Linear Subspace over $\mathbb{K}$.

$x, y \in V \Rightarrow x+y \in V$

$\alpha \in \mathbb{K}, x \in V \Rightarrow \alpha x \in V.$

Properties:
(1) $x+y=y+x$
(2) $x+(y+z)=(x+y)+z$
(3) $\exists 0 \in V,\ 0+x=x+0=x$
(4) $\forall x \in V,\ \exists (-x) \in V,\ \ x+(-x)=0$
(5) $(\alpha+\beta)x=\alpha x+\beta x,\ \forall \alpha, \beta \in \mathbb{K},\ \forall x \in V$
(6) $\alpha(x+y)=\alpha x+\alpha y$
(7) $(\alpha\beta)x=\alpha(\beta x)$

$\mathbb{F}$ is a field.
For $x, y \in \mathbb{F} \Rightarrow x \pm y \in \mathbb{F}$
$x \cdot y,\ x/y\ (y \neq 0) \in \mathbb{F}$

If $n=p$ (prime)
then $\mathbb{Z}_p$ is a field
$\mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$

$a \equiv b \mod m$
$m \mid (a-b)$

$\mathbb{Z}_2 = \{0, 1\}$ (remainders)

$-4\ -3\ -2\ -1\ 0\ \ 1\ 2\ 3\ 4\ 5$
$\ 1\ 0\ 1\ 0\ 1$

$\mathbb{Z}_3 = \{0, 1, 2\}$

$$|x_i+y_i|^p \le (|x_i|+|y_i|)^p \le (2\max\{|x_i|, |y_i|\})^p$$
$$|x_i| \le \max\{|x_i|, |y_i|\},\quad |y_i| \le \max\{|x_i|, |y_i|\}$$
$$= 2^p \cdot |x_i|^p \le 2^p(|x_i|^p+|y_i|^p)$$

For linear subspaces $U_1, U_2, \cdots, U_m$ of V,

Direct Sum: $\oplus$: $U_1 \oplus U_2 \oplus \cdots \oplus U_m$ is a linear space s.t. any of its elements can be <u>uniquely</u> represented as

$$u_1+\cdots+u_m,\quad u_i \in U_i,\ i=1, \cdots, m$$

* Consider: $\mathbb{R}^3 = \{\bar{x}=(x_1, x_2, x_3),\ x_i \in \mathbb{R}\}$

1. $U=\{(x, y, 0) \cdots\}$, $W=\{(0, 0, z), \cdots\}$ $\quad \mathbb{R}^3 = U \oplus W$
2. $U_i = \{(0, 0, \cdots, x_i, 0, \cdots) \in \mathbb{R}^n,\ x_i \in \mathbb{R}\}$, $\mathbb{R}^n = U_1 \oplus \cdots \oplus U_n$
3. $U_1 = \{(x, y, 0) \cdots\}$
   $U_2 = \{(0, 0, z) \cdots\}$ $\Rightarrow \mathbb{R}^3 \neq U_1 \oplus U_2 \oplus U_3$
   $U_3 = \{(0, y, y) \cdots\}$