8
Case study: traffic modelling

8.1 Simple models for traffic flow

Mathematicians and physicists have long been interested in the problem of traffic, and the area is one of active research. A variety of models have been suggested with a view to understanding, for example, how and why traffic jams form, how to maximise carrying capacity of roads, or how best to use signals, speed limits and other controls to reduce journey times (the feedback effect whereby quicker journeys encourage more people to take to the roads is strangely absent from these analyses). Some models are based on discrete simulations of the movement of individual cars; as you may imagine, such models can be very large and complicated, and indeed they fall into the trendy area of 'complex systems'. There is, however, a strand of traffic research that treats the cars as a continuum with a local number density and velocity that are more or less smooth functions of space and time, much as in the treatment of charged particles in the case study in Chapter 6. Models of this kind are unlikely ever to forecast the fine details of gridlock in New York City or indeed Oxford; but they do offer insights into the way in which traffic can behave, and to some extent they can be calibrated to, or at least compared with, observations. On the scale from parsimony (as few parameters and mechanisms as possible) to complexity, these models are very much at the parsimonious end; the cost, a lack of realism, is balanced by a gain in understanding. They fit in well with my recommended philosophy of always trying to do the easiest problem first.

Let us, then, start with a toy model for cars travelling in one direction down a single-lane road (no overtaking) that is long and straight.

Suppose that x measures distance along the road and that we work on a large enough lengthscale, or we look from far enough away, that the cars can be treated as a continuum with number density $\rho(x, t)$ (cars per kilometre) and speed $u(x, t)$. Supposing further that no cars join or leave the road, we immediately write down 'conservation of cars' in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

as the flux of cars is equal to ρu.

Given the continuum assumption, this equation is uncontroversial; but it is only one equation for two unknowns. We need some kind of 'constitutive relation' to close the system.

Blinkered drivers

As the basis for a very simple model we might say that, as they enter the road, drivers choose the constant speed they want to drive at, and then they drive at that speed no matter what happens. Of course, this is ludicrously unrealistic, but let's see what features it predicts. If the speed u of an individual car is constant, then the derivative of u following that car is zero:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

This kinematic wave equation is easy to solve by characteristics with initial data $u(x, 0) = u_0(x)$, say, corresponding to a snapshot at $t = 0$ of the speeds all along the road. The characteristic equations are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = 0,$$

so that u remains constant along a characteristic whose projection has slope $\mathrm{d}x/\mathrm{d}t = u$. This simply says that the cars move along characteristics with constant speed u. So, to construct the solution, we simply draw all the characteristic projections through the initial line $t = 0$ and read off the value of u at any point x and later time t. This procedure works fine if $u_0(x)$ is increasing, since then the characteristics spread out as in Figure 7.3. But if $u_0(x)$ is decreasing, we inevitably have a collision of characteristic projections – and cars – after a finite time, as in Figure 7.4. This is an example of the solution blow-up we discussed in Chapter 7, and here it has an obvious physical interpretation that fast cars have caught up with slow ones and are trying to occupy the same bit of road. That is, the model predicts that cars with different speeds will end up in the same place. Clearly, this model is inadequate as a description of how real traffic behaves. Its predictions are realistic within its severe limitations, but they are so far off the mark that we need to do something more sophisticated.

Local speed–density laws

In our quest for greater realism, we should try to describe how drivers respond to the traffic around them. A simple way to do this is to propose a (constitutive) relation between the speed of cars at a point x and their density there. That is, we assume that

$$u = U(\rho)$$

for a suitable function U. This function should be determined experimentally from observations of local speed and density, or at least written down in a parametric form and the parameters calibrated (fitted) to observations of global features of the traffic flow (an example of an inverse problem). Before going too far down this road, let us see what happens when we put a simple U into the model. As heavy traffic generally moves more slowly than light traffic, we want $U(\rho)$ to be a decreasing function of ρ. We may assume a maximum car speed $u_{\max}$ and that cars are driven at this speed on an empty road, when $\rho = 0$. Conversely, we can assume a maximum bumper-to-bumper density $\rho_{\max}$ at which the traffic comes to a complete halt, so that $u = 0$. This suggests that the speed–density law

Observation suggests that $u_{\max}$ is greater than the speed limit . . .

It is an implicit assumption of the model that all drivers behave in the same way, and it is also assumed that they drive as fast as is consistent with the ambient traffic density.

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

should be a reasonable qualitative description.

We can make an immediate and interesting observation. The flux of cars is

$$Q = u\rho = u_{\max}\rho_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)\frac{\rho}{\rho_{\max}},$$

and it is greatest when $\rho = \frac{1}{2}\rho_{\max}$, so that $u = \frac{1}{2}u_{\max}$. In this model the assumed free-market individual desire of drivers to minimise their journey time by always driving as fast as possible does not necessarily deliver the maximum-flux solution for drivers as a whole.

Leaving this aside, let us see whether we still have blow-up. Making the trivial scalings $u = u_{\max}u'$, $\rho = \rho_{\max}\rho'$, with suitable scalings for x and t, and dropping the primes, we have the dimensionless equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho(1-\rho)) = \frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0 \qquad (8.1)$$

(this is, of course, just a conservation law). The characteristic equations are

$$\frac{dt}{d\tau} = 1, \qquad \frac{dx}{d\tau} = 1 - 2\rho, \qquad \frac{d\rho}{d\tau} = 0,$$

so the characteristics are again straight, as ρ is constant on them. However, bearing in mind that $0 < \rho < 1$, we see that we can easily prescribe initial data for ρ that will again lead to finite-time blow-up: the characteristic projections can have slopes of either sign and they can easily cross. Indeed, the substitution $v = 1 - 2\rho$ reduces (8.1) to $\partial v/\partial t + v\partial v/\partial x = 0$, so blow-up is inevitable.

Note that the characteristic speed, $dx/dt = 1 - 2\rho$, is *not* equal to the car speed – that is $u = 1 - \rho$. Information always propagates more slowly than the cars and can indeed move backwards, if $\rho > \frac{1}{2}$.

Clearly, we must either tinker further with the model so that blow-up is forbidden or face up to the fact that it *will* happen in realistic models, and decide what to do about it.

8.2 Traffic jams and other discontinuous solutions

Red lights and shocks

We saw in Section 7.3 that the notion of a solution to the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

can be extended to allow jump discontinuities across curves $x = S(t)$, provided that $S(t)$ satisfies the Rankine–Hugoniot relation

$$\frac{dS}{dt} = \frac{[Q]_-^+}{[\rho]_-^+}.$$

Shocks can originate spontaneously, when characteristic projections cross, but a situation in which it is easy to see them is if a stream of traffic with speed u_0 and density ρ_0 is brought to a halt by a traffic light at, say, $x = 0$ and at time $t = 0$.

First, let us look at the cars that did not get through the light, the ones that are in $x < 0$ at $t = 0$; their density is $\rho(x, t)$, which satisfies (8.1). At the moment the light goes red, they are all travelling towards the light with speed u_0 and density ρ_0. These cars therefore see the initial condition $\rho(x, 0) = \rho_0$, $x < 0$. Because $u = 0$ at $x = 0$, the density there takes its maximum value, so $\rho(0, t) = 1$. There are two families of characteristics to consider. Those starting from the initial data on $t = 0$ have characteristic speed $1 - 2\rho_0$ (which may be negative), and they carry the value $\rho = \rho_0$. Those starting from the light at $x = 0$ have speed $1 - 2 = -1$ and carry the value $\rho = 1$. The two families therefore cross immediately and, as shown in Figure 8.1(a), a shock must originate at

Remember that dx/dt is the reciprocal of the gradient in the xt-plane.

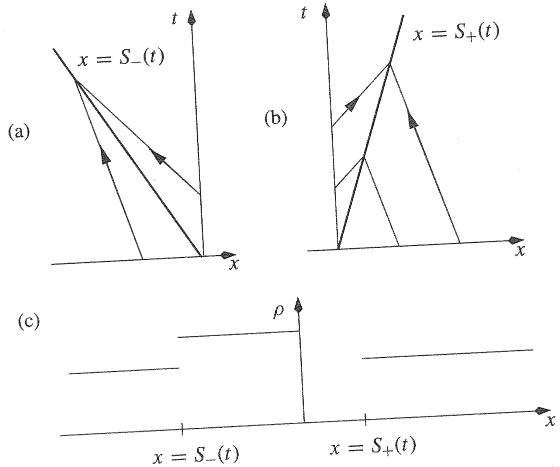

Figure 8.1 (a) Shock formation in traffic arriving at a red light. (b) A shock in traffic ahead of a red light. (c) Traffic density profile after the light turns red. In all three cases $\rho_0 = \frac{2}{3}$.

$x = 0$, $t = 0$. Its speed is given by

$$\begin{aligned}\frac{\mathrm{d}S_-}{\mathrm{d}t} &= \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+} \\ &= \frac{0 - \rho_0(1-\rho_0)}{1-\rho_0} \\ &= -\rho_0.\end{aligned}$$

So, for $x < 0$ there is a shock on $x = -\rho_0 t$ that propagates backwards into the oncoming traffic, bringing it to a halt.

Now consider the traffic that gets through the light before it turns red. We expect these cars to continue on their way at speed u_0, leaving a stretch of empty road behind them. The density satisfies (8.1) for $x > 0$, $t > 0$, with

$$\rho(x, 0) = \rho_0, \qquad \rho(0, t) = 0.$$

As shown in Figure 8.1(b), the characteristics starting from the lights at $x = 0$ all have $\mathrm{d}x/\mathrm{d}t = 1$ and they carry the value $\rho = 0$, while those starting from the initial data on $t = 0$ all have $\mathrm{d}x/\mathrm{d}t = 1 - 2\rho_0 < 1$ and $\rho = \rho_0$. Again, a shock must originate at $x = 0$, $t = 0$. Its speed is given by

$$\begin{aligned}\frac{\mathrm{d}S_+}{\mathrm{d}t} &= \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+} \\ &= 1 - \rho_0.\end{aligned}$$

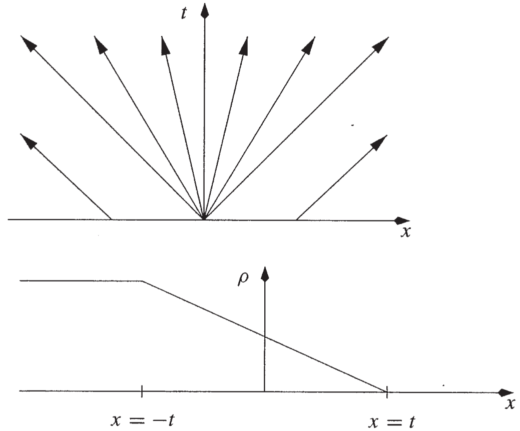

Figure 8.2 Expansion fan at a green light: characteristic projections and density profile.

This is just the speed of the last car to get through the lights, and our intuition is confirmed. (Note that both our shocks are causal, as defined in Exercise 7 at the end of Chapter 7.)

Green lights and expansion fans

What happens when the light turns green and a queue of stationary traffic ($\rho = 1$) moves off? In this case the initial data $\rho(x, 0)$ is discontinuous, being equal to 1 for $x < 0$ and 0 for $x > 0$. If we were to smooth out this discontinuity, say with a tanh function, we would see characteristics with all speeds between -1 (corresponding to $\rho = 1$) and $+1$ (corresponding to $\rho = 0$ as in Figure 7.3 on p. 88. This motivates the idea of an *expansion fan*, a collection of characteristic projections all emanating from a single point, as shown in Figure 8.2. It allows the solution to make a continuous transition from $\rho = 1$ to $\rho = 0$.

For our problem, the characteristic projections are all straight and ρ is constant along them. This means that ρ is a function of x/t alone (a similarity solution). On any characteristic, ρ is a constant equal to α, say, where $0 < \alpha < 1$, and the equation of the characteristic projection is $x = (1 - 2\alpha)t$. We can write this as $\rho = \frac{1}{2}(1 - \beta)$ on $x = \beta t$, or explicitly as

In general, the characteristic projections are only locally straight, and the fan curves over.

$$\rho(x, t) = \begin{cases} 1, & x < -t, \\ \frac{1}{2}(1 - x/t), & -t \le x \le t, \\ 0, & x > t. \end{cases}$$

Notice that discontinuities in the derivative of ρ propagate along the characteristics $x = -t$, $\rho = 1$ and $x = t$, $\rho = 0$.

It is, in principle, possible to construct the solution for most initial-value problems using a mixture of shocks and expansion fans, but it can become complicated, as we will see in Exercise 3.

8.3 More sophisticated models

The models above have their appeal, but they are rather limited in ambition and realism; we may indeed see fairly abrupt changes in traffic density, but never true jumps. Moreover, the drivers in these models are very myopic: they do not look ahead at all to anticipate future traffic developments.

There are several things we could do about this. One way to go is to introduce a non-local speed–density law, so that $u(x,t)$ depends on $\rho(x+h,t)$ as well as $\rho(x,t)$, where the driver looks a distance h ahead.[1] A limiting case (as $h \to 0$) of this law is the model

$$u = 1 - \rho - \epsilon \frac{\partial \rho}{\partial x},$$

where ϵ is a small positive constant. This says that drivers take into account whether traffic density is increasing or decreasing and slow down if it is increasing. Putting this into $\partial\rho/\partial t + \partial Q/\partial x = 0$ leads to the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho(1-\rho)) = \epsilon \frac{\partial}{\partial x}\left(\rho \frac{\partial \rho}{\partial x}\right),$$

a nonlinear diffusion equation with some interesting properties. Because it is parabolic when $\epsilon > 0$, albeit nonlinear (and degenerate because the 'diffusion coefficient' $\epsilon\rho$ vanishes when $\rho = 0$), its solutions may be smoother than is the case when $\epsilon = 0$. In Exercise 5 you are asked to show that travelling-wave solutions of this equation are consistent with the Rankine–Hugoniot conditions that apply when $\epsilon = 0$.

Further models involve an evolution equation for u rather than just a constitutive equation. For example, we might replace $u = U(\rho)$ by an equation such as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{u(\rho,t) - U(\rho)}{\tau},$$

which says that the rate of change of u following a car is proportional to the difference between u and the equilibrium speed U; here τ represents the time over which a driver changes speed to reach equilibrium. To this we might also add an anticipatory term $-\epsilon\, \partial\rho/\partial x$ as above, modelling drivers' tendency to speed up if they see light traffic ahead, and slow down

[1] One could also introduce a 'reaction time' delay in the t-variable.

if the traffic is getting worse. All these, and many more, possibilities have been discussed in the traffic literature (see [25] for more details). Roughly speaking, many of the models do a good job in describing generic features such as jams and abrupt changes in traffic density, but they are less successful in forecasting the evolution of traffic from a given starting density (which is, of course, the big question). Only recently has reliable empirical data, gathered by induction loops buried in roads, become available, and I have no doubt that there are many interesting developments to come.

8.4 Sources and further reading

The kinematic wave model for traffic flow is usually attributed to Whitham and Lighthill. An excellent survey of a huge variety of approaches to traffic modelling can be found in [25]; see also `www.trafficforum.org`.

8.5 Exercises

1 Blinkered cars. Consider the kinematic wave equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

where $u(x,0) = u_0(x)$ is a smooth decreasing function of x. Find the solution in parametric form. Look at the relevant Jacobian to show that the earliest time at which the characteristics cross is

$$t_{\min} = -\frac{1}{\min_{-\infty<x<\infty} u_0'(x)}.$$

Show that the rate at which neighbouring cars get closer to each other is $\partial u/\partial x$ and interpret the blow-up result mentioned on p. 108 in this light.

2 Traffic jams. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} = 0,$$

where $u = 1 - \rho$ for $0 \le \rho \le 1$.

(a) A tractor is travelling along the road at a quarter of the maximum speed and a very long queue of cars travelling at the same speed has built up behind it. At time $t = 0$ the tractor passes the origin $x = 0$ and immediately turns off the road. Sketch the characteristic diagram; show that there is an expansion fan for ρ centred at $x = 0$, $t = 0$ and find $\rho(x,t)$ for $t > 0$.

(b) A queue is building up at a traffic light at $x = 1$ so that, when the light turns to green at $t = 0$,

$$\rho(x, 0) = \begin{cases} 0 & \text{for } x < 0 \text{ and } x > 1, \\ x & \text{for } 0 < x < 1. \end{cases}$$

Show that the characteristics, labelled by s and starting from $(s, 0)$, are given by $t = \tau$ and

$$\begin{aligned} x - s &= \tau && \text{in } x < \tau \text{ and } x > \tau + 1, \text{ on which } \rho = 0, \\ x - s &= (1 - 2s)\tau && \text{in } \tau < x < 1 - \tau, \text{ on which } \rho = s, \\ x - 1 &= (1 - 2\rho_0)\tau && \text{in } 1 - \tau < x < 1 + \tau \end{aligned}$$

(these last characteristics, on which $\rho = \rho_0 = (\tau - x + 1)/(2\tau)$, form an expansion fan starting from the light). Draw the characteristic projections in the xt-plane; show that all those starting with $0 < s < 1$ pass through one point and deduce that a collision first occurs at $x = \frac{1}{2}$ and $t = \frac{1}{2}$.

Harder: Show that thereafter there is a shock $x = S(t)$, starting from $(\frac{1}{2}, \frac{1}{2})$, where

$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

Write $S(t) = 1 + \tilde{S}(t)$ to reduce this equation to one that is homogeneous in $\tilde{S}$ and t, and hence solve it.

3 Red light, green light. Continue the solution discussed in Section 8.2 as follows. Suppose that the light turns green after time T. Move the time origin to this moment and neglect the traffic that has already passed the light and is in $x > 0$. Find the solution of (8.1) with the initial data (at the new time origin)

$$\rho(x, 0) = \begin{cases} \rho_0, & x < -\rho_0 T, \\ 1, & -\rho_0 T < x < 0, \\ 0, & x > 0. \end{cases}$$

Show that the shock that is initially at $x = -\rho_0 T$ continues to propagate at speed ρ_0 until it is caught up by the characteristic projection $x = -t$ at time $t = \rho_0 T/(1 - \rho_0)$. Show that thereafter there is a shock at $x = S(t)$ where

Switching to the variable $v = \rho - \frac{1}{2}$ helps you to spot the simplification.

$$\begin{aligned} \frac{dS}{dt} &= \frac{\frac{1}{4}(1 - S^2/t^2) + \rho_0(1 - \rho_0)}{\frac{1}{2}(1 - S/t) - \rho_0} \\ &= \frac{(\rho_0 - \frac{1}{2})t + S}{2t}. \end{aligned}$$

Solve this equation by the substitution $S(t) = ts(t)$.

Harder: Show that the shock initially propagates to the right if $\rho_0 < \frac{1}{2}$ and to the left if $\rho_0 > \frac{1}{2}$. Calculate ρ for $x = S(t)+$ and show that the jump in ρ across the shock is equal to $(\rho_0(1 - \rho_0))^{1/2}\,(T/t)^{1/2}$ and hence decreases as t increases.

4 Two-lane traffic. Explain why the one-lane model above might be extended to a two-lane model in the form

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 u_1) &= F_{12}(\rho_1, \rho_2, u_1, u_2), \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}(\rho_2 u_2) &= -F_{12}(\rho_1, \rho_2, u_1, u_2), \end{aligned}$$

and explain where F_{12} comes from. What general properties should F_{12} have, in your opinion? How would it differ for an American freeway, in which overtaking is allowed on the inside lane, and for British case in which (in principle if not in practice) it is not?

5 Smoothed traffic equation. Consider the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho(1 - \rho)) = \epsilon \frac{\partial}{\partial x}\left(\rho \frac{\partial \rho}{\partial x}\right),$$

a model for anticipatory drivers. Suppose we look for a solution $\rho = f(x - Vt)$, $-\infty < x < \infty$, with $\rho \to \rho_\pm$ as $x \to \pm\infty$. Show that

$$V = \frac{[\rho(1 - \rho)]_{-\infty}^{\infty}}{[\rho]_{-\infty}^{\infty}}.$$

Compare this with the Rankine–Hugoniot condition. What do you think happens as $\epsilon \to 0$? We will return to this issue in Chapter 16.

Carry out the same procedure for the smoothed kinematic wave equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2},$$

known as Burgers' equation. (Amazingly, it can be reduced to the heat equation by the Cole–Hopf transformation $u = -2\partial \log v/\partial x$, taking $\epsilon = 1$ without loss of generality: try it!)

'The mass of this thing is about 1 kilometre.'