t) = p'(t)$

$$p(t) = a_0 + a_1 t + a_2 t^2 \Rightarrow Tp(t) = p'(t) = a_1 + 2a_2 t$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ker\ T = \{p(t)\colon\ Tp = 0\} = \{a_0\} = span(1),\ Im\ T = span(1, t)$$

Image and Kernel of a Linear Transformation

Lemma 1: Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be defined by the matrix $A_{n\times m}$. The columns of the matrix A are linearly independent iff

$$Ker\, A_{n\times m} = \{\bar{0}\} \iff rank\, A = m \Rightarrow m \leq n$$

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Lemma 2: Let $T\colon V \to W$ be a linear operator. $Im\, T$ and $Ker\, T$ are linear subspaces of V and W $\Rightarrow$ there exist bases of the kernel and the image of a linear transformation.

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- Let T be injective. As $\{0\} \subset Ker\ T$, so we need to show that $Ker\ T \subset \{0\}$.
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- Let $Ker\ T = \{0\}$. If $Tu = Tv$, then
  $T(u - v) = 0 \Rightarrow u - v \in Ker\ T \Rightarrow u - v = 0 \Rightarrow u = v$

Example: $T\colon \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (2x, 3y, x + 2y)$

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Image and Kernel of a Linear Transformation: Example

$$T: \mathbb{R}^6 \to \mathbb{R}^4,\ A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow Ker\ A = \{\bar{x} \in \mathbb{R}^6 : A\bar{x} = 0\}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 + 3x_6 = 0 & \Rightarrow x_2 = -2x_3 - 3x_6 \\ x_4 + 4x_6 = 0 & \Rightarrow x_4 = -4x_6 \\ x_5 + 5x_6 = 0 & \Rightarrow x_5 = -5x_6 \end{cases}$$

$$\bar{x} = r \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{pmatrix} \Rightarrow \dim Ker\ A = 3$$

Rank-Nullity Theorem

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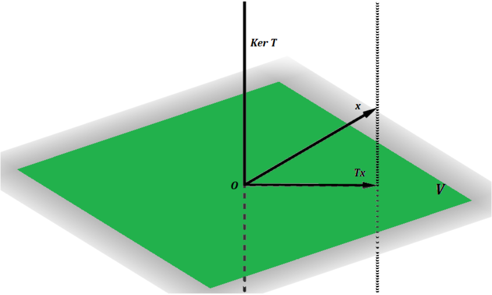
$$Ker\ T = \{\bar{x} \in \mathbb{R}^3 : proj_V \bar{x} = \bar{0}\},\ Im\ T = V$$



$$Ker\ T = \text{line orthogonal to } V$$

$$\underbrace{m}_{3} - \underbrace{\dim (Ker\ T)}_{1} = \underbrace{\dim Im\ T}_{2}$$

Rank-Nullity Theorem: Proof

Let $\dim(Ker\ T) = n$ and $\dim Ker\ T = k \Rightarrow k \le n$.
$\Rightarrow$ there exists a basis $v_1, \dots, v_k$, of $Ker\ T$. Complete this basis up to the basis of V: $v_1, \dots, v_k,\ v_{k+1}, \dots, v_n$
We are to prove that $Tv_{k+1}, \dots,\ Tv_n$ form the basis for $Im\ T$:

1. $Tv_{k+1}, \dots,\ Tv_n$ are linearly independent:

   $$\alpha_1 Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n = 0 \Rightarrow T(\alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n) = 0$$

   $\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in Ker\ T$
   $\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in span(v_1, \dots, v_k)$
   But $v_1, \dots, v_k,\ v_{k+1}, \dots, v_n$ are linearly independent
   $\Rightarrow \alpha_1 = \ldots = \alpha_{n-k} = 0$

2. $span(Tv_{k+1}, \dots,\ Tv_n) = Im\ T$

   A. $w \in Im\ T \Rightarrow \exists v \in V:\ Tv = w \Rightarrow T(\beta_1 w_1 + \ldots + \beta_n v_n) = w$

   $$w = \beta_1 \underbrace{Tv_1}_{=0} + \ldots + \beta_k \underbrace{Tv_k}_{=0} + \beta_{k+1} Tv_{k+1} + \ldots + \beta_n Tv_n$$

   $$w \in span(Tv_{k+1}, \dots,\ Tv_n) \Rightarrow Im\ T \subset span(Tv_{k+1}, \dots,\ Tv_n)$$

   B. $w \in span(Tv_{k+1}, \dots,\ Tv_n) \Rightarrow w = \alpha_{k+1} Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n$
   $w = T(\alpha_{k+1} v_{k+1} + \ldots + \alpha_{n-k} v_n) \Rightarrow w \in Im\ T$

Inverse Linear Transformations

Definition: Let V, W be linear spaces.
A linear operator $T\colon V \to W$ is called invertible if there exists a linear operator $S\colon W \to V$ such that ST equals the identity map on V and TS equals the identity map on W.

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Here the first I is the identity map on V and the second I is the identity map on W. We shall denote the inverse linear operator by T^{-1}.

$$T^{-1}(Tv) = v \quad \text{and} \quad T(T^{-1}w) = w \quad \forall v \in V\, \forall w \in W$$

Inverse Linear Transformations

Lemma: A linear operator is invertible iff it is one-to-one (injective) and onto (surjective).

- Let T^{-1} exists.
  - A Let $u, v \in V$ and $Tu = Tv$

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v \Rightarrow \ T \text{ is injective}$$

  - B Let $w \in W \Rightarrow w = T(T^{-1}w) \Rightarrow w \in Im\,T \Rightarrow W \subset Im\,T$
    As also $Im\,T \subset W$, so $W = Im\,T$
- Let T be injective and surjective. For any $w \in W$, define Sw be a unique element of V such that $T(Sw) = w$. This element exists since T is one-to-one and onto.
  - A From the definition, $TS = I$. Also

$$T((ST)v) = (TS)(Tv) = ITv = Tv \Rightarrow STv = v \Rightarrow ST = I$$

  - B S is linear:

$$w_1,\ w_2 \in W \Rightarrow T(Sw_1 + Sw_2) = TSw_1 + TSw_2 = w_1 + w_2$$

    Apply the definition of $S \Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2$
    Similarly, $S(\alpha w) = \alpha Sw \,\forall w \in W \,\forall \alpha \in \mathbb{K}$

Inverse Linear Transformations

Remarks:

1. $(T^{-1})^{-1} = T$
2. Let $V, W = \mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the system $A\bar{x} = \bar{y}$ has a unique solution

$$\iff \textit{rank } A = n \iff \textit{rref } A = I_n$$

Definition: A square matrix A is invertible if the linear transformation $T\bar{x} = A\bar{x}$ is invertible.

Inverse Linear Transformations

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$$\left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$I\bar{y} = A^{-1}\bar{x} \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 2 \\ -7 & 3 & -1 \end{pmatrix}$$

Inverse Linear Transformations

1. Let $A_{n\times n}$. If A^{-1} exists, then the system $A\bar{x} = \bar{0}$ has a unique solution

$$\Rightarrow rank\, A = n \Rightarrow \text{ columns of } A \text{ are linearly independent.}$$

2. If A^{-1} exists, then $A^{-1}A = AA^{-1} = I$.
3. $(AB)^{-1} = B^{-1}A^{-1}$

1. Review: basis, dimension,linear operators on finite dimensional linear spaces.
2. Coordinates of a vector in different bases of $\mathbb{R}^n$.
3. The change of basis matrix.
4. The $\mathfrak{B}$-matrix of a linear transformation.
5. Similar matrices.
6. Isomorphism. Isomorphic spaces.

Review: bases of linear subspaces.

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   What about the dimension of $M_{2\times 2}, l,\ l^\infty, l^2$?

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   What about the dimension of $M_{2\times 2}, l, l^\infty, l^2$?
5. Any linear operator $T : \mathbb{R}^m \to \mathbb{R}^n$ is described by a matrix $A_{n\times m}$.

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b. $\{t^n - 1, \ldots, t - 1\}$ is the basis of $M \Rightarrow \dim M = n$.

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2. $P_2(\mathbb{R}) = \{a_0 + a_1 t + a_2 t^2,\ a_i \in \mathbb{R}\} \Rightarrow \dim P_2(\mathbb{R}) = 3$

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Consider a subset M of $P_n(\mathbb{R})$

$$M = \{p(t) \in P_n(\mathbb{R}) \colon p(1) = 0\}$$

a. M is a linear subspace
b. $\{t^n - 1, \ldots, t - 1\}$ is the basis of $M \Rightarrow \dim M = n$.

2. $P_2(\mathbb{R}) = \{a_0 + a_1 t + a_2 t^2,\ a_i \in \mathbb{R}\} \Rightarrow \dim P_2(\mathbb{R}) = 3$

The basis is $\mathfrak{B} = \{1,\ t,\ t^2\}$. Let
$p_1(t) = 1,\ p_2(t) = 3t^2,\ p_3(t) = t^2 + 2t + 1,\ p_4(t) = t^2 - 2t + 2$
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$$\dim V = 2 \Rightarrow V \text{ is finite dimensional}$$

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$\mathfrak{B}$-matrix of a linear transformation

Definition: Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A matrix $B_{n\times n}$ that transforms $\bar{x}_{\mathfrak{B}}$ into $(T\bar{x})_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of T:

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$$\begin{array}{ccc} \bar{x} & \xrightarrow{T} & \frac{1}{3^2+1^2}\begin{pmatrix} 3^2 & 3\cdot 1 \\ 3\cdot 1 & 1^2 \end{pmatrix}\bar{x} \\ \downarrow L \uparrow S & & \downarrow L \uparrow S \\ \bar{x}_{\mathfrak{B}} & \xrightarrow{B} & (T\bar{x})_{\mathfrak{B}} \end{array}$$

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Definition: Let V be a linear space, $\dim V = n$ and $T\colon V \to V$ be a linear transformation. Let $\mathfrak{B}$ be a basis of V. The matrix B of a linear transformation

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$$T\colon V \to V,\ V = span(\cos x, \sin x) \subset C^{\infty},\ Tf = 3f + 2f' - f''$$

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Change of basis matrix of a linear transformation

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two bases of a linear space V, $\dim V = n$.

$$\mathfrak{B}_1 = \{b_1,\ b_2,\ \dots b_n\}$$

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Example: Let $V = span(e^x,\ e^{-x}) \in C^\infty$
The systems $\mathfrak{B}_1 = \{e^x + e^{-x},\ e^x - e^{-x}\}$ and $\mathfrak{B}_2 = \{e^x,\ e^{-x}\}$ are the bases of V.

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Definition: Let V, W be linear spaces.
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2. $L: M_{2\times 2} \to \mathbb{R}^4,\ L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\ \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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1. $T\colon M_{2\times 2} \to M_{2\times 2}$, $T(A) = S^{-1}AS$ with $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
2. $L : M_{2\times 2} \to \mathbb{R}^4$, $L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
3. To generalize 2, consider a linear space with a *finite* basis $V = span(f_1, f_2 \ldots, f_n)$ and define the coordinate transformation $L_{\mathfrak{B}}\colon V \to \mathbb{R}^n$ is $L(f) = f_{\mathfrak{B}}$

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More Examples

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6. If S is the set of all students in your linear algebra class. Can one define operations on S that make S into a real linear space?

No.