- For SISO systems the poles + zeros are defined by the "co-prime" numerator + denominator polynomials of the TF.

Co-prime: Two polynomials are co-prime if they have no common roots.

Co-prime:

$a(s) = (s+1)(s+2)$ $\qquad$ $b(s) = (s+3)(s+4)(s+5)$

Not Co-prime:

$a(s) = (s+1)(s+2)$ $\qquad$ $b(s) = (s+2)(s+3)(s+4)$

A SISO TF can be written:

$$g(s) = \frac{n(s)}{d(s)}$$

Poles = where the system becomes unbounded (division by zero)

zeros = values of $s \in \mathbb{C}$ for which $g(s) = 0$

i.e. $\quad g(s) = \dfrac{s+3}{(s+1)(s+2)}$ $\qquad$ zeros $(-3, \infty)$ ← true for strictly proper TF

Poles $= (-1, -2)$

If $n(s)$ + $d(s)$ are co-prime then the zeros are roots of $n(s)$ (and possibly infinity) and the poles are roots of $d(s)$.

SISO Alternative Definition:

Let $g(s) = \dfrac{s+1}{(s+2)(s+3)}$ Have zero @ $z_1 = -1$

Choose input: $u(t) = e^{-t}\,\mathbb{1}(t)$ w/ $\mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$U(s) = \frac{1}{s+1}$$

$$Y(s) = g(s)U(s) = \frac{\cancel{s+1}}{(s+2)(s+3)} \frac{1}{\cancel{(s+1)}}$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t} \qquad t \geq 0$$

could figure out coeff w/ partial frac. expansion.

Note: The output consists only of the modes of the TF. The input (e^{-t}) does not appear.

i.e. The input is absorbed or blocked & doesn't make it to the output.

Look @ the differential Eq:

$$\ddot{y} + 5\dot{y} + 6y = \dot{u} + u$$

$\downarrow$ w/ I.C.

$$(s^2 Y(s) - s y_0 - \dot{y}_0) + 5(sY(s) - y_0) + 6Y(s) = s(U(s) - u_0) + U(s)$$

w/ $u_0 = u(0^-)$
$y_0 = y(0^-)$
$\dot{y}_0 = \dot{y}(0^-)$

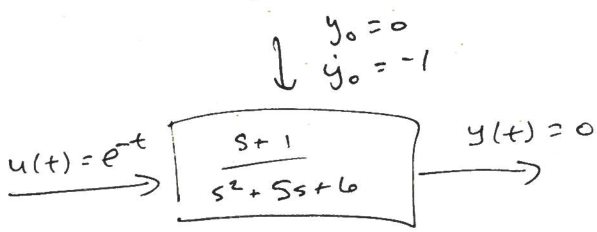
$$(s^2+5s+6)\,Y(s) = (s+1)\,U(s) - u_0 - s y_0 + \dot{y}_0 + 5y_0$$

If $u(t) = e^{-t}\mathbb{1}(t)$ let $u_0 = 0$

$$U(s) = \frac{1}{s+1}$$

$$(s^2+5s+6)\,Y(s) = \underbrace{\left(\frac{s+1}{s+1}\right)}_{=1} + s y_0 + \dot{y}_0 + 5y_0$$

Choose a specific I.C. $y_0 = 0$, $\dot{y}_0 = -1$



$$(s^2+5s+6)\,Y(s) = 0$$

$Y(s) = 0 \Rightarrow y(t) = 0$ for all time $(t \geq 0)$

Note: w/ SISO a zero is a number s.t. w/ input $u(t) = e^{z_i t}\mathbb{1}(t)$ it becomes possible to find I.C. $y_0, \dot{y}_0, \ldots, y_0^{(n-1)}$ so that the output will be $y(t) = 0 \;\forall t \geq 0$

(i.e zero output w/ specific I.C.)

MIMO Systems:

1) Poles of $G(s)$ are values of $s \in \mathbb{C}$ for which at least one entry becomes unbounded.

2) The transmission zeros of $G(s)$ are values of $s \in \mathbb{C}$ where $\hat{G}(s)$ loses rank. (Freq. domain)

3) An invariant zero of $P(s)$ are values of $s \in \mathbb{C}$ where $P(s)$ loses rank. (State-space domain)

where $P(s)$ = Rosenbrock's system matrix.

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n\times k)}$$

Invariant zero: The invariant zero of $P(s)$ is the monic greatest common divisor $z_p(s)$ of all non-zero minors of order $r = \text{rank}(P(s))$.

Note: Rosenbrock's system matrix is obtained by taking the LaPlace transform of the state space equations.

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$s\hat{X}(s) - x(0) = A\hat{X}(s) + B\hat{U}(s)$$
$$\hat{Y}(s) = C\hat{X}(s) + D\hat{U}(s)$$

rewrite as:

$$\underbrace{\begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}}_{P(s) = \text{Rosenbrock's system matrix}} \begin{bmatrix} -\hat{X}(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} -X(0) \\ \hat{Y}(s) \end{bmatrix}$$

Example:

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} s & 1 & -1 & 1 & 0 \\ -1 & s+2 & -1 & 1 & 1 \\ 0 & -1 & s+1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

rank is at most 4

4th order minors

(1st 4 columns) → leave out last column

expand along bottom row

$$\begin{vmatrix} s & 1 & -1 & 1 \\ -1 & s+2 & -1 & 1 \\ 0 & -1 & s+1 & 1 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0 \cdot \left|\quad\right| + (-1)(-1) \begin{vmatrix} s & -1 & 1 \\ -1 & -1 & 1 \\ 0 & s+1 & 1 \end{vmatrix} + 0 \cdot \left|\quad\right| + (-1)0 \left|\quad\right|$$

expand along this ~~row~~ column

$$= s \begin{vmatrix} -1 & 1 \\ s+1 & 1 \end{vmatrix} + (-1)(-1) \begin{vmatrix} -1 & 1 \\ s+1 & 1 \end{vmatrix}$$

$$= s\left(-1 - (s+1)\right) + \left(-1 - (s+1)\right)$$

$$= -s^2 - 3s - 2 = -(s+2)(s+1)$$

Compute for all 4th order minors:

$$\underbrace{-(s+1)(s+2)}_{\text{minus 5th col}} \qquad \underbrace{+(s+1)(s+2)}_{\text{minus 4th col}} \qquad \underbrace{-(s+2)}_{\text{minus 3rd col}} \qquad \underbrace{0}_{\text{minus 2nd col}} \qquad \underbrace{s+2}_{\text{minus 1st col}}$$

→ The least common divisor = Invariant zero

⟹ Single invariant zero @ $\{-2\}$

- Invariant zeros have a blocking property ($y(t)=0$)
If the invariant zero is not an e-value of A then we can use a non-zero input and initial state for which the output is identically zero.

Thm 19.2

$$\{\text{transmission zeros of } \hat{G}(s)\} \subset \{\text{invariant zeros of LTI}\}$$

→ i.e. there are more ways the $P(s)$ matrix can drop rank than $\hat{G}(s)$ can drop rank. But any time $\hat{G}(s)$ drops rank than $P(s)$ will drop rank.

→ This could happen w/ cancellations in $\hat{G}(s)$ to make the polynomials co-prime.

system both observable & controllable.

Note: If we have <u>minimal</u> realization then

$$\{\text{transmission zeros of } \hat{G}(s)\} = \{\text{Invariant zeros of LTI}\}$$

sets are the same.

MIMO zeros: For MIMO systems zeros have both a number z_i and direction u_0, s.t. input $u(t) = e^{z_i t} u_0 \mathbb{1}(t)$ it is possible to find an I.C. x_0 so that output $y(t) = 0 \quad \forall t \geq 0$

A MIMO transmission zero exists if:

$$(z_i I - A) x_0 - B u_0 = 0$$

$$C x_0 + D u_0 = 0$$

Proof: $\dot{x} = Ax + Bu$

$$sI\hat{X}(s) - x_0 = A\hat{X}(s) + B\hat{U}(s)$$

$$(sI - A)\hat{x}(s) = x_0 + B\hat{u}(s)$$

$$u(t) = e^{z_i t} u_0 \mathbb{1}(t) \Rightarrow \hat{U}(s) = \frac{1}{s - z_i} u_0$$

(plug-in)

$$(sI - A)\hat{x}(s) = x_0 + Bu_0\left(\frac{1}{s - z_i}\right) = \frac{x_0(s - z_i) + Bu_0}{(s - z_i)}$$

$$= \frac{s x_0 - z_i x_0 + Bu_0}{(s - z_i)}$$

(Add in + subtract Ax_0)

$$(sI - A)\hat{x}(s) = \frac{(sI - A)x_0 - \overbrace{(z_i I - A)x_0 + Bu_0}^{= 0 \text{ (from 1st assumption)}}}{(s - z_i)}$$

$$\Rightarrow \hat{x}(s) = \frac{x_0}{s - z_i} \Rightarrow x_0 e^{z_i t} \quad t \geq 0$$

$$\hat{Y}(s) = C\hat{X}(s) + D\hat{U}(s) = \frac{C X_0}{(s-z_i)} + \frac{D u_0}{(S-z_i)} = \frac{1}{(s-z_i)} \underbrace{(C X_0 + D u_0)}_{=0 \text{ from 2nd requirement}}$$

Eq. to the system:

$$\begin{bmatrix} z_i I - A & +B \\ -C & D \end{bmatrix} \begin{bmatrix} -X_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

How do we solve this system?

Note: This is not quite an e-vector/e-value problem because don't have $(z_i - D)$ only D. Called a "generalized e-value problem". (matlab tzero, null)

If the system has a solution then:

$$\det\left(\begin{bmatrix} z_i I - A & B \\ -C & D \end{bmatrix}\right) = 0 \quad \text{(singular - i.e. non-zero nullspace)}$$

Using partitioning tricks:

$$\Rightarrow \det(z_i I - A)\det\left(C(z_i - A)^{-1}B + D\right)$$

$\rightarrow$ If z_i is not also a pole of the system then $\det(z_i I - A) \neq 0$

$\rightarrow$ $\det\left(C(z_i - A)^{-1}B + D\right) = \det(G(z_i)) = 0 \quad \text{rank}(z_i) < m$

i.e. If we have a solution to this system (that is not a pole of A) then we have a transmission zero: z_i w/ initial condition X_0 + input direction u_0 s.t. w/ $u(t) = u_0 e^{z_i t} \Longrightarrow y(t) = 0 \ \forall t \geq 0$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \qquad D = [\phi]$$

$$P = \begin{bmatrix} s & 0 & 0 & 0 & 2 & 0 \\ -1 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 2 \\ 0 & 0 & -1 & s & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -1 & 0 & 0 \end{bmatrix}$$

$\det P = (s+1)(s+2)$ ← zeros of P not in det $(sI-A)$

$\text{eig}\, A = s^4$

$(z_1 = -1,\ z_2 = -2)$

$$G(z_1) = C(\cancel{z_1}I - A)^{-1}B + D = \begin{bmatrix} \frac{s+1}{s^2} & 0 \\ 0 & \frac{s+2}{s^2} \end{bmatrix}$$

→ Easily see will lose rank for $[s = -1, -2]$

Let $\cancel{z}$ $s = z_1 = -1$

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_{01} \\ -x_{02} \\ -x_{03} \\ -x_{04} \\ u_{01} \\ u_{02} \end{bmatrix} = [\phi]$$

$+x_{01} + 2u_{01} = 0$ $\qquad$ $+x_{03} + x_{04} = 0$

$+x_{01} + x_{02} = 0$ $\qquad$ $+\frac{1}{2}x_{01} + \frac{1}{2}x_{02} = 0$

$+x_{03} + 2u_{02} = 0$ $\qquad$ $+\frac{1}{2}x_{03} + x_{04} = 0$

$$+x_{01} + 2u_{01} = 0$$
$$x_{01} + x_{02} = 0$$
$$x_{03} + 2u_{02} = 0$$
$$x_{03} + x_{04} = 0$$
$$\tfrac{1}{2}x_{01} + \tfrac{1}{2}x_{02} = 0$$
$$\tfrac{1}{2}x_{03} + x_{04} = 0$$

let $u_{01} = 1$, $u_{02} = 0$

$$x_{01} = -2$$
$$x_{02} = 2$$
$$x_{03} = 0$$
$$x_{04} = 0$$

$$x_{03} + 2u_{02} = 0$$
$$x_{03} + x_{04} = 0$$
$$\tfrac{1}{2}x_{03} + x_{04} = 0$$

$$x_0 = [-2; 2; 0; 0]$$
$$u_0 = [1; 0]$$

Input $u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$

– can just jump to here from prev. analysis.

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau = e^{-t}x_0$$

$$y(t) = Cx(t) + Du(t)$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} e^{-t} \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \phi$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} e^{-t} = \phi$$