$z \in \mathbb{C} \Rightarrow z = |z| e^{i\varphi} \quad \varphi = \arg z$

$(x,y) = |(x,y)| \cdot e^{i\varphi}, \quad \varphi = \arg(x,y)$

$\tilde{x} = x \cdot e^{-i\varphi} \Rightarrow (\tilde{x},\tilde{x}) = (x e^{-i\varphi}, x e^{-i\varphi}) = e^{-i\varphi} e^{i\varphi} (x,x) =$
$= (x,x)$

$(\tilde{x},y) = (x e^{-i\varphi}, y) = e^{-i\varphi}(x,y) = e^{-i\varphi} |(x,y)| e^{i\varphi} = |(x,y)|$
$\in \mathbb{R}$

$\Rightarrow (x,y) \in \mathbb{R} \Rightarrow \mathrm{Re}(\tilde{x},y) = (\tilde{x},y) \Rightarrow |(\tilde{x},y)| \le \sqrt{(\tilde{x},\tilde{x})}\sqrt{(y,y)}$

$X, (\cdot,\cdot)$

$\|x\| = \sqrt{(x,x)} \quad \forall x$

1) $\|x\| \ge 0 \qquad \|x\| = 0 \Leftrightarrow (x,x) \Leftrightarrow x = 0$

2) $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{\alpha \cdot \bar{\alpha}(x,x)} = \sqrt{|\alpha|^2 (x,x)} = |\alpha|\sqrt{(x,x)} = |\alpha|\|x\|$

3) $\|x+y\|^2 = (x+y, x+y) = \underbrace{(x,x)}_{\|x\|^2} + \underbrace{(x,y) + (y,x)}_{2\,\mathrm{Re}(x,y)} + (y,y)$

$\le \|x\|^2 + 2\sqrt{(x,x)}\sqrt{(y,y)} + \|y^2\| = \|x^2\| + 2\|x\|\,\|y\| + \|y^2\| =$
$= (\|x\| + \|y\|)^2$

$\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y) =$
$= (x,x) + \cancel{(x,y)} + \cancel{(y,x)} + (y,y) + (x,x) - \cancel{(x,y)} - \cancel{(y,x)} + (y,y)$
$= 2(x,x) + 2(y,y) = 2(\|x\|^2 + \|y\|^2)$

Orthonormality

$\mathbb{R}^d \qquad \sum_{i=1}^{d} x_i y_i = (\bar{x}, \bar{y})$

$\|\bar{x}\| = \sqrt{\sum_{i=1}^{2} x_i^2} = \sqrt{x_1^2 + x_d^2}$

$x = cy + w$
$x = cy + (x - cy)$

$(x - cy, x) = (x,x) - c(y,x) = 0 \qquad c = \frac{(x,x)}{(y,x)}$

$(x - cy, y) = (x,y) - c(y,y) = 0 \qquad c = \frac{(x,y)}{(y,y)}$

$(v_i, v_j) = \begin{cases} 0, & i \ne j \\ 1 & i = j \end{cases}$

$\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \quad | \; v_i, \forall i$
$\alpha_1 (v_i, v_i) + \dots \alpha_n (v_i, v_j) = 0$

$V = \mathrm{span}(v) \qquad u = \frac{v}{\|v\|} \qquad V = \mathrm{span}(v_1, v_2)$

a) $u_1 = \frac{v_1}{\|v_1\|} \qquad u_2 = \perp u_1$

$x = cy + w = x^{\parallel} + x^{\perp}$

Coordinates

Let's say we have two vectors $\vec{v}_1 \vec{v}_2 \Rightarrow$ basis of V by "$\mathfrak{B}$" then the coordinate vector of $\vec{x}$ withe respect to $\mathfrak{B}$ denoted by $[\vec{x}]_{\mathfrak{B}}$

If $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = 3\vec{v}_1 + 2\vec{v}_2$ then $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ means $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \dots c_m\vec{v}_m]$

$\vec{x} = S[\vec{x}]_{\mathfrak{B}}$ where $S = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$

Properties

① $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}}$ ② $[\vec{x} \cdot k]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}}$

$$\begin{array}{ccc} \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 & \xrightarrow{T} & T(\vec{x}) = c_1\vec{v}_1 \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} & [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \end{array}$$

Definition: The matrix of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$; $\mathfrak{B}$ of $\mathbb{R}^n$ $\Rightarrow$ $n \times n$ matrix B that $[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}} \Rightarrow B$ is a $\mathfrak{B}$ matrix of T

If $\mathfrak{B} = (\vec{v}_1 \vec{v}_2 \dots \vec{v}_n)$ then $B = \left[[T(\vec{v}_1)]_{\mathfrak{B}} \; [T(\vec{v}_n)]_{\mathfrak{B}} \right]$

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & T(\vec{x}) \\ S \uparrow & & \uparrow S \\ [\vec{x}]_{\mathfrak{B}} & \xrightarrow{B} & [T(\vec{x})]_{\mathfrak{B}} \end{array}$$

$AS = SB$

$B = S^{-1}AS$

$A = SBS^{-1}$

where

$S = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_n]$

$T(\vec{x}) = A(\vec{x})$

Similar matrices

$AS = SB \Rightarrow B = S^{-1}AS$ (matrices are similar if they represent the same linear transformation with respect to different bases)

Linear transformation and Isomorphism

- An invertible linear transformation T is called an isomorphism. We say that the linear space V is isomorphic to the linear space W if there exists an isomorphism T from V to W

- Coordinate transformation $L_{\mathfrak{B}}(f) = [f]_{\mathfrak{B}}$ from V to $\mathbb{R}^n$ is an isomorphism.

Properties of isomorphisms.

a) T is isph. from V to W if and only if $\ker(T) = 0$ and $\mathrm{im}(T) = W$

b) $\dim(V) = \dim(W)$ isph.

c) if T is a linear transformation from V to W with $\ker(T) = 0$ then T is an isomorphism

Ex. Find $u \in P_2(\mathbb{R})$ s.t

$$\int_{-1}^{1} p(t)\cos \pi t \, dt = \int_{-1}^{1} p(t) u(t) \, dt$$

$\forall p(t) \in P_2(\mathbb{R})$

$P_2(\mathbb{R})$: $1, t, t^2$ - the usial basis ; $\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{45}{8}}(t^2-\frac{1}{3})$

- the orthonormal basis

$$u = \frac{1}{\sqrt{2}}\varphi\left(\frac{1}{\sqrt{2}}\right) + \sqrt{\frac{3}{2}}t\varphi\left(\sqrt{\frac{3}{2}}t\right) + \sqrt{\frac{45}{8}}(t^2-\frac{1}{3})\varphi\left(\sqrt{\frac{45}{8}}(t^2-\frac{1}{3})\right)$$

$$= -\frac{45}{2\pi^2}(t^2-\frac{1}{3})$$

$$\boxed{\forall \varphi \in V' \ \exists \underline{u} \in V \quad \varphi(v) = (v, u) \quad \forall v \in V}$$

$$\Rightarrow u = u_1 \overline{\varphi(u_1)} + \dots + u_n \overline{\varphi(u_n)}$$

Theorem $(\operatorname{Im} A)^\perp = \ker(A^*)$

$A: \mathbb{R}^m \to \mathbb{R}^n \quad T\bar{x} = A\bar{x} \quad \Rightarrow A_{n \times m}$

$A = (\bar{a}_1, \bar{a}_2, \bar{a}_3 \dots \bar{a}_m)$

$\operatorname{Im} A = \{c_1\bar{a}_1 + c_2\bar{a}_2 + \dots c_m\bar{a}_m\}$

$(\operatorname{Im} A)^\perp = \{\bar{v} \in \mathbb{R} \ ; \ (\bar{x}, \bar{v}) = 0 \quad \forall \bar{x} \in \operatorname{Im} A\}$

$(c_1a_1 + c_2a_2 + \dots + c_m\bar{a}_m, \bar{v}) = 0 \quad \forall c_i$

V, $\dim V = n$, $U \in V$

$\dim V = \dim U + \dim U^\perp$

$A_{3\times 2} \Rightarrow \mathbb{R}^2 \to \mathbb{R}^3$

Least-squares solution

$A\bar{x} = \bar{b}$ $\quad b \in \operatorname{Im} A$

$A\ \mathbb{R}^m \to \mathbb{R}^n \quad \operatorname{Im} A \in \mathbb{R}^n$

$(\operatorname{Im} A)^\perp = \operatorname{Ker} A^T$

$(\bar{b} - Ax^*) \in \operatorname{Ker} A^T \Rightarrow A^T\bar{b} - A^TAx^* = \bar{0}$

$A\vec{x} = \bar{b}$ axact $\quad A^TA\bar{x} = A^T\bar{b}$

Ex) $A\bar{x} = \bar{b} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$

$\operatorname{Ker} A = \{\bar{x} = \mathbb{R}^2 \quad \begin{matrix} x_1 + x_2 = 0 \\ x_1 + 2x_2 = 0 \\ x_1 + 3x_2 = 0 \end{matrix} \quad * \{0, 0, 0\}$

$\Rightarrow \exists A^{-1}$ and $\exists (A^TA)^{-1}$ $\quad \det = 42 - 36 = 6$

$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$

$(A^TA)^{-1} = \frac{1}{\det(A^TA)}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 14/6 & -1 \\ -1 & 0.5 \end{pmatrix}$

$\bar{x}^* = (A^TA)^{-1}A^T\bar{b} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$\bar{b} - A\bar{x}^* = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \to \|b - Ax^*\| = \sqrt{6}$

Let X be an inner product space and let X be complete

The series $\sum_{i=1}^{\infty}(x;e_i)e_i$, where $x \in X$

is called a Fourier series

$C[-\pi,\pi]$

$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}$

$C[-\pi,\pi]$: $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$, $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$

$L_2[-\pi;\pi]$ $\int_{-\pi}^{\pi} f^2 d\mu$

μ is a measure

a) $\operatorname{rank} A = \operatorname{rank} A^T$

b) $\operatorname{rank} A = \operatorname{rank}(A^T A)$

$C[-1,-1]$: $(f,g) = \int_{-1}^{1} \sqrt{1-t^2}\, f(t)g(t)\, dt$

$(f,f) = \int_{-1}^{1}\sqrt{1-t^2}\, f^2(t)dt \Rightarrow f=0 \Rightarrow (f,f)=0$

$\Rightarrow \int_{-1}^{1} \sqrt{1-t^2}\, f^2(t)\, dt = 0$; $(f,g)=\overline{(g,f)}$

; $(\alpha f + \alpha g, h)$ $\int_{-1}^{1} \sqrt{1-t^2}\,(\alpha f(t) + \beta g(t))\, h(t)\, dt$

$= \alpha (f,h) + \beta (f,t)$

$P_3(\mathbb{R})$ $1, t, t^2, t^3$

find the orthonormal basis

$\int_{-1}^{1}\sqrt{1-t^2}\,dt =$

$t=\cos\alpha$
$dt=-\sin\alpha$
$1=\cos\alpha$ $\alpha=0$
$-1=$ π 0 $-\frac{\pi}{2}$

$1-$ $\int_0^{\pi}\cos\alpha\, d\alpha$

$\cos\pi - \cos 0$

$-1-$

1) $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\Rightarrow \exists A^{-1}$ iff $\det A \neq 0$

2) Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ $A: \mathbb{R}^3 \to \mathbb{R}^3$

$\exists A^{-1}$ $\mathbb{R}^3 \to \mathbb{R}^3$

A $\dim = m$ $\dim = n$

$T: V \to W$ | $\operatorname{Ker} A = \{0\}$
$\operatorname{Im} A = W$

$\boxed{7-2}$

$\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \cdots \neq 0$ iff $\exists \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}$