

vv214: Cayley-Hamilton Theorem. Symmetric matrices. Quadratic forms. Singular Value Decomposition.

Dr.Olga Danilkina

UM-SJTU Joint Institute

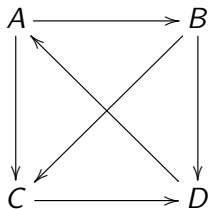


July 25, 2019

1. Ranking problem.
2. Cayley-Hamilton Theorem and its applications.
3. Orthogonally diagonalizable matrices.

Ranking Problem

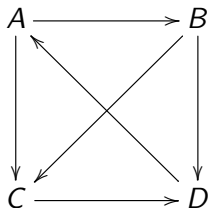
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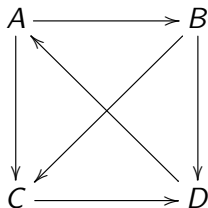


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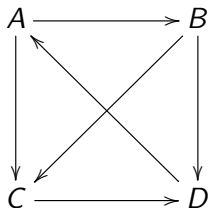
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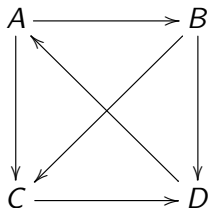
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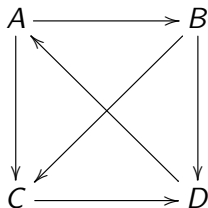
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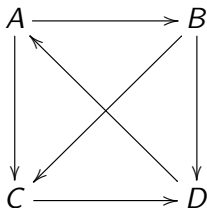
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How do we know who is better before ranking them?

- * Define recursion!

Ranking Problem

- * Give everyone the initial score of 1

$$\bar{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

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- * Define for all $n \geq 0$

$$\bar{x}_{n+1} = A\bar{x}_n,$$

where

$$A = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Ranking Problem

$$\bar{x}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

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The $(n+1)$ th score of a player A is the sum of the n th scores of the players that the player A defeated.

Ranking Problem

$$\bar{x}_5 = \begin{pmatrix} 8 \\ 6 \\ 3 \\ 5 \end{pmatrix}, \bar{x}_{10} = \begin{pmatrix} 35 \\ 34 \\ 21 \\ 26 \end{pmatrix}, \bar{x}_{100} = \begin{pmatrix} 1037 \\ 933 \\ 547 \\ 731 \end{pmatrix}$$

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Is $A > B > D > C? \Rightarrow$

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Theorem (Perron-Frobenius): There exists a *largest positive* eigenvalue λ_{PF} for a nonnegative matrix A such that the rescaled system

$$\bar{x}_n = \left(\frac{1}{\lambda_{PF}} A \right)^n \bar{x}_0$$

converges to an equilibrium state \bar{x}_∞ .

$$\bar{x}_\infty = \bar{x}_{\infty+1} = \frac{1}{\lambda_{PF}} A \bar{x}_\infty \Rightarrow A \bar{x}_\infty = \lambda_{PF} \bar{x}_\infty$$

The equilibrium state is the eigenvector associated with λ_{PF} !!!

Ranking Problem

The largest positive eigenvalue is

$$\lambda_{PF} = 1.3953369\dots$$

and

$$\bar{x}_{\infty} = \begin{pmatrix} 0.321\dots \\ 0.288\dots \\ 0.165\dots \\ 0.230\dots \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

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$B_m \rightarrow A \Rightarrow f_{B_m} \rightarrow f_A$ and determinant is a continuous function

$$f_A(A) = \lim_{m \rightarrow \infty} f_{B_m}(B_m) = 0$$

Cayley-Hamilton Theorem

Theorem: Any $A \in M_{n \times n}(\mathbb{K})$ satisfies its own characteristic equation, i.e.

$$f_A(A) = (-A)^n + (\operatorname{tr} A)(-A)^{n-1} + \dots + (\det A)I_n = 0$$

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Examples:

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$$g(t) = e^{tx} \Rightarrow e^{Ax} = \sum_{k=0}^{n-1} b_k A^k, \quad e^{\lambda_i x} = \sum_{k=0}^{n-1} b_k \lambda_i^k$$

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Remark: Let $g(t)$ be a polynomial with coefficients from \mathbb{K} . g has multiple roots iff $\gcd(g, g')$ is not a constant.

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Definition: A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix S such that

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Theorem (Spectral Theorem): A matrix A is orthogonally diagonalizable iff A is symmetric ($A^T = A$).

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Definition: Let $T: V \rightarrow W$ be linear, i.e. $T \in L(V, W)$.
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Properties of Adjoint Operators

1. $T \in L(V, W) \Rightarrow T^* \in L(W, V)$

$$(v, T^*(w_1 + w_2)) = (Tv, w_1 + w_2) = (Tv, w_1) + (Tv, w_2)$$

$$= (v, T^*w_1) + (v, T^*w_2) = (v, T^*w_1 + T^*w_2)$$

$$(v, T^*(\lambda w)) = (Tv, \lambda w) = \bar{\lambda}(Tv, w) = \bar{\lambda}(v, T^*w) = (v, \lambda T^*w)$$

2. $(T_1 + T_2)^* = T_1^* + T_2^* \quad \forall T_1, T_2 \in L(V, W)$

3. $(\lambda T)^* = \bar{\lambda} T^* \quad \forall \lambda \in \mathbb{K} \forall T \in L(V, W)$

4. $(T^*)^* = T \quad \forall T \in L(V, W)$

$$(w, (T^*)^*v) = (T^*w, v) = \overline{(v, T^*w)} = \overline{(Tv, w)} = (w, Tv) \quad \forall v \in V$$

5. $I^* = I$

6. $(T_1 T_2)^* = T_2^* T_1^* \quad \forall T_1 \in L(W, U), T_2 \in L(V, W)$

Properties of Adjoint Operators

7. $\text{Ker } T^* = (\text{Im } T)^\perp$

$$w \in \text{Ker } T^* \Leftrightarrow T^*w = 0 \Leftrightarrow (v, T^*w) = 0 \quad \forall v \in V$$

$$\Leftrightarrow (Tv, w) = 0 \quad \forall v \in V \Leftrightarrow w \in (\text{Im } T)^\perp$$

8. The matrix of the adjoint T^* w.r.t orthonormal bases $e_1, \dots, e_m \in V$; $f_1, \dots, f_n \in W$ is the conjugate transpose of the matrix of T .

$$A_T = (Te_1 \ Te_2 \ \dots \ Te_m) \quad Te_k \in W, \ f_1, \dots, f_n \text{ is orthonormal}$$

$$\Rightarrow Te_k = (Te_k, f_1)f_1 + \dots + (Te_k, f_n)f_n \Rightarrow (A_T)_{jk} = (Te_k, f_j)$$

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$$\begin{aligned} (Tv, v) - \overline{(Tv, v)} &= (Tv, v) - (v, Tv) = (Tv, v) - (T^*v, v) \\ &= ((T - T^*)v, v) \end{aligned}$$

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3. Let $Tv = \lambda v$. T is normal $\Leftrightarrow T - \lambda I$ is also normal.
4. Let $Tv = \lambda v$. Then $T^*v = \bar{\lambda}v$

$$0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^*v\| = \|(T^* - \bar{\lambda}I)v\| \Rightarrow T^*v = \bar{\lambda}v$$

5. Suppose $T \in L(V, V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

$$Tu = \alpha u, Tv = \beta v \Rightarrow T^*v = \bar{\beta}v$$

$$(\alpha - \beta)(u, v) = \alpha(u, v) - \beta(u, v) = (Tu, v) - (u, T^*v) = 0$$

Complex Spectral Theorem

Let $\mathbb{K} = \mathbb{C}$ and $T \in L(V, V)$. Then the following are equivalent:

- (a) T is normal.
- (b) V has an orthonormal basis consisting of eigenvectors of T .
- (c) T has a diagonal matrix with respect to some orthonormal basis of V .

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$q(0,0) = 0$ is the global minimum

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Definition: A function $q(x_1, \dots, x_n): \mathbb{R}^n \rightarrow \mathbb{R}$ is called a **quadratic form** if it is a linear combination of products $x_i x_j$.

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Remark 2: Let

$$Q_n = \{\text{quadratic } q(x_1, \dots, x_n): \mathbb{R}^n \rightarrow \mathbb{R}\}$$

$$\forall q \in Q_n \quad q = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 + c_4 x_3^2 + c_5 x_1 x_3 + c_6 x_2 x_3 + \dots$$

$$\Rightarrow \{x_i x_j\}, i, j = 1 \dots n, \text{ is the basis for } Q_n$$

$$n + (n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n+1)}{2} \quad \text{basis elements}$$

Theorem

If

$$q(\bar{x}) = \bar{x}^T A \bar{x}$$

is a quadratic form with a *symmetric matrix* A and

$$\mathfrak{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$$

is the eigenbasis for A with the associated eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$q(\bar{x}) = \lambda_1 \alpha_1^2 + \dots + \lambda_n \alpha_n^2,$$

where $\alpha_1, \dots, \alpha_n$ are the coordinates of \bar{x} in \mathfrak{B} .

Positive Definite Quadratic Forms

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is said to be **positive definite** if

$$q(\bar{x}) > 0 \quad \forall \bar{x} \neq \bar{0}, \bar{x} \in \mathbb{R}^n$$

If $q(\bar{x}) \geq 0 \forall \bar{x} \in \mathbb{R}^n$ the q is **positive semi-definite**.

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$$q(\bar{x}) = (A\bar{x}, A\bar{x}) = (A\bar{x})^T A\bar{x} = \bar{x}^T \underbrace{A^T A}_{\text{symmetric}} \bar{x}$$

$$q(\bar{x}) = \|A\bar{x}\|^2 \Rightarrow q(\bar{x}) \geq 0 \quad \forall \bar{x} \in \mathbb{R}^n \Rightarrow q \text{ is positive semidefinite}$$

$$q(\bar{x}) = 0 \text{ iff } \bar{x} \in \text{Ker } A$$

Remarks

Remark1: A quadratic form

$$q(\bar{x}) = \lambda_1 \alpha_1^2 + \dots \lambda_n \alpha_n^2$$

is positive semi-definite iff all $\lambda_1, \dots, \lambda_n$ are positive or zero.

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Remark 2: If $q(\bar{x}) = \bar{x}^T A \bar{x}$ is positive definite then

$$\det A = \lambda_1 \cdots \lambda_n > 0$$

BUT the converse is not true: Let $A_{3 \times 3}$

$$\lambda_1 > 0, \lambda_2, \lambda_3 < 0 \Rightarrow \det A = \lambda_1 \lambda_2 \lambda_3 > 0$$

$$q(\bar{x}) = \underbrace{\lambda_1 \alpha_1^2}_{>0} + \underbrace{\lambda_2 \alpha_2^2}_{<0} + \underbrace{\lambda_3 \alpha_3^2}_{<0} \Rightarrow q \text{ is indefinite}$$

Definition: A symmetric matrix is called **positive definite** provided all of its eigenvalues are positive.

Theorem

Let $A^{(m)}$ be a *principal submatrix* of a symmetric matrix A obtained by omitting all rows and columns of A past the m th. Then A is positive definite iff

$$\det A^{(m)} > 0 \quad \forall m = 1, \dots, n$$

Example:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 4 & 3 \end{pmatrix} \quad |A^{(1)}| = 1, |A^{(2)}| = 2, |A^{(3)}| = -18$$

A is not positive definite

$$B = \begin{pmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{pmatrix} \quad |B^{(1)}| = 9, |B^{(2)}| = 62, |B^{(3)}| = 89$$

B is positive definite

Motivation

Consider the equation

$$q(x_1, x_2) = 1$$

for the quadratic form

$$q: \mathbb{R}^2 \rightarrow \mathbb{R}, q(x_1, x_2) = ax_1^2 + bx_2^2.$$

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1. $b > a > 0 \Rightarrow$

$$\frac{x_1^2}{(1/\sqrt{a})^2} + \frac{x_2^2}{(1/\sqrt{b})^2} = 1$$

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2. $a > 0, b < 0 \Rightarrow$

$$\frac{x_1^2}{(1/\sqrt{a})^2} - \frac{x_2^2}{(1/\sqrt{-b})^2} = 1$$

is a hyperbola.

Theorem

Let λ_1, λ_2 be distinct eigenvalues of the matrix

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

of the quadratic form

$$q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2.$$

If $\lambda_1 \cdot \lambda_2 > 0$, then the curve

$$C \subset \mathbb{R}^2: q(x_1, x_2) = 1$$

is an *ellipse*.

If $\lambda_1 \cdot \lambda_2 < 0$, the C is *hyperbola*.

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$$\bar{x} = \bar{v}_1 \cos t + \bar{v}_2 \sin t$$

What is its image under the linear map L ?

$$L\bar{x} = A\bar{v}_1 \cos t + A\bar{v}_2 \sin t, A\bar{v}_1 \perp A\bar{v}_2$$

$\Rightarrow L\bar{x}$ is an ellipse with semi-axes $\|A\bar{v}_1\|$, $\|A\bar{v}_2\|$

$$\|A\bar{v}_1\|^2 = (A\bar{v}_1, A\bar{v}_1) = \lambda_1(\bar{v}_1, \bar{v}_1) = \lambda_1 \Rightarrow \|A\bar{v}_1\| = \sqrt{100} = 10$$

$$\|A\bar{v}_2\| = \sqrt{25} = 5$$

The eigenvalues of $A^T A$ define the ellipse as the image of the unit circle.

Singular Values

Definition: The **singular values** of a matrix $A_{n \times m}$ are the square roots of the eigenvalues of the symmetric matrix $(A^T A)_{m \times m}$ listed with their algebraic multiplicities:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$$

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$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$$

Theorem: Let $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $L\bar{x} = A\bar{x}$ be invertible. The image of the unit circle under the map L is an ellipse E . Singular values of A are the length of semi-axes of E .

Example

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2, L\bar{x} = A\bar{x} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \bar{x}$$

$$A^T A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$(1 - \lambda)^2(2 - \lambda) - 2 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$$

$$\sigma_1 = \sqrt{3} > \sigma_2 = 1 > \sigma_3 = 0$$

$$\bar{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \bar{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \bar{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Example

$$A\bar{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 3 \\ 3 \end{pmatrix}, A\bar{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, A\bar{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\|A\bar{v}_1\| = \sqrt{3} = \sigma_1, \|A\bar{v}_2\| = 1 = \sigma_2, \|A\bar{v}_3\| = 0 = \sigma_3$$

The unit sphere in \mathbb{R}^3 is defined by

$$\bar{x} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + c_3 \bar{v}_3, \quad c_1^2 + c_2^2 + c_3^2 = 1$$

The image of the unit sphere is

$$L\bar{x} = c_1 L\bar{v}_1 + c_2 L\bar{v}_2 = c_1 \lambda_1 \bar{v}_1 + c_2 \lambda_2 \bar{v}_2$$

$$c_1^2 + c_2^2 \leq 1$$

an ellipse

Singular Value Decomposition

Lemma: If $\text{rank } A_{n \times m} = r$, then its singular values

$$\sigma_1, \dots, \sigma_r \neq 0 \quad \text{and} \quad \sigma_{r+1}, \dots, \sigma_m = 0$$

Theorem (SVD): Any matrix $A_{n \times m}$ can be represented in the form

$$A = U \Sigma V^T,$$

U is an orthogonal $n \times n$ matrix

V is an orthogonal $m \times m$ matrix

Σ is an $n \times n$ matrix whose first r diagonal entries are nonzero singular values of A , $r = \text{rank } A$, and all other entries vanish

Singular Value Decomposition: Remarks

Remark 1:

$$\begin{array}{ll} A\bar{v}_i = \sigma_i \bar{u}_i, \quad i = 1, \dots, r & A\bar{v}_i = \bar{0}, \quad i = r + 1, \dots, m \\ \Downarrow & \Downarrow \\ \text{Im } A = \text{span}(\bar{u}_1, \dots, \bar{u}_r) & \text{Ker } A = \text{span}(\bar{v}_{r+1}, \dots, \bar{v}_m) \end{array}$$

$$\text{Remark 2: } A = U\Sigma V^T \Rightarrow A^T = V\Sigma \underbrace{U^T}_{U^{-1}} \Rightarrow A^T U = V\Sigma^T$$

$$\begin{array}{ll} A^T \bar{u}_i = \sigma_i \bar{v}_i, \quad i = 1, \dots, r & A^T \bar{u}_i = \bar{0}, \quad i = r + 1, \dots, m \\ \Downarrow & \Downarrow \\ \text{Im } A^T = \text{span}(\bar{v}_1, \dots, \bar{v}_r) & \text{Ker } A^T = \text{span}(\bar{u}_{r+1}, \dots, \bar{u}_m) \end{array}$$

Singular Value Decomposition: Example 1

$$A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$$

$$\bar{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \bar{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow V = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{\sigma_1} A \bar{v}_1 = \frac{1}{10\sqrt{5}} \begin{pmatrix} 10 \\ -20 \end{pmatrix}, \bar{u}_2 = \frac{1}{\sigma_2} A \bar{v}_2 = \frac{1}{5\sqrt{5}} \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Singular Value Decomposition: Example 2

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\bar{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \bar{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \bar{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{\sigma_1} A \bar{v}_1 = \frac{1}{\sqrt{3}\sqrt{6}} \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \bar{u}_2 = \frac{1}{\sigma_2} A \bar{v}_2 = \frac{1}{1 \cdot \sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$