Moments of a Random Variable

The "moments" of a random variable (or of its distribution) are expected values of powers or related functions of the random variable.

The r^{th} moment of X is $E(X^r)$.
In particular, the first moment is the **mean**, $\mu_X = E(X)$.
The mean is a measure of the "center" or "location" of a distribution.

Another measure of the "center" or "location" is a **median**, defined as a value m such that $P(X < m) \leq 1/2$ and $P(X \leq m) \geq 1/2$. If there is only one such value, then it is called *the* median.

The r^{th} central moment of X is $E[(X - \mu_X)^r]$.
In particular, the second central moment is the **variance**, $\sigma_X^2 = Var(X) = E[(X - \mu_X)^2]$.
The **standard deviation** of a random variable is the (nonnegative) square root of the variance: $\sigma_X = Sd(X) = \sqrt{\sigma_X^2}$
The variance and standard deviation are measures of the spread or dispersion of a distribution. The standard deviation is measured in the same units as X, while the variance is in X-units squared.

Another measure of spread is the mean (absolute) deviation: M.A.D. $= E(|X - m|)$ where m is the **median** of the distribution of X. It would seem that this is a more natural measure of spread than the standard deviation or the variance, but it is more difficult to deal with analytically.

The r^{th} factorial moment of X is $E[X^{\underline{r}}]$.
Recall that for a positive integer r, $X^{\underline{r}} = X \cdot (X-1) \cdot (X-2) \cdots (X-r+1) = \prod_{j=1}^{r} (X-j+1)$.

One measure of **skewness** is $E[(X - \mu_X)^3]/\sigma_X^3$.
One measure of **kurtosis** is $E[(X - \mu_X)^4]/\sigma_X^4$.

Properties of the variance. Let X and Y be random variables. Let a, b, c be constants.

- $0 \leq Var(X) \leq E(X^2)$.
- $Var(cX) = c^2 Var(X)$.
- $Var(X) = E(X^2) - [E(X)]^2$.
- $Var(aX + bY) = a^2 Var(X) + 2ab\, Cov(X, Y) + b^2 Var(Y)$.
- $Var(X + Y) = Var(X) + Var(Y)$ if X and Y are independent or uncorrelated.