vv214: Eigenvalue problems. Diagonalization.

Dr.Olga Danilkina

UM-SJTU Joint Institute

July 15, 2019

This week

1. Discrete dynamical systems
2. Eigenvectors/eigenfunctions and eigenvalues
3. Algebraic and geometric multiplicities
4. Eigenbasis
5. Diagonalization

Discrete Dynamical Systems

Consider a mathematical model

$$\begin{array}{r} \text{predator} \\ \text{prey} \end{array}\begin{cases} c(t+1) = a_1 c(t) + b_1 r(t), \\ r(t+1) = a_2 c(t) + b_2 r(t) \end{cases}$$

that describes the dynamic of two populations as time changes (say, a predator-prey model).

$t = t_0, t_1, t_2$

$\bar{x}(1) = A\bar{x}(0)$

$\bar{x}(2) = A\bar{x}(1) = A^2\bar{x}(0)$

$\bar{x}(3) = A^3\bar{x}(0)$

$\Rightarrow \bar{x}(t) = A^t \bar{x}(0) = A^t \begin{pmatrix} c_0 \\ r_0 \end{pmatrix}$

$$\Rightarrow 0 < a_1,\ b_1 < 1,\ b_2 > 1,\ a_2 < 0$$

$$a_1 = 0.86,\ b_1 = 0.08,\ a_2 = -0.12,\ b_2 = 1.14$$

$$\bar{x}(t) = \begin{pmatrix} c(t) \\ r(t) \end{pmatrix} \Rightarrow \bar{x}(t+1) = A\bar{x}(t),\ A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}$$

$$\bar{x}_0(t) = \begin{pmatrix} c_0 \\ r_0 \end{pmatrix} \xrightarrow{A} \bar{x}_1 = \bar{x}(1) = \begin{pmatrix} c(1) \\ r(1) \end{pmatrix} \xrightarrow{A} \ldots \xrightarrow{A} \bar{x}(t) \xrightarrow{A} \ldots$$

$$\bar{x}_1 = A\bar{x}_0,\ \bar{x}_2 = A\bar{x}_1 = A^2\bar{x}_0,\ \ldots,\ \boxed{\bar{x}(t) = A^t\bar{x}_0}$$

Discrete Dynamical Systems

Case 1: Let $c_0 = 100,\ r_0 = 300.$

$$\bar{x}(1) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1\bar{x}_0 \Rightarrow$$

$$\bar{x}(t) = (1.1)^t \bar{x}_0 \Rightarrow \begin{array}{c} c(t) = (1.1)^t 100 \\ r(t) = (1.1)^t 300 \end{array} \Rightarrow 10\% growth$$

Case 2: Let $c_0 = 200,\ r_0 = 100.$

$$\bar{x}(1) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 180 \\ 90 \end{pmatrix} = 0.9\bar{x}_0 \Rightarrow$$

$$\bar{x}(t) = (0.9)^t \bar{x}_0 \Rightarrow \begin{array}{c} c(t) = (0.9)^t 200 \\ r(t) = (0.9)^t 100 \end{array} \Rightarrow 10\% declination$$

Discrete Dynamical Systems

Case 3: Let $c_0 = 1000$, $r_0 = 1000$.

∗ find c.

$C_1 \bar{x}_{01} + C_2 \bar{x}_{02}$

$$\bar{x}(1) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = \begin{pmatrix} 940 \\ 1020 \end{pmatrix} \stackrel{?}{=} C\bar{x}_0$$

Since it worked in Cases 1,2, so let

$$\bar{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = C_1 \underbrace{\begin{pmatrix} 100 \\ 300 \end{pmatrix}}_{\bar{v}_1} + C_2 \underbrace{\begin{pmatrix} 200 \\ 100 \end{pmatrix}}_{\bar{v}_2} = 2\bar{v}_1 + 4\bar{v}_2$$

$$\bar{x}(1) = A\bar{x}_0 = A(2\bar{v}_1 + 4\bar{v}_2) = 2(1.1)\bar{v}_1 + 4(0.9)\bar{v}_2 \Rightarrow$$

$(A\bar{v}_1 = (1.1)\bar{v}_1)$

$$\bar{x}(t) = 2(1.1)^t\bar{v}_1 + 4(0.9)^t\bar{v}_2$$

$$c(t) = 200(1.1)^t + 800(0.9)^t$$
$$r(t) = 600(1.1)^t + 400(0.9)^t$$

Eigenvectors and Eigenvalues

Definition: A non-zero vector $\bar{v} \in \mathbb{R}^n$ is called an eigenvector of the matrix $A_{n\times n}$ if

$$\exists \lambda \in \mathbb{K} \colon A\bar{v} = \lambda \bar{v}$$

The scalar λ is called the eigenvalue associated with the eigenvector $\bar{v}$.

Remarks:

1. $A\bar{v} || \bar{v} \Rightarrow \bar{v}$ is the eigenvector.
2. $\bar{v}$ is the eigenvector of $A \Rightarrow \bar{v}$ is the eigenvector of $A^2, A^3, \ldots$
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto a line L

$$\Rightarrow \bar{w} \perp L,\ \bar{v} \subset L \quad \text{are eigenvectors} \,(\lambda = 0,\ \lambda = 1)$$

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation through $\frac{\pi}{2}$

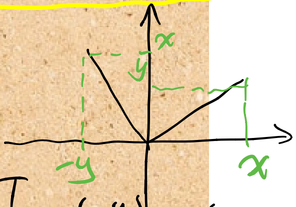
$$\Rightarrow \quad \text{there are no real eigenvalues and eigenvectors}$$

5. $A_{n\times n}$ is orthogonal $\Rightarrow$ the eigenvalues are ± 1

$A\bar{v} = \Omega \bar{v}$

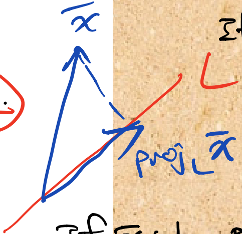
$A^2\bar{v} = A(A\bar{v})$
$= A(\Omega\bar{v}) = \Omega^2\bar{v}$

$\Rightarrow \Omega^2 \longrightarrow A^n$
(eigenvalue)



$\text{proj}_L \bar{x} = A\bar{x} = \Omega\bar{x}$

If $\bar{x} \in L$
$\Downarrow$
$\text{proj}_L \bar{x} = \bar{x} = A\bar{x}$
$\Downarrow$
$\Omega = 1$



If $\bar{x} \perp L$, $\text{proj}_L \bar{x} = 0$,
$= 0 \cdot \bar{x} = A\bar{x}$
$\Omega = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\exists \lambda (?)$

$A\bar{v} = \lambda \bar{v} \Rightarrow \begin{cases} -y = \lambda x \\ x = \lambda y \end{cases}$

$\Rightarrow -y = \lambda^2 y$

$\lambda^2 = -1$

$\lambda_{1,2} = \pm i$

Discrete Dynamical Systems

Definition: If a state $\bar{x}(t)$ of a physical system at any given time t is described by n values $x_1(t), \dots, x_n(t)$, then the *law of change* of the state of the system from time t to time $t+1$ in the linear form

$$\bar{x}(t+1) = A_{n \times n}\bar{x}(t)$$

is called a discrete dynamical system.

Remarks:

1. $\bar{x}(t) = A^t \bar{x}_0$ if the initial state $\bar{x}_0$ is known.
2. If a basis $\{\bar{v}_1, \dots, \bar{v}_n\} \subset \mathbb{R}^n$ consists of eigenvalues of A:

$$A\bar{v}_i = \lambda_i \bar{v}_i, \; i = 1..n,$$

then

$$\bar{x}(t) = \underbrace{(\bar{v}_1 \quad \bar{v}_2 \dots \bar{v}_n)}_{S} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}^t S^{-1}\bar{x}_0$$

If $\bar{x}(0) = x_0 = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \dots + \alpha_n \bar{v}_n$

$\bar{x}(t) = A^t \bar{x}_0 = A^t(\alpha_1 \bar{v}_1 + \dots + \alpha_n \bar{v}_n)$

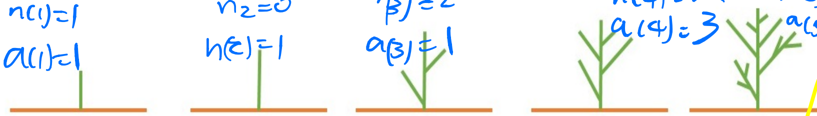
$= \alpha_1 \underbrace{A^t \bar{v}_1}_{\lambda_1^t \bar{v}_1} + \dots + \alpha_n \underbrace{A^t \bar{v}_n}_{\lambda_n^t \bar{v}_n}$

$= \alpha_1 \lambda_1^t \bar{v}_1 + \dots + \alpha_n \lambda_n^t \bar{v}_n$

$= (\bar{v}_1 \dots \bar{v}_n) \begin{pmatrix} \lambda_1^t & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_n^t \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

Lilac Bush Growth Model

$(\bar{v}_1 \cdots \bar{v}_n)^{-1}\bar{x}_0$



$n(1)=1$, $a(1)=1$

$n_2=0$, $n(2)=1$

$n(3)=2$, $a(3)=1$

$n(4)=2$, $a(4)=3$

$n(5)=6$, $a(5)=5$

$\bar{x}_0 = \alpha_1\bar{v}_1 + \cdots + \alpha_n\bar{v}_n$

$(\bar{x}_0)_{\underbrace{\{\bar{v}_1,\ldots,\bar{v}_n\}}_{S}} = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

$(\bar{x}_0)_{st} = (\bar{v}_1\ \bar{v}_2 \cdots \bar{v}_n)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

$n(t)$ is the number of new branches and
$a(t)$ is the number of old branches

$$a(t+1) = n(t) + a(t), \quad n(t+1) = 2a(t)$$

$$\begin{pmatrix} n(t+1)\\ a(t+1)\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2\\ 1 & 1\end{pmatrix}}_{A}\begin{pmatrix} n(t)\\ a(t)\end{pmatrix}$$

Lilac Bush Growth Model

$$A\begin{pmatrix}1\\1\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix},\quad A\begin{pmatrix}2\\-1\end{pmatrix}=-1\begin{pmatrix}2\\-1\end{pmatrix}$$

Since $n(0)=1$ and $a(0)=0$, so

$$\begin{pmatrix}1\\0\end{pmatrix}=c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}2\\-1\end{pmatrix}\Rightarrow c_1=c_2=\frac{1}{3}$$

$$\begin{pmatrix}n(t+1)\\a(t+1)\end{pmatrix}=\frac{2^t}{3}\begin{pmatrix}1\\1\end{pmatrix}+\frac{(-1)^t}{3}\begin{pmatrix}2\\-1\end{pmatrix}$$

Invariant Subspaces

Definition: Let $A_{n\times n}$. A linear subspace $V \subset \mathbb{R}^n$ is A-invariant if

$$A\bar{v} \in V \quad \forall v \in V$$

Let $\dim V = 1$.

1. If $\bar{v} \in V$, $\bar{v} \neq 0$, then

$$A\bar{v} \in V \Rightarrow A\bar{v} \in span\, \bar{v} \Rightarrow A\bar{v} = \lambda\bar{v}$$

   $\Rightarrow \quad \bar{v}$ is an eigenvector of A.

2. If $A\bar{v} = \lambda\bar{v}$ then

$$A\bar{v} = \lambda\bar{v} \in span\, \bar{v} = V$$

   $\Rightarrow$ as $\dim V = 1$, so V is A-invariant.

Therefore, one-dimensional A- invariant subspaces V of $\mathbb{R}^n$ are

$$V = \{span\, \bar{v} : A\bar{v} = \lambda\bar{v}\}$$

Eigenvalues and eigenfunctions

Definition: A scalar λ is called an eigenvalue of a linear operator $T: V \to V$ if

$$\exists f \in V,\ f \neq 0: \boxed{Tf = \lambda f.}$$

The element $f \in V$ is called an eigenfunction.

1. $T(x_1, x_2, x_3, \ldots) = (x_1, x_3, x_5, x_7, \ldots)$

$$T(\bar{x}) = (x_1, x_3, x_5, x_7, \ldots) = \lambda(x_1, x_2, x_3, x_4, x_5, \ldots)$$

$$\bar{x} = (x_1 = \lambda x_1, x_2, x_3 = \lambda x_2, x_4,\ x_5 = \lambda x_3, \ldots)$$

x_{2n} is arbitrary, $\quad x_{2n+1} = \lambda x_{n+1}$

$$\lambda = 5 \Rightarrow \bar{x} = (0, x_2, 5x_2, x_4, 25x_2, x_5, \ldots)$$

All real λ are eigenvalues with infinite number of eigensequences .

Eigenvalues and eigenfunctions

Shift operator

2. $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$

$$T(\bar{x}) = (x_2, x_3, x_4, x_5, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5, \ldots)$$

$$x_2 = \lambda x_1, \quad x_3 = \lambda x_2 = \lambda^2 x_1, \quad x_4 = \lambda x_3 = \lambda^3 x_1, \ldots$$

$$\bar{x} = (x_1, x_2, x_3, \ldots) = (x_1, \lambda x_1, \lambda^2 x_1, \ldots) = x_1(1, \lambda, \lambda^2, \ldots)$$

All real λ are eigenvalues and $(1, \lambda, \lambda^2, \ldots)$ is the eigensequence.

$A_{n \times n}$

$\bar{v} \neq 0,\ \exists \lambda \in \mathbb{K},$

$A\bar{v} = \lambda \bar{v}$

$\to A\bar{v} - \lambda I_n \bar{v} = \bar{0}$

$(A - \lambda I_n)\bar{v} = 0$

$n \times n$

$\Rightarrow$ since $\bar{v} = 0$,

$\det(A - \lambda I_n) = 0.$

Eigenvalues of a Matrix

Theorem: A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix $A_{n\times n}$ iff

$$\det(A - \lambda I_n) = 0.$$

The equation $f_A(\lambda) = \det(A - \lambda I_n) = 0$ is called the characteristic equation of the matrix A.

Remarks:

1. The eigenvalues of a diagonal matrix $A = diag(d_1, \ldots, d_n)$ are $d_1, \ldots, d_n$.
2. $\det(A - \lambda I_n) = (-\lambda)^n + (tr\, A)(-\lambda)^{n-1} + \ldots + \det A$
3. There are n eigenvalues of a $n \times n$ matrix A.
4. For $A_{n\times n}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$ listed with their algebraic multiplicities

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n, \quad tr\, A = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Algebraic Multiplicity

Definition: An eigenvalue λ_0 of a square matrix A has algebraic multiplicity k if λ_0 is the root of multiplicity k of the characteristic polynomial $f_A(\lambda) \Rightarrow f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, $g(\lambda_0) \neq 0$

Examples:

1. $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I_3) = 0 \Rightarrow \lambda_{1,2} = 0,\ \lambda_3 = 3$$

2. $$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \det(A - \lambda I_2) = 0$$

$$\Rightarrow \lambda^2 - \underbrace{(1+3)}_{tr\,A}\lambda + \underbrace{1 \cdot 3 - 2 \cdot 4}_{\det A} = 0$$

Eigenspace

Definition: The kernel of the matrix $A - \lambda I_n$ is called the eigenspace E_λ associated with λ :

$$E_\lambda = Ker\,(A - \lambda I_n) = \{\bar{v} \in \mathbb{R}^n : A\bar{v} = \lambda\bar{v},\ \bar{v} \neq 0\}$$

Examples:

1. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal projection onto a plane V in $\mathbb{R}^3$

$$\lambda = 1 \Rightarrow E_1 = V \quad \text{OR} \quad \lambda = 0 \Rightarrow E_0 = V^\perp$$

2. $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 5,\ \lambda_2 = -1$

$$E_5 = \{\bar{v} \in \mathbb{R}^2 : A\bar{v} = 5\bar{v}\} = Ker \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} = span \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$E_{-1} = \{\bar{v} \in \mathbb{R}^2 : A\bar{v} = -\bar{v}\} = Ker \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} = span \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenspace

3. $A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} \Rightarrow \lambda^2 = -1 \Rightarrow \lambda_{1,2} = \pm i$

$$E_i = \{\bar{v} \in \mathbb{R}^2 : A\bar{v} = i\bar{v}\} = Ker \begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix}$$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-(2+i)v_1 - v_2 = 0 \Rightarrow \bar{v} = \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$$

$$E_i = span \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$$

Remark: If λ is an eigenvalue of a real matrix $A_{n \times n}$ with the associated eigenvector $\bar{v}$, then $\bar{\lambda}$ is also an eigenvalue of A whose associated eigenvector $\bar{v}^{\star}$is the complex conjugate of $\bar{v}$.

$$|A-\lambda I| = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow (-\lambda)^2(1-\lambda)=0$$

$$\lambda_{1,2}=0,\ \lambda_3=1.$$

$\boxed{\lambda_{1,2}=0}$ $(A-\lambda I)\bar{v}=0$

$$\Rightarrow A\bar{v}=0$$

$$\Rightarrow \begin{pmatrix} -v_2-v_3 \\ v_2 \\ v_3 \end{pmatrix} = v_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow E_0 = \{\bar{v} : A\bar{v} = 0\bar{v}\}$$

$$= \left\{\bar{v} : \bar{v} = c_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$\dim E_0 = 2$, $GM_0 = 2$

$\boxed{\lambda_3=1}$ $(A-1I)\bar{v}=0$

Geometric Multiplicity

4. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \lambda_{1,2} = 1,\ \lambda_3 = 0$

$$E_1 = Ker\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_0 = Ker\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = span\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Definition: The dimension of eigenspace E_λ is called the geometric multiplicity of the eigenvalue λ.

$$G.M. = n - rank\,(A - \lambda I_n)$$

Let $\bar{v} = \alpha \bar{v}_1 + \dots + \alpha_{m-1}\bar{v}_{m-1}$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\bar{v} = 0$$

$$\Rightarrow E_1 = \left\{\bar{v} : \bar{v} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$\dim E_1 = G.M._1 = 1.$

$\dim E_0 + \dim E_1 = 3 = \dim \mathbb{R}^3$

Eigenbasis

Theorem:

1. The system $\bar{v}_1, \ldots, \bar{v}_s$ consisting of (all basis vectors) of each eigenspace of $A_{n\times n}$ is linearly independent; s is the sum of all geometric multiplicities of eigenvalues of A.

2. There exists an eigenbasis for $A_{n\times n}$ iff the geometric multiplicities add up to $n \Rightarrow s = n$.

Remark: If $A_{n\times n}$ has n distinct eigenvalues then there exists an eigenbasis for A.

Example: Let

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T\bar{x} = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_{1,2} = 1 \quad \text{BUT} \quad Ker(A - \lambda I_2) = span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\dim E_1 = 1 \Rightarrow G.M. = 1 < 2 = n$$

Let $\bar{v}_m = \alpha_1 \bar{v}_1 + \ldots$

be the 1st reduced ...

If $\bar{v}_k, \bar{v}_m \in E_{\lambda^*}$, $1 \le k \le m-1 \Rightarrow \bar{v}_m \ne \alpha_i \bar{v}_i$

$\Rightarrow \bar{v}_1, \ldots, \bar{v}_{m-1} \Rightarrow \exists \bar{v}_r : \lambda_r \ne \lambda^*$

$A\bar{v}_r = \lambda_r \bar{v}_r$, $A\bar{v}_k = \lambda^* \bar{v}_k$

$(A - \lambda^* I)\bar{v}_m = (A - \lambda^* I)(\alpha_1 \bar{v}_1 + \ldots + \alpha_{m-1}\bar{v}_{m-1}) \ne 0$

$$\underbrace{A\bar{v}_m - \lambda^* \bar{v}_m}_{=0} = \alpha_1(\lambda_1 - \lambda^*)\bar{v}_1 + \ldots + \alpha_r(\lambda_r - \lambda^*)\bar{v}_r + \ldots + \alpha_{m-1}(\lambda_{m-1} - \lambda^*)\bar{v}_{m-1}$$

Similar Matrices

$$\begin{tikzcd} x \arrow[r, "A"] & f(x) \\ {[x]_B} \arrow[u, "S"] \arrow[r, "B"] & {[f(x)]_B} \arrow[u, "S"'] \end{tikzcd}$$

Theorem: Let $B = S^{-1}AS$, i.e. A, B be similar matrices. Then

1. $f_A(\lambda) = f_B(\lambda)$
2. $rank(A) = rank(B), \quad nullity(A) = nullity(B)$
3. A, B have the same eigenvalues with the same AM, GM
4. $\det A = \det B, \quad tr(A) = tr(B)$

1. $f_B(\lambda) = \det(B - \lambda I) = \det(S^{-1}AS - \lambda S^{-1}S)$
$= \det(S^{-1}(A - \lambda I)S)$
$= \det(S^{-1})\det(A - \lambda I)\det S$
$= f_A(\lambda)$

2. Let $\bar{x} \in \ker B \Rightarrow B\bar{x} = \bar{0} \Rightarrow S^{-1}AS\bar{x} = \bar{0}$
$S\bar{x} = \ker A$

$S^{-1}AS\bar{x} = 0 \,|\, S$

$AS\bar{x} = \underbrace{S\bar{0}}_{=0}$

$T: \ker B \to \ker A, \quad T\bar{x} = S\bar{x}$

T is linear, one-to-one & onto

$\Rightarrow$ T is an $\boxed{\text{isomorphism}}$

$\ker T = \{0\}$,

Also, if $A_{n\times n}$, $B_{n\times n}$,
then $n = \dim \ker A + \dim \operatorname{im} A$
$= \dim \ker B + \dim \operatorname{im} B$

Motivation

$\ker T = \ker A$

$\boxed{\dim(\ker B) = \dim(\ker A)}$

$\ker A$

$T\bar{x} = S\bar{x}$

$T(\alpha_1\bar{x}_1 + \beta\bar{x}_2) = S(\alpha\bar{x}_1 + \beta\bar{x}_2)$

$= \alpha S\bar{x}_1 + \beta S\bar{x}_2$

$S\bar{x}_1 = S\bar{x}_2 \Rightarrow S(\bar{x}_1 - \bar{x}_2) = \bar{0}$

$\Rightarrow \underbrace{S^{-1}S}_{I}(\bar{x}_1 - \bar{x}_2) = S^{-1}\bar{0} = 0$

$(\bar{x}_1 = \bar{x}_2)$

Let $y = \ker A$ (onto, one-to-one)

$\exists \bar{x} = S^{-1}\bar{y} \in \ker B$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = 5 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \bar{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow \mathfrak{B} = \{\bar{v}_1,\ \bar{v}_2\}$ is the eigenbasis for A.

$$\begin{array}{ccc} \bar{x} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 & \xrightarrow{A} & T\bar{x} = -\alpha_1\bar{v}_1 + 5\alpha_2\bar{v}_2 \\ \downarrow & & \downarrow \\ \bar{x}_{\mathfrak{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} & \xrightarrow{B} & (T\bar{x})_{\mathfrak{B}} = \begin{pmatrix} -\alpha_1 \\ 5\alpha_2 \end{pmatrix} \end{array}$$

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

B is diagonal $\Rightarrow$ we denote $B = D$, $\mathfrak{D} = \{\bar{v}_1,\ \bar{v}_2\}$

Diagonalizable Matrices

Theorem: Consider a linear transformation $T\bar{x} = A\bar{x}$, $A_{n\times n}$.
Let $\mathfrak{D} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ be an eigenbasis for T: $A\bar{v}_i = \lambda_i \bar{v}_i$.
Then the $\mathfrak{D}$-matrix D of T is

$$D = S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Definition: An $n \times n$ matrix A is called diagonalizable if A is similar to a diagonal matrix D:

$$\exists \text{ invertible } S_{n\times n} : D = S^{-1}AS$$

Remark: The matrix $S^{-1}AS$ is diagonal iff the columns of S form an eigenbasis of A.

Theorem

1. A matrix A is diagonalizable iff there exists an eigenbasis for A.
2. If $A_{n \times n}$ has n distinct eigenvalues then A is diagonalizable.

Example 1

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 0,\ \lambda_{2,3} = 1$$

$$E_0 \overset{\ker A}{=} \Rightarrow \dim E_0 = \dim Ker\, A = 1$$

$$E_1 \overset{\ker}{=} \begin{pmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim E_1 = 2 \text{ iff } b = -ac$$

$$\dim E_0 + \dim E_1 = 3 = n$$

$$\bar{v}_1 = \begin{pmatrix} -a \\ 1 \\ 0 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 0 \\ c \\ 1 \end{pmatrix} \Rightarrow S = \begin{pmatrix} -a & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

特征根

Example 2

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = 1,\ \lambda_{2,3} = 0$$

$$E_0 = A \Rightarrow \dim E_0 = \dim Ker\, A = 2$$

$$E_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow Ker\, E_1 = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \dim E_1 = 1$$

$$\Rightarrow \dim E_0 + \dim E_1 = 3 = n \Rightarrow A \text{ is diagonalizable}$$

Example 2

$$\bar{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow S = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark

$$A_{n\times n} \quad \text{is diagonalizable } \Rightarrow \exists \,\text{invertible}\, S_{n\times n}: \; D = S^{-1}AS$$

$$A = SDS^{-1}$$

$$\Downarrow$$

$$A^t = (SDS^{-1})^t = SDS^{-1}SDS^{-1}\dots SDS^{-1} = SD^T S^{-1}$$

$$\Downarrow$$

$$A^t = SD^t S^{-1}$$

Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 5,\ \lambda_2 = -1 \Rightarrow A \text{ is diagonalizable}$$

$$E_5 = Ker \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \dim E_5 = 1$$

$$E_{-1} = Ker \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ \dim E_{-1} = 1$$

$$A = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}}_{S^{-1}}$$

$$A^t = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5^t & 0 \\ 0 & (-1)^t \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5^t + 2(-1)^t & 5^t + (-1)^{t+1} \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{pmatrix}$$

Diagonalizable Linear Operators

Definition: A scalar λ is called an eigenvalue of a linear operator $T: V \to V$ if

$$\exists f \in V,\ f \neq 0 \colon\ Tf = \lambda f.$$

The element $f \in V$ is called an eigenfunction.

If $\dim V < +\infty$, then a basis $\mathfrak{D}$ consisting of eigenfunctions of V is called an eigenbasis for T.

A linear operator T is diagonalizable if the matrix of T w.r.t. some basis is diagonal $\Leftrightarrow$ there exists an eigenbasis for T.

Example 1

$$T\colon P_2 \to P_2,\ Tf(x) = f(2x-1), \quad \mathfrak{B} = \{1, x, x^2\}$$

$$\begin{array}{ccc}
a + bx + cx^2 & \xrightarrow{T} & a - b + c + (2b-4c)x + 4cx^2 \\
\downarrow L_{\mathfrak{B}} & & \downarrow L_{\mathfrak{B}} \\
\begin{pmatrix} a \\ b \\ c \end{pmatrix} & \xrightarrow{A} & \begin{pmatrix} a-b+c \\ 2b-4c \\ 4c \end{pmatrix}
\end{array}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 4$$

$$\Rightarrow A \text{ is diagonalizable}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ is the eighebasis for } T$$

Example 1

$$S_{\mathfrak{D}\to\mathfrak{B}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow f_1 = 1,\ f_2 = -1 + x,\ f_3 = 1 - 2x + x^2$$

$$T(1) = \underbrace{1}_{\lambda_1},\ T(x-1) = \underbrace{2}_{\lambda_2}(x-1),\ T(x-1)^2 = \underbrace{4}_{\lambda_3}(x-1)^2$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Example 2

$$L\colon \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}, \quad L(A) = A - A^T$$

$$L(A) = \lambda A$$

1. A is symmetric $\Rightarrow A^T = A \Rightarrow L(A) = 0 = 0 \cdot A \Rightarrow \lambda = 0$

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\dim E_0 = 3$$

2. A is skew-symmetric $\Rightarrow A^T = -A \Rightarrow L(A) = 2A \Rightarrow \lambda = 2$

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\dim E_2 = 1$$

Ranking Problem

- Consider the results of a tournament

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl] \\ C \arrow[r] & D \arrow[ul] \end{tikzcd}$$

  where "$A \to B$" means A defeated B.

  How to rank the players fairly?

- The Page-Brin idea:it should *be worth more* to defeat a *better player* $\Leftarrow$ based on Perron (1907)-Frobenius (1912) Theorem

  How do we know who is better before ranking them?

- Define recursion!

Ranking Problem

* Give everyone the initial score of 1

$$\bar{x}_0 = \begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{array}{c} DU \\ SHAOXIONG \\ C \\ OLGA \end{array} \end{array}$$

* Define for all $n \geq 0$

$$\bar{x}_{n+1} = A\bar{x}_n,$$

where

$$A = \begin{array}{cc} & \begin{array}{cccc} DU & SHAOXIONG & C & O \end{array} \\ \begin{array}{c} DU \\ SHAOXIONG \\ C \\ O \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Ranking Problem

$$\bar{x}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

$$\bar{x}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

$$\bar{x}_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 3 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

The (n+1)th score of a player A is the sum of the nth scores of the players that the player A defeated.

Ranking Problem

$$\bar{x}_5 = \begin{pmatrix} 8 \\ 6 \\ 3 \\ 5 \end{pmatrix}, \bar{x}_{10} = \begin{pmatrix} 35 \\ 34 \\ 21 \\ 26 \end{pmatrix}, \bar{x}_{100} = \begin{pmatrix} 1037 \\ 933 \\ 547 \\ 731 \end{pmatrix}$$

Is $A > B > D > C$? $\Rightarrow$ analyze $\bar{x}_n = A^n \bar{x}_0$ as $n \to \infty$

Theorem (Perron-Frobenius): There exists a *largest positive* eigenvalue λ_{PF} for a nonnegative matrix A such that the rescaled system

$$\bar{x}_n = \left(\frac{1}{\lambda_{PF}} A\right)^n \bar{x}_0$$

converges to an equilibrium state $\bar{x}_\infty$.

$$\bar{x}_\infty = \bar{x}_{\infty+1} = \frac{1}{\lambda_{PF}} A\bar{x}_\infty \Rightarrow A\bar{x}_\infty = \lambda_{PF}\bar{x}_\infty$$

The equilibrium state is the eigenvector associated with λ_{PF}*!!!*

Ranking Problem

The largest positive eigenvalue is

$$\lambda_{PF} = 1.3953369...$$

and

$$\bar{x}_\infty = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{pmatrix} 0.321... \\ 0.288... \\ 0.165... \\ 0.230... \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

$$A > B > D > C$$