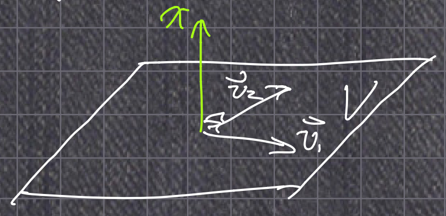
5.1. Orthogonality, length, unit vectors

5.1.1. a. $\vec{v} \perp \vec{w}$ if $\vec{v} \cdot \vec{w} = 0$.
b. Length: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
c. Unit vector $\vec{u}$: $(\|\vec{u}\| = 1)$ $\qquad \vec{u} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$

5.1.2. $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

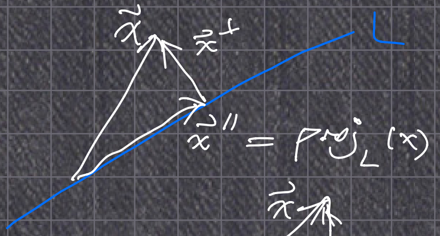
$\vec{x} \in \mathbb{R}^n$ is orthogonal to a subspace $V \subset \mathbb{R}^n$ if $\vec{x} \perp$ all the vectors $\vec{v} \in V$. $\Rightarrow \vec{x} \cdot \vec{v} = 0$ for $\forall \vec{v} \in V$.



5.1.3 a. Orth. vectors are linearly indep.
b. Orth. vec. $\vec{u}_1, \cdots, \vec{u}_n \in \mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

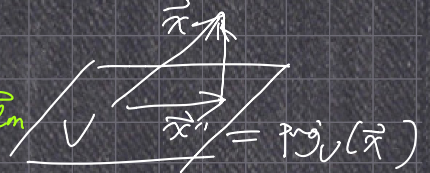
Orthogonal Projection

5.1.4 $\quad \vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$



For basis $\{\vec{u}_1, \cdots, \vec{u}_m\}$ span $V \in \mathbb{R}^n$,

5.1.5 $\quad \mathrm{proj}_V(\vec{x}) = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x}_m)\vec{u}_m$



5.1.6. For orth. basis $\vec{u}_1, \cdots, \vec{u}_n$ of $\mathbb{R}^n$,

$$\vec{x} = (\vec{u}_1 \cdot \vec{x}) \cdot \vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\, \vec{u}_n$$

5.1.7. Orthogonal complement $V^{\perp}$.

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^n ;\ \vec{v} \cdot \vec{x} = 0 \text{ for } \forall \vec{v} \in V\}.$$

(properties : $P \frac{193}{205}$)

5.1.9. Pythagorean Th
For $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\quad \|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$.

5.1.10 $\quad \|\mathrm{proj}_V \vec{x}\| \leq \|\vec{x}\|$

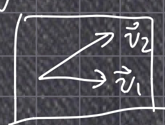
5.1.11. Cauchy - Schwarz Inequality $\quad |\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$

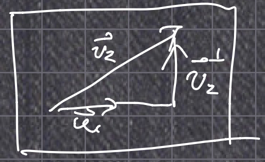
5.1.12 $\quad \theta = \cos^{-1} \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$

5.2. Gram-schmidt Process

- How to construct orthonormal basis?
$\{\vec{v}_1, \cdots, \vec{v}_m\} \longrightarrow$ orth. basis $\{\vec{u}_1, \cdots, \vec{u}_m\}$.

① Construct $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$





② (Let: $\vec{v}_2 = \vec{v}_2^{\,\parallel} + \vec{v}_2^{\,\perp}$)

$$\vec{v}_2^{\,\perp} = \vec{v}_2 - \vec{v}_2^{\,\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1$$

5.2.1 The Gram-Schmidt Process

Get $\vec{v}_j = \vec{v}_j^{\,\parallel} + \vec{v}_j^{\,\perp}$ w.r.t. $\mathrm{span}(\vec{v}_1, \cdots, \vec{v}_{j-1})$

$$\Rightarrow \vec{v}_j^{\,\perp} = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j)\vec{u}_{j-1}$$

QR Factorization

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}}_{Q} R \qquad \mathfrak{B} \to \mathfrak{U}$$

Old basis : $\mathfrak{B}$ New orth. basis $\mathfrak{U}$

$$\begin{tikzcd} {[\vec{x}]_{\mathfrak{U}}} \arrow[dr, "Q"] & \\ & \vec{x} \\ {[\vec{x}]_{\mathfrak{B}}} \arrow[uu, "R"] \arrow[ur, "M"'] & \end{tikzcd}$$

$\mathfrak{U} = (\vec{u}_1, \cdots, \vec{u}_m)$, use theorem 5.2.1

$$\Rightarrow \vec{v}_j = \vec{v}_j^{\,\parallel} + \vec{v}_j^{\,\perp} = \underbrace{\underbrace{(\vec{u}_1 \cdot \vec{v}_j)}_{r_{1j}}\vec{u}_1 + \cdots + \underbrace{(\vec{u}_i \cdot \vec{v}_j)}_{r_{ij}}\vec{u}_i + \cdots + \underbrace{(\vec{u}_{j-1} \cdot \vec{v}_j)}_{r_{j-1,j}}\vec{u}_{j-1}}_{\vec{v}_j^{\,\parallel}} + \underbrace{\underbrace{\|\vec{v}_j^{\,\perp}\|}_{r_{jj}}\vec{u}_j}_{\vec{v}_j^{\,\perp}}$$

Follows : $\boxed{\begin{cases} r_{ij} = \vec{u}_i \cdot \vec{v}_j & (i < j) \\ \quad = 0 & (i > j) \\ r_{jj}. \end{cases}}$ R: upper triangular

5.2.2. $M_{n\times m}$: linearly indep. $\vec{v}_1, \cdots, \vec{v}_m$
$Q_{n\times m}$: ortho: $\vec{u}_1, \cdots, \vec{u}_m$.
Upper triangular: R, (+) diagonal entries. } $M = QR$

Example: $M = [\vec{v}_1 \ \vec{v}_2] = \underbrace{[\vec{u}_1 \ \vec{u}_2]}_{Q} \underbrace{\begin{bmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \|\vec{v}_2^{\,\perp}\| \end{bmatrix}}_{R}$? Why? (P220: $R_{3\times3}$)

5.3. Orthogonal Transformations and Orthogonal Matrices.

5.3.1. $T: \mathbb{R}^n \to \mathbb{R}^n$ orthogonal : Preserves length of vectors.

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$$

5.3.2. If $\vec{v} \perp \vec{w}$ in $\mathbb{R}^n \Rightarrow T(\vec{v}) \perp T(\vec{w})$

5.3.3. a. $T: \mathbb{R}^n \to \mathbb{R}^n$ is ortho $\Longleftrightarrow$ $T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$.

b. $A_{n\times n}$ is ortho $\Longleftrightarrow$ Its columns form an orthonormal basis for $\mathbb{R}^n$.

5.3.8 **Orthogonal Matrices:** $A_{n\times n}$

i. A is an orth. mat.
ii. $L(\vec{x}) = A\vec{x}$ preserves length: $\|A\vec{x}\| = \|\vec{x}\| \ \forall x \in \mathbb{R}^n$
iii. Columns of A form orth. basis for $\mathbb{R}^n$.
iv. $A^TA = I_n$
v. $A^{-1} = A^T$

5.3.9. Properties of transpose. (P228)

a. $A: n\times p$, $B: p\times m$

$$\Rightarrow (AB)^T = B^TA^T$$

b. $A_{n\times n}$, invertible, so is A^T,

$$(A^T)^{-1} = (A^{-1})^T.$$

c. $\forall A$, $\quad \text{rank}(A) = \text{rank}(A^T)$

5.4. Least Squares & Data Fitting.

*One Characterization of Orthogonal Complements.

$\rightarrow$ Take $V = \text{image}(A)$ of $\mathbb{R}^n$, $A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_m]$.

$$V^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{v}\cdot\vec{x} = 0 \ \forall \vec{v} \in V\} = \{\vec{x} \in \mathbb{R}^n : \vec{v}_i^T\vec{x} = 0 \text{ for } i = \overline{1,m}\}$$

$\Rightarrow V^\perp = (\text{im}A)^\perp$ is the kernel of the matrix

$$A^T = \begin{bmatrix} - \vec{v}_1^T - \\ - \vec{v}_2^T - \\ \vdots \\ - \vec{v}_m^T - \end{bmatrix}$$

5.4.1 $\forall$ matrix A,

$$(\text{im}A)^\perp = \ker(A^T)$$

e.g. $V = \text{im}\begin{bmatrix}1\\2\\3\end{bmatrix}$

$V^\perp = \ker[1\ 2\ 3]$

is the plane with eq. $x_1 + 2x_2 + 3x_3 = 0$

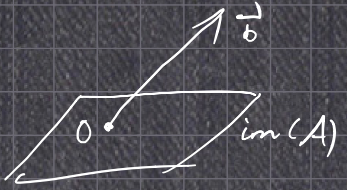
5.4.2. (a). If $A_{n\times m}$, then $\ker(A) = \ker(A^TA)$

[b]. If A $n\times m$ & $\ker(A)=\{\vec{0}\}$, $\Rightarrow A^TA$ is invertible.

5.4.3. Consider $\vec{x}\in\mathbb{R}^n$, $V\subset\mathbb{R}^n$

$$\|\vec{x}-\mathrm{proj}_V\vec{x}\| < \|\vec{x}-\vec{v}\| \quad \text{Closest.}$$

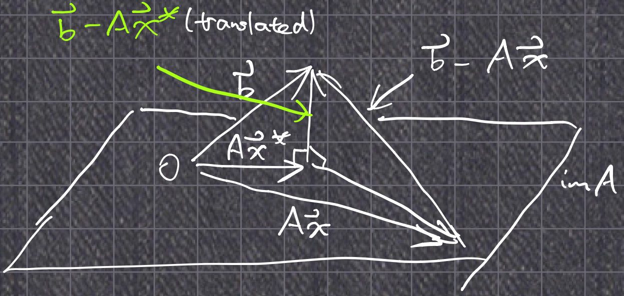
Least Square approximations.



5.4.4. For a linear system $A\vec{x}=\vec{b}$,

A vector $\vec{x}^*\in\mathbb{R}^m$ is called a least-squares solution of this system if $\|\vec{b}-A\vec{x}^*\|\le\|\vec{b}-A\vec{x}\|$ for all $\vec{x}\in\mathbb{R}^m$

minimize the sum of the squares of the components of $(\vec{b}-A\vec{x})$



How to find least-squares solutions of $A\vec{x}=\vec{b}$?

5.4.5 For sys. $A\vec{x}=\vec{b}$, the least-sq-- solutions are the exact solutions of the system:

$$A^TA\vec{x}=A^T\vec{b}. \quad (\text{normal equations of } A\vec{x}=\vec{b})$$

5.4.6. If $\ker(A)=\{\vec{0}\}$, then system $A\vec{x}=\vec{b}$ has the unique least-square solution:

$$\vec{x}^*=(A^TA)^{-1}A^T\vec{b}.$$

5.4.7. For $V\subset\mathbb{R}^n$, with basis $\{v_1, v_2, ---, v_m\}$.

Let $A=[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_m]$

Then the matrix of the orthogonal projection onto V is

$$A(A^TA)^{-1}A^T$$

* If vectors $\vec{v}_1 \cdots \vec{v}_m$ are orthogonal,

$A^TA=I_m$, formula $\Rightarrow$ AA^T

5.5. Inner product Spaces.

☆ $(p,q)=\int_{-1}^{1} p(t)q(t)\,dt$

5.5.1. Properties: - - - . (P 246)

* $T(f)=\langle f, g\rangle$ is a linear transformation: $V\to\mathbb{R}$

5.5.2. Norm, orth.--.
(Magnitute).

$$\|f\|=\sqrt{\langle f, f\rangle}.$$

5.5.3. Orth. projection.

$$\text{proj}_W f = \langle g_1, f\rangle g_1 + \cdots + \langle g_m, f\rangle g_m \quad \forall f \in V.$$

5.5.4. Orthnormal basis of T_n. (The space of all trigonometric polynomials order $\le n$

Let $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$

Then functions $\frac{1}{\sqrt{2}}, \sin(t), \cos(t), \sin(2t), \cdots, \sin(nt), \cos(nt)$ form an orthonormal basis of T_n.

※ 5.5.5. Fourier coefficients.