Second Recitation Class
Linear Algebra

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Finding extension of basis

Any set of linear independent set of elements can be extended to a basis of a linear space.

$$\mathbb{R}^3 : (2,3,4),\ (9,6,8) \text{ are linearly independent}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is a basis for } \mathbb{R}^3$$

Linear Transformation

Definition: A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is called a linear transformation if there exists an $n \times m$ matrix A such that $T\bar{x} = A\bar{x} \quad \forall \bar{x} \in \mathbb{R}^m$.

Remark: A linear transformation is a special case of a linear operator:

Let V, U be linear spaces over $\mathbb{K}$. A map $T: V \to U$ is a linear operator if

1. $T(v_1 + v_2) = Tv_1 + Tv_2 \quad \forall v_1, v_2 \in V$
2. $T(\alpha v) = \alpha Tv \quad \forall \alpha \in \mathbb{K}, \forall v \in V$

Correction for last RC

Linear Transformation

- Exercise
- Determinant
- Finding Inverse

Geographical Interpretation

- Scaling
- Orthogonal Projection
- Reflection
- Rotation
- Shears

Review Exercise

Reference

Exercise

Consider the linear transformation $T(\vec{x}) = A\vec{x}$, with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where for simplicity we write $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ instead of $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

Determinant

Definition: For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the quantity $ad - bc$ is called the determinant of A:

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If $\det A \neq 0$, then the inverse linear transformation exists and

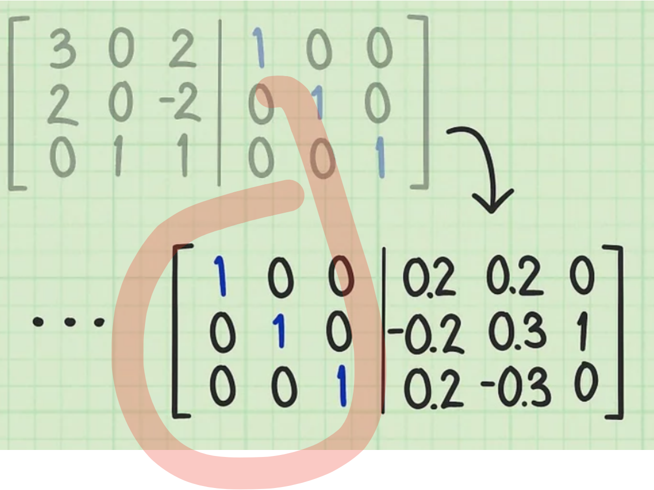
$$B = A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Finding Inverse: Method 1

$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} ; \quad M^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Method 1

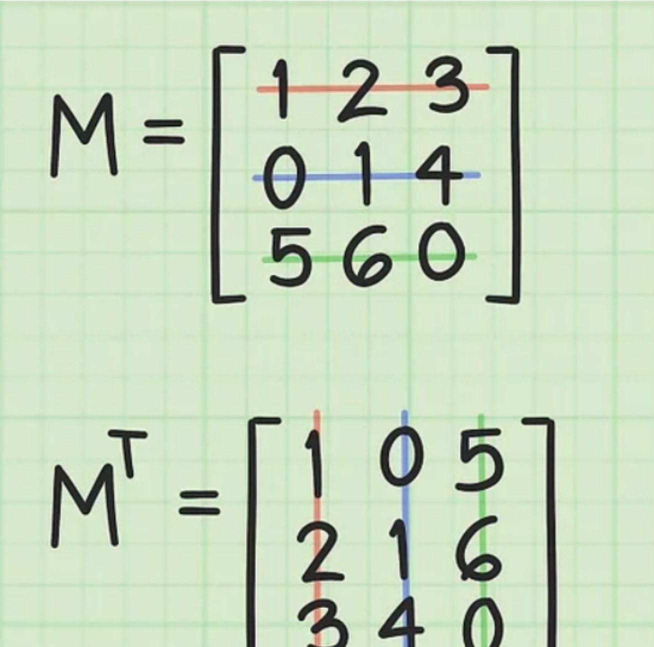


Finding Inverse: Method 2

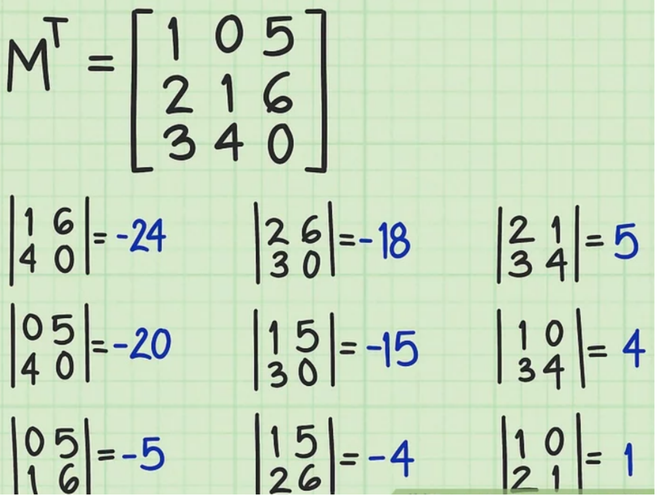
$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\det(M) = 1(0-24) - 2(0-20) + 3(0-5) = 1$$

Method 2



Method 2


$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$
$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24$
$\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18$
$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$
$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20$
$\begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15$
$\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$
$\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5$
$\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4$
$\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

Method 2

$$\mathrm{Adj}(M) = \begin{bmatrix} -24 & -18 & 5 \\ -20 & -15 & 4 \\ -5 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\mathrm{Adj}(M) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Method 2

$$\operatorname{Adj}(M) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \quad ; \quad \det(M) = 1$$

$$M^{-1} = \frac{1}{\det(M)} \times \operatorname{Adj}(M)$$

$$\boxed{M^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}}$$

Scaling

Scalings

For any positive constant k, the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k, since

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

This is a *dilation* (or enlargement) if k exceeds 1, and it is a *contraction* (or shrinking) for values of k between 0 and 1. (What happens when k is negative or zero?)

Definition

Orthogonal Projections[4]

Consider a line L in the plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L. See Figure 1.

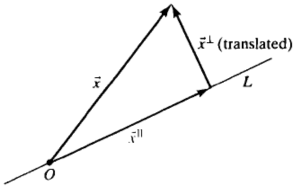


Figure 1

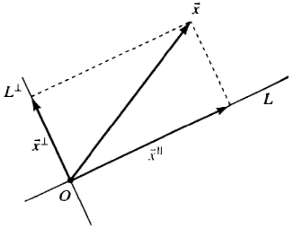


Figure 2

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the *orthogonal projection of $\vec{x}$ onto L*, often denoted by $\text{proj}_L(\vec{x})$:

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}.$$

Calculation Method

Definition 2.2.1 Orthogonal Projections

Consider a line L in the coordinate plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L.

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the *orthogonal projection of $\vec{x}$ onto L*, often denoted by $\text{proj}_L(\vec{x})$. If $\vec{w}$ is a nonzero vector parallel to L, then

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}.$$

In particular, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a *unit* vector parallel to L, then

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}.$$

The transformation $T(\vec{x}) = \text{proj}_L(\vec{x})$ is linear, with matrix

$$\frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}.$$

Exercise

Find the matrix A of the orthogonal projection onto the line L spanned by $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Solution

Solution

$$A = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$$

Definition

Reflections

Consider a line L in the coordinate plane, running through the origin, and let $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ be a vector in $\mathbb{R}^2$. The linear transformation $T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ is called the *reflection of* $\vec{x}$ *about* L, often denoted by $\mathrm{ref}_L(\vec{x})$:

$$\mathrm{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

We have a formula relating $\mathrm{ref}_L(\vec{x})$ to $\mathrm{proj}_L(\vec{x})$:

$$\mathrm{ref}_L(\vec{x}) = 2\mathrm{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}.$$

The matrix of T is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a reflection about a line.

Use Figure 6 to explain the formula $\mathrm{ref}_L(\vec{x}) = 2\mathrm{proj}_L(\vec{x}) - \vec{x}$ geometrically.

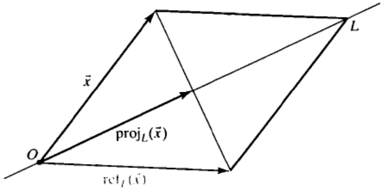


Figure 6

Exercise

Let V be the plane defined by $2x_1 + x_2 - 2x_3 = 0$, and let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$. Find $\mathrm{ref}_V(\vec{x})$.

Solution

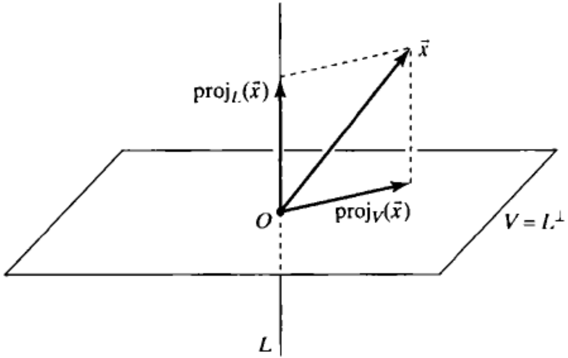


Figure 7

Solution

Solution

Note that the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to plane V (the components of $\vec{v}$ are the coefficients of the variables in the given equation of the plane: 2, 1, and -2). Thus

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Solution

is a unit vector perpendicular to V, and we can use the formula we derived earlier:

$$\begin{aligned}
\operatorname{ref}_V(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u} &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \frac{2}{9}\left(\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \\
&= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}. \qquad \blacksquare
\end{aligned}$$

Definition

Rotations

The matrix of a counterclockwise rotation in $\mathbb{R}^2$ through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that this matrix is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a rotation. ■

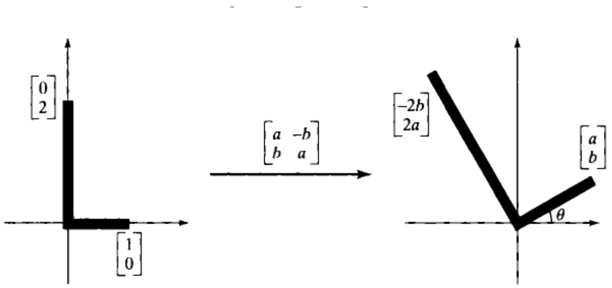
Exercise

Examine how the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

affects our standard letter L. Here a and b are arbitrary constants.

Solution



$\begin{bmatrix}0\\2\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix}$
$\begin{bmatrix}a & -b\\b & a\end{bmatrix}$
$\begin{bmatrix}-2b\\2a\end{bmatrix}$
$\begin{bmatrix}a\\b\end{bmatrix}$
θ

Solution

Solution

Figure 10 suggests that T represents a *rotation combined with a scaling*. Think polar coordinates: This is a rotation through the phase angle θ of vector $\begin{bmatrix} a \\ b \end{bmatrix}$, combined with a scaling by the magnitude $r = \sqrt{a^2 + b^2}$ of vector $\begin{bmatrix} a \\ b \end{bmatrix}$. To verify this claim algebraically, we can write the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates, as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix},$$

Solution

as illustrated in Figure 11. Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

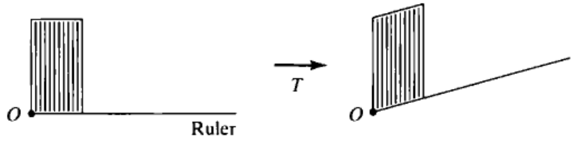
It turns out that matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a scalar multiple of a rotation matrix, as claimed. ∎

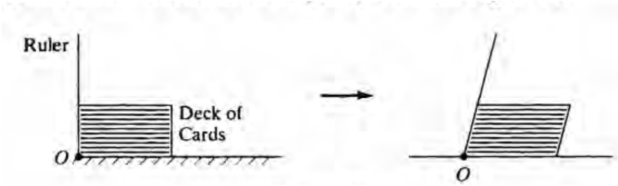
Definition

Horizontal and vertical shears

The matrix of a *horizontal shear* is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a *vertical shear* is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is an arbitrary constant.

Graph





Exercise

(a) (4 points) Write down the matrix of any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies
$T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$.

(b) (4 points) Can you find a matrix A with 3 rows and 5 columns such that $A\vec{x} = \vec{0}$ has a unique solution? If so, write down such a matrix. If not, explain why not.

(c) (4 points) Is the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$ invertible? Justify your answer with a computation.

(d) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and maps $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the image of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ under this linear transformation.

(e) (4 points) Can you find a matrix A with 4 rows and 3 columns that satisfies $rank(A) = 2$ and $nullity(A) = 2$? If so, write down such a matrix. If not, explain why not.

Solution

(a) (4 points) Write down the matrix of any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies $T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}4\\8\\12\end{bmatrix}$.

Solution:
One example of such a matrix is

$$\begin{bmatrix}0 & 2\\0 & 4\\0 & 6\end{bmatrix}.$$

(b) (4 points) Can you find a matrix A with 3 rows and 5 columns such that $A\vec{x} = \vec{0}$ has a unique solution? If so, write down such a matrix. If not, explain why not.

Solution:
$n < m$
No. When there are more columns than rows, there can only be no solutions or infinitely many solutions. Namely, the maximum value the rank can be is 3 (the number of rows), which means any solution to $A\vec{x} = \vec{0}$ will have at least two free parameters.

(c) (4 points) Is the matrix $\begin{bmatrix}1 & 1 & 1\\0 & 0 & 2\\0 & 3 & 3\end{bmatrix}$ invertible? Justify your answer with a computation.

Solution:
Yes. When we swap the second and third row, it becomes clear that the rank of this matrix is 3 and so the reduced row echelon form of this matrix will be I_3.

Solution

(d) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps $\begin{bmatrix}1\\1\end{bmatrix}$ to $\begin{bmatrix}4\\3\end{bmatrix}$ and maps $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}-1\\1\end{bmatrix}$. Find the image of $\begin{bmatrix}6\\3\end{bmatrix}$ under this linear transformation.

Solution:

The second column of the matrix of this linear transformation is $T(\vec{e}_2) = \begin{bmatrix}-1\\1\end{bmatrix}$. Since $\vec{e}_1 = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}$, the first column is $T(\vec{e}_1) = \begin{bmatrix}4\\3\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}$. Therefore, we have

$$T\left(\begin{bmatrix}6\\3\end{bmatrix}\right) = \begin{bmatrix}5 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}27\\15\end{bmatrix}.$$

(e) (4 points) Can you find a matrix A with 4 rows and 3 columns that satisfies $rank(A) = 2$ and $nullity(A) = 2$? If so, write down such a matrix. If not, explain why not.

Solution:

No. The rank-nullity theorem says that $rank(A) + nullity(A)$ is equal to the number of columns of A. In this case, the number of columns is 3, but the proposed $rank(A) + nullity(A) = 4$.

Reference

1. https://www.wikihow.com/Find-the-Inverse-of-a-3x3-Matrix