vv214: Determinants

Dr.Olga Danilkina

UM-SJTU Joint Institute

July 9, 2019

1. Review: Orthonormal bases, Least squares solution, Fourier series, correlation.
2. Determinants of 2×2 and 3×3 matrices.
3. Properties of determinants.
4. Patterns, inversions, determinants of $n \times n$ matrices.
5. Multiplicative property of a determinant.
6. Laplace expansions.

Review

1. The Cauchy-Schwarz inequality

$$|(x, y)| \leq \sqrt{(x, x)} \cdot \sqrt{(y, y)}$$

Review

1. The Cauchy-Schwarz inequality

$$|(x, y)| \leq \sqrt{(x, x)} \cdot \sqrt{(y, y)}$$

2. $(Im\, A)^{\perp} = Ker\ \left(A^{T}\right)$

Review

1. The Cauchy-Schwarz inequality

$$|(x,y)| \leq \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$

2. $(Im\, A)^{\perp} = Ker\ \left(A^T\right)$
3. $Ker\, A = Ker(A^T A), \quad Ker\, A = \{\bar{0}\} \Rightarrow \exists \left(A^T A\right)^{-1}$

Review

1. The Cauchy-Schwarz inequality

$$|(x,y)| \leq \sqrt{(x,x)} \cdot \sqrt{(y,y)}$$

2. $(Im\, A)^{\perp} = Ker\ \left(A^T\right)$
3. $Ker\ A = Ker(A^T A), \quad Ker\ A = \{\bar{0}\} \Rightarrow \exists \left(A^T A\right)^{-1}$
4. The least-squares solutions of the system

$$A\bar{x} = \bar{b}$$

Review

1. The Cauchy-Schwarz inequality

$$|(x, y)| \leq \sqrt{(x, x)} \cdot \sqrt{(y, y)}$$

2. $(Im\, A)^{\perp} = Ker\ \left(A^T\right)$
3. $Ker\ A = Ker(A^T A), \quad Ker\ A = \{\bar{0}\} \Rightarrow \exists \left(A^T A\right)^{-1}$
4. The least-squares solutions of the system

$$A\bar{x} = \bar{b}$$

   are the exact solutions of

$$A^T A\bar{x} = A^T \bar{b}$$

Correlation

Def: Let $\bar{x}, \bar{y} \in \mathbb{R}^n$. There is a positive correlation between $\bar{x}$ and $\bar{y}$ if and only if $(\bar{x}, \bar{y}) > 0$.

Correlation

Def: Let $\bar{x}, \bar{y} \in \mathbb{R}^n$. There is a positive correlation between $\bar{x}$ and $\bar{y}$ if and only if $(\bar{x}, \bar{y}) > 0$.

Def: The correlation coefficient r of two vectors $\bar{x}$ and $\bar{y}$ is

$$r = \cos(\bar{x}, \bar{y}) = \frac{(\bar{x}, \bar{y})}{|\bar{x}||\bar{y}|}$$

Remark: By the Cauchy-Schwarz inequality,

$$|(\bar{x}, \bar{y})| \leq |\bar{x}||\bar{y}| \Rightarrow -1 \leq r \leq 1$$

Correlation: Example

Consider meat consumption and incidence of cancer rate in the following countries:

Country	Consumption	Rate	Deviation: Cons	Deviation: Rate
Japan	26	7.5	-122	-10.7
Finland	101	9.8	-47	-8.4
Israel	124	16.4	-24	-1.8
GB	205	23.3	57	5.1
US	284	34	136	15.8
Mean	148	18.2		

The correlation coefficient is

$$r = \frac{122 \cdot 10.7 + 47 \cdot 8.4 + 24 \cdot 1.8 + 57 \cdot 5.1 + 136 \cdot 15.8}{198.53 \cdot 21.539} \approx 0.9782$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$

   If $Im\, A \neq \mathbb{R}^3$,

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist

   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$

   If $Im\,A \neq \mathbb{R}^3$, then A^{-1} does not exist

   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

   $$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist
   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

   $$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,\ \bar{a}_3 \Rightarrow$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$

2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist

   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

$$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,\ \bar{a}_3 \Rightarrow \bar{a}_2 \times \bar{a}_3 \perp \bar{a}_1 \Rightarrow$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{pmatrix}$
   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist
   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

$$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,\ \bar{a}_3 \Rightarrow \bar{a}_2 \times \bar{a}_3 \perp \bar{a}_1 \Rightarrow \left(\bar{a}_1, \bar{a}_2 \times \bar{a}_3\right) = 0$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1}$ iff $\det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist

   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

$$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,\ \bar{a}_3 \Rightarrow \bar{a}_2 \times \bar{a}_3 \perp \bar{a}_1 \Rightarrow (\bar{a}_1, \bar{a}_2 \times \bar{a}_3) = 0$$

$$\exists A^{-1} \text{ iff } (\bar{a}_1, \bar{a}_2 \times \bar{a}_3) \neq 0$$

Determinants

1. $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \exists A^{-1} \text{ iff } \det A \neq 0$
2. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$

   If $Im\, A \neq \mathbb{R}^3$, then A^{-1} does not exist

   $\Rightarrow$ $span\,(\bar{a}_1, \bar{a}_2, \bar{a}_3) \subset \mathbb{R}^2 \Rightarrow \bar{a}_1, \bar{a}_2, \bar{a}_3$ are on the same plane

$$\bar{a}_2 \times \bar{a}_3 \perp \bar{a}_2,\ \bar{a}_3 \Rightarrow \bar{a}_2 \times \bar{a}_3 \perp \bar{a}_1 \Rightarrow (\bar{a}_1, \bar{a}_2 \times \bar{a}_3) = 0$$

$$\exists A^{-1} \text{ iff } (\bar{a}_1, \bar{a}_2 \times \bar{a}_3) \neq 0$$

Definition: The determinant of $A = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3)$ is

$$\det A = (\bar{a}_1, \bar{a}_2 \times \bar{a}_3)$$

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$.

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$.
2. Alternating Property

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$.
2. Alternating Property

$$\det(\bar{a}_1 \quad \bar{a}_3 \quad \bar{a}_2) = (\bar{a}_1, \bar{a}_3 \times \bar{a}_2) = (\bar{a}_1, -(\bar{a}_2 \times \bar{a}_3)) = -\det A$$

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$.
2. Alternating Property

$$\det(\bar{a}_1 \quad \bar{a}_3 \quad \bar{a}_2) = (\bar{a}_1, \bar{a}_3 \times \bar{a}_2) = (\bar{a}_1, -(\bar{a}_2 \times \bar{a}_3)) = -\det A$$

3. Let $T \colon \mathbb{R}^3 \to \mathbb{R}$

$$T\bar{x} = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{vmatrix} 2 & x_1 & 5 \\ 3 & x_2 & 6 \\ 4 & x_3 & 7 \end{vmatrix} = 3x_1 - 6x_2 + 3x_3$$

T is a linear transformation $\Rightarrow$

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$.
2. Alternating Property

$$\det(\bar{a}_1 \quad \bar{a}_3 \quad \bar{a}_2) = (\bar{a}_1, \bar{a}_3 \times \bar{a}_2) = (\bar{a}_1, -(\bar{a}_2 \times \bar{a}_3)) = -\det A$$

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}$

$$T\bar{x} = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{vmatrix} 2 & x_1 & 5 \\ 3 & x_2 & 6 \\ 4 & x_3 & 7 \end{vmatrix} = 3x_1 - 6x_2 + 3x_3$$

T is a linear transformation $\Rightarrow$ The determinant is linear in the second column.
BUT

Remarks

1. $\det A =$
$a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}.$
2. Alternating Property
$$\det(\bar{a}_1 \quad \bar{a}_3 \quad \bar{a}_2) = (\bar{a}_1, \bar{a}_3 \times \bar{a}_2) = (\bar{a}_1, -(\bar{a}_2 \times \bar{a}_3)) = -\det A$$
3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}$
$$T\bar{x} = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{vmatrix} 2 & x_1 & 5 \\ 3 & x_2 & 6 \\ 4 & x_3 & 7 \end{vmatrix} = 3x_1 - 6x_2 + 3x_3$$

T is a linear transformation $\Rightarrow$ The determinant is linear in the second column.
BUT the map $M_{3\times 3} \to \mathbb{R}$ such that $A_{3\times 3} \to \det A$ is not linear.

Remarks

4.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Remarks

4.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Definition: A choice of an element in each row and each column of a square matrix is called a pattern P in the matrix. The product of all entries in the pattern is denoted by *prod P*.

Remarks

By the Alternating property,

$$\det \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} =$$

Remarks

By the Alternating property,

$$\det\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{pmatrix} =$$

Remarks

By the Alternating property,

$$\det\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{pmatrix} =$$

$$= \det\begin{pmatrix} a_{31} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \end{pmatrix} = a_{12}a_{23}a_{31} \text{ (two row swaps)}$$

BUT

$$\det\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} =$$

Remarks

By the Alternating property,

$$\det\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{pmatrix} =$$

$$= \det\begin{pmatrix} a_{31} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \end{pmatrix} = a_{12}a_{23}a_{31} \text{ (two row swaps)}$$

BUT

$$\det\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = -\det\begin{pmatrix} a_{31} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{13} \end{pmatrix} =$$

$$= -a_{13}a_{22}a_{31} \text{ (one row swap)}$$

Inversions

Definition: Two entries in the pattern are inverted if one of them is to the right and above the other.

Inversions

Definition: Two entries in the pattern are inverted if one of them is to the right and above the other.

Remark: In $P = \{a_{12},\, a_{23},\, a_{31}\}$, there are 2 inversions:

$$\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} = a_{12}a_{23}a_{31} = (-1)^2 a_{12}a_{23}a_{31} > 0$$

Inversions

Definition: Two entries in the pattern are inverted if one of them is to the right and above the other.

Remark: In $P = \{a_{12},\, a_{23},\, a_{31}\}$, there are 2 inversions:

$$\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} = a_{12}a_{23}a_{31} = (-1)^2 a_{12}a_{23}a_{31} > 0$$

In $P = \{a_{13},\, a_{22},\, a_{31}\}$, there are 3 inversions:

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} = -a_{13}a_{22}a_{31} = (-1)^3 a_{12}a_{23}a_{31} < 0$$

The sign of Prod P depends on the number of inversions in the pattern.

Verify it for $A_{3\times 3}$.

Determinant

Definition: Let $N^\star$ be the number of inversions in the pattern P.

Determinant

Definition: Let $N^\star$ be the number of inversions in the pattern P.

$$(-1)^{N^\star}$$

is called the signature $sgn\, P$ of the pattern P.

Determinant

Definition: Let $N^\star$ be the number of inversions in the pattern P.

$$(-1)^{N^\star}$$

is called the signature $sgn\,P$ of the pattern P.

Definition: The determinant of a matrix $A_{n\times n}$ is defined by

$$\det A = \sum_P (sgn\,P)(prod\,P)$$

Examples

1. $$A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}:$$

Examples

1. $A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$: 3rd column: 2, 2nd column: 3

$$\begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

Examples

1. $A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$: 3rd column: 2, 2nd column: 3

$$\begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

Examples

1. $A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$: 3rd column: 2, 2nd column: 3

$$\begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

$$(-1)^2\, 6 \cdot 3 \cdot 2 \cdot 4$$

Examples

1. $$A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$: 3rd column: 2, 2nd column: 3

$$\begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

$$(-1)^2\, 6 \cdot 3 \cdot 2 \cdot 4 \qquad (-1)^3\, 7 \cdot 3 \cdot 2 \cdot 3$$

Examples

1. $A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$: 3rd column: 2, 2nd column: 3

$$\begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}$$

$$(-1)^2\, 6 \cdot 3 \cdot 2 \cdot 4 \qquad (-1)^3\, 7 \cdot 3 \cdot 2 \cdot 3$$

$$144 \qquad -126 \qquad = 18$$

Examples

2. $$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}:$$

Examples

2. $A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$: 3rd: 5, 4th: 1, 5th: 1

Examples

2. $A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$: 3rd: 5, 4th: 1, 5th: 1

$$\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Examples

2. $$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$: 3rd: 5, 4th: 1, 5th: 1

$$\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Examples

2. $$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$: 3rd: 5, 4th: 1, 5th: 1

$$\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(-1)^0\, 5 \cdot 7 \cdot 5 \cdot 1 \cdot 1$$

Examples

2. $$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$: 3rd: 5, 4th: 1, 5th: 1

$$\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(-1)^0\, 5 \cdot 7 \cdot 5 \cdot 1 \cdot 1 \qquad (-1)^1\, 6 \cdot 4 \cdot 5 \cdot 1 \cdot 1$$

Examples

2. $$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$: 3rd: 5, 4th: 1, 5th: 1

$$\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(-1)^0\, 5 \cdot 7 \cdot 5 \cdot 1 \cdot 1 \qquad (-1)^1\, 6 \cdot 4 \cdot 5 \cdot 1 \cdot 1$$

$$175 \qquad -106 \qquad = 55$$

Examples

3. $$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{pmatrix} =$$

Examples

3. $$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{pmatrix} = -48 + 72 = 24$$

4. $$\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} =$$

Examples

3. $$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{pmatrix} = \; -48 + 72 = 24$$

4. $$\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot 4$$

Examples

3. $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{pmatrix} = \; -48 + 72 = 24$

4. $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot 4$

The determinant of a triangular matrix is the product of its diagonal elements.

Examples

5. Let

$$M = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ d_{11} & d_{12} & c_{11} & c_{12} \\ d_{21} & d_{22} & c_{21} & c_{22} \end{pmatrix} \Rightarrow$$

Examples

5. Let

$$M = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ d_{11} & d_{12} & c_{11} & c_{12} \\ d_{21} & d_{22} & c_{21} & c_{22} \end{pmatrix} \Rightarrow$$

$$M = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

is the block representation

Examples

5. Let

$$M = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ d_{11} & d_{12} & c_{11} & c_{12} \\ d_{21} & d_{22} & c_{21} & c_{22} \end{pmatrix} \Rightarrow \quad M = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

is the block representation

Consider the case $d_{ij} = 0,\ i,\ j = 1,\ 2 \Rightarrow \det M = \det A \cdot \det C$

Examples

5. Let

$$M = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ d_{11} & d_{12} & c_{11} & c_{12} \\ d_{21} & d_{22} & c_{21} & c_{22} \end{pmatrix} \Rightarrow \qquad M = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

is the block representation

Consider the case $d_{ij} = 0,\ i,\ j = 1,\ 2 \Rightarrow \det M = \det A \cdot \det C$

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 & 0 & 0 \\ 3 & 8 & 6 & 0 & 0 & 0 \\ 4 & 9 & 5 & 2 & 1 & 4 \\ 5 & 8 & 4 & 0 & 2 & 5 \\ 6 & 7 & 3 & 0 & 3 & 6 \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 3 & 8 & 6 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} =$$

$$= (1 \cdot 7 \cdot 6)(2 \cdot 2 \cdot 6 - 2 \cdot 3 \cdot 5) = -252$$

Examples

6. Is $\det \begin{pmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 7 & 1000 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1000 \\ 1 & 2 & 1000 & 3 & 4 \end{pmatrix}$ positive or negative?

The number of all patterns is...

The maximal pattern is...

Evaluate the values of other patterns.

Positive.

Properties of determinants

1. $\det A = \det A^T$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$

$$\det B = \frac{1}{k} \det A$$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$

$$\det B = \frac{1}{k} \det A$$

4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$
7. $\exists A^{-1}$ iff $\det A \neq 0$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$
7. $\exists A^{-1}$ iff $\det A \neq 0$
8. $\det(AB) = (\det A)(\det B)$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$

$$\det B = \frac{1}{k} \det A$$

4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$
7. $\exists A^{-1}$ iff $\det A \neq 0$
8. $\det(AB) = (\det A)(\det B)$
9. $B = S^{-1}AS \Rightarrow \det A = \det B$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$
7. $\exists A^{-1}$ iff $\det A \neq 0$
8. $\det(AB) = (\det A)(\det B)$
9. $B = S^{-1}AS \Rightarrow \det A = \det B$
10. $\det A^{-1} = \dfrac{1}{\det A}$

Properties of determinants

1. $\det A = \det A^T$
2. The determinant is linear in all columns and rows.
3. B is obtained from A by dividing a row of A by a scalar $k \Rightarrow$
$$\det B = \frac{1}{k} \det A$$
4. B is obtained from A by a row swap $\Rightarrow \det B = -\det A$
5. A has two identical rows $\Rightarrow \det A = 0$
6. B is obtained from A by adding a multiple of a row of A to another row $\Rightarrow \det A = \det B$
7. $\exists A^{-1}$ iff $\det A \neq 0$
8. $\det(AB) = (\det A)(\det B)$
9. $B = S^{-1}AS \Rightarrow \det A = \det B$
10. $\det A^{-1} = \dfrac{1}{\det A}$
11. Laplace expansions.

Exercises:

1. Use elementary row transformations to compute the determinant of
$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$
2. Consider a map $\mathbb{R}^3 \to \mathbb{R}$ such that
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \det \begin{pmatrix} x & 1 & 2 \\ y & 2 & 1 \\ z & -1 & 0 \end{pmatrix}$$
Show that the map is linear and find its matrix.
3. Let
$$\det \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = 2$$
Evaluate
$$\det \begin{pmatrix} 3a+2c & c \\ 2d+3b & d \end{pmatrix}$$

Exercises:

4. Let $\bar{v}_1, \ldots, \bar{v}_{n-1} \in \mathbb{R}^n$ and

$$\det\left(\bar{v}_1 \ldots \bar{v}_{n-1}\, \bar{x}\right) = 1,\ \det\left(\bar{v}_1 \ldots \bar{v}_{n-1}\, \bar{y}\right) = 2,$$

$$\det\left(\bar{v}_1 \ldots \bar{v}_{n-1}\, \bar{z}\right) = 3$$

Find

$$\det\left(\bar{v}_1 \ldots \bar{v}_{n-1}\left(-\bar{x} + 2\bar{y} + 3\bar{z} + 5\bar{v}_3\right)\right)$$

Commutative Rings

Definition: A ring is a set R equipped with two operations

$$\forall x, y \in R \quad x + y \in R, \quad x \cdot y \in R$$

satisfying

a. R is an abelian group under addition.
b. $x(yz) = (xy)z$ (multiplication is associative)
c. $x(y + z) = xy + xz$, $(y + z)x = yx + zx$ (the two distributive laws hold)

If $xy = yx \, \forall x, y \in R$, we say that the ring R is commutative.
If there is an element $1 \in R$ such that $1x = x1 = x$ for each $x \in R$, R is said to be a ring with identity, and 1 is called the identity for R.

Examples of commutative rings with identity:

1. The set of integers $\mathbb{Z}$ with usual operations (but it is not a field)
2. The set of all polynomials over a field with usual operations. We shall denote the polynomial ring by $R[x]$.

Determinants

If R is a commutative ring with identity, we define an $m \times n$ matrix over R as a function A from the set of pairs (i, j) of integers into R.

Recall that we have considered matrices with real or complex entries, that is, matrices over the fields $\mathbb{R}$, $\mathbb{C}$.

All what we did with matrices over fields can be applied to matrices over a ring, including the evaluation of the determinant function.

Example: Let $R = \mathbb{R}[x]$ be the ring of all polynomials over the field of real numbers. Let

$$A = \begin{pmatrix} x^2 + x & x + 1 \\ x - 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} x^2 - 1 & x + 2 \\ x^2 - 2x + 3 & x \end{pmatrix}$$

$$\Rightarrow \det A = x + 1, \quad \det B = -6$$

A is not invertible over $\mathbb{R}[x]$, whereas B is invertible over $\mathbb{R}[x]$.

Properties of determinants

Definition: The scalar $(-1)^{i+j}\det A_{ij}$ is called the cofactor of the entree a_{ij}.

$$c_{ij} = (-1)^{i+j}\det A_{ij} \Rightarrow \det A = \sum_{i=1}^{n} a_{ij}c_{ij},$$

$$\sum_{i=1}^{n} a_{ik}c_{ij} = 0 \,\text{if}\, j \neq k$$

Replace the jth column of A by its kth column, and call the resulting matrix B.

$$\det B = 0, \quad B_{ij} = A_{ij}$$

$$0 = \det B = \sum_{i=1}^{n}(-1)^{i+j}B_{ij}b_{ij} = \sum_{i=1}^{n}(-1)^{i+j}A_{ij}a_{ik} = \sum_{i=1}^{n} a_{ik}c_{ij}$$

12. $\sum_{i=1}^{n} a_{ik}c_{ij} = \delta_{jk}\det A$

Properties of determinants

Definition: The classical adjoint *adj* A of A is the transpose of the matrix of cofactors of A:

$$(adj\, A)_{ij} = c_{ji} = (-1)^{i+j} \det A_{ji}$$

Summarize formulas from the property 12 into the matrix equation

$$(adj\, A)A = (\det A)I$$

$$(-1)^{i+j} A^T_{ij} = (-1)^{i+j} A_{ji} \Rightarrow adj\, A^T = (adj\, A)^T$$

$$(adj\, A^T)A^T = (\det A^T)I = (\det A)I$$

transposing $\Rightarrow A(adj\, A^T)^T = (\det A)I \Rightarrow A(adj\, A) = (\det A)I$

13. Let A be an $n \times n$ matrix over R. A is invertible over R iff $\det A$ is invertible in R. If A is invertible, the unique inverse is defined by

$$A^{-1} = (\det A)^{-1}(adj\, A)$$

In the example above, $\det A = x + 1 \Rightarrow A$ is not invertible.

Properties of determinants: Cramer's Rule

Consider a linear system of equations $Ax = y$

$$(adj\ A)Ax = (adj\ A)y$$

$$(\det A)x = (adj\ A)y \Rightarrow (\det A)x_j = \sum_{i=1}^{n} (adj\ A)_{ji} y_i = \sum_{i=1}^{n} (-1)^{i+j} \det A_{ij} y_i$$

The last expression is the determinant of the matrix obtained from A by replacing the jth column by y.

$$\det A \neq 0 \Rightarrow x_j = \frac{\det B_j}{\det A}, \quad i = 1, \ldots, n$$

Exercises

1. Find the classical adjoint of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

   and use it to compute A^{-1}.

Exercises

1. Find the classical adjoint of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

   and use it to compute A^{-1}.

2. Let $\mathit{rank}\, A = n - 1$.
   Use $A(\mathit{adj}\, A) = (\det A)I$ to find $\mathit{rank}\, (\mathit{adj}\, A)$.

Orthogonal Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if

$$||T\bar{x}|| = ||\bar{x}|| \quad \forall \bar{x} \in \mathbb{R}^n$$

The matrix of an orthogonal transformation is called orthogonal.

Example: The rotation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \\ sin\theta & \cos\theta \end{pmatrix}$$

preserves the length $\Rightarrow$ A is the orthogonal matrix.

Properties of Orthogonal Transformations

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $\{T\bar{e}_1,\ T\bar{e}_2, \ldots, T\bar{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

Properties of Orthogonal Transformations

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $\{T\bar{e}_1,\ T\bar{e}_2, \ldots, T\bar{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

2. $A_{n\times n}$ is orthogonal iff its columns form an orthonormal basis for $\mathbb{R}^n$.

Properties of Orthogonal Transformations

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $\{T\bar{e}_1,\ T\bar{e}_2, \ldots, T\bar{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

2. $A_{n\times n}$ is orthogonal iff its columns form an orthonormal basis for $\mathbb{R}^n$.

3. $A_{n\times n}$ is orthogonal iff $A^T A = I_n$

Properties of Orthogonal Transformations

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $\{\, T\bar{e}_1,\ T\bar{e}_2, \ldots, T\bar{e}_n \}$ is an orthonormal basis for $\mathbb{R}^n$.

2. $A_{n\times n}$ is orthogonal iff its columns form an orthonormal basis for $\mathbb{R}^n$.

3. $A_{n\times n}$ is orthogonal iff $A^T A = I_n$

4. $A^T A = I_n \Rightarrow \det A = \pm 1$

Properties of Orthogonal Transformations

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $\{T\bar{e}_1, T\bar{e}_2, \ldots, T\bar{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

2. $A_{n\times n}$ is orthogonal iff its columns form an orthonormal basis for $\mathbb{R}^n$.

3. $A_{n\times n}$ is orthogonal iff $A^TA = I_n$

4. $A^TA = I_n \Rightarrow \det A = \pm 1$

Definition: An orthogonal matrix $A_{n\times n}$ with $\det A = 1$ is called a rotation matrix.
The corresponding linear transformation $T\bar{x} = A\bar{x}$ is called a rotation.

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$

2. Find the length of the base and the height of the obtained parallelogram.

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$
2. Find the length of the base and the height of the obtained parallelogram.
3. Represent the length of the base and the height in the form of the inner product.

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$
2. Find the length of the base and the height of the obtained parallelogram.
3. Represent the length of the base and the height in the form of the inner product.
4. Compute the area of the parallelogram.

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$
2. Find the length of the base and the height of the obtained parallelogram.
3. Represent the length of the base and the height in the form of the inner product.
4. Compute the area of the parallelogram.
5. Obtain QR-factorization of the matrix A.

Motivational Example

1. Find the image of the unit square under a linear map defined by the matrix
$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$
2. Find the length of the base and the height of the obtained parallelogram.
3. Represent the length of the base and the height in the form of the inner product.
4. Compute the area of the parallelogram.
5. Obtain QR-factorization of the matrix A.
6. Evaluate $\det A$ using QR-factorization.

Properties of Orthogonal Transformations

5. $A_{2\times 2} = (\bar{a}_1 \quad \bar{a}_2) \Rightarrow \det A = (\bar{a}_{1\,rot}, \bar{a}_2) = ||\bar{a}_1|| \cdot ||\bar{a}_2|| \sin\theta$

$$|\det A| = ||\bar{a}_1|| \cdot ||\bar{a}_2^{\perp}||$$

Properties of Orthogonal Transformations

5. $A_{2\times 2} = (\bar{a}_1 \quad \bar{a}_2) \Rightarrow \det A = (\bar{a}_{1\,rot}, \bar{a}_2) = ||\bar{a}_1|| \cdot ||\bar{a}_2|| \sin\theta$

$$|\det A| = ||\bar{a}_1|| \cdot ||\bar{a}_2^{\perp}||$$

6. $A_{n\times n} = (\bar{a}_1 \ldots \bar{a}_n)$ and $\exists A^{-1} \Rightarrow A = QR$

$$|\det Q| = 1, \quad r_{11} = ||\bar{a}_1||,\ r_{jj} = ||\bar{a}_j^{\perp}||\,\forall j \geq 2$$

$$|\det A| = ||\bar{a}_1||||\bar{a}_2^{\perp}||\ldots||\bar{a}_n^{\perp}||, \quad \bar{a}_i^{\perp} \perp span(\bar{a}_1, \ldots \bar{a}_{i-1})$$

If A^{-1} does not exist $\Rightarrow \exists \bar{a}_i$ redundant $\Rightarrow \bar{a}_i^{\perp} = 0 \Rightarrow \det A = 0$

Properties of Orthogonal Transformations

5. $A_{2\times 2} = (\bar{a}_1 \quad \bar{a}_2) \Rightarrow \det A = (\bar{a}_{1\,rot}, \bar{a}_2) = ||\bar{a}_1|| \cdot ||\bar{a}_2|| \sin\theta$

$$|\det A| = ||\bar{a}_1|| \cdot ||\bar{a}_2^{\perp}||$$

6. $A_{n\times n} = (\bar{a}_1 \ldots \bar{a}_n)$ and $\exists A^{-1} \Rightarrow A = QR$

$$|\det Q| = 1, \quad r_{11} = ||\bar{a}_1||,\ r_{jj} = ||\bar{a}_j^{\perp}||\,\forall j \geq 2$$

$$|\det A| = ||\bar{a}_1||||\bar{a}_2^{\perp}|| \ldots ||\bar{a}_n^{\perp}||, \quad \bar{a}_i^{\perp} \perp span(\bar{a}_1, \ldots \bar{a}_{i-1})$$

If A^{-1} does not exist $\Rightarrow \exists \bar{a}_i$ redundant $\Rightarrow \bar{a}_i^{\perp} = 0 \Rightarrow \det A = 0$

Example: $A_{3\times 3} = (\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3) \Rightarrow$
$|\det A| =$ the volume of the parallelepiped spanned by $\bar{a}_1\ \bar{a}_2\ \bar{a}_3$.

m-volume

Definition: Let $\bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$.
The *m-parallelepiped* defined by the vectors $\bar{a}_1, \ldots, \bar{a}_m$ is

$$\{\bar{x} \in \mathbb{R}^n \colon \bar{x} = c_1\bar{a}_1 + \ldots + c_m\bar{a}_m,\ 0 \leq c_i \leq 1\, \forall i = 1..m\}$$

m-volume

Definition: Let $\bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$.
The *m-parallelepiped* defined by the vectors $\bar{a}_1, \ldots, \bar{a}_m$ is

$$\{\bar{x} \in \mathbb{R}^n \colon \bar{x} = c_1\bar{a}_1 + \ldots + c_m\bar{a}_m,\ 0 \leq c_i \leq 1\, \forall i = 1..m\}$$

The *m-volume* $V(\bar{a}_1, \ldots, \bar{a}_m)$ is

$$V(\bar{a}_1) = ||\bar{a}_1||, \quad V(\bar{a}_1, \ldots, \bar{a}_m) = V(\bar{a}_1, \ldots, \bar{a}_{m-1})||\bar{a}_m^{\perp}||$$

$$\text{OR} \quad V(\bar{a}_1, \ldots, \bar{a}_m) = ||\bar{a}_1||||\bar{a}_2^{\perp}|| \ldots ||\bar{a}_m^{\perp}||$$

m-volume

Definition: Let $\bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$.
The m-parallelepiped defined by the vectors $\bar{a}_1, \ldots, \bar{a}_m$ is

$$\{\bar{x} \in \mathbb{R}^n \colon \bar{x} = c_1 \bar{a}_1 + \ldots + c_m \bar{a}_m,\ 0 \leq c_i \leq 1\, \forall i = 1..m\}$$

The m-volume $V(\bar{a}_1, \ldots, \bar{a}_m)$ is

$$V(\bar{a}_1) = ||\bar{a}_1||, \quad V(\bar{a}_1, \ldots, \bar{a}_m) = V(\bar{a}_1, \ldots, \bar{a}_{m-1})||\bar{a}_m^{\perp}||$$

$$\text{OR} \quad V(\bar{a}_1, \ldots, \bar{a}_m) = ||\bar{a}_1||||\bar{a}_2^{\perp}|| \ldots ||\bar{a}_m^{\perp}||$$

Remark: $A_{n \times m} = (\bar{a}_1 \ldots \bar{a}_m)$ and $\exists A^{-1} \Rightarrow A = QR$

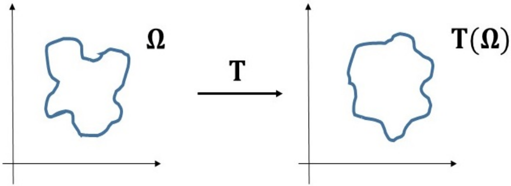
$$A^T A = R^T Q^T QR = R^T R \Rightarrow \det A^T A = (\det R)^2 = (V(\bar{a}_1, \ldots, \bar{a}_m))^2$$

$$V(\bar{a}_1, \ldots, \bar{a}_m) = \sqrt{\det A^T A}$$

$$m = n \Rightarrow V(\bar{a}_1, \ldots, \bar{a}_m) = |\det A|$$

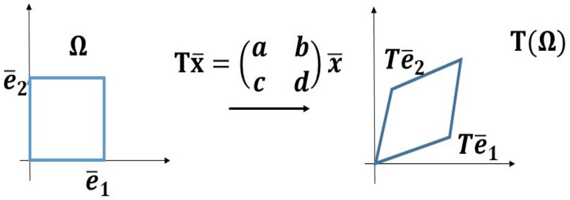
Expansion Factor

Definition: Consider a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$



$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \text{the expansion factor}$$

Expansion Factor

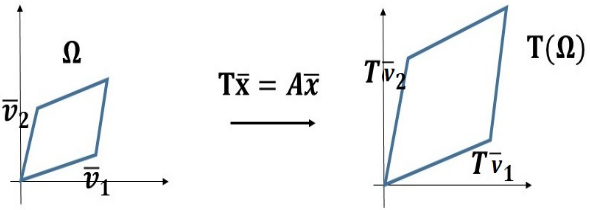


$$\text{The expansion factor} = \frac{\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|}{1} = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

$$\text{Exp.factor} = |\det A|$$

Expansion Factor

If $\Omega = span\,(\bar{v}_1,\ \bar{v}_2)$, $B = (\bar{v}_1 \quad \bar{v}_2)$ and $T\bar{x} = A\bar{x}$, then



$$\text{The expansion factor} = \frac{|\det\,(T\bar{v}_1 \quad T\bar{v}_2)|}{|\det B|} = |\det A|$$

$$\text{In } \mathbb{R}^n,\ |\det A| = \frac{V(A\bar{v}_1, \ldots, A\bar{v}_n)}{V(\bar{v}_1, \ldots, \bar{v}_n)}$$