CHAPTER 3

Specific Problems

In this chapter the basic ideas developed so far are applied in more detail to the particular cases raised in Section 2.2. At the same time, the general ideas can be taken further on the basis of specific sets of equations.

3.1 Traffic Flow

The application of these ideas to traffic flow was formulated and discussed independently by Lighthill and Whitham (1955) and Richards (1956). It is clear in this case that the flow velocity

$$V(\rho)=\frac{Q(\rho)}{\rho}$$

must be a decreasing function of ρ which starts from a finite maximum value at $\rho=0$ and decreases to zero as $\rho\to\rho_j$, the value for which the cars are bumper to bumper. Thus $Q(\rho)$ is zero at both $\rho=0$ and $\rho=\rho_j$, and has a maximum value q_m at some intermediate density ρ_m. It has the general convex form shown in Fig. 3.1. Actual observations of traffic flow indicate that typical values for a single lane are $\rho_j \sim 225$ vehicles per mile, $\rho_m \sim 80$ vehicles per mile, $q_m \sim 1500$ vehicles per hour. It appears to be roughly correct to multiply these values by the number of lanes for multilane highways. It is interesting that, according to these figures, the maximum flow rate q_m is attained at a low velocity in the neighborhood of 20 miles per hour.

The propagation velocity for the waves is

$$c(\rho)=Q'(\rho)=V(\rho)+\rho V'(\rho).$$

Since $V'(\rho)<0$, the propagation velocity is less than the car velocity; waves propagate backward through the stream of traffic and drivers are warned of disturbances ahead. The velocity c is the slope of the (q,ρ) curve so the

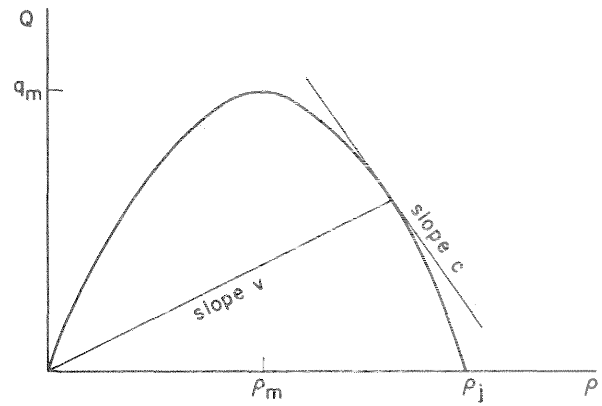

Fig. 3.1. Flow-density curve in traffic flow.

waves move forward or backward relative to the road depending on whether $\rho<\rho_m$ or $\rho>\rho_m$. At the maximum flow rate, $\rho=\rho_m$, the waves are stationary relative to the road, so the propagation velocity relative to the cars is then the same as $q_m/\rho_m \sim 20$ mph.

Near $\rho=\rho_j$, we can make a rough estimate on the basis of a simple reaction time argument. If we assume that a driver and his car take a time δ to react to any change ahead, then the gap between cars should be kept at $V\delta$ for safety. If h is the headway, defined as the distance between the front ends of successive cars, and L is the typical car length, this leads to

$$V=\frac{h-L}{\delta}.$$

Since $h=1/\rho$, $L=1/\rho_j$, we have

$$V(\rho)=\frac{L}{\delta}\left(\frac{\rho_j}{\rho}-1\right), \qquad Q(\rho)=\frac{L}{\delta}(\rho_j-\rho).$$

One should probably interpret this as an estimate of the slope of the $Q(\rho)$ curve at ρ_j, rather than as a realistic prediction of a linear dependence on ρ. In any event, it gives $c_j=-L/\delta$ for the propagation velocity there. In the traffic flow context δ is usually estimated in the range 0.5–1.5 sec, although in other circumstances the human reaction time can be much faster. With $L=20$ ft, $\delta=1$ sec, we have $c_j \sim -14$ mph.

Greenberg (1959) found a good fit with data for the Lincoln Tunnel in

New York by taking

$$Q(\rho)=a\rho\log\frac{\rho_j}{\rho},$$

with $a=17.2$ mph, $\rho_j=228$ vpm (vehicles per mile). For this formula, the relative propagation velocity $V-c$ is equal to the constant value a at all densities. The values of ρ_m and q_m are $\rho_m=83$ vpm, $q_m=1430$ vph (vehicles per hour). The logarithmic formula does not give a finite value for V as $\rho\to 0$, but the theory would be on dubious ground for very light traffic so this point alone is not important. With a finite maximum V and a finite $V'(\rho)$, we have $c\to V$ as $\rho\to 0$, so one should expect $V-c$ to decrease at the lighter densities.

Since $Q(\rho)$ is convex with $Q''(\rho)<0$, c itself is always a decreasing function of ρ. This means that a local increase of density propagates as shown in Fig. 3.2 with a shock forming at the back. Individual cars move faster than the waves, so that a driver enters such a local density increase from behind; he must decelerate rapidly through the shock but speeds up only slowly as he leaves the congestion. This seems to accord with experience. The details can be analyzed by the theory of Chapter 2. In particular the final asymptotic behavior is the triangular wave which is the last profile in Fig. 3.2. The length of the wave increases like $t^{1/2}$ and the shock decays like $t^{-1/2}$. The actual analytic expressions are

$$c\sim\frac{x}{t},\qquad \rho-\rho_0\sim\frac{x-c_0t}{c'(\rho_0)\,t}\qquad\text{for}\quad c_0t-\sqrt{2Bt}<x<c_0t,$$

where

$$B=|c'(\rho_0)|\int_{-\infty}^{\infty}(\rho-\rho_0)\,dx.$$

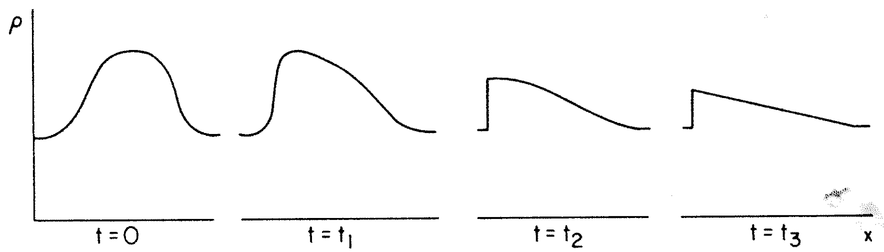

Fig. 3.2. Breaking wave in traffic flow.

The shock is at

$$x=c_0t-\sqrt{2Bt}\,,$$

and the jumps of c and ρ at the shock are

$$c-c_0\sim-\sqrt{\frac{2B}{t}}\,,\qquad \rho-\rho_0\sim\frac{1}{|c'(\rho_0)|}\sqrt{\frac{2B}{t}}\,.$$

Traffic Light Problem.

A more complicated problem is the analysis of the flow at a traffic light. We construct the characteristics in the (x,t) diagram. These are lines of constant density and their slopes $c(\rho)$ determine the corresponding values of ρ on them. So the problem is solved once the (x,t) diagram has been obtained.

Suppose first that the red period of the light is long enough to allow the incoming traffic to flow freely at some value $\rho_i<\rho_m$. Then we may start with characteristics of slope $c(\rho_i)$ intersecting the t axis in the interval AB in Fig. 3.3; AB is part of a green period. [The (x,t) diagram is plotted with x vertical and t horizontal since this is the usual practice in the references on traffic flow.] Just below the red period BC, the cars are stationary with $\rho=\rho_j$; hence the characteristics have the negative slope $c(\rho_j)$. The line of separation between the stopped queue at the traffic light and the free flow must be a shock BP, and from the shock condition its velocity is

$$-\frac{q(\rho_i)}{\rho_j-\rho_i}.$$

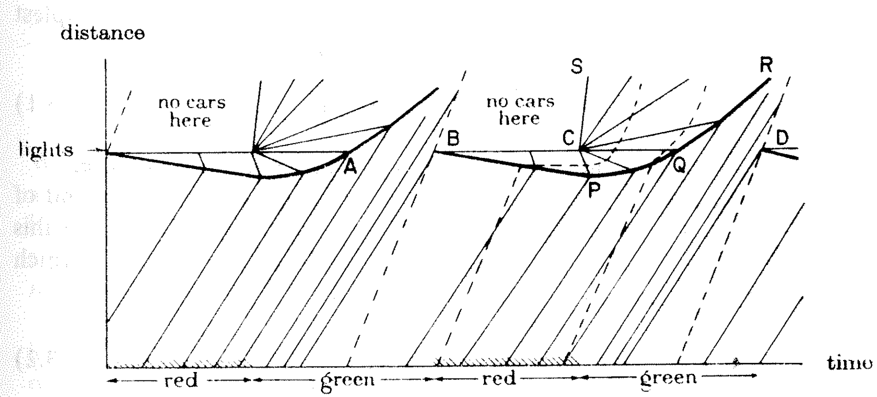

Fig. 3.3. Wave diagram for an efficient traffic light.

When the light turns green at C, the leading cars can go at the maximum speed since $\rho=0$ ahead of them. (The finite acceleration could be allowed for roughly by extending the effective red period.) This is represented by the characteristic CS with maximum slope $c(0)$. Between CS and CP we have an expansion fan with all values of c being taken. Exactly at the intersection CQ, the slope c must be zero. But this corresponds to the maximum $q=q_m$. Therefore we have the interesting result that q attains its maximum value right at the traffic light. The shock $BPQR$ is weakened by the expansion fan and ultimately accelerates through the intersection, provided the green period is long enough. The criterion for whether the shock gets through is easily established. The total incoming flow for the time BQ is $(t_r+t_s)q_i$ where t_r is the red period BC and t_s is the part of the green period before the shock gets through. The flow across the intersection in this time is $t_s q_m$. These two must be equal; therefore

$$t_s=\frac{t_r q_i}{q_m-q_i}.$$

For the shock to get through and the light to operate freely, the green period must exceed this critical value.

If the shock does not get through, the flow never becomes free and the notorious traffic crawl develops. It is perhaps sufficient to show the corresponding (x,t) diagram Fig. 3.4 without comment!

Higher Order Effects; Diffusion and Response Time.

There are two obvious additional effects one may wish to include in the theory. One was mentioned in Section 2.4: the dependence of q on ρ_x as well as ρ. This introduces in a rough way the drivers' awareness of conditions ahead, and it produces a *diffusion* of the waves. The simplest assumption with the correct qualitative behavior is

$$q=Q(\rho)-\nu\rho_x, \qquad v=V(\rho)-\frac{\nu}{\rho}\rho_x, \tag{3.1}$$

and one does not have much basis for any more complicated choice.

The second effect is the time lag in the response of the driver and of his car to any changes in the flow conditions. One way to introduce this effect is to consider the expression for v in (3.1) as a desired velocity which the driver accelerates toward; therefore the equation

$$v_t+vv_x=-\frac{1}{\tau}\left\{v-V(\rho)+\frac{\nu}{\rho}\rho_x\right\} \tag{3.2}$$

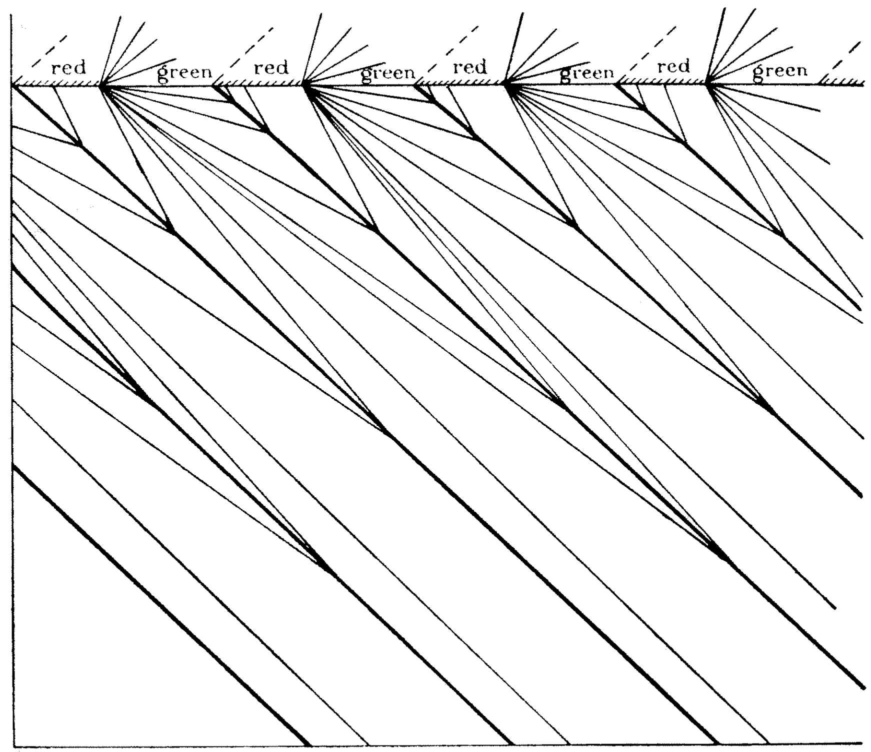

Fig. 3.4. Wave diagram for the slow crawl at an overcrowded traffic light.

may be introduced for the acceleration. The coefficient τ is a measure of the response time and is akin to the quantity δ mentioned earlier. Equation 3.2 is to be solved together with the conservation equation

$$\rho_t+(\rho v)_x=0. \tag{3.3}$$

When ν, τ are both small in a suitable nondimensional measure, (3.2) is approximated by $v=V(\rho)$, and we have the simpler theory. With the higher order terms included in (3.2), we expect shocks to appear as smooth steps and so on. This is true on the whole, but the situation turns out to be more complicated.

It is always helpful to get a first feel for a nonlinear equation by looking at the linearized theory, even though the linearization may have its own shortcomings, as we discussed in Section 2.10. If (3.2) and (3.3) are linearized for small perturbations about $\rho=\rho_0$, $v=v_0=V(\rho_0)$, by substituting

$$\rho=\rho_0+r, \qquad v=v_0+w,$$

and retaining only first powers of r and w, we have

$$\tau(w_t+v_0w_x)=-\left\{w-V'(\rho_0)r+\frac{\nu}{\rho_0}r_x\right\},$$

$$r_t+v_0r_x+\rho_0w_x=0.$$

The kinematic wave speed is

$$c_0=\rho_0V'(\rho_0)+V(\rho_0);$$

hence $V'(\rho_0)=-(v_0-c_0)/\rho_0$. Introducing this expression and then eliminating w, we have

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=\nu\frac{\partial^2 r}{\partial x^2}-\tau\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2 r. \tag{3.4}$$

When $\nu=\tau=0$, we have the linearized approximation to the kinematic waves: $r=f(x-c_0t)$. The term proportional to ν introduces typical diffusion of the heat equation type. The effect of the finite response time τ is more complicated, but a quick insight can be gained as follows. In the basic wave motion governed by the left hand side, $r=f(x-c_0t)$, so that t derivatives are approximately equal to $-c_0$ multiplied by x derivatives:

$$\frac{\partial}{\partial t}\simeq -c_0\frac{\partial}{\partial x}. \tag{3.5}$$

If this approximation is used in the right hand side of (3.4), the equation reduces to

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=\left\{\nu-(v_0-c_0)^2\tau\right\}\frac{\partial^2 r}{\partial x^2}. \tag{3.6}$$

There is a combined diffusion when

$$\nu>(v_0-c_0)^2\tau \tag{3.7}$$

but instability if

$$\nu<(v_0-c_0)^2\tau. \tag{3.8}$$

This is reasonable; for stability a driver should look far enough ahead to make up for his response time.

The stability criterion can be verified directly from the complete equation (3.4) in the traditional way. There are exponential solutions of (3.4) with

$$r\propto e^{ikx-i\omega t}$$

provided that

$$\tau(\omega-v_0k)^2+i(\omega-c_0k)-\nu k^2=0.$$

The exponential solutions will be stable provided $\mathfrak{I}\,\omega<0$ for both of the roots ω. It is easily verified that the requirement for this is (3.7), so the result of the approximate procedure is confirmed and extended to all wavelengths.

Higher Order Waves.

It is important to note that the right hand side of (3.4) is itself a wave operator and we may write the equation as

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=-\tau\left(\frac{\partial}{\partial t}+c_+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_-\frac{\partial}{\partial x}\right)r, \tag{3.9}$$

where

$$c_+=v_0+\sqrt{\nu/\tau}\,, \qquad c_-=v_0-\sqrt{\nu/\tau}\,.$$

It would be expected therefore that waves traveling with speeds c_+ and c_- also play some role. It would be premature to go deeply into this question at this stage, but one remark has great significance in interpreting the stability condition. We shall see later in our discussion of higher order equations that the propagation speeds in the highest order derivatives always determine the fastest and slowest signals. Thus in the present case however small τ may be provided it is nonzero, the fastest signal travels with speed c_+ and the slowest with speed c_-. It is clear therefore that the approximation

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=0 \tag{3.11}$$

could only make sense if

$$c_-<c_0<c_+. \tag{3.12}$$

But this is exactly the stability criterion (3.7). So the flow is stable only if (3.12) holds, and then it is appropriate to approximate (3.9) by (3.11) for small τ. There is a nice correspondence between stability and wave interaction.

Equation 3.9 arises in several applications and a full discussion is given in Chapter 10.

Shock Structure.

The more complicated form of the higher order corrections introduces a new possibility in the shock structure. For the simple diffusion term used in Section 2.4 with $\nu>0$, a continuous shock structure was obtained. We shall see now that this is not always the case when there are additional higher terms. We look for a steady profile solution of (3.2)–(3.3) with

$$\rho=\rho(X), \qquad v=v(X), \qquad X=x-Ut,$$

where U is the constant translational velocity. Equation 3.3 becomes

$$-U\rho_X+(v\rho)_X=0 \tag{3.13}$$

and may be integrated to

$$\rho(U-v)=A, \tag{3.14}$$

where A is a constant. Equation 3.2 becomes

$$\tau\rho(v-U)v_X+\nu\rho_X+\rho v-Q(\rho)=0. \tag{3.15}$$

Since $v=U-A/\rho$, this may be reduced to

$$\left(\nu-\frac{A^2}{\rho^2}\tau\right)\rho_X=Q(\rho)-\rho U+A. \tag{3.16}$$

For $\tau=0$, it is the same as (2.21), as it should be. For $\tau\neq0$, the possibility that $\nu-A^2\tau/\rho^2$ may vanish introduces the new effects.

As before we are interested in solution curves between ρ_1 at $X=+\infty$ and ρ_2 at $X=-\infty$. These values will be zeros of the right hand side of (3.16). For traffic flow $c'(\rho)=Q''(\rho)<0$, so $\rho_2<\rho_1$ and the right hand side of (3.16) is positive for $\rho_2<\rho<\rho_1$. If $\nu-A^2\tau/\rho^2$ remains positive in this range, then $\rho_X>0$ and we have a smooth profile as in Fig. 3.5. In view of (3.14), the condition for $\nu-A^2\tau/\rho^2$ to remain positive may be written

$$\nu>(v-U)^2\tau, \qquad \text{that is,} \quad v-\sqrt{\nu/\tau}<U<v+\sqrt{\nu/\tau}. \tag{3.17}$$

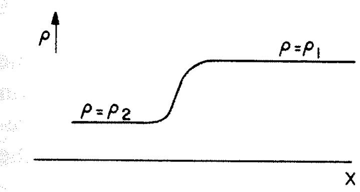

Fig. 3.5. Continuous shock structure.

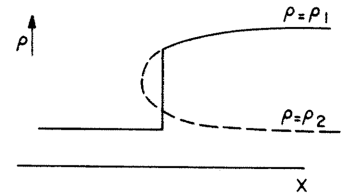

Fig. 3.6. Shock structure with an inner discontinuity.

This is similar in form to the linearized stability criterion (3.7) with v_0 replaced by the local velocity v and c_0 replaced by the shock velocity U. We might also interpret it in a way similar to (3.12) as a warning of possible complications if a shock tries to violate the higher order signal speeds. However, it is not necessarily an unstable situation. The conditions for the uniform states at $\pm\infty$ to be stable are

$$v_1-\sqrt{\nu/\tau}<c_1<v_1+\sqrt{\nu/\tau}, \qquad v_2-\sqrt{\nu/\tau}<c_2<v_2+\sqrt{\nu/\tau}. \tag{3.18}$$

It is possible, in general, for these to be satisfied and yet (3.17) to be violated. When this is the case, $\nu-A^2\tau/\rho^2$ changes sign in the profile, as in Fig. 3.6, and a single-valued continuous profile is no longer possible.

In most problems of shock structure, when the profile turns back on itself in this way, it is rectified by fitting in an appropriate discontinuity. The situation again corresponds, strictly speaking, to a breakdown of the assumptions for the particular level of description, but the introduction of a discontinuity, *provided it corresponds to a valid integrated form of the basic equations*, avoids an explicit discussion of yet higher order effects. In the case of (3.2) and (3.3), it is not clear which conservation forms are appropriate for the discontinuity conditions nor what additional effects should be introduced. One expects a discontinuous profile shown by the full curve in Fig. 3.6, but the precise determination of the discontinuity is not clear for this case. In other cases discussed later the details can be completed. The point to stress here is that the discontinuities in the simple theory using

$$\rho_t+c(\rho)\rho_x=0$$

may be only partially resolved into continuous transitions in a more accurate formulation.

A Note on Car-following Theories.

Considerable work has been done on discrete models where the motion of the nth car in a line of cars is prescribed in terms of the motion of the other cars. [See, for example, Newell (1961) and the earlier references given there.] If the position of the nth car is $s_n(t)$ at time t, the assumed laws of motion usually take the form

$$\dot{s}_n(t+\Delta) = G\{s_{n-1}(t) - s_n(t)\}, \tag{3.19}$$

between velocity $\dot{s}_n$ and headway $h_n = s_{n-1} - s_n$ with a time lag Δ to account for the driver response time. If $G(h_n)$ is chosen to be linear in h_n, or if the equation is linearized to study fluctuations about a uniform state, solutions can be obtained by Laplace transforms. In general, however, one must appeal to computer studies.

This type of model takes a more rigid view of how each individual car moves, so it is narrower in scope than the continuum theory, where the whole complicated behavior of the individuals is lumped together in the function $Q(\rho)$ and the parameters ν and τ. But each model leads to a particular form for these quantities, which may be helpful in interpreting observational data. Moreover, such models may lead to additional effects that cannot be seen in the continuum theory.

To see the correspondence of the particular car-following model in (3.19) with the continuum theory, we first note the relation of $G(h)$ to $Q(\rho)$. In a uniform stream with equal spacing h, the velocities in (3.19) are all equal and are given by the relation $v = G(h)$. Since $h = 1/\rho$, $v = q/\rho$, the function $Q(\rho)$ in the corresponding continuum equations is

$$Q(\rho) = \rho G\left(\frac{1}{\rho}\right).$$

If empirical or other information is known about $G(h)$, it may be transferred to information about $Q(\rho)$ near $\rho = \rho_j$. Of course at lower densities $Q(\rho)$ will be affected more by cars overtaking and changing lanes.

The wave propagation described by (3.19), in which the motion of a lead car is transmitted successively back through the stream, should be a typical finite difference version of the earlier continuum results with this choice of $Q(\rho)$. The finite difference form of (3.19) also introduces higher order effects equivalent to those in (3.2) and we can make a detailed comparison. If we let

$$v_n(t) = \dot{s}_n(t), \qquad s_{n-1}(t) - s_n(t) = h_n(t), \tag{3.20}$$

(3.19) is equivalent to the pair of equations

$$v_n(t+\Delta) = G(h_n), \tag{3.21}$$

$$\frac{dh_n}{dt} = v_{n-1}(t) - v_n(t). \tag{3.22}$$

In this form we introduce continuous functions $v(x,t)$ and $h(x,t)$ such that

$$v(s_n, t) = v_n(t), \tag{3.23}$$

$$h\left(\frac{s_{n-1}+s_n}{2}, t\right) = h_n(t), \tag{3.24}$$

and obtain corresponding partial differential equations in the approximations of small Δ and small h_n. Equation 3.21 may be written

$$v\{s_n(t+\Delta), t+\Delta\} = G\left\{h\left(s_n + \frac{1}{2}h_n, t\right)\right\},$$

and it may be approximated by

$$v + (v_t + vv_x)\Delta = G(h) + \frac{1}{2}hG'(h)h_x, \tag{3.25}$$

where the functions are evaluated at $x = s_n(t)$ and the errors are of order Δ^2, h^2. Equation 3.22 may be written

$$\frac{d}{dt}h\left(\frac{s_{n-1}+s_n}{2}, t\right) = v(s_{n-1}, t) - v(s_n, t)$$

and approximated by

$$h_t + vh_x = hv_x \qquad \text{at } x = \frac{s_{n-1}+s_n}{2}. \tag{3.26}$$

The error in (3.26) is *third* order in h [due to the centering of h at the midpoint $(s_{n-1}+s_n)/2$], so the equation is correct to both first and second orders. In terms of $\rho = 1/h$, $V(\rho) = G(h)$, (3.25)–(3.26) become

$$v + (v_t + vv_x)\Delta = V(\rho) + \frac{1}{2}\frac{V'(\rho)}{\rho}\rho_x, \tag{3.27}$$

$$\rho_t + (\rho v)_x = 0. \tag{3.28}$$

To lowest order in Δ and h, we would have

$$v = V(\rho), \qquad \rho_t + (\rho v)_x = 0,$$

which is just the kinematic theory. The differencing has been arranged so that the next order corrections leave the conservation equation (3.28) unchanged.

Equations 3.27 and 3.28 are identical with (3.2) and (3.3) if we take

$$\tau = \Delta, \qquad \nu = -\frac{1}{2}V'(\rho).$$

Since $V - c = -\rho V'(\rho)$, the stability criterion (3.7) may be written

$$2\rho^2 |V'(\rho)| \Delta < 1,$$

or, equivalently,

$$2G'(h)\Delta < 1.$$

This is exactly the condition found in the car-following theories (Chandler, Herman, and Montroll, 1958; Kometani and Sasaki, 1958). Similarly the shock structures discussed earlier on the basis of (3.2) should be close to those discussed by Newell (1961) on the basis of (3.19).

An effect that cannot be covered by the continuum theory is the actual collision of cars. In a queue described by (3.19), this occurs if $s_{n-1} - s_n$ ever drops to the car length L. In the special case

$$\dot{s}_n(t+\Delta) = \alpha\{s_{n-1}(t) - s_n(t) - L\},$$

which can be solved by Laplace transforms, it may be shown that the criterion for avoiding collision is

$$\alpha\Delta < \frac{1}{e};$$

this is slightly more stringent than the stability criterion $2\alpha\Delta < 1$ found above. The analysis would take us too far afield and the reader is referred to the discussion of local stability in the paper by Herman, Montroll, Potts, and Rothery (1959).

3.2 Flood Waves

For flood waves, the "density" in the sense of the general theory presented in Chapter 2 is the cross-sectional area of the riverbed, $A(x,t)$, at

position x along the river at time t. If the volume flow across the section is $q(x,t)$ per unit time, the conservation equation is

$$\frac{d}{dt}\int_{x_2}^{x_1} A(x,t)\,dx + q(x_1,t) - q(x_2,t) = 0,$$

or, in differentiated form,

$$\frac{\partial A}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{3.29}$$

Flow in a river is obviously so complicated that any flow model for the second relation between q and A must be extremely approximate, giving only qualitative effects and general order of magnitude results for propagation speeds, wave profiles, and so on. However, observations during slow changes in the river level may be used also to establish the dependence of depth and the area A on the flow q. These provide empirical curves for the function

$$q = Q(A,x) \tag{3.30}$$

in steady flows. This relation can be combined with (3.29) to give a first approximation for unsteady flows which vary slowly. Then $A(x,t)$ satisfies

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial A}\frac{\partial A}{\partial x} = -\frac{\partial Q}{\partial x}. \tag{3.31}$$

We have again the theory discussed in Chapter 2 with the propagation velocity

$$c = \frac{\partial Q}{\partial A} = \frac{1}{b}\frac{\partial Q}{\partial h}. \tag{3.32}$$

[The second form introduces the breadth b and depth h, and $dA = b\,dh$.] This is the Kleitz-Seddon formula for flood waves, apparently established first by Kleitz (1858, unpublished) and thoroughly discussed and used effectively by Seddon (1900).

Empirical relations for (3.30) can be viewed against simple theoretical models. The relation is an expression of the balance between the frictional force of the river bed and the gravitational force. In theoretical models, the frictional force is usually assumed to be proportional to v^2, where v is the average velocity

$$v = \frac{q}{A},$$

and also proportional to the wetted perimeter P of the cross-section at