Inverse Differential Kinematics

- Joint space variables <--(highly nonlinear)--> operational space variables
- J is time variant

J invertible -- Numerical Solution

when $\mathbf{J}$ is square matrix and of full rank --> invertible. use Newton-Euler Integration Method for IK.

$$\mathbf{q}(t) = \int_0^t \dot{\mathbf{q}}(t)dt + \mathbf{q}(0)$$

Geometric Jacobian $\mathbf{J}$	Analytical Jacobian $\mathbf{J}_A$

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{J}^{-1}(\mathbf{q})\mathbf{v}_e \\ \Rightarrow \mathbf{q}(t_{k+1}) &= \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t \\ &= \mathbf{q}(t_k) + \mathbf{J}^{-1}(\mathbf{q}(t_k))\mathbf{v}_e(t_k)\Delta t \end{aligned}$$

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$$\begin{aligned} \dot{\mathbf{q}} &= {\mathbf{J}_A}^{-1}(\mathbf{q})\dot{\mathbf{x}}_e \\ \Rightarrow \mathbf{q}(t_{k+1}) &= \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t \\ &= \mathbf{q}(t_k) + {\mathbf{J}_A}^{-1}(\mathbf{q}(t_k))\dot{\mathbf{x}}_e(t_k)\Delta t \end{aligned}$$

当然可以用其他数值积分方法, 以达到更高的精度.

1.st Order Numerical Solution

$\mathbf{J}$ square, and of full rank, choose $\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{v}_e + \mathbf{K}e)$

Redundant Manipulators

Problem: $r < n, \mathbf{J}$ **not square** 不可逆. given $\mathbf{v}_e, \mathbf{J}$, find $\dot{\mathbf{q}}$ and minimize the quadratic cost function of joint velocities $g(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}}, \mathbf{W}$ is a suitable $(n \times n)$ symmetric positive definite weighting matrix.

Solving: Using Lagrange Methods:

- modify cost function:
$$g(\dot{\mathbf{q}}, \lambda) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}} + \lambda^T(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})$$
where λ is an $r \times 1$ vector of unknown lagrange multipliers that allows the incorporation of the constraints to minimize.
- neccessary conditions:
$$\left(\frac{\partial g}{\partial \dot{\mathbf{q}}}\right)^T = \mathbf{0}, \left(\frac{\partial g}{\partial \lambda}\right)^T = \mathbf{0}$$
- the $???$ leads to
$$\mathbf{W}\dot{\mathbf{q}} - \mathbf{J}^T\lambda = 0 \rightarrow \dot{\mathbf{q}} = \mathbf{W}^{-1}\mathbf{J}^T\lambda$$
.
- Notice that solution is minimum, since $\frac{\partial^2 g}{\partial^2 \dot{\mathbf{q}}} = \mathbf{W}$ is positive definite.
- **under the assumption that $\mathbf{J}$ has full rank**:
$$\mathbf{v}_e = \mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T\lambda$$
where $\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T$ is an $r \times r$ square matrix of rank r and can be inverted.
$$\lambda = (\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}\mathbf{v}_e$$
$$\dot{\mathbf{q}} = \mathbf{W}^{-1}\mathbf{J}^T(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}\mathbf{v}_e, \rightarrow \mathbf{J}\cdot$$
$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}\mathbf{v}_e = \mathbf{v}_e$$

以上计算显示, $\mathbf{W}^{-1}\mathbf{J}^T(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}$ 可以充当 $\mathbf{J}$ 的逆.

Solution: Right Pseudo-Inverse of Jacobian
$$\mathbf{J}^\dagger = \mathbf{W}^{-1}\mathbf{J}^T(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}$$
$$\mathbf{J}^\dagger = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}, \text{ when } \mathbf{W} = \mathbf{I}$$
$$\dot{\mathbf{q}} = \mathbf{J}^\dagger\mathbf{v}_e$$

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same for Analytical Jacobian $\mathbf{J}_A$ with $\dot{\mathbf{x}}_e$

Null space IK

Null Space solutions of IK:

if $\dot{\mathbf{q}}^*$ is a solution, the $\dot{\mathbf{q}}^* + \mathbf{P}\dot{\mathbf{q}}_0$ is also a solution, where $\dot{\mathbf{q}}_0$ is a vector of arbitrary joint velocities and P is a projector in the null space of $\mathbf{J}$.

- Reconstruct the cost function: and minimize the norm of $\dot{\mathbf{q}} - \dot{\mathbf{q}}_0$, 也就是, 解$\dot{\mathbf{q}}$ 要满足??? 的同时, 尽可能的靠近$\mathbf{q}_0$

$$g'(\dot{\mathbf{q}}) = \frac{1}{2}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_0)^T(\dot{\mathbf{q}} - \dot{\mathbf{q}}_0) + \lambda^T(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})$$

??? leads to:

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}_0 + \mathbf{J}^T\lambda$$
$$\lambda = (\mathbf{J}\mathbf{J}^T)^{-1}(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})$$
$$\dot{\mathbf{q}} = \dot{\mathbf{q}}_0 + \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}}) = \mathbf{J}^\dagger\mathbf{v}_e + (\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})\dot{\mathbf{q}}_0$$

first term: minimum norm joint velocities second term: Homogeneous solution, the null space projector is $\mathbf{P} = \mathbf{I} - \mathbf{J}^\dagger\mathbf{J}$

in the case $\mathbf{v}_e = \mathbf{0}$, it is possible to generate **internal motion** described by $(\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})\dot{\mathbf{q}}_0$ that reconfigure the manipulator structure without changing the EEF pose.

Null Space control

IK with Kinematic Singularities

- $\mathbf{J}$ **does not have full rank**. $\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}}$ has linearly dependent equations , $\mathbf{J}$ has linearly dependent rows or columns.
- if $\mathbf{J}$ is square (not redundant): at singularity or in its neighborhood,

$$\mathbf{J} \to \text{singular}, \det(\mathbf{J}) \to 0, \mathbf{J}^{-1} \to \infty, \Rightarrow \dot{\mathbf{q}} = \mathbf{J}^{-1}\mathbf{v}_e$$

即, small $\mathbf{v}_e$ leads to large joint velocities.

Solution

1. SVD: singular value decomposition of Matrix $\mathbf{J} \Rightarrow \mathbf{J}^\dagger$, (在矩阵不满秩的情况下, 计算pseudo inverse)
2. DLS: damped least-squares inverse

$$\mathbf{J}^* = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T + k^2\mathbf{I})^{-1}$$

where k is a damping factor that renders the inversion better conditioned from a numerical point of view. Reconstruct the cost function

$$g''(\dot{\mathbf{q}}) = \frac{1}{2}(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})^T(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}}) + \frac{1}{2}k^2\dot{\mathbf{q}}^T\dot{\mathbf{q}}$$

where the first term allows a finite inversion error to be tolerated, with the advantage of norm-bounded velocities. The factor k establishes the relative weight between the two objectives.

Multi-task

With Equal Priority

Stack of Jacobian method:

$$\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_k \end{bmatrix}^{\dagger} \cdot \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{bmatrix} = \bar{\mathbf{J}} \cdot \bar{\mathbf{w}}$$

In case the row-rank of the stacked Jacobian matrix $\bar{\mathbf{J}}$ is larger than the column rank,the tasks are only fulfilled in a least square optimal sense $\|\bar{\mathbf{w}} - \bar{\mathbf{J}}\dot{\mathbf{q}}\|_2$.

we can use weighted pseudo-inv to weight each tasks. then we have $\|\mathbf{W}^{\frac{1}{2}}(\bar{\mathbf{w}} - \bar{\mathbf{J}}\dot{\mathbf{q}})\|_2$. Since $\mathbf{W}$ is diagonal, the Cholesky factor $\frac{1}{2}$ corresponds to element-wise square root.

With Prioritization

we use consecutive null-space projection. For task with decreasing priority $T_1 > T_2 > \cdots > T_k$.

- task 1
$$\dot{\mathbf{q}} = \mathbf{J}_1^{\dagger}\mathbf{w}_1 + \mathbf{P}_1\dot{\mathbf{q}}_0$$
- task 2 must be achieved in the nullspace of task 1 and don't violate the above
$$\begin{aligned} \mathbf{w}_2 &= \mathbf{J}_2\dot{\mathbf{q}} = \mathbf{J}_2(\mathbf{J}_1^{\dagger}\mathbf{w}_1 + \mathbf{P}_1\dot{\mathbf{q}}_0) \\ \Rightarrow \dot{\mathbf{q}}_0 &= (\mathbf{J}_2\mathbf{P}_1)^{\dagger}(\mathbf{w}_2 - \mathbf{J}_2\mathbf{J}_1^{\dagger}\mathbf{w}_1) \\ \Rightarrow \dot{\mathbf{q}} &= \mathbf{J}_1^{\dagger}\mathbf{w}_1 + \mathbf{P}_1(\mathbf{J}_2\mathbf{P}_1)^{\dagger}(\mathbf{w}_2 - \mathbf{J}_2\mathbf{J}_1^{\dagger}\mathbf{w}_1) \end{aligned}$$
- task n can be written in a recursive way
$$\dot{\mathbf{q}} = \sum_{i=1}^{k} \bar{P}_i\dot{\mathbf{q}}_i$$
$$\dot{\mathbf{q}}_i = (\mathbf{J}_i\bar{\mathbf{P}}_i)^{\dagger}\left(\mathbf{w}_i - \mathbf{J}_i\sum_{j=1}^{i-1}\bar{\mathbf{P}}_j\dot{\mathbf{q}}_j\right)$$
$$\bar{\mathbf{P}}_i = Nullspace(\bar{\mathbf{J}}_i) = Nullspace\left(\begin{bmatrix} \mathbf{J}_1^T & \dots & \mathbf{J}_{i-1}^T \end{bmatrix}^T\right)$$

6-DOF Manipulator with spherical wrist

- a 6-DOF kinematic structure has closed form IK solutions, IF:
  - three consecutive revolute joint axes intersect at a common point, like the spherical wrist.
  - three consecutive revolute joint axes are parallel
- Articulating the IK problem into 2 subproblems, since solution for **position** is ***decoupled*** from that for **orientation**:
  - compute the wrist position $\mathbf{p}_{wrist} = \mathbf{p}_e - d_6\mathbf{a}_e$
  - solve inverse kinematics for q_1, q_2, q_3
  - compute $\mathbf{R}_{o,3}(q_1, q_2, q_3)$
  - compute $\mathbf{R}_{3,6}(q_4, q_5, q_6) = \backslash\mathrm{bRT}_{o,3}\mathbf{R}_e$
  - solve inverse kinematics for q_4, q_5, q_6

Numerical IK

> - $\mathbf{q} \leftarrow \mathbf{q}_0$
> - while $\|\mathbf{x}_d - \mathbf{x}(\mathbf{q})\| > \epsilon\|$ do:
>   - $\mathbf{J} \leftarrow \mathbf{J}(\mathbf{q}) = \frac{\partial\mathbf{x}}{\partial\mathbf{q}}(\mathbf{q})$
>   - $\Delta\mathbf{x} \leftarrow \mathbf{x}_d - \mathbf{x}(\mathbf{q})$
>   - $\mathbf{q} \leftarrow \mathbf{q} + \alpha\mathbf{J}^{\dagger}\Delta\mathbf{x}$
> - end while

Note that: $0 < \alpha < 1$ is the step size to cope with the divergence problem when the error $\Delta\mathbf{x}$ is very big. However, small α could lead to slow convergence.

Another problem is when the target is close to the singularity, the Jacobian inversion is ill-conditioned. One could use damped solution. Another approach is to use Jacobian transpose instead of inverse $\mathbf{q} \leftarrow \mathbf{q} + \alpha\mathbf{J}^T\Delta\mathbf{x}$ and choose α small enogh.

Statics

manipulator at **equilibrium configuration**, the relationship between

- the **generalized forces $\boldsymbol{\gamma}_e$ applied to EEF [Forces/Moments]** and
- the **generalized forces $\boldsymbol{\tau}$ applied to the joints, [Torques]**

principle of virtual work:

- elementary work by joint torques $\boldsymbol{\tau}$:
$$dW_{\boldsymbol{\tau}} = \boldsymbol{\tau}^T d\mathbf{q}$$
- EEF forces $\boldsymbol{\gamma}_e$, with force contribution $\mathbf{f}_e$ and moment contribution μ_e:
$$dW_{\gamma_e} = (\mathbf{f}_e^T d\mathbf{p}_e + \mu_e^T \mathbf{w}_e)dt = \mathbf{f}_e^T \mathbf{J}_P(\mathbf{q})d\mathbf{q} + \mu_e^T \mathbf{J}_O(\mathbf{q})d\mathbf{q} = \boldsymbol{\gamma}_e^T \mathbf{J}(\mathbf{q})d\mathbf{q}$$

since virtual and elementar displacement coincide, with δ the symbol for virtual quantities
$$\begin{aligned}\delta W_{\boldsymbol{\tau}} &= \boldsymbol{\tau}^T \delta\mathbf{q} \\ \delta W_{\gamma_e} &= \boldsymbol{\gamma}_e^T \mathbf{J}(\mathbf{q})\delta\mathbf{q} \\ \delta W_{\boldsymbol{\tau}} &= \delta W_{\gamma_e} \Rightarrow \\ \boldsymbol{\tau} &= \mathbf{J}^T(\mathbf{q})\boldsymbol{\gamma}_e\end{aligned}$$

Note that: EEF力的数值是表达在World Frame里的

Kineto-Statics Duality

- the **range space** of $\mathbf{J}^T$ is the subspace $\mathscr{R}(\mathbf{J}^{\mathscr{T}})$ in $\mathbb{R}^n$ of the joint torques that can balance the EEF forces, in the given manipulator posture.
- the **null space** of $\mathbf{J}^T$ is the subspace $\mathcal{N}(\mathbf{J}^T)$ in $\mathbb{R}^r$ of the EEF forces that do not require any balancing joint torques, in the given manipulator posture.
$$\mathcal{N}(\mathbf{J}) \equiv \mathcal{R}^{\perp}(\mathbf{J}^T), \mathcal{R}(\mathbf{J}) \equiv \mathcal{N}^{\perp}(\mathbf{J}^T)$$