vv214: Linear transformations II.

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This week

Today

1. Kerenel and image of a linear transpromation.
2. Rank-Nullity Theorem.
3. Inverse linear transformations.

Next class

Coordinates.

Image and Kernel of a Linear Transformation

Definition: The kernel and image of a linear operator $T: V \to W$ are defined by

$$Ker\ T = \{v \in V : Tv = 0\} \quad Im\ T = \{w \in W : w = Tv,\ v \in V\}$$

Examples: 1. $T: \mathbb{R} \to \mathbb{R}$, $Tx = x^2$ (not linear)

$$Ker\ T = \{0\},\ Im\ T = \mathbb{R}_+ \cup \{0\}$$

2. $T: \mathbb{R} \to \mathbb{R}^2$, $T(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ (not linear)

$$Ker\ T = \varnothing,\ Im\ T = \text{unit circle}$$

3. $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

$$Ker\ T = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ k = const,\ Im\ T = xy \text{ plane}$$

Image and Kernel of a Linear Transformation

4. $T\colon \mathbb{R}^2 \to \mathbb{R}^2,\ T\bar{x} = A\bar{x},\ A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

$$Ker\ A = t \begin{pmatrix} -3/2 \\ 1 \end{pmatrix},\ Im\ A = span \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

5. $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ Tp(t) = p'(t)$

$$p(t) = a_0 + a_1 t + a_2 t^2 \Rightarrow Tp(t) = p'(t) = a_1 + 2a_2 t$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ker\ T = \{p(t)\colon\ Tp = 0\} = \{a_0\} = span(1),\ Im\ T = span(1, t)$$

Image and Kernel of a Linear Transformation

Lemma 1: Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be defined by the matrix $A_{n\times m}$. The columns of the matrix A are linearly independent iff

$$Ker\, A_{n\times m} = \{\bar{0}\} \iff rank\, A = m \Rightarrow m \leq n$$

Lemma 2: Let $T\colon V \to W$ be a linear operator. $Im\, T$ and $Ker\, T$ are linear subspaces of V and $W \Rightarrow$ there exist bases of the kernel and the image of a linear transformation.

Image and Kernel of a Linear Transformation

Definition: A map $T\colon V \to W$ is called injective if $Tu = Tv$ implies $u = v$.
"distinct inputs to distinct outputs"

Lemma: A linear operator $T\colon V \to W$ is injective iff $Ker\ T = \{0\}$.

- Let T be injective. As $\{0\} \subset Ker\ T$, so we need to show that $Ker\ T \subset \{0\}$.
  Let $v \in Ker\ T \Rightarrow Tv = 0 = T(0) \Rightarrow v = 0 \Rightarrow Ker\ T \subset \{0\}$.
- Let $Ker\ T = \{0\}$. If $Tu = Tv$, then
  $T(u - v) = 0 \Rightarrow u - v \in Ker\ T \Rightarrow u - v = 0 \Rightarrow u = v$

Example: $T\colon \mathbb{R}^2 \to \mathbb{R}^3,\ T(x,y) = (2x, 3y, x + 2y)$

Definition: A map $T\colon V \to W$ is called surjective if $Im\ T = W$.

Example: $T : P_5(\mathbb{R}) \to P_5(\mathbb{R}),\ Tp(t) = p'(t)$ is not surjective.

Image and Kernel of a Linear Transformation: Example

$$T\colon \mathbb{R}^6 \to \mathbb{R}^4,\ A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathit{Ker}\ A = \{\bar{x} \in \mathbb{R}^6 \colon A\bar{x} = 0\}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 + 3x_6 = 0 & \Rightarrow x_2 = -2x_3 - 3x_6 \\ x_4 + 4x_6 = 0 & \Rightarrow x_4 = -4x_6 \\ x_5 + 5x_6 = 0 & \Rightarrow x_5 = -5x_6 \end{cases}$$

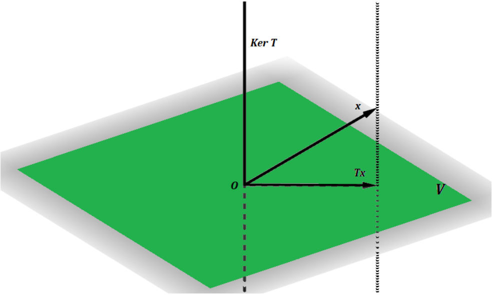
$$\bar{x} = r \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{pmatrix} \Rightarrow \dim \mathit{Ker}\ A = 3$$

Rank-Nullity Theorem

$$\boxed{\dim Ker\ T + \dim Im\ T = \dim V}$$

Example: $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\bar{x} = proj_V \bar{x},\ V \subset \mathbb{R}^3$

$$Ker\ T = \{\bar{x} \in \mathbb{R}^3 : proj_V \bar{x} = \bar{0}\},\ Im\ T = V$$



$$Ker\ T = \text{line orthogonal to } V$$

$$\underbrace{m}_{3} - \underbrace{\dim\left(Ker\ T\right)}_{1} = \underbrace{\dim Im\ T}_{2}$$

Rank-Nullity Theorem: Proof

Let $\dim(Ker\ T) = n$ and $\dim Ker\ T = k \Rightarrow k \leq n$.

$\Rightarrow$ there exists a basis $v_1, \ldots, v_k$, of $Ker\ T$. Complete this basis up to the basis of V: $v_1, \ldots, v_k,\ v_{k+1}, \ldots, v_n$

We are to prove that $Tv_{k+1}, \ldots,\ Tv_n$ form the basis for $Im\ T$:

1 $Tv_{k+1}, \ldots,\ Tv_n$ are linearly independent:

$$\alpha_1 Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n = 0 \Rightarrow T(\alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n) = 0$$

$\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in Ker\ T$

$\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in span(v_1, \ldots, v_k)$

But $v_1, \ldots, v_k,\ v_{k+1}, \ldots, v_n$ are linearly independent

$\Rightarrow \alpha_1 = \ldots = \alpha_{n-k} = 0$

2 $span(Tv_{k+1}, \ldots,\ Tv_n) = Im\ T$

A $w \in Im\ T \Rightarrow \exists v \in V\colon\ Tv = w \Rightarrow T(\beta_1 w_1 + \ldots + \beta_n v_n) = w$

$$w = \beta_1 \underbrace{Tv_1}_{=0} + \ldots + \beta_k \underbrace{Tv_k}_{=0} + \beta_{k+1} Tv_{k+1} + \ldots + \beta_n Tv_n$$

$$w \in span(Tv_{k+1}, \ldots,\ Tv_n) \Rightarrow Im\ T \subset span(Tv_{k+1}, \ldots,\ Tv_n)$$

B $w \in span(Tv_{k+1}, \ldots,\ Tv_n) \Rightarrow w = \alpha_{k+1} Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n$

$w = T(\alpha_{k+1} v_{k+1} + \ldots + \alpha_{n-k} v_n) \Rightarrow w \in Im\ T$

Inverse Linear Transformations

Definition: Let V, W be linear spaces.
A linear operator $T\colon V \to W$ is called invertible if there exists a linear operator $S\colon W \to V$ such that ST equals the identity map on V and TS equals the identity map on W.

A linear operator $S\colon W \to V$ satisfying $ST = I$ and $TS = I$ is called an inverse of T.
Here the first I is the identity map on V and the second I is the identity map on W. We shall denote the inverse linear operator by T^{-1}.

$$T^{-1}(Tv) = v \quad \text{and} \quad T(T^{-1}w) = w \quad \forall v \in V \, \forall w \in W$$

Inverse Linear Transformations

Lemma: A linear operator is invertible iff it is one-to-one (injective) and onto (surjective).

- Let T^{-1} exists.
  - A Let $u, v \in V$ and $Tu = Tv$

    $$u = T^{-1}(Tu) = T^{-1}(Tv) = v \Rightarrow \ T \text{ is injective}$$

  - B Let $w \in W \Rightarrow w = T(T^{-1}w) \Rightarrow w \in Im\, T \Rightarrow W \subset Im\, T$
    As also $Im\, T \subset W$, so $W = Im\, T$
- Let T be injective and surjective. For any $w \in W$, define Sw be a unique element of V such that $T(Sw) = w$. This element exists since T is one-to-one and onto.
  - A From the definition, $TS = I$. Also

    $$T((ST)v) = (TS)(Tv) = ITv = Tv \Rightarrow STv = v \Rightarrow ST = I$$

  - B S is linear:

    $$w_1,\ w_2 \in W \Rightarrow T(Sw_1 + Sw_2) = TSw_1 + TSw_2 = w_1 + w_2$$

    Apply the definition of $S \Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2$
    Similarly, $S(\alpha w) = \alpha Sw \, \forall w \in W \, \forall \alpha \in \mathbb{K}$

Inverse Linear Transformations

Remarks:

1. $(T^{-1})^{-1} = T$
2. Let $V, W = \mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the system $A\bar{x} = \bar{y}$ has a unique solution

$$\iff \textit{rank}\, A = n \iff \textit{rref}\, A = I_n$$

Definition: A square matrix A is invertible if the linear transformation $T\bar{x} = A\bar{x}$ is invertible.

Inverse Linear Transformations

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix} \Rightarrow A\bar{x} = I\bar{y}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$I\bar{y} = A^{-1}\bar{x} \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 2 \\ -7 & 3 & -1 \end{pmatrix}$$

Inverse Linear Transformations

1. Let $A_{n \times n}$. If A^{-1} exists, then the system $A\bar{x} = \bar{0}$ has a unique solution

   $$\Rightarrow rank\, A = n \Rightarrow \text{ columns of } A \text{ are linearly independent.}$$

2. If A^{-1} exists, then $A^{-1}A = AA^{-1} = I$.
3. $(AB)^{-1} = B^{-1}A^{-1}$