Vv214 Linear Algebra

Mid2

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Definition

Coordinates in a subspace of $\mathbb{R}^n$

Consider a basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ of a subspace V of $\mathbb{R}^n$. By Theorem 3.2.10, any vector $\vec{x}$ in V can be written uniquely as

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m.$$

The scalars $c_1, c_2, \ldots, c_m$ are called the $\mathfrak{B}$-*coordinates* of $\vec{x}$, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

is the $\mathfrak{B}$-*coordinate vector* of $\vec{x}$, denoted by $[\vec{x}]_{\mathfrak{B}}$. Thus

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{means that } \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m.$$

Note that

$$\vec{x} = S\,[\vec{x}]_{\mathfrak{B}}, \quad \text{where } S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}, \text{ an } n \times m \text{ matrix.}$$

Relation between them

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] & T(\vec{x}) \\
{[\vec{x}]_{\mathfrak{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, "S"]
\end{tikzcd}$$

Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$. Let B be the $\mathfrak{B}$-matrix of T, and let A be the standard matrix of T (such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$). Then

$$AS = SB, \quad B = S^{-1}AS, \quad \text{and} \quad A = SBS^{-1}, \quad \text{where} \quad S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}. \blacksquare$$

Change of basis

Consider two bases $\mathfrak{A}$ and $\mathfrak{B}$ of an n-dimensional linear space V. Consider the linear transformation $L_{\mathfrak{A}} \circ L_{\mathfrak{B}}^{-1}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, with standard matrix S, meaning that $S\vec{x} = L_{\mathfrak{A}}\left(L_{\mathfrak{B}}^{-1}(\vec{x})\right)$ for all $\vec{x}$ in $\mathbb{R}^n$. This invertible matrix S is called the *change of basis matrix* from $\mathfrak{B}$ to $\mathfrak{A}$, sometimes denoted by $S_{\mathfrak{B}\to\mathfrak{A}}$. See the accompanying diagrams. Letting $f = L_{\mathfrak{B}}^{-1}(\vec{x})$ and $\vec{x} = [f]_{\mathfrak{B}}$, we find that

$$[f]_{\mathfrak{A}} = S\,[f]_{\mathfrak{B}}, \quad \text{for all } f \text{ in } V.$$

If $\mathfrak{B} = (b_1, \ldots, b_i, \ldots, b_n)$, then

$$[b_i]_{\mathfrak{A}} = S\,[b_i]_{\mathfrak{B}} = S\vec{e}_i = (i\text{th column of } S),$$

so that

$$S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix} [b_1]_{\mathfrak{A}} & \qquad & [b_n]_{\mathfrak{A}} \end{bmatrix}$$

$$\begin{tikzcd} & \mathbb{R}^n \\ V \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\ & \mathbb{R}^n \arrow[uu, "S"'] \end{tikzcd} \qquad \begin{tikzcd} & {[f]_{\mathfrak{A}}} \\ f \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\ & {[f]_{\mathfrak{B}}} \arrow[uu, "S"'] \end{tikzcd}$$

Exercise

Let V be the subspace of C^∞ spanned by the functions e^x and e^{-x}, with the bases $\mathfrak{A} = (e^x, e^{-x})$ and $\mathfrak{B} = (e^x + e^{-x}, e^x - e^{-x})$. Find the change of basis matrix $S_{\mathfrak{B}\to\mathfrak{A}}$.

Answer

$$S = \begin{bmatrix} [e^x + e^{-x}]_{\mathfrak{A}} & [e^x - e^{-x}]_{\mathfrak{A}} \end{bmatrix}.$$

$$e^x + e^{-x} = 1 \cdot e^x + 1 \cdot e^{-x} \qquad e^x - e^{-x} = 1 \cdot e^x + (-1) \cdot e^{-x}$$

$$[e^x + e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [e^x - e^{-x}]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S_{\mathfrak{B} \to \mathfrak{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What you should know about this part

1. What function S,B,A has?
2. How to find them?
3. If x is not a simple vector, instead it is a function, a matrix, a complex number...Just handle it in the same way!
4. Be clear what the question demands you to get.

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Isomorphism

An invertible linear transformation T is called an *isomorphism*. We say that the linear space V is isomorphic to the linear space W if there exists an isomorphism T from V to W.

Properties

a. A linear transformation T from V to W is an isomorphism if (and only if) $\ker(T) = \{0\}$ and $\operatorname{im}(T) = W$.

In parts (b) through (d), the linear spaces V and W are assumed to be finite dimensional.

b. If V is isomorphic to W, then $\dim(V) = \dim(W)$.

c. Suppose T is a linear transformation from V to W with $\ker(T) = \{0\}$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

d. Suppose T is a linear transformation from V to W with $\operatorname{im}(T) = W$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

To prove a transformation is a isomorphism, first prove it is linear!

What you should know about this part

1. Definition of isomorphism
2. Prove it via properties(Im, Ker, Dim...)

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Inner Product

An *inner product* in a linear space V is a rule that assigns a real scalar (denoted by $\langle f, g \rangle$) to any pair f, g of elements of V, such that the following properties hold for all f, g, h in V, and all c in $\mathbb{R}$:

a. $\langle f, g \rangle = \langle g, f \rangle$ (symmetry)

b. $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$

c. $\langle cf, g \rangle = c\langle f, g \rangle$

d. $\langle f, f \rangle > 0$, for all nonzero f in V (positive definiteness)

A linear space endowed with an inner product is called an *inner product space*.

Properties b and c express the fact that $T(f) = \langle f, g \rangle$ is a linear transformation from V to $\mathbb{R}$, for a fixed g in V.

Compare these rules with those for the dot product in $\mathbb{R}^n$, listed in the Appendix, Theorem A.5. Roughly speaking, an inner product space behaves like $\mathbb{R}^n$ as far as addition, scalar multiplication, and the dot product are concerned.

Norm and orthogonality

The *norm* (or magnitude) of an element f of an inner product space is

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

Two elements f and g of an inner product space are called *orthogonal* (or perpendicular) if

$$\langle f, g\rangle = 0.$$

We can define the *distance* between two elements of an inner product space as the norm of their difference:

$$\text{dist}(f, g) = \|f - g\|.$$

What you should know about this part

1. A redefined inner product changes the definition of everything including norm(magnitude), orthogonality, distance...

Exercise

Consider a linear space $P_2(\mathbb{R})$.

a. Prove that $(f_1, f_2) = f_1(-1)f_2(-1) + f_1(0)f_2(0) + f_1(1)f_2(1)$ satisfies all the requirements for an inner product.

b. Take any nonzero element $f \in P_2(\mathbb{R})$ and calculate $||f||$.

c. Find a pair of orthonormal elements $u_1(t), u_2(t) \in P_2(\mathbb{R})$. Calculate the distance between $u_1(t)$ and $u_2(t)$.

d. Let $V = span\ (u_1(t), u_2(t))$. Describe $V^\perp$, calculate $\dim V^\perp$ and find any nonzero element in $V^\perp$.

e. Find an element $f \in P_2(\mathbb{R})$ such that $f \notin V \cup V^\perp$ and calculate its orthogonal projection onto V.

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Orthogonality

a. Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are called *perpendicular* or *orthogonal*[1] if $\vec{v} \cdot \vec{w} = 0$.

b. The *length* (or magnitude or norm) of a vector $\vec{v}$ in $\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

c. A vector $\vec{u}$ in $\mathbb{R}^n$ is called a *unit vector* if its length is 1, (i.e., $\|\vec{u}\| = 1$, or $\vec{u} \cdot \vec{u} = 1$).

The vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ in $\mathbb{R}^n$ are called *orthonormal* if they are all unit vectors and orthogonal to one another:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

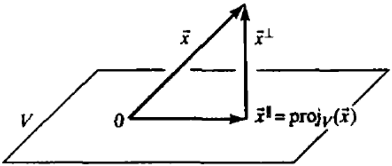
a. Orthonormal vectors are linearly independent.

b. Orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.

Orthogonal projection

The vector $\vec{x}^{\parallel}$ is called the *orthogonal projection* of $\vec{x}$ onto V, denoted by $\text{proj}_V(\vec{x})$. See Figure 4.

The transformation $T(\vec{x}) = \text{proj}_V(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is linear.



If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$, then

$$\text{proj}_V(\vec{x}) = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m.$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Consider an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_n$ of $\mathbb{R}^n$. Then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n.$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Orthogonal complement

Consider a subspace V of $\mathbb{R}^n$. The *orthogonal complement* $V^\perp$ of V is the set of those vectors $\vec{x}$ in $\mathbb{R}^n$ that are orthogonal to all vectors in V:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\}.$$

Note that $V^\perp$ is the kernel of the orthogonal projection onto V.

Consider a subspace V of $\mathbb{R}^n$.

a. The orthogonal complement $V^\perp$ of V is a subspace of $\mathbb{R}^n$.

b. The intersection of V and $V^\perp$ consists of the zero vector alone: $V \cap V^\perp = \{\vec{0}\}$.

c. $\dim(V) + \dim(V^\perp) = n$

d. $(V^\perp)^\perp = V$

Gram-Schmidt process

Definition

It gives the way to construct an orthonormal bases given an arbitrary basis of the space.
The construction process is as follow:

1. $$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}, \qquad \text{with respect to } \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}).$$

$$\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{\parallel} = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\,\vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j)\,\vec{u}_{j-1}$$

2. $$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1, \quad \vec{u}_2 = \frac{1}{\left\|\overrightarrow{v_2}^{\perp}\right\|}\vec{v}_2^{\perp}, \ldots, \overline{u}_m = \frac{1}{\left\|\overrightarrow{v_m}^{\perp}\right\|}\vec{v}_m^{\perp}$$

The calculation will not be more than 4 dimension.

QR factorization

Definition

Consider an $n \times m$ matrix M with linearly independent columns $\vec{v}_1, \dots, \vec{v}_m$. Then there exists an $n \times m$ matrix Q whose columns $\vec{u}_1, \dots, \vec{u}_m$ are orthonormal and an upper triangular matrix R with positive diagonal entries such that

$$M = QR.$$

This representation is unique. Furthermore, $r_{11} = \|\vec{v}_1\|$, $r_{jj} = \left\|\vec{v}_j^{\perp}\right\|$ (for $j = 2, \dots, m$), and $r_{ij} = \vec{u}_i \cdot \vec{v}_j$ (for $i < j$). ■

Just perform the Gram-Schmidt process for the set of columns of M and all the information about matrix Q and R will be shown in the process.

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Least squares approximation

Method

The least-squares solutions of the system

$$A\vec{x} = \vec{b}$$

are the exact solutions of the (consistent) system

$$A^T A\vec{x} = A^T\vec{b}.$$

The system $A^T A\vec{x} = A^T\vec{b}$ is called the *normal equation* of $A\vec{x} = \vec{b}$.

If $\ker(A) = \{\vec{0}\}$, then the linear system

$$A\vec{x} = \vec{b}$$

has the unique least-squares solution

$$\vec{x}^* = (A^T A)^{-1} A^T\vec{b}.$$

Note that $\vec{x}^*$ is not the orthogonal projection of $\vec{b}$, $A\vec{x}^*$ is.

Least squares approximation

Exercise

Consider the data in the following table.

Reading score, r	Shoe size, s
73	9
92	10
67	5
48	4

Suppose we model the relationship by $r = a + bs + cs^2$ for unknown a, b, c.

(a) (3 points) Set up a system of linear equations for a, b, c that uses all the data above. (Your system may be consistent or inconsistent, depending on the data. Your technique should result in the correct a, b, c when the data allows.)

(b) (3 points) Sketch a graph of r against s.

(c) (3 points) Is the system above consistent? Explain. Hint: You might want to refer to the graph and to consider $92 - 73, 73 - 67$, and $67 - 48$.

Least squares approximation

Exercise

(d) (3 points) Set up a system of linear equations whose solution is the least squares solution for a, b, c. Show work. You need not solve the system. Explain why there is a unique solution.

(e) (3 points) Instead of the above, now assume that s is a constant that doesn't depend on r, so $s = d$. Find the least squares solution for d, showing work.

Least squares approximation

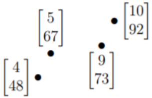
Answer

Solution:
We have

$$\begin{bmatrix} 1 & 9 & 81 \\ 1 & 10 & 100 \\ 1 & 5 & 25 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 73 \\ 92 \\ 67 \\ 48 \end{bmatrix}.$$

(b) (3 points) Sketch a graph of r against s.

Solution:



(c) (3 points) Is the system above consistent? Explain. Hint: You might want to refer to the graph and to consider $92 - 73, 73 - 67$, and $67 - 48$.

Solution:
No. As s increases, r increases, but with the pattern "big jump, little jump, big jump." This is inconsistent with r a quadratic function of s. That is, r is neither convex nor concave.

Least squares approximation

Answer

(d) (3 points) Set up a system of linear equations whose solution is the least squares solution for a, b, c. Show work. You need not solve the system. Explain why there is a unique solution.

Solution:

With A equal to

$$\begin{bmatrix} 1 & 9 & 81 \\ 1 & 10 & 100 \\ 1 & 5 & 25 \\ 1 & 4 & 16 \end{bmatrix},$$

the normal equation is

$$(A^T A) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \begin{bmatrix} 73 \\ 92 \\ 67 \\ 48 \end{bmatrix}.$$

Since the matrix A is Vandermonde with different parameters and at least as many rows as columns (i.e., since three distinct points determine a quadratic and we have data for at least three distinct points), the matrix A has full rank, 3. It follows that

Least squares approximation

Answer

A^TA is invertible. Thus

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^TA)^{-1}A^T \begin{bmatrix} 73 \\ 92 \\ 67 \\ 48 \end{bmatrix}$$

is the unique solution.

(e) (3 points) Instead of the above, now assume that s is a constant that doesn't depend on r, so $s = d$. Find the least squares solution for d, showing work.

Solution:
Note: There was a typo in the question and so it was not scored in Fall 2016. It should read "$r = d$."
Algebraicly, we have $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, so $A^TA = 4$ and

$$d = (A^TA)^{-1}A^T \begin{bmatrix} 73 \\ 92 \\ 67 \\ 48 \end{bmatrix}.$$

This is the average, $\frac{(73+67)+(92+48)}{4} = 70$.

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Riesz-Fischer

Refer to the slide "Riesz-Fischer" on Canvas. This is important!

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Remark

1. Know how to find determinant using properties of determinant, patterns, Laplace expansion...Properties are especially important!!
2. Try to make the determinant easy to solve: using Laplace expansion to reduce the calculation of 4×4 matrix to 3×3 matrix, use properties of determinant and elementary row operation to make the determinant obvious...

Exercise

$$\det\begin{pmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{pmatrix}, \qquad \det\begin{pmatrix} 3 & 2 & 2 & 2 & \dots & 2 & 2 \\ 2 & 3 & 2 & 2 & \dots & 2 & 2 \\ 2 & 2 & 3 & 2 & \dots & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & 2 & \dots & 2 & 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 2 & 1 \\ 2 & 2 & 2 & \dots & 2 & 2 & 2 \\ 2 & 2 & 2 & \dots & 3 & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & n-1 & 2 & \dots & 2 & 2 & 2 \\ n & 2 & 2 & \dots & 2 & 2 & 2 \end{pmatrix}$$