vv214: Eigenvalue problems. Diagonalization.

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UM-SJTU Joint Institute

July 18, 2019

This week

1. Discrete dynamical systems
2. Eigenvectors/eigenfunctions and eigenvalues
3. Algebraic and geometric multiplicities
4. Eigenbasis
5. Diagonalization

Discrete Dynamical Systems

Consider a mathematical model

$$\begin{cases} c(t+1) = a_1 c(t) + b_1 r(t), \\ r(t+1) = a_2 c(t) + b_2 r(t) \end{cases}$$

that describes the dynamic of two populations as time changes (say, a predator-prey model).

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Lilac Bush Growth Model

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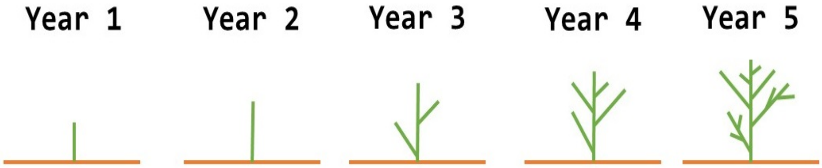

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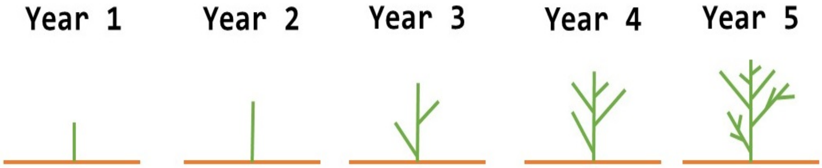

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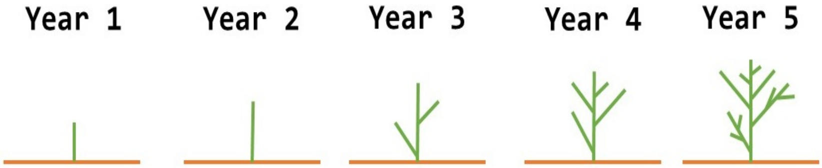

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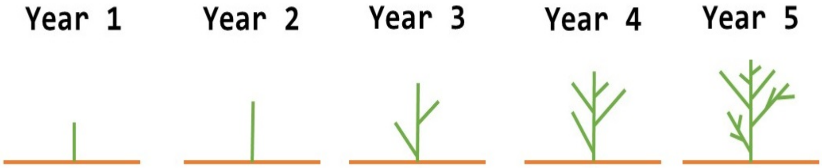

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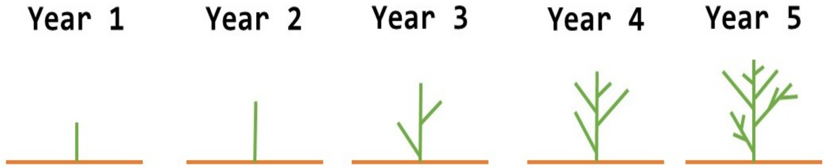

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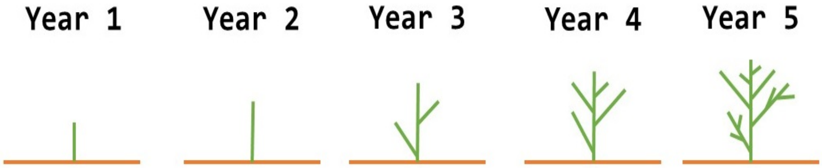

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$$\begin{pmatrix} n(t+1) \\ a(t+1) \end{pmatrix} = \frac{2^t}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{(-1)^t}{3}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

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$$A\bar{v} = \lambda\bar{v} \in span\,\bar{v} = V$$

   $\Rightarrow$ as $\dim V = 1$, so V is A-invariant.

Therefore, one-dimensional A- invariant subspaces V of $\mathbb{R}^n$ are

$$V = \{span\,\bar{v} \colon A\bar{v} = \lambda\bar{v}\}$$

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Theorem: A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix $A_{n\times n}$ iff

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The equation $f_A(\lambda) = \det(A - \lambda I_n) = 0$ is called the characteristic equation of the matrix A.

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4. For $A_{n\times n}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$ listed with their algebraic multiplicities

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n, \quad tr\, A = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

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Remark: If λ is an eigenvalue of a real matrix $A_{n\times n}$ with the associated eigenvector $\bar{v}$, then $\bar{\lambda}$ is also an eigenvalue of A whose associated eigenvector $\bar{v}^{\star}$is the complex conjugate of $\bar{v}$.

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$$G.M. = n - rank\,(A - \lambda I_n)$$

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Theorem:
1. The system $\bar{v}_1, \ldots, \bar{v}_s$ consisting of all basis vectors of each eigenspace of $A_{n \times n}$ is linearly independent; s is the sum of all geometric multiplicities of eigenvalues of A.
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Similar Matrices

Theorem: Let $B = S^{-1}AS$, i.e. A, B be similar matrices. Then

1. $f_A(\lambda) = f_B(\lambda)$
2. $rank(A) = rank(B), \quad nullity(A) = nullity(B)$
3. A, B have the same eigenvalues with the same AM, GM
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$= \det(S^{-1}(A - \lambda I)S) = \det S^{-1} \det(A - \lambda I) \det S = f_A(\lambda)$
2. Let $\bar{x} \in Ker\, B \Rightarrow B\bar{x} = \bar{0} \Rightarrow S^{-1}AS\bar{x} = \bar{0} \Rightarrow S\bar{x} \in Ker\, A$

$$T : Ker\, B \to Ker\, A \quad T\bar{x} = S\bar{x}$$

T is linear, one-to-one and onto $\Rightarrow$ T is an isomorphism.

Similar Matrices

Theorem: Let $B = S^{-1}AS$, i.e. A, B be similar matrices. Then

1. $f_A(\lambda) = f_B(\lambda)$
2. $rank(A) = rank(B), \quad nullity(A) = nullity(B)$
3. A, B have the same eigenvalues with the same AM, GM
4. $\det A = \det B, \quad tr(A) = tr(B)$

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Also, if $A_{n\times n}$, $B_{n\times n}$, then $n = \dim Ker\ A + \dim Im\ A$

Motivation

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = 5 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \bar{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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$$B = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

B is diagonal $\Rightarrow$ we denote $B = D$, $\mathfrak{D} = \{\bar{v}_1,\ \bar{v}_2\}$

Diagonalizable Matrices

Theorem: Consider a linear transformation $T\bar{x} = A\bar{x}$, $A_{n\times n}$.
Let $\mathfrak{D} = \{\bar{v}_1,\ \bar{v}_2, \dots, \bar{v}_n\}$ be an eigenbasis for T: $A\bar{v}_i = \lambda_i \bar{v}_i$.
Then the $\mathfrak{D}$-matrix D of T is

$$D = S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

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Definition: An $n \times n$ matrix A is called diagonalizable if A is

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Remark: The matrix $S^{-1}AS$ is diagonal iff the columns of S form an eigenbasis of A.

Theorem

1. A matrix A is diagonalizable iff there exists an eigenbasis for A.

2. If $A_{n \times n}$ has n distinct eigenvalues then A is diagonalizable.

Matrix Diagonalization: Example 1

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 0,\ \lambda_{2,3} = 1$$

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Remark

$$A_{n\times n} \quad \text{is diagonalizable } \Rightarrow \exists \, \text{invertible}\, S_{n\times n}:\ D = S^{-1}AS$$

$$A = SDS^{-1}$$

$$\Downarrow$$

$$A^t = (SDS^{-1})^t = SDS^{-1}SDS^{-1}\ldots SDS^{-1} = SD^T S^{-1}$$

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Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 5,\ \lambda_2 = -1 \Rightarrow A \text{ is diagonalizable}$$

$$E_5 = Ker \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \dim E_5 = 1$$

$$E_{-1} = Ker \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ \dim E_{-1} = 1$$

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$$A = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}}_{S^{-1}}$$

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$$A^t = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5^t & 0 \\ 0 & (-1)^t \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5^t + 2(-1)^t & 5^t + (-1)^{t+1} \\ 2 \cdot 5^t - 2(-1)^t & 2 \cdot 5^t + (-1)^t \end{pmatrix}$$

Diagonalizable Linear Operators

Definition: A scalar λ is called an eigenvalue of a linear operator $T\colon V \to V$ if

$$\exists f \in V,\ f \neq 0\colon\ Tf = \lambda f.$$

The element $f \in V$ is called an eigenfunction.

If $\dim V < +\infty$, then a basis $\mathfrak{D}$ consisting of eigenfunctions of V is called an eigenbasis for T.

A linear operator T is diagonalizable if the matrix of T w.r.t. some basis is diagonal $\Leftrightarrow$ there exists an eigenbasis for T.

Diagonalizable Linear Operators: Example 1

$$T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ Tp(t) = p(2t-1), \quad \mathfrak{B} = \{1, t, t^2\}$$

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\end{array}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow$$

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Diagonalizable Linear Operators: Example 2

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$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \dim E_2 = 1$$

Ranking Problem

* Consider the results of a tournament

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl] \\
C \arrow[r] & D \arrow[ul]
\end{tikzcd}$$

where "$A \rightarrow B$" means A defeated B.

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How do we know who is better before ranking them?

- Define recursion!

Ranking Problem

* Give everyone the initial score of 1

$$\bar{x}_0 = \begin{array}{cc} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

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* Define for all $n \geq 0$

$$\bar{x}_{n+1} = A\bar{x}_n,$$

where

$$A = \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Ranking Problem

$$\bar{x}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

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The (n+1)th score of a player A is the sum of the nth scores of the players that the player A defeated.

Ranking Problem

$$\bar{x}_5 = \begin{pmatrix} 8 \\ 6 \\ 3 \\ 5 \end{pmatrix}, \bar{x}_{10} = \begin{pmatrix} 35 \\ 34 \\ 21 \\ 26 \end{pmatrix}, \bar{x}_{100} = \begin{pmatrix} 1037 \\ 933 \\ 547 \\ 731 \end{pmatrix}$$

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Is $A > B > D > C$? $\Rightarrow$

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Theorem (Perron-Frobenius): There exists a *largest positive* eigenvalue λ_{PF} for a nonnegative matrix A such that the rescaled system

$$\bar{x}_n = \left(\frac{1}{\lambda_{PF}} A \right)^n \bar{x}_0$$

converges to an equilibrium state $\bar{x}_\infty$.

$$\bar{x}_\infty = \bar{x}_{\infty+1} = \frac{1}{\lambda_{PF}} A \bar{x}_\infty \Rightarrow A \bar{x}_\infty = \lambda_{PF} \bar{x}_\infty$$

The equilibrium state is the eigenvector associated with λ_{PF}!!!

Ranking Problem

The largest positive eigenvalue is

$$\lambda_{PF} = 1.3953369...$$

and

$$\bar{x}_\infty = \begin{array}{c} \left(\begin{array}{c} 0.321... \\ 0.288... \\ 0.165... \\ 0.230... \end{array} \right) \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

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