CHAPTER 2

Waves and First Order Equations

We start the detailed discussion of hyperbolic waves with a study of first order equations. As noted in Chapter 1, the simplest wave equation is

$$\rho_t + c_0 \rho_x = 0, \qquad c_0 = \text{constant}. \tag{2.1}$$

When this equation arises, the dependent variable is usually the density of something so we now use the symbol ρ rather than the all-purpose symbol φ of the introduction. The general solution of (2.1) is $\rho = f(x - c_0 t)$, where $f(x)$ is an arbitrary function, and the solution of any particular problem consists merely of matching the function f to initial or boundary values. It clearly describes a wave motion since an initial profile $f(x)$ would be translated unchanged in shape a distance $c_0 t$ to the right at time t. At two observation points a distance s apart, exactly the same disturbance would be recorded with a time delay of s/c_0.

Although this linear case is almost trivial, the nonlinear counterpart

$$\rho_t + c(\rho)\rho_x = 0, \tag{2.2}$$

where $c(\rho)$ is a given function of ρ, is certainly not and a study of it leads to most of the essential ideas for nonlinear hyperbolic waves. As remarked earlier, many of the classical examples of wave propagation are described by second or higher order equations such as the wave equation $c_0^2 \nabla^2 \varphi = \varphi_{tt}$, but a surprising number of physical problems do lead directly to (2.2) or extensions of it. Examples will be given after a preliminary discussion of the solution. Even in higher order problems, one often searches for special solutions or approximations that involve (2.2).

2.1 Continuous Solutions

One approach to the solution of (2.2) is to consider the function $\rho(x, t)$ at each point of the (x, t) plane and to note that $\rho_t + c(\rho)\rho_x$ is the total derivative of ρ along a curve which has slope

$$\frac{dx}{dt}=c(\rho) \tag{2.3}$$

at every point of it. For along any curve in the (x,t) plane, we may consider x and ρ to be functions of t, and the total derivative of ρ is

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\frac{dx}{dt}\frac{\partial\rho}{\partial x}.$$

The total derivative notation should be sufficient to indicate when x and ρ are being treated as functions of t on a certain curve; the introduction of new symbols each time this is done eventually becomes confusing. We now consider a curve $\mathcal{C}$ in the (x,t) plane which satisfies (2.3). Of course such a curve cannot be determined explicitly in advance since the defining equation (2.3) involves the unknown values of ρ on the curve. However, its consideration will lead us to a simultaneous determination of a possible curve $\mathcal{C}$ and the solution ρ on it. On $\mathcal{C}$ we deduce from the total derivative relation and from (2.2) that

$$\frac{d\rho}{dt}=0, \qquad \frac{dx}{dt}=c(\rho). \tag{2.4}$$

We first observe that ρ remains constant on $\mathcal{C}$. It then follows that $c(\rho)$ remains constant on $\mathcal{C}$, and therefore that the curve $\mathcal{C}$ must be a straight line in the (x,t) plane with slope $c(\rho)$. Thus the general solution of (2.2) depends on the construction of a family of straight lines in the (x,t) plane, each line with slope $c(\rho)$ corresponding to the value of ρ on it. This is easily done in any specific problem.

Let us take for example the initial value problem

$$\rho=f(x), \quad t=0, \quad -\infty<x<\infty,$$

and refer to the (x,t) diagram in Fig. 2.1. If one of the curves $\mathcal{C}$ intersects $t=0$ at $x=\xi$ then $\rho=f(\xi)$ on the whole of that curve. The corresponding slope of the curve is $c(f(\xi))$, which we will denote by $F(\xi)$; it is a known function of ξ calculated from the function $c(\rho)$ in the equation and the given initial function $f(\xi)$. The equation of the curve then is

$$x=\xi+F(\xi)t.$$

This determines one typical curve and the value of ρ on it is $f(\xi)$. Allowing ξ to vary, we obtain the whole family:

$$\rho=f(\xi), \qquad c=F(\xi)=c(f(\xi)) \tag{2.5}$$

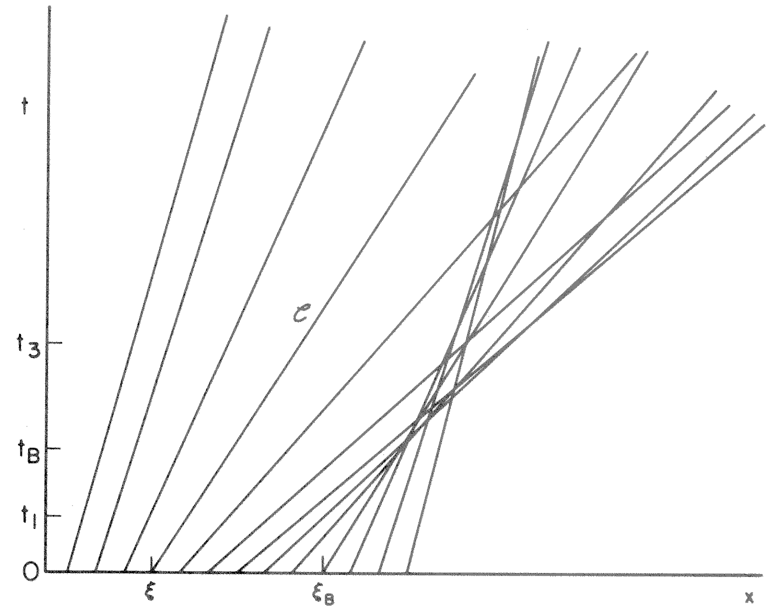

Fig. 2.1. Characteristic diagram for nonlinear waves.

on

$$x=\xi+F(\xi)t. \tag{2.6}$$

We may now change the emphasis and use (2.5) and (2.6) as an analytic expression for the solution, free of the particular construction. That is, ρ is given by (2.5) where $\xi(x,t)$ is defined implicitly by (2.6). Let us check that this gives the solution. From (2.5),

$$\rho_t=f'(\xi)\xi_t, \qquad \rho_x=f'(\xi)\xi_x,$$

and from the t and x derivatives of (2.6),

$$0=F(\xi)+\{1+F'(\xi)t\}\xi_t,$$

$$1=\{1+F'(\xi)t\}\xi_x.$$

Therefore

$$\rho_t=-\frac{F(\xi)f'(\xi)}{1+F'(\xi)t}, \qquad \rho_x=\frac{f'(\xi)}{1+F'(\xi)t}, \tag{2.7}$$

and we see that

$$\rho_t+c(\rho)\rho_x=0$$

since $c(\rho)=F(\xi)$. The initial condition $\rho=f(x)$ is satisfied because $\xi=x$ when $t=0$.

The curves used in the construction of the solution are the *characteristic curves* for this special problem. Similar characteristics play an important role in all problems involving hyperbolic differential equations. In general, characteristic curves do not have the property that the solution remains constant along them. This happens to be true in the special case of (2.2); it is not the defining property of characteristics. The general definitions will be considered later, but it will be convenient now to refer to the curves defined by (2.3) as characteristics.

The basic idea of wave propagation is that some recognizable feature of the disturbance moves with a finite velocity. For hyperbolic equations, the characteristics correspond to this idea. Each characteristic curve in (x,t) space represents a moving wavelet in x space, and the behavior of the solution on a characteristic curve corresponds to the idea that information is carried by that wavelet. The mathematical statement in (2.4) may be given this type of emphasis by saying that different values of ρ "propagate" with velocity $c(\rho)$. Indeed, the solution at time t can be constructed by moving each point on the initial curve $\rho=f(x)$ a distance $c(\rho)t$ to the right; the distance moved is different for the different values of ρ. This is shown in Fig. 2.2 for the case $c'(\rho)>0$; the corresponding time levels are indicated in Fig. 2.1. The dependence of c on ρ produces the typical nonlinear distortion of the wave as it propagates. When $c'(\rho)>0$, higher values of ρ propagate faster than lower ones. When $c'(\rho)<0$, higher values of ρ propagate slower and the distortion has the opposite tendency to that shown in Fig. 2.2. For the linear case, c is constant and the profile is translated through a distance ct without any change of shape.

It is immediately apparent from Fig. 2.2 that the discussion is far from complete. Any compressive part of the wave, where the propagation velocity is a decreasing function of x, ultimately "breaks" to give a

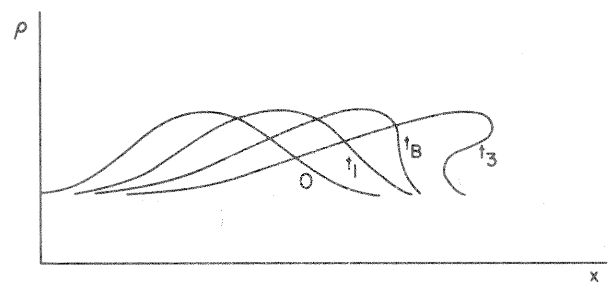

Fig. 2.2. Breaking wave: successive profiles corresponding to the times 0, t_1, t_B, t_3 in Fig. 2.1.

triple-valued solution for $\rho(x,t)$. The breaking starts at the time indicated by $t=t_B$ in Fig. 2.2, when the profile of ρ first develops an infinite slope. The analytic solution (2.7) confirms this and allows us to determine the breaking time t_B. On any characteristic for which $F'(\xi)<0$, ρ_x and ρ_t become infinite when

$$t=-\frac{1}{F'(\xi)}.$$

Therefore breaking first occurs on the characteristic $\xi=\xi_B$ for which $F'(\xi)<0$ and $|F'(\xi)|$ is a maximum; the time of first breaking is

$$t_B=-\frac{1}{F'(\xi_B)}. \tag{2.8}$$

This development can also be followed in the (x,t) plane. A compressive part of the wave with $F'(\xi)<0$ has converging characteristics; since the characteristics are straight lines, they must eventually overlap to give a region where the solution is multivalued, as in Fig. 2.1. This region may be considered as a fold in the (x,t) plane made up of three sheets, with different values of ρ on each sheet. The boundary of the region is an envelope of characteristics. The family of characteristics is given by (2.6) with ξ as parameter. The condition that two neighboring characteristics ξ, $\xi+\delta\xi$ intersect at a point (x,t) is that

$$x=\xi+F(\xi)t$$

and

$$x=\xi+\delta\xi+F(\xi+\delta\xi)t$$

hold simultaneously. In the limit $\delta\xi\to 0$, these give

$$x=\xi+F(\xi)t \quad \text{and} \quad 0=1+F'(\xi)t$$

for the implicit equations of an envelope. The second of these relations shows that an envelope is formed in $t>0$ by those characteristics for which $F'(\xi)<0$. The minimum value of t on the envelope occurs for the value of ξ for which $-F'(\xi)$ is maximum. This is the first time of breaking in agreement with (2.8). If $F''(\xi)$ is continuous, the envelope has a cusp at $t=t_B$, $\xi=\xi_B$, as shown in Fig. 2.1.

An extreme case of breaking arises when the initial distribution has a discontinuous step with the value of $c(\rho)$ behind the discontinuity greater than that ahead. If we have the initial functions

$$f(x)=\begin{cases}\rho_1, & x>0\\ \rho_2, & x<0\end{cases}$$

and

$$F(x)=\begin{cases} c_1=c(\rho_1), & x>0 \\ c_2=c(\rho_2), & x<0 \end{cases}$$

with $c_2>c_1$, then breaking occurs immediately. This is shown in Fig. 2.3 for the case $c'(\rho)>0$, $\rho_2>\rho_1$. The multivalued region starts right at the origin and is bounded by the characteristics $x=c_1t$ and $x=c_2t$; the boundary is no longer a cusped envelope since F and its derivatives are not continuous. Nevertheless, the result may be considered as the limit of a series of smoothed-out steps, and the breaking point moves closer to the origin as the initial profile approaches the discontinuous step.

On the other hand, if the initial step function is expansive with $c_2<c_1$, there is a perfectly good continuous solution. It may be obtained as the limit of (2.5) and (2.6) in which all the values of F between c_2 and c_1 are taken on characteristics through the origin $\xi=0$. This corresponds to a fan of characteristics in the (x,t) plane as in Fig. 2.4. Each member of the fan has a different slope F but the same ξ. The function F is a step function but we use all the values of F between c_2 and c_1 on the face of the step and take them all to correspond to $\xi=0$. In the fan, the solution (2.5), (2.6) then reads

$$c=F, \qquad x=Ft, \qquad \text{for } c_2<F<c_1,$$

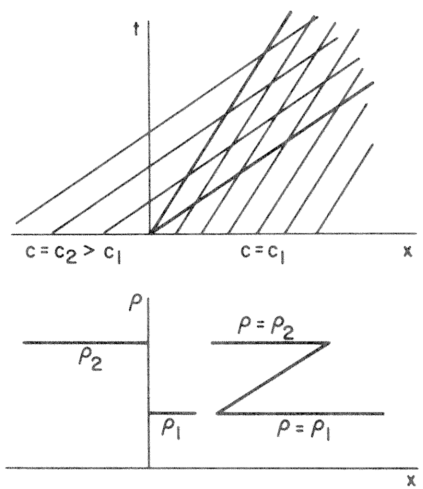

Fig. 2.3. Centered compression wave with overlap.

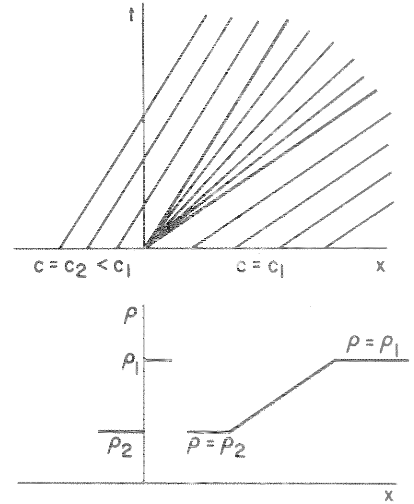

Fig. 2.4. Centered expansion wave.

and by elimination of F we have the simple explicit solution for c:

$$c=\frac{x}{t}, \qquad c_2<\frac{x}{t}<c_1.$$

The complete solution for c is

$$c=\begin{cases} c_1, & c_1<\dfrac{x}{t}, \\ \dfrac{x}{t}, & c_2<\dfrac{x}{t}<c_1, \\ c_2, & \dfrac{x}{t}<c_2. \end{cases} \tag{2.9}$$

The relation $c=c(\rho)$ can be solved to determine ρ. For the compressive step, $c_2>c_1$, the fan in the (x,t) plane is reversed to produce the overlap shown in Fig. 2.3.

In most physical problems where this theory arises, $\rho(x,t)$ is just the density of some medium and is inherently single-valued. Therefore when breaking occurs (2.2) must cease to be valid as a description of the physical problem. Even in cases such as water waves where a multivalued solution for the height of the surface could at least be interpreted, it is still found that (2.2) is inadequate to describe the process. Thus the situation is that some assumption or approximate relation in the formulation leading to (2.2) is no longer valid. In principle one must return to the physics of the problem, see what went wrong, and formulate an improved theory. However, it turns out, as we shall see, that the foregoing solution can be saved by allowing discontinuities into the solution; there is then a single-valued solution with a simple jump discontinuity to replace the multivalued continuous solution. This requires some mathematical extension of what we mean by a "solution" to (2.2), since strictly speaking the derivatives of ρ will not exist at a discontinuity. It can be done through the concept of a "weak solution." But it is important to appreciate that the real issue is not just a mathematical question of extending the solution of (2.2). The breakdown of the continuous solution is associated with the breakdown of some approximate relation in the physics, and the two aspects must be considered together. It is found, for example, that there are several possible families of discontinuous solutions, all satisfactory mathematically; the nonuniqueness can be resolved only by appeal to the physics.

Clearly then, we cannot proceed further without discussion of some physical problems. The prototype is the nonlinear theory of waves in a gas and the formation of shock waves. When viscosity and heat conduction are ignored, the equations of gas dynamics have breaking solutions similar to the preceding ones. As the gradients become steep, just before breaking, the effects of viscosity and heat conduction are no longer negligible. These effects can be included to give an improved theory and waves no longer break in that theory. There is a thin region, a shock wave, in which viscosity and heat conduction are crucially important; outside the shock wave, viscosity and heat conduction may still be neglected. The flow variables change rapidly in the shock. This shock region is idealized into a discontinuity in the "extended" inviscid theory, and only shock conditions relating the jumps of the flow variables across the discontinuity need to be added to the inviscid theory.

We will study all these various aspects in detail. However, gas dynamics is not the simplest example, since it involves higher order equations, and we shall discuss the essential ideas first in the context of the simpler first order problems. It should be remembered, though, that these ideas were developed for gas dynamics, and we are reversing the chronological order. The basic ideas were elucidated by Poisson (1807), Stokes (1848), Riemann (1858), Earnshaw (1858), Rankine (1870), Hugoniot (1889), Rayleigh (1910), Taylor (1910)—a most impressive list. The time required indicates that putting the different aspects together was quite a complicated affair.

2.2 Kinematic Waves

In many problems of wave propagation there is a continuous distribution of either material or some state of the medium, and (for a one

dimensional problem) we can define a density $\rho(x,t)$ per unit length and a flux $q(x,t)$ per unit time. We can then define a flow velocity $v(x,t)$ by

$$v=\frac{q}{\rho}.$$

Assuming that the material (or state) is conserved, we can stipulate that the rate of change of the total amount of it in any section $x_1>x>x_2$ must be balanced by the net inflow across x_1 and x_2. That is,

$$\frac{d}{dt}\int_{x_2}^{x_1}\rho(x,t)\,dx+q(x_1,t)-q(x_2,t)=0. \tag{2.10}$$

If $\rho(x,t)$ has continuous derivatives, we may take the limit as $x_1\to x_2$ and obtain the conservation equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0. \tag{2.11}$$

The simplest wave problems arise when it is reasonable, on either theoretical or empirical grounds, to postulate (in a first approximation!) a functional relation between q and ρ. If this is written as

$$q=Q(\rho), \tag{2.12}$$

(2.11) and (2.12) form a complete system. On substitution we have

$$\rho_t+c(\rho)\rho_x=0 \tag{2.13}$$

where

$$c(\rho)=Q'(\rho). \tag{2.14}$$

This leads to our (2.2) and a typical solution is given by (2.5) to (2.6). The breaking requires us to reconsider both the mathematical assumption that ρ and q have derivatives and the physical assumption that $q=Q(\rho)$ is a good approximation. To fix ideas for the further development of the theory some specific examples are noted briefly here. We shall return to them in Chapter 3 for a more detailed discussion after the theoretical ideas are complete.

An amusing case (which is also important) concerns traffic flow. It is reasonable to suppose that some essential features of fairly heavy traffic flow may be obtained by treating a stream of traffic as a continuum with an observable density $\rho(x,t)$, equal to the number of cars per unit length, and a flow $q(x,t)$, equal to the number of cars crossing the position x per unit time. For a stretch of highway with no entries or exits, cars are conserved! So we stipulate (2.10). For traffic it also seems reasonable to argue that the traffic flow q is determined primarily by the local density ρ and to propose (2.12) as a first approximation. Such functional relations have been studied and documented to some extent by traffic engineers. We can then apply the theory. But it is clear in this case that when breaking occurs there is no lack of possible explanations for some breakdown in the formulation. Certainly the assumption $q = Q(\rho)$ is a very simplified view of a very complicated phenomenon. For example, if the density is changing rapidly (as it is near breaking), one expects the drivers to react to more than the local density and one also expects that there will be a time lag before they respond adequately to the changing conditions. One might also question the continuum assumption itself.

Another example is flood waves in long rivers. Here ρ is replaced by the cross-sectional area of the channel, A, and this varies with x and t as the level of the river rises. If q is the volume flux across the section, then (2.10) between A and q expresses the conservation of water. Although the fluid flow is extremely complicated, it seems reasonable to start with a functional relation $q = Q(A)$ as a first approximation to express the increase in flow as the level rises. Such relations have been plotted from empirical observations on various rivers. But it is again clear that this assumption is an oversimplification which may well have to be corrected if troubles arise in the theory.

A similar example, proposed and studied extensively by Nye (1960), is the example of glacier flow. The flow velocity is expected to increase with the thickness of the ice, and it seems reasonable to assume a functional dependence between the two.

In chromatography and in similar exchange processes studied in problems of chemical engineering, the same theory arises. The formulation is a little more complicated. The situation is that a fluid carrying dissolved substances or particles or ions flows through a fixed bed and the material being carried is partially adsorbed on the fixed solid material in the bed. The fluid flow is idealized to have a constant velocity V. Then if ρ_f is the density of the material carried in the fluid, and ρ_s is the density deposited on the solid,

$$\rho = \rho_f + \rho_s, \qquad q = V\rho_f.$$

Hence the conservation equation (2.11) reads

$$\frac{\partial}{\partial t}(\rho_f + \rho_s) + \frac{\partial}{\partial x}(V\rho_f) = 0.$$

A second relation concerns the rate of deposition on the solid bed. The

exchange equation

$$\frac{\partial \rho_s}{\partial t} = k_1(A - \rho_s)\rho_f - k_2\rho_s(B - \rho_f)$$

is apparently the simplest equation with the required properties. The first term represents deposition from the fluid to the solid at a rate proportional to the amount in the fluid, but limited by the amount already on the solid up to a capacity A. The second term is the reverse transfer from the solid to the fluid. (In some processes, the second term is just proportional to ρ_s; this is the limit $B \to \infty$, $k_2 B$ finite.) In equilibrium, the right hand side of the equation vanishes and ρ_s is a definite function of ρ_f. In slowly varying conditions, with relatively large reaction rates k_1 and k_2, we may take a first approximation in which the right hand side still vanishes ("quasi-equilibrium") and we have

$$\rho_s = A\frac{k_1\rho_f}{k_2 B + (k_1 - k_2)\rho_f}.$$

Thus ρ_s is a function of ρ_f; hence q is a function of ρ. When changes become rapid, just before breaking, the term $\partial\rho_s/\partial t$ in the rate equation can no longer be neglected.

As a different type of example, the concept of group velocity can be fitted into this general scheme. In linear dispersive waves, as already noted following (1.26), there are oscillatory solutions with a local wave number $k(x,t)$ and a local frequency $\omega(x,t)$. Thus k is the density of the waves—the number of wave crests per unit length—and ω is the flux—number of wave crests crossing the position x per unit time. If we expect that wave crests will be conserved in the propagation, we have, in differential form, the conservation equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0.$$

In addition, k and ω are related by the dispersion relation

$$\omega = \omega(k).$$

Hence

$$\frac{\partial k}{\partial t} + \omega'(k)\frac{\partial k}{\partial x} = 0.$$

We have a wave propagation for the variations of the local wave number of the "carrier" wavetrain, and the propagation velocity is $d\omega/dk$. This is the group velocity. These ideas will be considered in full detail in the later discussion of dispersive waves.

The wave problems listed here depend primarily on the conservation equation (2.11), and for this reason they were given the name *kinematic waves* (Lighthill and Whitham, 1955) in contrast to the usual acoustic or elastic waves which depend strongly on how the acceleration is determined through the laws of dynamics.

After this review of some of the physical problems, we return to the study of breaking and shock waves in order to complete the theory. Further details of the physical problems are pursued in Chapter 3.

2.3 Shock Waves

When breaking occurs we question the assumption $q = Q(\rho)$ in (2.12) and also the differentiability of ρ and q in (2.11). But, provided the continuum assumption is adequate, we still insist on the conservation equation (2.10).

Consider first the mathematical question of whether discontinuities are possible. Certainly a simple jump discontinuity in ρ and in q is feasible as far as (2.10) is concerned; all the expressions in (2.10) have a meaning. Does (2.10) provide any restriction? To answer this, suppose there is a discontinuity at $x = s(t)$ and that x_1 and x_2 are chosen so that $x_1 > s(t) > x_2$. Suppose ρ and q and their first derivatives are continuous in $x_1 \geqslant x > s(t)$ and in $s(t) > x \geqslant x_2$, and have finite limits as $x \to s(t)$ from above and below. Then (2.10) may be written

$$q(x_2,t) - q(x_1,t) = \frac{d}{dt}\int_{x_2}^{s(t)} \rho(x,t)\,dx + \frac{d}{dt}\int_{s(t)}^{x_1} \rho(x,t)\,dx$$

$$= \rho(s^-,t)\dot{s} - \rho(s^+,t)\dot{s} + \int_{x_2}^{s(t)} \rho_t(x,t)\,dx + \int_{s(t)}^{x_1} \rho_t(x,t)\,dx,$$

where $\rho(s^-,t)$, $\rho(s^+,t)$ are the value of $\rho(x,t)$ as $x \to s(t)$ from below and above, respectively, and $\dot{s} = ds/dt$. Since ρ_t is bounded in each of the intervals separately, the integrals tend to zero in the limit as $x_1 \to s^+$, $x_2 \to s^-$. Therefore

$$q(s^-,t) - q(s^+,t) = \{\rho(s^-,t) - \rho(s^+,t)\}\dot{s}.$$

A conventional notation is to use a subscript 1 for the values ahead of the shock and a subscript 2 for values behind. Then if U is the shock velocity, $\dot{s}$,

$$q_2 - q_1 = U(\rho_2 - \rho_1). \tag{2.15}$$

The condition may also be written in the form

$$-U[\rho] + [q] = 0, \tag{2.16}$$

where the brackets indicate the jump in the quantity. This form gives a nice correspondence between the shock condition and the differential equation (2.11), the correspondence being

$$\frac{\partial}{\partial t} \leftrightarrow -U[\ \ ], \qquad \frac{\partial}{\partial x} \leftrightarrow [\ \ ]. \tag{2.17}$$

We can now extend our solutions of (2.10) to allow such discontinuities. In any continuous part of the solution, (2.11) will still be satisfied and the assumption (2.12) may be retained. Since $q = Q(\rho)$ in the continuous parts, we have $q_2 = Q(\rho_2)$ and $q_1 = Q(\rho_1)$ on the two sides of any shock, and the shock condition (2.15) may be written

$$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}. \tag{2.18}$$

The problem then reduces to fitting shock discontinuities into the solution (2.5), (2.6) in such a way that (2.18) is satisfied and multivalued solutions are avoided.

The simplest case is the problem

$$\left.\begin{array}{lll} \rho = \rho_1, & c = c(\rho_1) = c_1, & x > 0, \\ \rho = \rho_2, & c = c(\rho_2) = c_2, & x < 0, \end{array}\right\} \quad t = 0,$$

with $c_2 > c_1$. The breaking solution was indicated in Fig. 2.3. Now a single-valued solution is possible which is just a shock moving with velocity (2.18):

$$\rho = \rho_1, \quad x > Ut,$$
$$\rho = \rho_2, \quad x < Ut.$$

This is represented schematically in Fig. 2.5.

A popular way to derive the shock condition is to view this particular solution from a frame of reference in which the shock is at rest, as shown

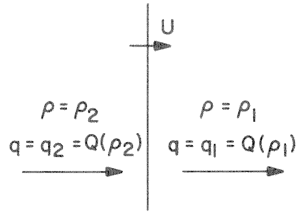

Fig. 2.5. Flow quantities for moving shock.

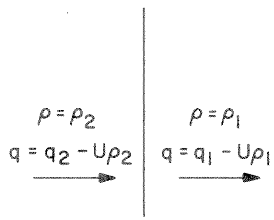

Fig. 2.6. Flow quantities relative to stationary shock.

in Fig. 2.6. The relative flows become $q_1 - U\rho_1$ and $q_2 - U\rho_2$. The conservation law may be stated immediately in the form

$$q_1 - U\rho_1 = q_2 - U\rho_2,$$

and (2.15) follows.

Before proceeding with the general problem of shock fitting, we consider the alternative view that the differential equation (2.11) is adequate but that the assumed relation (2.12) is insufficient.

2.4 Shock Structure

As a particular case, we need to find and examine a more accurate description of the simple discontinuous solution represented in Fig. 2.5. This is the problem of finding the "shock structure."

In many problems of kinematic waves, it would be a better approximation to suppose that q is a function of the density gradient ρ_x as well as ρ. A simple assumption is to take

$$q = Q(\rho) - \nu\rho_x, \tag{2.19}$$

where ν is a constant. In traffic flow, for example, we may argue that drivers will reduce their speed to account for an increasing density ahead,

and conversely. This argument would propose a positive value for ν, and we see below that the sign is important. If ν is small, in some suitable dimensionless measure, (2.12) is a good approximation provided ρ_x is not relatively large. At breaking, ρ_x becomes large and the correction term becomes crucial, however small ν may be. Now in *this* formulation, consider continuous solutions. From (2.11) and (2.19), they satisfy

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx}, \qquad c(\rho) = Q'(\rho). \tag{2.20}$$

The term $c(\rho)\rho_x$ in (2.20) leads to steepening and breaking. On the other hand, the term $\nu\rho_{xx}$ introduces diffusion typical of the heat equation

$$\rho_t = \nu\rho_{xx}.$$

For the heat equation, the solution of the initial step function problem

$$\left.\begin{aligned} \rho &= \rho_1, & x &> 0, \\ \rho &= \rho_2, & x &< 0, \end{aligned}\right\} \qquad t = 0$$

is

$$\rho = \rho_2 + \frac{\rho_1 - \rho_2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\nu t}} e^{-\zeta^2}\, d\zeta.$$

This represents a smoothed-out step approaching values ρ_1, ρ_2 as $x \to \pm\infty$, and with slope decreasing like $(\nu t)^{-1/2}$. The two opposite tendencies of nonlinear steepening and diffusion are combined in (2.20). The significance of $\nu > 0$ can be seen from the heat equation; solutions are unstable if $\nu < 0$.

We now look within the framework of this more accurate theory for the solution to replace the one shown in Fig. 2.5. One obvious idea is to look for a steady profile solution in which

$$\rho = \rho(X), \qquad X = x - Ut,$$

where U is a constant still to be determined. Then from (2.20),

$$\{c(\rho) - U\}\rho_X = \nu\rho_{XX}.$$

Integrating once, we have

$$Q(\rho) - U\rho + A = \nu\rho_X, \tag{2.21}$$

where A is a constant of integration. An implicit relation for $\rho(X)$ is obtained in the form

$$\frac{X}{\nu} = \int \frac{d\rho}{Q(\rho) - U\rho + A}, \tag{2.22}$$

but the qualitative behavior is more readily seen directly from (2.21). We are interested in the possibility of a solution which tends to constant states $\rho \to \rho_1$ as $X \to +\infty$, $\rho \to \rho_2$ as $X \to -\infty$. If such a solution exists with $\rho_X \to 0$ as $X \to \pm\infty$, the arbitrary parameters U, A must satisfy

$$Q(\rho_1) - U\rho_1 + A = Q(\rho_2) - U\rho_2 + A = 0.$$

In particular,

$$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}. \tag{2.23}$$

In such a solution, the relation between the velocity U and the two states at $\pm\infty$ is exactly the same as in the shock condition!

The values ρ_1, ρ_2 are zeros of $Q(\rho) - U\rho + A$, and in general they are simple zeros. As $\rho \to \rho_1$ or ρ_2 in (2.22), the integral diverges and $X \to \pm\infty$ as required. If $Q(\rho) - U\rho + A < 0$ between the two zeros, and if ν is positive, we have $\rho_X < 0$ and the solution is as shown in Fig. 2.7 with ρ increasing monotonically from ρ_1 at $+\infty$ to ρ_2 at $-\infty$. If $Q(\rho) - U\rho + A > 0$ and $\nu > 0$, the solution increases from ρ_2 at $-\infty$ to ρ_1 at $+\infty$. It is clear from (2.21) that if ρ_1, ρ_2 are kept fixed (so that U, A are fixed), a change in ν can be absorbed by a change in the X scale. As $\nu \to 0$, the profile in Fig. 2.7 is compressed in the X direction and tends in the limit to a step function increasing ρ from ρ_1 to ρ_2 and traveling with the velocity given by (2.23). This is exactly the discontinuous shock solution seen in Fig. 2.5. For small nonzero ν the shock is a rapid but continuous increase taking place over a narrow region. The breaking due to the nonlinearity is balanced by the diffusion in this narrow region to give a steady profile.

One very important point is the sign of the change in ρ. A continuous wave carrying an increase of ρ will break forward and require a shock with $\rho_2 > \rho_1$ if $c'(\rho) > 0$; it will break backward and require a shock with $\rho_2 < \rho_1$ if $c'(\rho) < 0$. The shock structure given by (2.21) must agree. As remarked above, ν is always positive for stability, so the direction of increase of ρ depends on the sign of $Q(\rho) - U\rho + A$ between the two zeros ρ_1 and ρ_2. But

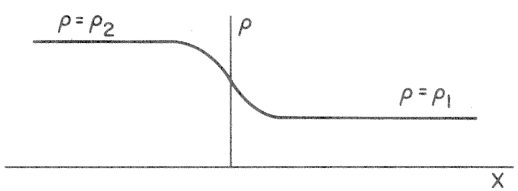

Fig. 2.7. Shock structure.

$c'(\rho) = Q''(\rho)$. Hence when $c'(\rho) > 0$, $Q(\rho) - U\rho + A < 0$ between zeros and the solution is as seen in Fig. 2.7 with $\rho_2 > \rho_1$ as required. If $c'(\rho) < 0$, the step is reversed and $\rho_2 < \rho_1$. The breaking argument and the shock structure agree.

In the special case of a quadratic expression for $Q(\rho)$, taken as

$$Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma, \tag{2.24}$$

the integral in (2.22) is easily evaluated. The sign of α determines the sign of $c'(\rho) = Q''(\rho)$ and we consider $\alpha > 0$, for definiteness. We may write

$$Q - U\rho + A = -\alpha(\rho - \rho_1)(\rho_2 - \rho),$$

where

$$U = \beta + \alpha(\rho_1 + \rho_2), \qquad A = \alpha\rho_1\rho_2 - \gamma.$$

Then (2.22) becomes

$$\frac{X}{\nu} = -\int \frac{d\rho}{\alpha(\rho - \rho_1)(\rho_2 - \rho)} = \frac{1}{\alpha(\rho_2 - \rho_1)} \log \frac{\rho_2 - \rho}{\rho - \rho_1}. \tag{2.25}$$

As $X \to \infty$, $\rho \to \rho_1$ exponentially, and as $X \to -\infty$, $\rho \to \rho_2$ exponentially. There is no precise thickness to the transition region, but we can introduce various measures of the scale, such as the length over which 90% of the change occurs or $(\rho_2 - \rho_1)$ divided by the maximum slope $|\rho_X|$. Clearly all such measures of thickness are proportional to

$$\frac{\nu}{\alpha(\rho_2 - \rho_1)}. \tag{2.26}$$

If this is small compared with other typical lengths in the problem, the rapid shock transition is satisfactorily approximated by a discontinuity. We confirm that the thickness tends to zero as $\nu \to 0$ for fixed ρ_1, ρ_2, but it also should be noted that sufficiently weak shocks with $(\rho_2 - \rho_1)/\rho_1 \to 0$ ultimately become thick for fixed ν, however small. For weak shocks $Q(\rho)$ can always be approximated by a suitable quadratic over the range ρ_1 to ρ_2, so that (2.25) applies. Even for moderately strong shocks it is a good overall approximation to the shape.

The shock structure is only one special solution of (2.20), but from it we might expect in general that when $\nu \to 0$ in some suitable nondimensional form, solutions of (2.20) tend to solutions of

$$\rho_t + c(\rho)\rho_x = 0$$

together with discontinuous shocks satisfying

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}.$$

This is true when the solutions are compared at fixed (x,t) with $\nu\to 0$. However, the fact that the shock transition becomes very wide as $(\rho_2-\rho_1)/\rho_1\to 0$, for fixed ν, means that in any problem where the shocks ultimately tend to zero strength as $t\to\infty$, there may be some final stage with extremely weak shocks when the discontinuous theory will be invalid. This is often a very uninteresting stage, since the shocks must be very weak.

Otherwise, we can say that the two alternative ways of improving on the unacceptable multivalued solutions agree. The use of discontinuous shocks is the easier analytically and can be carried further in more complicated problems.

Confirmation in more detail would require some explicit solutions of (2.20) which involve shocks of varying strength. Although solutions are not known for a general $Q(\rho)$, it turns out that (2.20) can be solved explicitly when $Q(\rho)$ is once again a quadratic in ρ. If (2.20) is multiplied by $c'(\rho)$, it may be written

$$\begin{aligned} c_t+cc_x &= \nu c'(\rho)\rho_{xx} \\ &= \nu c_{xx}-\nu c''(\rho)\rho_x^2. \end{aligned} \tag{2.27}$$

If $Q(\rho)$ is quadratic, $c(\rho)$ is linear in ρ, then $c''(\rho)=0$ and we have

$$c_t+cc_x=\nu c_{xx}. \tag{2.28}$$

This is Burgers' equation and it can be solved explicitly. The main results are given in Chapter 4. For the present, we accept the evidence for pursuing discontinuous solutions of (2.2) bearing in mind that for extremely weak shocks it will not be appropriate. For the extremely weak shocks, $Q(\rho)$ can be approximated by a quadratic and Burgers' equation can be used.

The arguments in this section depend strongly on $\nu>0$. As noted previously, this is required for stability of the problem. Interesting cases of instability do occur, however, in traffic flow and flood waves. They are discussed in Chapter 3.

2.5 Weak Shock Waves

In a number of situations the shocks are weak in that $(\rho_2-\rho_1)/\rho_1$ is small, but they are not so extremely weak that they may no longer be treated as discontinuities. It is useful to note some approximations for such cases.

The shock velocity

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}$$

tends to the characteristic velocity

$$c(\rho)=\frac{dQ}{d\rho}$$

in the limit as the shock strength $(\rho_2-\rho_1)/\rho_1\to 0$. For weak shocks the expression for the shock velocity U may be expanded in a Taylor series in $(\rho_2-\rho_1)/\rho_1$ as

$$U=Q'(\rho_1)+\frac{1}{2}(\rho_2-\rho_1)Q''(\rho_1)+O(\rho_2-\rho_1)^2.$$

The propagation velocity $c(\rho_2)=Q'(\rho_2)$ may also be expanded as

$$c(\rho_2)=c(\rho_1)+(\rho_2-\rho_1)Q''(\rho_1)+O(\rho_2-\rho_1)^2.$$

Therefore

$$U=\frac{1}{2}(c_1+c_2)+O(\rho_2-\rho_1)^2, \tag{2.29}$$

where $c_1=c(\rho_1)$ and $c_2=c(\rho_2)$. To this approximation, the shock velocity is the mean of the characteristic velocities on the two sides of it. In the (x,t) plane the shock curve bisects the angle between the characteristics which meet on the shock. This property is useful for sketching in the shocks, but it also simplifies the analytic determination of shock positions. Clearly the relation is exact when $Q(\rho)$ is a quadratic.

2.6 Breaking Condition

A continuous wave breaks and requires a shock if and only if the propagation velocity c decreases as x increases. Therefore when the shock is included we have

$$c_2>U>c_1, \tag{2.30}$$

where all velocities are measured positive in the direction of x increasing