- LQR required the existence of a symmetric solution $P$ to the ARE

$$A^TP + PA + Q - (PB+N)R^{-1}(B^TP+N^T) = 0$$

For which -

$$(A - BR^{-1}(B^TP + N^T)) \text{ - stability matrix}$$

(Expand ARE terms)

$$A^TP + PA + Q - PBR^{-1}B^TP - PBR^{-1}N^T - NR^{-1}B^TP - NR^{-1}N^T = 0$$

(group terms)

$$(A - BR^{-1}N^T)^TP + P(A - BR^{-1}N^T) + (Q - NR^{-1}N^T) - PBR^{-1}B^TP = 0$$

$$[P \quad -I]\underbrace{\begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix}}_{\triangleq H \in \mathbb{R}^{2n\times 2n}}\begin{bmatrix} I \\ P \end{bmatrix} = 0$$

(called the Hamiltonian matrix - associated w/ ARE.

$$\Rightarrow [P \quad -I]\, H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

* The solution of the ARE is linked to properties of the eigenspaces of the Hamiltonian matrix.

Claim #1 : If $\lambda$ is an eigenvalue of $H$, then $-\lambda$ is also an eigenvalue of $H$.

(E-values of real matrices appear in complex conjugate pairs, the e-values of $H$ are symmetric across both the real + imaginary axis)

Proof:

Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ & consider similarity

$JH + H^TJ = 0$
$JH = -H^TJ$
$JHJ^{-1} = -H^T$ → shows H is a Hamiltonian matrix

transformation $JHJ^{-1}$ —

$$JHJ^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \\ -(A - BR^{-1}N^T) & BR^{-1}B^T \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(A - BR^{-1}N^T)^T & Q - NR^{-1}N^T \\ BR^{-1}B^T & A - BR^{-1}N^T \end{pmatrix}$$

transpose matrix then transpose individual elements (most symmetric)

$$= -\begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{pmatrix}^T$$

$$= -H^T$$

→ E-values are preserved under a similarity transformation, if $\lambda$ is an e-value of $H$, then it is an e-value of $-H^T$, and since $H$ & $H^T$ have the same e-values, $\lambda$ is an eigenvalue of $-H$

i.e. $Hx = \lambda x \Rightarrow (-H)x = \lambda x$

$H(-x) = (-\lambda)(-x)$

$(-\lambda, -x)$ – E-pair of $H$

- When will $H$ not have e-values on the Im axis?

<u>Lemma 21.1</u>: Assume $Q - NR^{-1}N^T \geq 0$, & $R = R^T > 0$

If — ① $(A,B)$ - stabilizable

② $(A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ - detectable

Then $H$ will have no e-values on imaginary axis.

Proof by contradiction —

Assume that $j\omega$ is an e-value of $H$ w/ e-vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then
$$[x_2^* \ \ x_1^*] H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1^* \ \ x_2^*] H^T \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= (x_2^* \ \ x_1^*) Hx + (Hx)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= [x_2^* \ \ x_1^*] j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= j\omega (x_2^* x_1 \ \ x_1^* x_2) + -j\omega (x_1^* x_2 + x_2^* x_1)$$

$$= 0$$

$$\Rightarrow [x_2^* \ \ x_1^*] H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1^* \ \ x_2^*] H^T \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= [x_2^* \ \ x_1^*] \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ [x_1^* \ \ x_2^*] \begin{bmatrix} (A - BR^{-1}N^T)^T & -Q + NR^{-1}N^T \\ -BR^{-1}B^T & -(A - BR^{-1}N^T) \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= \left[ x_2^*(A-BR^{-1}N^T) + x_1^*(-Q+NR^{-1}N^T) \;\middle|\; -x_2^* BR^{-1}B^T - x_1^*(A-BR^{-1}N^T)^T \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \left[ x_1^*(A-BR^{-1}N^T)^T - x_2^* BR^{-1}B^T \;\middle|\; x_1^*(-Q+NR^{-1}N^T) - x_2^*(A-BR^{-1}N^T) \right] \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= x_2^*(A-BR^{-1}N^T)x_1 + x_1^*(-Q+NR^{-1}N^T)x_1 - x_2^* BR^{-1}B^T x_2 - x_1^*(A-BR^{-1}N^T)^T x_2$$

$$+ x_1^*(A-BR^{-1}N^T)^T x_2 + x_1^*(-Q+NR^{-1}N^T)x_1 - x_2^* BR^{-1}B^T x_2 - x_2^*(A-BR^{-1}N^T)x_1$$

$$= -2x_1^*(Q-NR^{-1}N^T)x_1 - 2x_2^* BR^{-1}B^T x_2$$

$$= 0$$

So, since $R \succ 0$ then $B^T x_2 = 0$ + $(Q-NR^{-1}N^T)x_1 = 0$

Also, since — $\left(j\omega, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ — e-pair of $H$

$$Hx = \lambda x \Rightarrow \begin{bmatrix} A-BR^{-1}N^T & -BR^{-1}B^T \\ -Q+NR^{-1}N^T & -(A-BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} j\omega-(A-BR^{-1}N^T) & BR^{-1}B^T \\ Q-NR^{-1}NT & j\omega+(A-BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \emptyset$$

→ ~~the~~ taking 1st eq.

$$\left(j\omega-(A-BR^{-1}N^T)\right)x_1 + \underbrace{BR^{-1}B^T x_2}_{=0} = 0$$

$$\left(j\omega-(A-BR^{-1}N^T)\right)x_1 = 0 \Rightarrow +(A-BR^{-1}N^T)x_1 = j\omega x_1$$

$(j\omega, x_1)$ - e-pair of $(A-BR^{-1}N^T)$

→ ~~taking 2nd eq.~~ So we have an ~~e-value~~ e-vector $x_1$ of $(A-BR^{-1}N^T)$ in the kernel of $(Q-NR^{-1}N^T)$

→ $(A-BR^{-1}N^T, Q-NR^{-1}N^T)$ — not detectable.

Similarly,

$$j\omega x_2 + (A - BR^{-1}N^T)^T x_2 = 0$$

$$\Rightarrow \quad j\omega x_2 + (A^T x_2 - \underbrace{NR^{-1}B^T x_2}_{=0}) = 0$$

$$A^T x_2 = -j\omega x_2$$

— $x_2$ is e-vector of $A^T$ in the ker of $B^T$ & so is an unstable e-value

$\Rightarrow$ $(A, B)$ is not stabilizable.

$\rightarrow$ Therefore, if —

① $Q - NR^{-1}N^T \geq 0$

② $(A, B)$ - stabilizable

③ $(A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ - detectable

$\rightarrow$ H will have no e-values on Im axis

## Stable Subspace:

- Given a sq. matrix $T$, factor the char. polynomial as —

$$\Delta(s) = \det(sI - T) = \Delta_-(s)\,\Delta_+(s)$$

where the roots of $\Delta_-(s)$ are in the open LHP & $\Delta_+(s)$ are in closed RHP.

- The stable subspace is

$$\mathcal{V}_- = \operatorname{Ker} \Delta_-(T)$$

Properties :

P21.1 – $\dim \mathcal{V}_- = \deg \Delta_-(s)$

P21.2 – For every matrix $V_-$ whose cols form a basis for $\mathcal{V}_-$, there exists a stability matrix $T_-$ s.t. ~~$TV_- = V_-M$~~ ~~$V_-T_-$~~

$$TV_- = V_-T_-$$

Proof –

– Decompose $T$ as $T\begin{bmatrix} W_- & W_+ \end{bmatrix} = \begin{bmatrix} W_- & W_+ \end{bmatrix}\left[\begin{array}{c|c} J_- & 0 \\ 0 & J_+ \end{array}\right]$

– where $J_-$ is the Jordan form associated w/ stable e-values and $W_-$ are associated w/ (e-values, e-vectors) that span $\mathcal{V}_-$

$\Rightarrow \therefore \; TW_- = W_-J_-$

Now, for an invertible $Z$ we have

$$TW_-Z^{-1} = W_-Z^{-1}ZJ_-Z^{-1}$$

Let $V_- = W_-Z^{-1}$ + $T_- = ZJ_-Z^{-1}$, then

$$TV_- = V_-T_-$$

## Domain of the Riccati Operator: $(H \in \mathbb{R}^{2n \times 2n})$

- The Hamiltonian matrix $H$ is said to be in the domain of the Riccati operator if there exists a stability matrix $H_- \in \mathbb{R}^{n\times n}$ and there exists a $P \in \mathbb{R}^{n \times n}$ s.t.

$$HM = MH_- \quad \text{where} \quad M = \begin{bmatrix} I \\ P \end{bmatrix}$$

Thm 21.1: If $H \in Ric$, then

① $P$ satisfies ARE (21.1)

② $A - BR^{-1}(B^TP + N^T) = H_-$ is a stability matrix

③ $P$ is a symmetric matrix

↓ This is the closed loop system derived by solving JLQR

**Proof:**

1)

$$HM = MH_-$$

$$H\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

$$\Rightarrow \begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} P & -I \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} H_- = (P-P)H_- = 0$$

- Recall that the ARE can be written as -

$$\begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

2)

$$H\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

$$\begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A-BR^{-1}N^T)^T \end{pmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

→ The top line:

$$A - BR^{-1}N^T - BR^{-1}B^T P = I H_- = H_-$$

3) $HM = MH_-$

$$\underbrace{[-P^T \;\; I]\, H \begin{bmatrix} I \\ P \end{bmatrix}} = [-P^T \;\; I] \begin{bmatrix} I \\ P \end{bmatrix} H_- = (P - P^T) H_-$$

– can show this is symmetric by working through the def. of H

$\therefore \; (P-P^T)H_-$ is symmetric

i.e $(P-P^T)H_- = H_-^T(P^T - P) = -H_-^T(P-P^T)$

$\Rightarrow \; (P-P^T)H_- + H_-^T(P-P^T) = 0$

$$e^{H_-^T t}(P-P^T)H_- e^{H_- t} + e^{H_-^T t} H_-^T (P-P^T) e^{H_- t} = 0$$

(multiples both sides by $e^{H_-^T t}$)

$$\frac{d}{dt}\left\{ e^{H_-^T t}(P-P^T)e^{H_- t} \right\} = 0 \quad \forall t$$

$\Rightarrow \; e^{H_-^T t}(P-P^T)e^{H_- t}$ is a constant for all time

→ $H_-$ is a stability matrix → 0 as $t \to \infty$

So $e^{H_-^T t}(P-P^T)e^{H_- t} = 0$ + $e^{H_- t} \neq 0 \;\forall t$

So $P = P^T$

## Thm 21.2 –

If ① $R > 0$

② $Q - NR^{-1}N^T \geq 0$

③ $(A, B)$ – Stabilizable

④ $(A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ – detectable

If $N = 0 \Rightarrow (A, Q)$ – detectable

Then

① $H$ is in the domain of the Riccati operator

② $P$ satisfies the ARE

③ $A - BR^{-1}(B^TP + N^T) = H_-$ is a stability matrix

④ $P$ is symmetric + pos-definite

where $P$, $H_- \in \mathbb{R}^{n\times n}$ ~~are~~ defined by (21.10) + e-values of $H_-$ are the stable e-values of $H$.

Proof:

1) Since $H$ has $n$ e-values in the LHP and $n$ e-values in the open RHP by claim #1 + lemma 21.1 we have —

$$H \begin{bmatrix} V_- & V_+ \end{bmatrix} = \begin{bmatrix} V_- & V_+ \end{bmatrix} \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix}$$

$$HV_- = V_- J_-$$

where $V_- = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ are the stable e-vectors

If $V_1^{-1}$ exists (prove this in a min.)

Then $H V_- V_1^{-1} = V_- \underbrace{V_1^{-1} V_1}_{I} J_- V_1^{-1}$

$$\Rightarrow \quad H \begin{bmatrix} I \\ V_2 V_1^{-1} \end{bmatrix} = \begin{bmatrix} I \\ V_2 V_1^{-1} \end{bmatrix} V_1 J_- V_1^{-1}$$

letting: $P = V_2 V_1^{-1}$ and $H_- = V_1 J_- V_1^{-1}$

gives $H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_- \quad \Rightarrow \quad H \in Ric$

2) Follows from Thm 21.2

3) we need to show that $V_1^{-1}$ exists, or in other words the solution to the ARE exists.

$(A, B)$ - stabilizable

$\Rightarrow \exists\, k$ s.t.

$\dot{x} = (A - Bk)x$ is exp. stable

$\Rightarrow x_k(t) \to 0$ exp.

$\Rightarrow u_k(t) = -k\, x_k(t) \to 0$ exp.

$$\Rightarrow \int_0^\infty x_k^T(t) Q x_k(t) + u_k^T(t) R u_k + u_k^T N x_k(t)\, dt < \infty$$

But,

$$x_0^T P x_0 = \min_u \int_0^\infty x^T Q x + u^T R u + u^T N x\, dt \le \int_0^\infty x_k^T Q x_k + u_k^T R u_k + u_k^T N x_k\, dt < \infty$$

$\rightarrow$ true for all $x_0 \Rightarrow \therefore P$ exists $\Rightarrow V_1^{-1}$ exists

4) Follows from Thm 21.1

5) P-symmetric follows from Thm 21.1

To show P is pos. definite

The ARE can be written as

$$(A - BR^{-1}N^T)^T P + P(A - BR^{-1}N^T) + Q - NR^{-1}N^T - PBR^{-1}B^TP = 0$$

$$\Rightarrow (A - BR^{-1}(B^TP + N^T))^T P + P(A - BR^{-1}(B^TP+N^T)) = -(Q - NR^{-1}N^T) - PBR^{-1}BP$$

$$H_-^T P + P H_- = -S$$

Recall from thm 15.10 that there exists a unique pos-def P iff $(H_-, S)$-observable.

- To show observability of $(H_-, S)$ use e-vector test + prove by contradiction.

Assume $x$ is e-vector of $H_-$ that lies in $\ker S$.

$$\left(A - BR^{-1}(B^TP + N^T)\right)x = \lambda x$$

$$Sx = \Big(\underbrace{(Q - NR^{-1}N^T)}_{\text{Symmetric \& pos-def}} + \underbrace{PBR^{-1}B^TP}_{\text{symmetric \& pos-def.}}\Big)x = 0$$

$$\Rightarrow \quad \cancel{x^TSx = 0} \Rightarrow (Q - NR^{-1}N^T)x = 0$$

$$\& \quad B^TPx = 0$$

Thus, $(A - BR^{-1}N^T)x = \lambda x$

$$(Q - NR^{-1}N^T)x = 0$$

which is a contradiction that the pair is observable.

- $(A - BR^{-1}N^T, Q - NR^{-1}N^T)$ is observable
- So $(H_-, S)$ must be observable.