LQR Optimization:

$$\min_u \int_0^\infty \left(x^T Q x + u^T R u\right) dt \qquad w/ \quad \dot{x} = Ax + Bu$$

- optimal control: $u = -Kx$
- This makes LQR problem:

$$\min_K \int_0^\infty (x^T Q x + x^T K^T R K x) dt = \min_K \int_0^\infty x^T (Q + K^T R K) x \, dt$$

$$\dot{x} = (A - BK)x \quad \text{— stability matrix}$$

If $(A - BK)$ – stable

Then $\displaystyle\int_0^\infty x^T (Q + K^T R K) x \, dt = x_0^T P x_0 = tr\{P x_0 x_0^T\}$

w/ P solving Lyap eq. $\quad$ P - symmetric + pos-def.

$$P(A - BK) + (A - BK)^T P + (Q + K^T R K) = 0$$

$\Rightarrow$ the LQR optimization is:

$$\min_k tr\{P x_0 x_0^T\}$$

s.t. $(A - Bk)$ - stability matrix

$$P(A - Bk) + (A - Bk)^T P + (Q + k^T R k) = 0$$

Observer optimization (Kalman-Bucy Filter):

$$\dot{x} = Ax + Bu + d \qquad \text{w/} \quad n \sim N(0, N)$$

$$y = Cx + n \qquad\qquad d \sim N(0, D)$$

→ ~~[illegible]~~ Want to find an optimal observer (estimator) to minimize process + sensor noise (in terms of variance of random variables)

→ Choose L to minimize ~~(mean sq. error)~~

$$V_f = \lim_{t \to \infty} E\{e^T C C^T e\} \qquad e \to \text{function of time } e(t)$$

→ This is the mean sq. error of the measured error states: ↳ as $t \to \infty$

$$\tilde{y} = Ce(t)$$

w/ $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

Error Dynamics: $e = x - \hat{x}$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu + d - A\hat{x} + - Bu - L(Cx + n - C\hat{x})$$

$$= (A - LC)e + \underbrace{d - Ln}_{\psi}$$

let $\psi = d - Ln$

$$E\{\psi(t)\psi^T(t)\} = E\{(d - Ln)(d^T - n^T L^T)\}$$

$$= E\{dd^T\} - E\{dn^T L^T\} - E\{Lnd^T\} + E\{Lnn^T L\}$$

$$E\{\psi\psi^T\} = D + LNL^T$$

$$\psi \sim N(0, D + LNL^T)$$

$$V_f = \lim_{t\to\infty} E\{e^T C C^T e\}$$

w/ $\dot{e} = (A - LC)e + \psi$

$\Rightarrow$ $V_f = \mathrm{tr}(SCC^T)$ (from prev. section)

Subject to:

$$(A-LC)S + S(A-LC)^T + (D + LNL^T) = 0$$

Comparison w/ LQR:

$$V_c = \mathrm{tr}(P x_0 x_0^T)$$

Subject to:

$$(A-BK)^T P + P(A-BK) + (Q + K^T R K) = 0$$

w/ Solution:

$$K = R^{-1}B^T P \quad \& \quad P \text{ satisfying } \underbrace{A^T P + PA + Q - PBR^{-1}B^T P = 0}_{\text{CARE}}$$

Duality:

CARE	FARE
R	N
B	C^T
K	L^T
A	A^T
Q	D

- By a simple substitution one problem becomes the other.

$\therefore$

$$K = R^{-1}B^T P \Rightarrow N^{-1}CS = L^T$$

$$\Rightarrow \quad L = SC^T N^{-1}$$

w/ S satisfying —

$$AS + SA^T + D - SC^T N^{-1} CS = 0$$

(Filter Algebraic Riccati Equation — FARE)

$\Rightarrow$ The combination of LQR w/ kalman filter is LQG (linear quadratic gaussian)

Optimal Filter:

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

$L = SC^T N^{-1}$ w/ S solving FARE:

$$AS + SA^T + D - SC^T N^{-1} CS = 0$$

LQR	Kalman
$[A, B]$ - stabilizable	$[A^T, C^T]$ - stabilizable $\Rightarrow$ $[A, C]$ - detectable
$[A, Q^{1/2}]$ - detectable	$[A^T, D^{1/2}]$ - detectable $\Rightarrow$ $[A, D^{1/2}]$ - stabilizable