(1) Find an orthonormal basis of $P_2(\mathbb{R})$ with

$$(p,q) = \int_{-1}^{1} p(t)q(t)\,dt$$

Solution: The standard basis of $P_2(\mathbb{R})$ is $1, t, t^2$

We apply the Gram-Schmidt procedure

To get started, $\|1\|^2 = \int_{-1}^{1} 1^2\,dt = 2$

$$\Rightarrow \|1\| = \sqrt{2} \Rightarrow u_1 = \frac{1}{\sqrt{2}}$$

Next, $t - (t,u_1)u_1 = t - \int_{-1}^{1} \frac{t}{\sqrt{2}}\,dt\ \frac{1}{\sqrt{2}} = t$

$$\Rightarrow \|t\|^2 = \int_{-1}^{1} t^2\,dt = \frac{2}{3} \Rightarrow \|t\| = \sqrt{\frac{2}{3}}$$

$$\Rightarrow u_2 = \sqrt{\frac{3}{2}}\,t$$

Finally, $t^2 - (t^2,u_1)u_1 - (t^2,u_2)u_2 = t^2 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{t^2}{\sqrt{2}}\,dt$

$$- \sqrt{\frac{3}{2}}\,t \int_{-1}^{1} t^2 \sqrt{\frac{3}{2}}\,t\,dt$$

$$= t^2 - \frac{1}{3}$$

$$\Rightarrow \|t^2 - \tfrac{1}{3}\|^2 = \int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right)dt = \frac{8}{45}$$

$$\Rightarrow \|t^2 - \tfrac{1}{3}\| = \sqrt{\frac{8}{45}} \Rightarrow u_3 = \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right)$$

$\Rightarrow$ $\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\,t,\ \sqrt{\frac{45}{8}}\left(t^2-\frac{1}{3}\right)$ is the orthonormal basis in $P_2(\mathbb{R})$

② Riesz - Fischer Theorem

Let $\dim V = n$ and $f \in V'$, i.e. $f : V \to \mathbb{R}(\mathbb{C})$
Then there exists a unique element $u \in V$ such that
$$f(v) = (v, u) \quad \forall v \in V$$

Proof: Let $u_1, \dots, u_n$ be an orthonormal basis of V

$\Rightarrow \forall v \in V \quad v = c_1 u_1 + \dots + c_n u_n$

$\Rightarrow (v, u_i) = c_1 \underbrace{(u_1, u_i)}_{=0} + c_2 \underbrace{(u_2, u_i)}_{=0} + \dots + c_i \underbrace{(u_i, u_i)}_{=1} + \dots + c_n \underbrace{(u_n, u_i)}_{=0}$

$\Rightarrow (v, u_i) = c_i$

$\Rightarrow \forall v \in V \quad v = (v, u_1) u_1 + \dots + (v, u_n) u_n$

$\Rightarrow f(v) = (v, u_1) f(u_1) + \dots + (v, u_n) f(u_n)$

$= (v, u_1 \overline{f(u_1)}) + \dots + (v, u_n \overline{f(u_n)})$

$= (v, \underbrace{u_1 \overline{f(u_1)} + \dots + u_n \overline{f(u_n)}}_{u})$

③ Find $u \in P_2(\mathbb{R})$ s.t. $\int_{-1}^{1} p(t) \cos \pi t \, dt = \int_{-1}^{1} p(t) u(t) dt$
$\forall p \in P_2(\mathbb{R})$

Solution: Let $f(p) = \int_{-1}^{1} p(t) \cos \pi t$, $f : P_2(\mathbb{R}) \to \mathbb{R}$

$\Rightarrow$ find $u \in P_2(\mathbb{R})$ s.t. $f(p) = (p, u)$

$\Rightarrow u = u_1 f(u_1) + u_2 f(u_2) + u_3 f(u_3)$

$= \left(\int_{-1}^{1} \sqrt{\frac{1}{2}} \cos \pi t \, dt \right) \frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} \sqrt{\frac{3}{2}} t \cos \pi t \, dt \right) \sqrt{\frac{3}{2}} t$

$+ \left(\int_{-1}^{1} \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3}\right) \cos \pi t \, dt \right) \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3}\right) = -\frac{45}{2\pi^2} \left(t^2 - \frac{1}{3}\right)$