Objectives:

1. Determine solution for state transition matrix for LTI system (matrix exponential)
2. Determine 3 methods for computing the matrix exponential

$$\left(\begin{array}{l} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{array}\right)$$

- Last class we learned that the solution to the CLTV system was:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

$$y = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

where $\Phi(t,t_0)$ is the state transition matrix defined by the peano-baker formula

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \cdots$$

→ this is an infinite series of integrals, can't solve for the state transition matrix unless we can see that it converges.

$$= I + A\int_{t_0}^{t} ds_1 + A^2 \int_{t_0}^{t}\int_{t_0}^{s_1} ds_2\, ds_1 + \dots$$

$$= I + A(t-t_0) + A^2 \int_{t_0}^{t} \underbrace{\left[(s_1 - t_0)\right] ds_1}_{} + \dots$$

$$\frac{s_1^2}{2} - t_0 s \Big|_{t_0}^{t} = \frac{t^2}{2} - t_0 t - \frac{t_0^2}{2} + t_0^2$$

$$\cancel{= \tfrac{1}{2}(t^2 - 2t_0 t)}$$

$$= \tfrac{1}{2}(t^2 - 2t_0 t + t_0^2)$$

$$= \frac{1}{2}(t-t_0)^2$$

~~= I + A(t − t₀) + A²~~

$$= I + A(t-t_0) + A^2 \frac{(t-t_0)^2}{2!} + \dots \frac{A^k (t-t_0)^k}{k!} + \dots$$

$$\Phi(t, t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k A^k}{k!}$$

$(t-t_0)^k$ = scalar

Power series of exponential:

$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$\Phi(t, t_0) = e^{A(t-t_0)}$ for LTI systems.

$\Phi(t, 0) = e^{At}$

Plugging into our Solution:

LTI. so constant

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$y(t) = C e^{A(t-t_0)} x_0 + \int C e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

we need an integrating factor w/ the properties —

(a) $$\frac{d}{dt} e^{-At} = e^{-At}(-A)$$

(b) $$e^{At} e^{A\tau} = e^{A(t+\tau)}$$

The scalar exponential has these properties —

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots$$

$$\frac{d}{dt} e^{at} = \frac{d}{dt}\left(1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots\right)$$

$$= 0 + a + a^2t + \frac{1}{2!}a^3t^2 + \cdots$$

$$= a\left(1 + at + \frac{1}{2!}a^2t^2 + \cdots\right) = \left(1 + at + \frac{1}{2!}a^2t^2 + \cdots\right)a$$

$$= a e^{at} \qquad = e^{at} a$$

and by direct multiplication —

$$e^{at} e^{a\tau} = \left(1 + at + \frac{1}{2!}a^2t^2 + \cdots\right)\left(1 + a\tau + \frac{1}{2!}a^2\tau^2 + \cdots\right)$$

$$= 1 + at + a\tau + \frac{1}{2!}a^2t^2 + a^2t\tau + \frac{1}{2!}a^2\tau^2 + \cdots$$

$$= \left(1 + a(t+\tau) + \frac{1}{2!}a^2(t+\tau)^2 + \cdots\right)$$

$$= e^{a(t+\tau)}$$

Define an equivalent matrix exponential —

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

these are now matrix multiplications

$A^2 = AA$
$A^3 = AAA$
etc.

Note that —

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)$$

$$= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \cdots$$

$$= A\left(I + At + \frac{1}{2!}A^2t^2\right) = \left(I + At + \frac{1}{2!}A^2t^2\right)A$$

$$= Ae^{At} = e^{At}A \quad \Rightarrow \text{A commutes w/ its exponential}$$

Also — same as w/ scalar case

$$e^{At}e^{A\tau} = e^{A(t+\tau)}$$

Use the matrix exp. as an integrating factor

$$e^{-At}(\dot{x} - Ax) = e^{-At}Bu$$

$$\frac{d}{dt}\left(e^{-At}x\right) = e^{-At}Bu$$

$$\Rightarrow \int_{t_0}^{t} d\left(e^{-At}x\right) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{-At_0}x(0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$\therefore \quad x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{+A(t-\tau)}Bu(\tau)d\tau$$

$$y = Cx + Du$$

Plug in for x

$$\therefore \quad y = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du$$

Note that:

If $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ $\quad e^{At} \neq \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-2t} & e^{-t} \end{bmatrix}$

Proof:

$$= I + \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} t + \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\frac{t^3}{3!} + \dots$$

$$e^{At} = \underbrace{I}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} + \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} t + \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 11 & -2 \\ 2 & 11 \end{bmatrix}\frac{t^3}{3!}$$

$$= \begin{bmatrix} 1 - t - \frac{1}{2!}3t^2 + \frac{1}{3!}11t^3 \dots & + 2t - \frac{1}{2!}4t^2 - \frac{1}{3!}2t^3 + \dots \\ - 2t + \frac{1}{2!}4t^2 + \frac{1}{3!}2t^3 \dots & 1 - t - \frac{1}{2!}3t^2 + \frac{1}{3!}11t^3 + \dots \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} & e^{2t} \\ e^{-2t} & e^{-t} \end{bmatrix} = \begin{bmatrix} 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \dots & 1 - 2t + \frac{1}{2!}4t^2 + \frac{1}{3!}8t^3 + \dots \\ 1 - 2t + \frac{1}{2!}4t^2 - \frac{1}{3!}8t^3 + \dots & 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \dots \end{bmatrix}$$

$\rightarrow$ Clearly not equal

$$e^{At} = e^{-t}\begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

– To gain a bit more insight into this solution – lets look @ the characteristic equation

$$\det(\lambda I - A) = 0$$

$$\det\begin{pmatrix} \lambda+1 & -2 \\ 2 & \lambda+1 \end{pmatrix} = 0 \Rightarrow (\lambda+1)^2 + 4 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm 2j \Rightarrow e^{\lambda t} = e^{(-1+2j)t}$$

$$= e^{-t}e^{2jt}$$

$$= e^{-t}(\cos 2t + j\sin 2t)$$

Common Matrix Exponentials —

If A is a diagonal matrix —

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{a_{11}t} & 0 \\ 0 & e^{a_{22}t} \end{bmatrix}$$

This generalizes to any $n \times n$ matrix

If $A = \text{diag}\{a_{ii}\}_{i=1}^{n} = \begin{bmatrix} a_{11} & & & & \phi \\ & a_{22} & & & \\ & & a_{33} & & \\ & \phi & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$

$$e^{At} = \text{diag}\{e^{a_{ii}t}\}_{i=1}^{n} = \begin{bmatrix} e^{a_{11}t} & & & \phi \\ & e^{a_{22}t} & & \\ & \phi & \ddots & \\ & & & e^{a_{nn}t} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Rightarrow e^{A_1 t} = e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \Rightarrow e^{A_2 t} = e^{at}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Generalized A_2

$$A_3 = \begin{bmatrix} a & 1 & \phi & \cdots & \phi \\ \phi & a & 1 & \cdots & \phi \\ \vdots & & \ddots & \ddots & 1 \\ \phi & & \phi & & a \end{bmatrix} \Rightarrow e^{A_3 t} = e^{at}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(k-1)!}t^{k-1} \\ 0 & 1 & t & \cdots & \frac{1}{(k-2)!}t^{k-2} \\ & & \ddots & \ddots & \vdots \\ & \phi & & \ddots & t \\ & & & & 1 \end{bmatrix}$$

$k \times k$ matrix

w/ a on diagonal &
1's directly to right &
0's elsewhere.

Jordan Form

- lets look @ this same problem from a slightly different perspective —

Recall the derivation of the 1st order scalar differential equation.

$$\dot{y} = ay + bu$$

Put y's & u's on same side

$$\dot{y} - ay = bu$$

Introduce an integrating factor

$$\underbrace{e^{-at}(\dot{y} - ay)} = e^{-at}bu$$

$$\frac{d}{dt}e^{-at}y = e^{-at}bu \qquad \text{because } \frac{d}{dt}e^{-at} = (-a)e^{-at}$$

Integrate both sides from t_o to t

$$\int_{t_o}^{t} \frac{d}{dt} e^{-at} y \, dt = \int_{t_o}^{t} e^{-a\tau} bu \, d\tau$$

$$\Rightarrow \quad e^{-at}y(t) - e^{-at_o}y(t_o) = \int_{t_o}^{t} e^{-a\tau} b\, u(\tau)\, d\tau$$

multiply both sides to get

$$y(t) = e^{a(t-t_o)}y(t_o) + \int_{t_o}^{t} e^{-a\tau} b\,u(\tau)\,d\tau$$

lets try to solve the LTI state space eq. the same way —

$$\dot{x} = Ax + Bu$$

Move variable to same side

$$\dot{x} - Ax = Bu$$

If A is block diagonal —

$$A = \begin{bmatrix} \boxed{A_1} & & & \phi \\ & \boxed{A_2} & & \\ & & \boxed{A_3} & \\ \phi & & & \ddots \end{bmatrix}$$

$e^{A_1 t}$ $e^{A_2 t}$ $e^{A_3 t}$

w/ each A^k as square matrices

$$e^{At} = \text{block diag}\left\{ e^{A_k t} \right\}_{k=1}^{k}$$

Example:

$$A_4 = \left[\begin{array}{cc|cc} -2 & 1 & & \phi \\ 0 & -2 & & \\ \hline & \phi & -1 & 3 \\ & & -3 & -1 \end{array}\right] \Rightarrow e^{A_4 t} = \left[\begin{array}{cc|cc} e^{-2t} & te^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ \hline 0 & 0 & e^{-t}\cos 3t & e^{-t}\sin 3t \\ 0 & 0 & -e^{-t}\sin 3t & e^{-t}\cos 3t \end{array}\right]$$

— ~~So~~ How do we solve generally? (Different procedures give us different ~~insight~~ insights into the matrix)

— Will discuss 3 different methods for solving e^{At}

1. LaPlace Transform
2. Cayley-Hamilton
3. Eigenvector - Eigenvalue