# Final

2019年8月4日 星期日　　下午7:25

## Cayley-Hamilton Theorem

$f_D(D)=0$.

$D=S^{-1}AS$

$f_A(A)=(-A)^n+\underbrace{(\mathrm{tr}A)}_{\lambda_1+\lambda_2+\cdots+\lambda_n}(-A)^{n-1}+\cdots+(\det A)I_n=0$.

## Order reduction

$g(t)=h(t)f_A(t)+r(t)$　　$[f_A(A)=0]$

例: $g(A)=A^5+2A^4-A^3+A^2-2A+I$. $A=\begin{pmatrix}-3&1\\0&-2\end{pmatrix}$

$f_A(\lambda)=\lambda^2+5\lambda+6$.

$g(\lambda)=(\lambda^3-3\lambda^2+8\lambda-21)(\lambda^2+5\lambda+6)+\underbrace{65\lambda+127}_{b_1t+b_0}$.

$g(A)=55A+127I$.

## Analytic functions

例1: $g(t)=\sin t$. $A=\begin{pmatrix}-3&1\\0&-2\end{pmatrix}$, $r(t)=b_0+b_1t$

$f_A(\lambda)=\lambda^2+5\lambda+6$. $\lambda_1=-2$, $\lambda_2=-3$.

$\begin{cases}\sin\lambda_1=b_0+b_1\lambda_1\\ \sin\lambda_2=b_0+b_1\lambda_2\end{cases}\Rightarrow\begin{cases}b_0=3\sin(-2)-2\sin(-3)\\ b_1=\sin(-2)-\sin(-3)\end{cases}$

$g(A)=r(A)=b_0I+b_1A=\sin A$.

例2: Matrix Exponential.

$g(t)=e^{tx}\Rightarrow e^{tx}=\sum_{k=0}^{n-1}b_kt^k$.

$\lambda_1=-2$, $\lambda_2=-3$,

$\begin{cases}e^{-2x}=b_0-2b_1\\ e^{-3x}=b_0-3b_1\end{cases}\Rightarrow e^{Ax}=b_0I+b_1A$.

## Eigenvalues & Eigenvectors

Eigen space. $E_\lambda=\ker(A-\lambda I)=\mathrm{span}\begin{pmatrix}\vdots\end{pmatrix}$

Geometric multiplicity: dim of $E_\lambda$.

* $A_{3\times3}$. $E_1=\mathrm{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$　$E_0=\mathrm{span}\begin{bmatrix}-1\\1\\0\end{bmatrix}$

not enough $\Rightarrow$ no eigenbasis.

Eigenbasis: $n\times n$ matrix has $n$ distinct eigen values

Algebraic multiplicity. 重根个数 for different $\lambda$.

$f_A(\lambda)=(1-\lambda)^3(2-\lambda)^2$

AM ≥ GM.

## Diagonalization

Diagonal: $S^{-1}AS$ iff column vectors of $S$ form an eigenbasis for $A$.

Diagonalization: ① $f_A(\lambda)=\det(A-\lambda I_n)=0$

② for each $\lambda$, find $E_\lambda=\ker(A-\lambda I_n)$

③ diagonalizable $\Leftrightarrow$ dim E add up to **n**.

$S=[\bar v_1\ \bar v_2\ \bar v_3]$. $D=S^{-1}AS=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$

### Orthogonal Diagonalization:

例: $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\Rightarrow\lambda_{1,2}=0$, $\lambda_3=3$,

$\lambda_{1,2}=0\Rightarrow\begin{cases}\bar v_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}\Rightarrow\hat u_1=\begin{pmatrix}-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{pmatrix}\\ \bar v_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}\Rightarrow\bar u_2=\begin{pmatrix}-\frac{1}{\sqrt6}\\-\frac{1}{\sqrt6}\\2\end{pmatrix}\end{cases}$

$\lambda_3=3\Rightarrow\bar v_3=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\bar u_3=\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$S=[\bar u_1\ \bar u_2\ \bar u_3]$. $D=S^{-1}AS=\begin{pmatrix}0&0&0\\0&0&0\\0&0&3\end{pmatrix}$

### Spectral Theorem

orthogonally diagonalizable iff $A^T=A$.

## Adjoint Operators

**Definition:** Let $T\colon V\to W$ be linear, i.e. $T\in L(V,W)$. The adjoint of $T$ is the operator $T^\star\colon W\to V$ such that

$$(Tv,w)=(v,T^\star w)\quad\forall v\in V\,\forall w\in W$$

例: Let $T:\mathbb{R}^3\to\mathbb{R}^2$

$T(x_1,x_2,x_3)=(x_2+3x_3,2x_1)$

$(T\bar x,\bar y)=((x_2+3x_3,2x_1),(y_1,y_2))=x_2y_1+3x_3y_1+2x_1y_2$

$=((x_1,x_2,x_3),(2y_2,y_1,3y_1))$

$\Rightarrow T^*\bar y=(2y_2,y_1,3y_1)$

### Properties:

1. $(v,T^*(\lambda w))=(Tv,\lambda w)=\bar\lambda(Tv,w)=(v,\lambda T^*w)$

   $T\in L(V,W)\Rightarrow T^*\in L(W,V)$

2. $(T_1+T_2)^*=T_1^*+T_2^*$
3. $(\lambda T)^*=\bar\lambda T^*$
4. $(T^*)^*=T$.

   $(w,(T^*)^*v)=(T^*w,v)=\overline{(v,T^*w)}=\overline{(Tv,w)}=(w,Tv)$

5. $I^*=I$.
6. $(T_1T_2)^*=T_2^*T_1^*$
7. $\mathrm{Ker}\,T^*=(\mathrm{Im}\,T)^\perp$.
8. 

8. The matrix of the adjoint $T^\star$ w.r.t orthonormal bases $e_1,\ldots,e_m\in V$; $f_1,\ldots f_n\in W$ is the conjugate transpose of the matrix of $T$.

$A_T=(Te_1\,Te_2\ldots Te_m)$　$Te_k\in W$, $f_1,\ldots,f_n$ is orthonormal

$\Rightarrow Te_k=(Te_k,f_1)f_1+\ldots+(Te_k,f_n)f_n\Rightarrow(A_T)_{jk}=(Te_k,f_j)$

$A_{T^\star}=(T^\star f_1\,T^\star f_2\ldots T^\star f_m)$　$T^\star f_k\in V$, $e_1,\ldots,e_m$ is orthonormal

$\Rightarrow T^\star f_k=(T^\star f_k,e_1)e_1+\ldots+(T^\star f_k,e_m)e_m$

$\Rightarrow(A_{T^\star})_{jk}=(T^\star f_k,e_j)=\overline{(e_j,T^\star f_k)}=\overline{(Te_j,f_k)}$

## Self-adjoint operators.

$T^*=T$.　$(Tv,w)=(v,Tw)$

① $T$ is symmetric
② Eigenvalues are real.
③ $(Tv,v)\in\mathbb{R}$

## Normal Operators.

$TT^*=T^*T$.

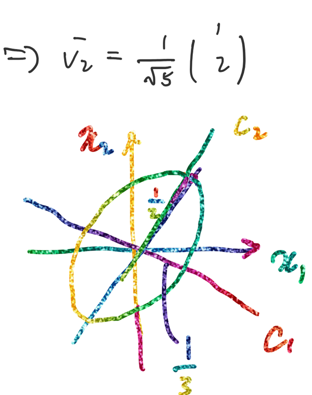
① self-adjoint is normal
② $\|Tv\|=\|T^*v\|$
③ Let $Tv=\lambda v\Leftrightarrow(T-\lambda I)$ is also normal.
④ Let $Tv=\lambda v$. then $T^*v=\bar\lambda v$
⑤ If $T$ is normal. its eigenvectors are $\perp$.

## Quadratic Forms

$q(\bar x)=\bar x\cdot A\bar x=\bar x^TA\bar x$　symmetric

$A=\begin{pmatrix}8&-2\\-2&5\end{pmatrix}$　$\begin{cases}\lambda_1=9\Rightarrow\bar v_1=\frac{1}{\sqrt5}\begin{pmatrix}2\\-1\end{pmatrix}\\ \lambda_2=4\Rightarrow\bar v_2=\frac{1}{\sqrt5}\begin{pmatrix}1\\2\end{pmatrix}\end{cases}$

$q(\bar x)=\lambda_1\alpha_1^2+\lambda_2\alpha_2^2\geq0$.




## SVD.

### Spectral Theorem:

$A$ is orthogonally diagonalizable $\Leftrightarrow A^T=A$

### * QR.




$r_{11}=\|\bar v_1\|$　$\bar u_1=\frac{1}{r_{11}}\bar v_1$

$r_{12}=\bar u_1\cdot\bar v_2$　$\bar v_2^\perp=\bar v_2-r_{12}\cdot\bar u_1$

$r_{22}=\|\bar v_2^\perp\|$　$\bar u_2=\frac{1}{r_{22}}\bar v_2^\perp$.

$\left.\begin{aligned}r_{13}&=\bar u_1\cdot\bar v_3\\ r_{23}&=\bar u_2\cdot\bar v_3\end{aligned}\right\}\ \bar v_3^\perp=\bar v_3-r_{13}\cdot\bar u_1-r_{23}\cdot\bar u_2$

$r_{33}=\|\bar v_3^\perp\|$.　$\bar u_3=\frac{1}{r_{33}}\bar v_3^\perp$

### Orthogonal diagonalization of a symmetric A.

① Find eigenvalues. $\Rightarrow$ space.
② Gram-Schmidt $\Rightarrow$ orthonormal basis
③ $S=[\bar u_1\ \bar u_2\ \bar u_3]$

$D=S^{-1}AS$

### Singular values

$\sigma_1\geq\sigma_2\geq\sigma_3\cdots\geq0$.

例: For $L(\bar x)=A\bar x$. where $A=\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}$

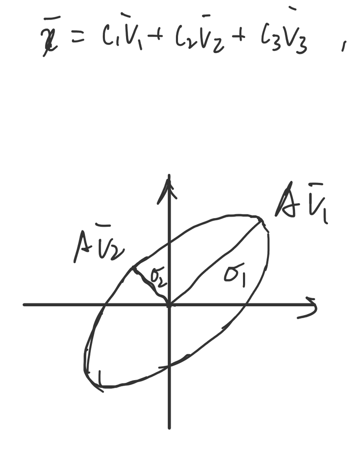
$A^TA=\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}$　$\lambda_1=3$, $\lambda_2=1$. $\lambda_3=0$.　$\Sigma=\begin{bmatrix}\sqrt3&0&0\\0&1&0\end{bmatrix}$

$\sigma_1=\sqrt3$　$\sigma_2=1$　$\sigma_3=0$.

| | | |
|---|---|---|
| $E_3=\mathrm{span}\begin{bmatrix}1\\2\\1\end{bmatrix}$ | $E_1=\mathrm{span}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ | $E_0=\mathrm{span}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ |
| $\bar v_1=\frac{1}{\sqrt6}\begin{bmatrix}1\\2\\1\end{bmatrix}$ | $\bar v_2=\frac{1}{\sqrt2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ | $\bar v_3=\frac{1}{\sqrt3}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ |
| $A\bar v_1=\frac{1}{\sqrt6}\begin{bmatrix}3\\3\end{bmatrix}$ | $A\bar v_2=\frac{1}{\sqrt2}\begin{bmatrix}-1\\1\end{bmatrix}$ | $A\bar v_3=\begin{bmatrix}0\\0\end{bmatrix}$ |

$V=[\bar v_1\ \bar v_2\ \bar v_3]$

$\bar x=c_1\bar v_1+c_2\bar v_2+c_3\bar v_3$, $c_1^2+c_2^2+c_3^2=1$




$\bar u_1=\frac{1}{\sigma_1}A\bar v_1$　$\bar u_2=\frac{1}{\sigma_1}A\bar v_2$　$\bar u_3=\frac{1}{\sigma_1}A\bar v_3$　$U=[\bar u_1\ \bar u_2]$

$A=U\Sigma V^T$

$U=[\bar u_1\ \bar u_2\cdots\bar u_n]$. $\Sigma=\begin{bmatrix}\sigma_1&&&0\\&\sigma_2&&\\&&\ddots\ \sigma_r&\\0&&&0\end{bmatrix}$

$A=\sigma_1\bar u_1\bar v_1^T+\sigma_2\bar u_2\bar v_2^T+\cdots$

$V=[\bar v_1\ \bar v_2]$