- Any LTI system is algebraically equivalent to the following form:

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \qquad x_c \in \mathbb{R}^{\bar{n}},\ x_u \in \mathbb{R}^{n-\bar{n}}$$

$$y = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + Du \qquad u \in \mathbb{R}^k,\ y \in \mathbb{R}^m$$

$\bar{n}$ = dim of controllable subsystem.

Def 14.1: The system (A, B) is stabilizable if $n = \bar{n}$ or A_u is a stability matrix.

So either:

1. The system is fully controllable + A_u doesnt exist.
2. A_u is a stability matrix and the states x_u will converge to zero exp. fast.

(~~Remember: if (A,B) is controllable $\Rightarrow$ A is stability matrix~~
~~(A_c, B_c) controllable $\Rightarrow$ A_c Hurwitz~~)

Eigenvector Test for Stabilizability:

Thm 14.1: $\dot{x} = Ax + Bu$ is stabilizable iff every e-vector of A^T corresponding to an e-value ~~iff~~ w/ positive or zero real part, is not in the kernel of B^T.

Discrete Systems: The D-LTI System is stabilizable iff every e-vector of A^T corresponding to an e-value w/ magnitute larger of equal to 1 is not in the kernel of B^T.

Proof: Use similarity transform to transform to controllable decomposition.

$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1}AT \qquad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B$$

① Show that if the (A,B) system is stabilizable then every "unstable" e-vector of A^T is not in the ker of B^T.

(prove by contradiction)

Assume (A,B) stabilizable $\rightarrow$ show unstable e-vector in null space of B^T.

i.e. (λ, x) = an unstable e-pair s.t. $A^T x = \lambda x$ + $B^T x = 0$ $\quad x \neq 0$

$$\Rightarrow (T^{-1}AT)^T x = \lambda x \quad + \quad (T^{-1}\bar{B})^T x = 0$$

$$\Rightarrow \begin{bmatrix} A_c^T & \\ A_{12}^T & A_u^T \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c^T & 0 \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = 0$$

Where $\begin{bmatrix} x_c \\ x_u \end{bmatrix} = T^{-1}x = \bar{x} \neq 0$

Since (A_c, B_c) – controllable +

$A_c^T x_c = \lambda x_c$ and $B_c^T x_c = 0$

Then $x_c = 0$ (not an e-vector) < From previous therom, if (A_c, B_c) is controllable then can't have a e-vector in $\text{Ker } B_c^T$

$\Rightarrow A_u^T x_u = \lambda x_u \Rightarrow \lambda$ is an unstable e-value which contradicts stabilizability assumption.

② – Suppose system is not stabilizable
– Assume every unstable e-vector of A^T is not in $\text{Ker } B^T$

Not stabilizable $\Rightarrow \exists \lambda \in \text{RHP}$ s.t. $A_u^T x_u = \lambda x_u$
$x_u \neq 0$

But

$$\begin{bmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix}$$

and $\bar{B}^T \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} B_c^T & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = 0$ for $\begin{bmatrix} 0 \\ x_u \end{bmatrix} \neq 0$

$\Rightarrow \begin{bmatrix} 0 \\ x_u \end{bmatrix}$ is an unstable e-vector of $\bar{A}^T$

and $\begin{bmatrix} 0 \\ x_u \end{bmatrix} \in \text{Ker } \bar{B}^T$

→ We found an "unstable" e-vector of $\bar{A}^T$ in the Ker of $\bar{B}^T$ so $(\bar{A}, \bar{B})$ cannot be stabilizable.

~~And~~ $\cancel{x = (T^T)^{-1} \begin{bmatrix} 0 \\ x_u \end{bmatrix}}$

→ contradiction since similarity transforms preserve controllability properties.

PBH Test for Stabilizability:

Thm 14.2 —

1. The cont-time LTI system is stabilizable iff

$$\text{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C} : \text{Re}[\lambda] \geq 0$$

→ Only difference is that we are looking @ the "unstable" e-values.

→ So, if unstable e-values are controllable, then we are okay. The uncontrollable e-values will all be neg and go to zero (modes) after some transient time.

Lyapunov Test for Stabilizability:

Thm 14.3 — The LTI system is stabilizable iff there exists a pos-def P s.t.

$$AP + PA^T - BB^T < 0$$

↑ opposite sign from controllability lyap test

Ⓐ Show that if P has pos-def solution then system is stabilizable.

(use e-vector test + assume $AP + PA^T - BB^T < 0$) w/ P being pos-definite

Let $A^T x = \lambda x$ (where λ = unstable e-value)

Then $\quad x^*(AP + PA^T)x < x^* BB^T x = \|B^T x\|^2$

$$(A^T x)^* Px + x^* PA^T x = \lambda^* x^* Px + \lambda x^* Px$$

$$= 2\text{Re}\{\lambda\}(x^* Px) = \|B^T x\|^2$$

P is pos-def and $\text{Re}\{\lambda\} \geq 0$ (assumed unstable)

~~pos-definite~~ ~~Pos.~~

$$0 \leq 2\text{Re}\{\lambda\} x^* Px < \|B^T x\|^2$$

$\Rightarrow x \notin \ker B^T$ (because $B^T x > 0$)

$\therefore$ System is stabilizable (by e-vector test)

(B) Assume AB-LTI is stabilizable – show that P is pos-def solution.

- Let T be the similarity transform that creates controllable decomposition.

$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1} A T$$

$$\bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1} B$$

(A_c, B_c) = controllable $\Rightarrow \exists\, P_c$ s.t.

$$A_c P_c + P_c A_c^T - B_c B_c^T = -Q_c < 0$$

(from feedback stabilization based off Lyap. test)

A_u - stable $\Rightarrow \exists\, P_u$ s.t.

$$A_u P_u + P_u A_u^T = -Q_u < 0$$ (std. Lyap. eq. for stability)

Let $\bar{P} = \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix}$ for $\rho > 0$

$\bar{A}\bar{P} + \bar{P}\bar{A}^T - \bar{B}\bar{B}^T$ (will show this equal a neg. def matrix)

$$= \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} + \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix}\begin{bmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{bmatrix} - \begin{bmatrix} B_c \\ 0 \end{bmatrix}\begin{bmatrix} B_c^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_c P_c & \rho A_{12} P_u \\ 0 & \rho A_u P_u \end{bmatrix} + \begin{bmatrix} P_c A_c^T & 0 \\ \rho P_u A_{12}^T & \rho P_u A_u^T \end{bmatrix} - \begin{bmatrix} B_c B_c^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{A_c P_c + P_c A_c^T - B_c B_c^T}^{-Q_c} & \rho A_{12} P_u + P_c A_c^T \\ \rho P_u A_{12}^T & \underbrace{\rho A_u P_u + \rho P_u A_u^T}_{-\rho Q_u} \end{bmatrix}$$

$$= \begin{pmatrix} -Q_c & \rho A_{12} P_u \\ \rho P_u A_{12}^T & \rho(-Q_u) \end{pmatrix} = -\begin{bmatrix} Q_c & -\rho A_{12} P_u \\ -\rho P_u A_{12}^T & \rho Q_u \end{bmatrix}$$

$\longrightarrow$ If ρ is small enough, this will be neg-definite.

<u>Feedback Stabilization Based on Lyapunov Test</u>:

Assume (AB-LTI) is stabilizable.

Then $AP + PA^T - BB^T < 0$ (from last ~~test~~ theorem)

~~Let $K = \frac{1}{2} B^T P^{-1}$~~

$\Rightarrow$ $AP + PA^T - \frac{1}{2} BB^T P^{-1} P - \frac{1}{2} PP^{-1} BB^T < 0$

$$(A - \tfrac{1}{2} BB^T P^{-1}) P + P (A - \tfrac{1}{2} BB^T P^{-1})^T < 0$$

$$(A - BK) P + P (A - BK)^T < 0$$

w/ $K = \frac{1}{2} B^T P^{-1}$

$\Rightarrow$ $PP^{-1}(A - BK)P + P(A - BK)^T P^{-1} P < 0$

$$P \left[P^{-1}(A - BK) + (A - BK)^T P^{-1} \right] P < 0$$

Let $Q = P^{-1}$ & mult both right & left side by Q

$\Rightarrow$ $Q(A - BK) + (A - BK)^T Q < 0$

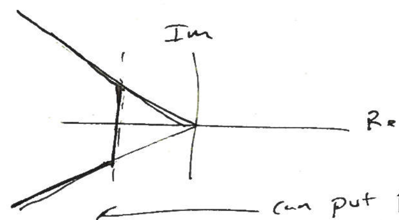
* $(A - BK)$ is a stability matrix by Lyap. theorem.
* state feedback control $u = -Kx$ asymptotically stabilizes the system.

Thm 14.4 - when the ~~stat~~ system (AB-LTI) is stabilizable can always find ~~k~~ $u=-kx$ that makes closed loop system $\dot{x}=(A-BK)x$ asymptotically stable.

→ This is a neccesary + sufficient condition.

(A,B) - stabilizable iff $\exists\, k$ s.t. $u=-kx$ makes $\dot{x}=(A-BK)x$ asymp. stable.

14.6 : Eigenvalue assignment :



can put ~~poles~~ e-values where we want to make system arbitrarily fast & eliminate overshoot/wish.

If $\dot{x}=Ax+Bu$ is controllable

Given any set of n complex #'s (appearing in complex conj. pairs), there exists a state feedback matrix K s.t. the closed-loop ~~system~~ system & $\dot{x}=(A-Bk)x$ has e-values equal to λ_i

(i.e. can put ~~poles~~ e-values pretty much anywhere)

Controllable

E-vector Test for controllability

- (A,B) is controllable iff there is no e-vector of A^T that is in the ker of B^T

Lyapunov Test for controllability

- A - stability matrix then (A,B) controllable iff $\exists$ a unique, pos-def sol. $W = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$ to eq. $AW + WA^T = -BB^T$

PBH Test for Controllability

(A,B) is controllable iff $\text{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n$

$\forall \lambda \in \mathbb{C}$

Stabilizable

Eigenvector Test for Stabilizability

- (A,B) is stabilizable iff every e-vector of A^T corresponding to an e-value w/ $Re\{\lambda\} \geq 0$, is not in ker of B^T

Lyapunov Test for Stabilizability

- LTI system is stabilizable iff $\exists$ a unique, pos-def solution P to eq. $AP + PA^T - BB^T < 0$

PBH Test for Stabilizability

- The LTI system is stabilizable iff $\text{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n$

$\forall \lambda \in \mathbb{C} : Re\{\lambda\} \geq 0$

complex plane

$\rightarrow$ only look @ "unstable" e-values to see if they are controllable

Pole Placement

(A,B) - controllable $\iff$ $A_{cc} = (A - BK)$ can have any "legal" set of e-values.

$$P_{cc}(s) = \prod_{i=1}^{n}(s-\mu_i) = s^n + \gamma_{n-1}s^{n-1} + \dots + \gamma_1 s + \gamma_0$$

Find K so that $\det(sI_n - A_{cc}) = \det(sI_n - A + BK) = \underbrace{P_{cc}(s)}_{\text{desired closed-loop polynomial.}}$

Assume a SISO system: (initially)

→ Put in controllable cannonical form, if (A,B) is controllable can put in this form.

$$A = A_c = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix}$$

char of polynomial of A matix

Where $\det(sI_n - A) = P_A(s) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0$

$$B = B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

w/ single input the gain matrix K will be a row vector

$$K = [k_0 \ \ k_1 \ \cdots \ k_{n-1}]$$

$$A_{cc} = A_c - B_c K = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ & & \ddots & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [k_0 \ \ k_1 \cdots k_{n-1}]}_{\begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & 0 \\ k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix}}$$

$$A_{cc} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \cdots & 1 \\ -(\alpha_0+k_0) & -(\alpha_1+k_1) & \cdots & & -(\alpha_{n-1}+k_{n-1}) \end{bmatrix}$$

→ This matrix is also in controller cannonical form

→ This means char polynomial

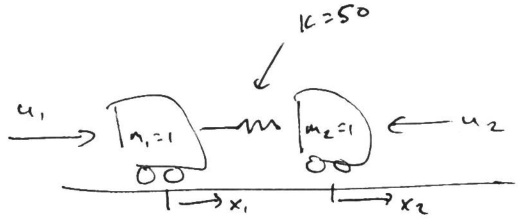
$$P_{A_{cc}}(s) = s^n + (\alpha_{n-1}+k_{n-1})s^{n-1} + \cdots + (\alpha_1+k_1)s + (\alpha_0+k_0)$$

$$k_1 = \gamma_1 - \alpha_1$$

Put in part want. Subtract off part don't want

Choose: $\left[\begin{array}{l} \gamma_i = \alpha_i + k_i \\ k_i = \gamma_i - \alpha_i \end{array}\right.$

What we want the e-value to be — coefficients of the desired polynomial.



→ want poles at $(-10, -10, 5\sqrt{2}(-1 \pm j))$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & 50 & 0 & 0 \\ 50 & -50 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

→ controllability matrix $\text{rank}(C) = 4$

→ Put into Multiple Input Controller Form

$$A_c = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -50 & 0 & -50 & 0 \\ \hline 0 & 0 & 0 & 1 \\ -50 & 0 & -50 & 0 \end{array}\right] \qquad B_c = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}$$

$$A_c + B_c K_c = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -50+k_{11} & k_{12} & -50+k_{13} & k_{14} \\ \hline 0 & 0 & 0 & 1 \\ -50+k_{21} & k_{22} & -50+k_{23} & k_{24} \end{array}\right]$$

→ Single Block structure

$k_{11} = 50$

$k_{13} = 51$

$k_{12} = k_{14} = 0$

$$A_c + B_c K_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50+k_{21} & k_{22} & -50+k_{23} & k_{24} \end{bmatrix}$$

desired char eq:

$$P_{des}(s) = s^4 + 34s^2 + 483s^2 + 3414s + 10000$$

$\longrightarrow$ current char eq.

$$P_{Act}(s) = s^4 - k_{24}s^3 + (+50 - k_{23})s^2 - k_{22}s + (+50 - k_{21})$$

$k_{21} = -9950$

$\cancel{k_{22} = -3414}$

$k_{22} = -3414$

$k_{23} = -433$

$k_{24} = -34$