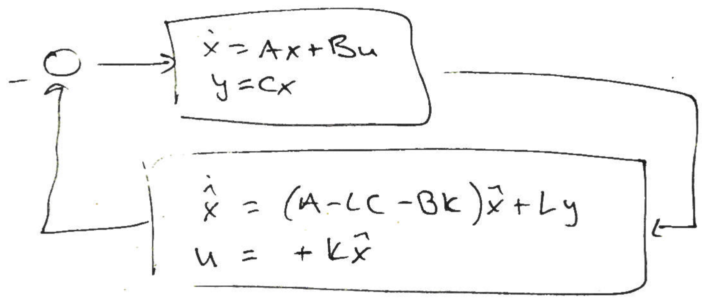
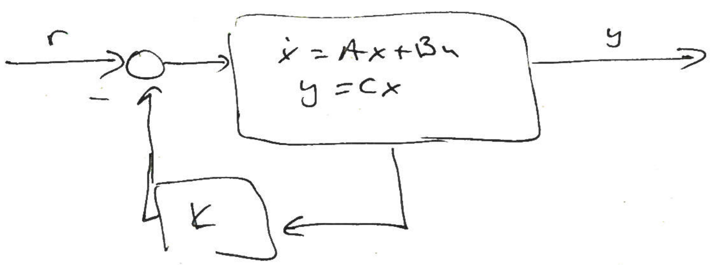
Observer / LQR



→ Have not specified a reference input. W/ state space equations need to know how to introduce the reference.

→ Lets go back to state feedback –



New System: $$\dot{x} = (A - BK)x + Br$$
$$y = Cx$$

w/ $$y(t) = \underbrace{Ce^{(A-BK)t}x_0}_{\substack{\to 0 \\ (A-BK)\text{ - stability matrix}}} + \underbrace{\int_0^t Ce^{(A-BK)(t-\tau)} Br(\tau)\, d\tau}_{\to \text{ this won't neccesarily go to } r.}$$

→ Need to do more to guarantee tracking.

→ Assume r is a constant and we want our controlled output to converge to it, i.e $(z = r)$

→ Let r correspond to an eq. point (x_{eq}, u_{eq})

Then LQR formulation:

$$\dot{x} = Ax + Bu, \quad z = Gx + Hu$$

$$J_{LQR} = \int_0^\infty \tilde{z}(t)^T \bar{Q} \tilde{z}(t) + \rho \tilde{u}(t)^T \bar{R} \tilde{u}(t)\, dt$$

w/ $\tilde{z} = z - r$ + $\tilde{u} = u - u_{eq}$

In steady-state or @ the eq. point:

$$\dot{x} = Ax + Bu \Rightarrow 0 = Ax_{eq} + Bu_{eq}$$

$$z = Gx + Hu \Rightarrow r = Gx_{eq} + Hu_{eq}$$

$$\underbrace{\begin{bmatrix} -A & B \\ -G & H \end{bmatrix}}_{(n+\ell)\times(n+k)} \begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

k = outputs
ℓ = inputs

→ Assuming equal inputs to outputs ($\ell = k$) have a square matrix then we can solve this eq. if there is not an invariant zero @ the origin.
(or transmission zero)

→ Can see this using Rosenbrock's System matrix:

$$P(s) = \begin{bmatrix} sI - A & B \\ -G & H \end{bmatrix}$$

– If $s = 0$ is a zero then will lose rank ⇒ nonsingular.

- Assume x_{eq} + u_{eq} are linearly dependent on r.

$$x_{eq} = Fr$$

$$u_{eq} = Nr$$

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{pmatrix} -A & B \\ -G & H \end{pmatrix}\begin{bmatrix} -Fr \\ Nr \end{bmatrix} = \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}\begin{bmatrix} -F \\ N \end{bmatrix} r$$

$$\Rightarrow \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{pmatrix} -A & B \\ -G & H \end{pmatrix}\begin{bmatrix} -F \\ N \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -F \\ N \end{bmatrix} = \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\tilde{x} = x - x_{eq}$$

Let $\dot{\tilde{x}} = Ax + Bu = A(x - x_{eq}) + B(u - u_{eq}) + \underbrace{Ax_{eq} + Bu_{eq}}_{=0}$

$$\tilde{z} = Gx + Hu - r = G(x - x_{eq}) + H(u - u_{eq}) + \underbrace{G(x_{eq}) + (Hu_{eq} - r)}_{=0}$$

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\tilde{z} = G\tilde{x} + H\tilde{u}$$

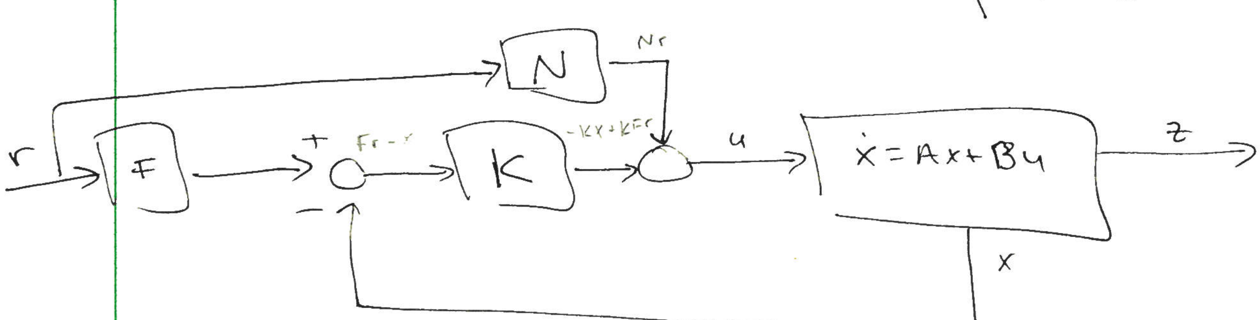
w/ $\tilde{u}(t) = -K\tilde{x}(t)$

$$u - u_{eq} = -K(x - x_{eq})$$

$$u = -K(x - x_{eq}) + u_{eq}$$

$$= -K(x - Fr) + Nr$$

$$= -Kx + (N + KF)r$$



From diagram :

$$\hat{U}(s) = N\hat{r}(s) + KF\hat{r}(s) - \underbrace{K(sI-A)^{-1}B}_{L(s)}\,\hat{U}(s)$$

$$\xrightarrow{U(s)} \boxed{\dot{x} = Ax + Bu} \xrightarrow{x} \boxed{K} \longrightarrow$$

$\left(I(sI-A)^{-1}B\right)$ mult. by gain $\underline{K(sI-A)^{-1}B}$

$$\hat{U}(s) + L(s)\,\hat{U}(s) = N\hat{r}(s) + KF\hat{r}(s)$$

$$(I + L(s))\,\hat{U}(s) = \cancel{N\hat{r}(s) + KF\hat{r}}\ (N + KF)\hat{r}(s)$$

$$\left[\cancel{\hat{U}(s) = (I + L(s))^{-1}\hat{r}(s)}\right] \quad \boxed{\hat{U}(s) = (I + L(s))^{-1}(N + KF)\,\hat{r}(s)}$$

$G_p(s)$ = plant TF

$$\dot{x} = Ax + Bu$$
$$z = Gx + Hu$$

$$G_p(s) = G(sI-A)^{-1}B + H$$

$\rightarrow$ TF from $\hat{U}(s)$ to $\hat{z}(s)$

$$\hat{z}(s) = \hat{G}_p(s)\,\hat{U}(s)$$

$\rightarrow$ TF from $\hat{z}(s)$ to $\hat{r}(s)$ (plug into $\hat{U}(s) = (\ )\hat{r}(s)$)

$$G_p^{-1}(s)\,\hat{z}(s) = (I + L(s))^{-1}\,\cancel{\hat{r}(s)}\,(N + KF)\,\hat{r}(s)$$

$$\boxed{\hat{z}(s) = \hat{T}(s)(I + L(s))^{-1}(N + KF)\,\hat{r}(s)}$$

– when state is inaccessible – need to replace w/ an observer —

$$u(t) = -k(\hat{x}(t) - x_{eq}) + u_{eq}$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + \underbrace{Bu}_{\text{Plug-in } u} + Ly$$

$$\dot{\hat{x}} = (A - LC)\hat{x} - Bk\hat{x} + BKx_{eq} + Bu_{eq} + Ly$$

let $\bar{x} = x_{eq} - \hat{x}$

$$\dot{\bar{x}} = \underbrace{\dot{x}_{eq}}_{=0} - \dot{\hat{x}} = -(A - LC - Bk)\hat{x} + -Ly - BKx_{eq} - Bu_{eq}$$

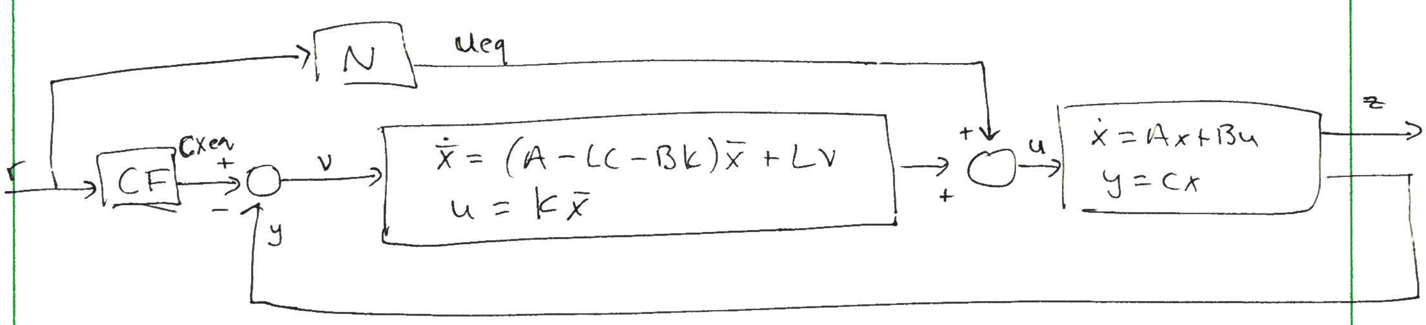
$$+ \underbrace{(Ax_{eq} + Bu_{eq})}_{=0}$$

→ Add in + subtract (LCx_{eq})

$$= -(A - LC - Bk)\hat{x} + (A - BK \cancel{x_{eq}} - LC)x_{eq} - L(y - Cx_{eq})$$

$$= (A - LC - Bk)\bar{x} - L(\underbrace{y - Cx_{eq}}_{Fr})$$

w/ $u = +K\bar{x} + u_{eq}$



$v = CFr - y$

$u = -k\bar{x} + u_{eq}$