Frequency Response:

SISO

$u(t) = e^{j\omega t}$

If $G(s)$ – stable $\Rightarrow y(t) = G(j\omega)e^{j\omega t} + \varepsilon(t)$

– where $\varepsilon(t) \to 0$ exp fast

– $G(j\omega)e^{j\omega t} = |G(j\omega)|\, e^{j(\omega t + \measuredangle G(j\omega))}$

i.e. if $u(t) = \sin(\omega t) \Rightarrow y_{ss}(t) = |G(j\omega)| \sin(\omega t + \measuredangle G(j\omega))$

MIMO

$u(t)$ = vector – If we assume same freq. for all comp, but different amplitude + phase.

$$u(t) = \underline{u}_o e^{j\omega t} \quad w/ \quad u_o = \begin{bmatrix} A_1 e^{j\varphi_1} \\ A_2 e^{j\varphi_2} \\ \vdots \\ A_k e^{j\varphi_k} \end{bmatrix}$$

(All k inputs are sinusoidal, but w/ different amplitudes A_i + phases φ_i)

$$y_{ss} = G(j\omega)\, u_o e^{j\omega t} = \underbrace{\begin{pmatrix} A_1 |G(j\omega)|\, e^{i(\omega t + \varphi_1 + \measuredangle G(j\omega))} \\ \vdots \\ A_k |G(j\omega)|\, e^{i(\omega t + \varphi_k + \measuredangle G(j\omega))} \end{pmatrix}}_{y_o}$$

$$y_o = \underset{(m \times k)}{\begin{bmatrix} G(j\omega) \end{bmatrix}} \underset{(k \times 1)}{\begin{bmatrix} u_o \end{bmatrix}} = \underset{(m \times 1)}{\begin{bmatrix} \quad \end{bmatrix}}$$

$$y_{o,i} = \sum_{\ell=1}^{k} g_{i\ell}(j\omega)\, u_{o,\ell} = \sum_{\ell=1}^{k} |g_{i\ell}(j\omega)|\, A_\ell\, e^{j(\varphi_\ell + \measuredangle g_{i\ell}(j\omega))}$$

$\rightarrow$ Difficult to look @ individual Bode diagrams for MIMO systems.

– Instead, quantify $\|y_o\|$ as a function of $\|u_o\|$

For complex system: $\|x\|_2^2 = x^* x$ $\quad * =$ complex conj. transpose.

$$\|y_o\|_2^2 = y_o^* y_o = [G(j\omega)u_o]^* G(j\omega)u_o = u_o^* \underbrace{G(j\omega)^* G(j\omega)}_{H(j\omega)} u_o$$

$\|y_o\|_2^2 = u_o^* H(j\omega) u_o$ → freq. dependent quad form w/ symmetric matrix $H(j\omega)$.

→ E-values of $H(j\omega)$ are real + non-negative.

→ use quadratic ~~identity~~ property:

$$0 \le \lambda_{min}(H(j\omega))\|u_o\|_2^2 \le \|y_o\|_2^2 \le \lambda_{max}(H(j\omega))\|u_o\|_2^2$$

or w/ $\|u_o\|_2 \neq 0$

$$(\lambda_{min}(H(j\omega)))^{1/2} \le \frac{\|y_o\|_2}{\|u_o\|_2} \le \lambda_{max}(H(j\omega))^{1/2}$$

$$\downarrow$$

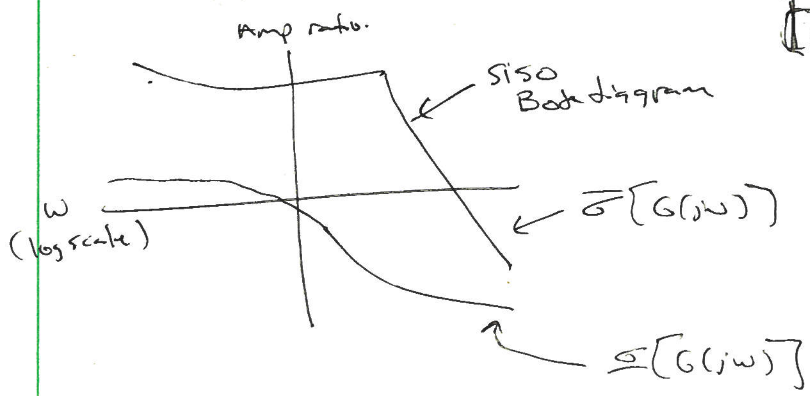
$$\lambda_{max}(G^*(j\omega)G(j\omega))^{1/2} = \text{2-norm of TFM } G(j\omega)$$

Let $\bar{\sigma}[G(j\omega)] = \max_i \sigma_i(G(j\omega))$

$\underline{\sigma}[G(j\omega)] = \min_i \sigma_i(G(j\omega))$

$\frac{\|y_o\|_2}{\|u_o\|_2}$ → amplitude ratio for MIMO system

w/ $\bar{\sigma}(G(j\omega)]$ + $\underline{\sigma}[G(j\omega)]$ → playing a role similar to $|G(j\omega)|$ for SISO.



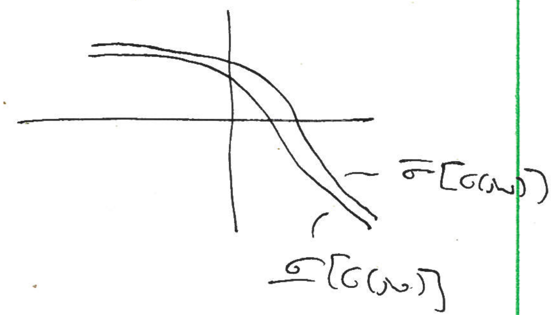
→ At every freq the amplitude ratio lies between the upper + lower bound.

→ Direction of the input determines if closer to lower or upper bound.

→ TF is "strongly directional" when there is a large difference between $\bar{\sigma}[G(j\omega)]$ + $\underline{\sigma}[G(j\omega)]$ over a range of freq.

i.e → Not Strongly Directional

(These are more like SISO Systems)



→ Direction is specific combination of amplitudes A_i + phase ψ_i

w/ $u_0 = \begin{bmatrix} A_1 e^{j\psi_1} \\ A_2 e^{j\psi_2} \\ \vdots \\ A_k e^{j k_k} \end{bmatrix}$

→ • USE singular value decomposition:

$G = L \Sigma R^*$ or $(U \Sigma V^*)$

Note: works for non-square matrices as well

Let $q = \min(k, m)$

$$\begin{cases} \bar{\sigma} = \sigma_1 \\ \underline{\sigma} = \begin{cases} \sigma_q & \overset{m \geq k}{\cancel{p \geq m}} \text{ (tall)} \\ 0 & \underset{m < k}{\cancel{p < m}} \text{ (wide)} \end{cases} \end{cases}$$

~~Let~~ Partition R + L matrices.
i.e.

$R = [r_1 \; r_2 \; \cdots \; r_k]$
$L = [l_1 \; l_2 \; \cdots \; l_m]$ } Cols are orthogonal

$$y_0 = \sum_{l=1}^{k} \sigma_l (r_l^* u_0) l_l$$

Suppose $u_0 = r_i$

→ Input = "right" singular direction
→ output = i^{th} singular value along corresponding left singular direction.

$$y_0 = \sum_{l=1}^{k} \sigma_l (r_l^* r_i) l_l = \cancel{\sigma_l l_l} \; \sigma_i l_i$$

$\therefore$ if $u_0 = r_1 \Rightarrow y_0 = \sigma_1 \ell_1 = \bar{\sigma}[G]\,\ell_1$

$$\|y_0\|_2 = (y_0^* y_0)^{1/2} = \bar{\sigma}[G]\,(\ell_1^* \ell_1)^{1/2} = \bar{\sigma}[G]$$

Example: evalfr( )

$$G(3j) = \begin{pmatrix} -.284+258j & -.28+.19j \\ .995+.59j & .1-.009j \end{pmatrix}$$

$[L, \Sigma, R] = \mathrm{svd}(G)$

$$L(3j) = \begin{pmatrix} \underbrace{\begin{matrix} -0.0011+1j \\ .0039+.0023j \end{matrix}}_{\ell_1} & \underbrace{\begin{matrix} .0045-.0002j \\ -.4739+.8805j \end{matrix}}_{\ell_2} \end{pmatrix} \qquad \overset{(3j)}{\Sigma} = \begin{bmatrix} 258.003 & 0 \\ 0 & 0.1005 \end{bmatrix}$$

$$R(3j) = \begin{pmatrix} \underbrace{\begin{matrix} 1 \\ .0007-.0011j \end{matrix}}_{r_1} & \underbrace{\begin{matrix} .0013 \\ -.5635+.8261j \end{matrix}}_{r_2} \end{pmatrix}$$

Put into maglangle format:

$$\ell_1 = \begin{bmatrix} 1 + \angle 1.5719 \\ 0.0045 + \angle 0.5351 \end{bmatrix} \qquad r_1 = \begin{bmatrix} 1 \\ 0.0013 + \angle -0.9722 \end{bmatrix}$$

Given input: r_1 amplitude $\quad$ r_1 phase

$$u = \begin{bmatrix} 1 \ \sin(3t) \\ 0.0013 \ \sin(3t + -0.9722) \end{bmatrix}$$

Then

$$y_{ss} = \underbrace{258}_{\Sigma(1)} \begin{bmatrix} \sin(3t + 1.5719) \\ \underbrace{0.0045}_{\ell_1 \text{ amplitude}} \ \sin(3t + \underbrace{0.5351}_{\ell_1 \text{ phase}}) \end{bmatrix}$$