~~12~~ Controllable Systems

	Cont.	Discrete
LTV System —	$\dot{x} = A(t)x + B(t)u$	$x(t+1) = A(t)x(t) + B(t)u(t)$

$t_1 > t_0 \geq 0$

Def. 12.1 : Reachable System — The system (CLTV or DLTV) ~~is~~ or the pair $(A(\cdot), B(\cdot))$ is reachable on $[t_0, t_1]$ if $\mathcal{R}[t_0, t_1] = \mathbb{R}^n$, i.e you can get to any state from the origin.

Def. 12.2 : Controllable System — The System (CLTV or DLTV) or pair $(A(\cdot), B(\cdot))$ is controllable on $[t_0, t_1]$ if $\mathcal{C}[t_0, t_1] = \mathbb{R}^n$, i.e you can go (transfer) from any state to the origin.

Thm 12.1 — For an LTI system, it is controllable iff $\operatorname{rank} \mathcal{C} = n$.

where $\mathcal{C} = [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$

~~Eigenvector Test~~ :

~~Eigenvector Test~~: A-Invariant

Def: Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is said to be A-invariant if for every $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$

Example: let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ w/ $A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ 0 \end{bmatrix}$

we have that $\mathcal{V} = \{ x = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \}$ is A-invariant

Since $v \in \mathcal{V} \Rightarrow \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} a\alpha \\ 0 \end{bmatrix} \in \mathcal{V}$

Properties: Given an $n \times n$ matrix A and a non-zero A-invariant subspace $\mathcal{V} \in \mathbb{R}^n$

P12.1 If the columns of $V \in \mathbb{R}^{n \times k}$ form a basis for $\mathcal{V}$, there exists a $k \times k$ $\bar{A}$ s.t.

$$\underset{(n\times n)(n\times k)}{AV} = \underset{(n\times k)(k\times k)}{V\bar{A}}$$

P.12.2 $\mathcal{V}$ contains at least 1 e-vector of A.

Proof (12.1): Let $V = [v_1 \dots v_k]$ form a basis for $\mathcal{V}$ ~~[illegible]~~ ~~[illegible]~~ singular values.

then $Av_i \in \mathcal{V}$ implies that $Av_i = \underbrace{\bar{a}_1 v_1 + \bar{a}_2 v_2 + \dots + \bar{a}_k v_k}$

$$= V\bar{a}_i$$

can write the basis vectors as a linear combination of cols of V mult. by another col vector $\bar{a}_i$

$$\Rightarrow A[v_1 \dots v_k] = [V\bar{a}_1 \; V\bar{a}_2 \dots V\bar{a}_k]$$

different col vectors.

$$\Rightarrow AV = V\bar{A}$$

Proof (12.2): Let $(\lambda, \bar{v})$ be the e-pair of $\bar{A}$,

Then $\bar{A}\bar{v} = \lambda \bar{v}$.

~~$\Rightarrow$~~ And $\quad AV\bar{v} = \overbrace{V\bar{A}}^{\text{from 12.1 property}}\bar{v} = V\underbrace{\bar{A}\bar{v}}_{\lambda\bar{v}} = V\lambda\bar{v}$

$\therefore$ $(\lambda, V\bar{v})$ is an e-pair of A and $\underbrace{V\bar{v}}\in\mathcal{V}$ — Because this is a linear combination of the cols of V and $V \in \mathcal{V}$.

E-vector test for Controllability:

The LTI system (cont. + discrete) (A, B) is controllable iff there is no eigenvector of A^T that is in the kernel of B^T.

Proof: (necessary cond.)

Assume (A,B)-controllable. To complete the proof by contradiction, assume that there is an e-vector of A^T that is in the kernel of B^T.

i.e. $\exists x \neq 0$ s.t. $A^T x = \lambda x$ and $B^T x = 0$

Then

$$\mathcal{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0$$

$\Rightarrow$ $\mathcal{C}$ is not full rank

$\Rightarrow$ (A,B) is not controllable

(Sufficient)

Show that $\ker C^T$ is A^T-invariant

Let $x \in \ker C^T$ i.e $C^T x = 0$ is $\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix} x = 0$

Need to show that $Ax \in \ker C^T$ or

$$C^T A^T x = \begin{bmatrix} B^T A^T \\ B^T (A^2)^T \\ \vdots \\ B^T (A^{n-1})^T \\ B^T (A^n)^T \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B^T (A^n)^T x \end{bmatrix}$$

But by the Cayley-Hamilton theorem –

$$(A^n)^T = \alpha_0 I + \alpha_1 A^T + \alpha_2 (A^2)^T + \cdots + \alpha_{n-1} (A^{n-1})^T$$

$$\Rightarrow B^T (A^n)^T x = 0$$

$$\Rightarrow Ax \in \ker C^T$$

From property 12.2 $\ker C^T$ contains at least one e-vector x of A^T i.e $C^T x = 0 \Rightarrow B^T x = 0$.

$\Rightarrow$ If no e-vector of A^T in $\ker B^T$ then there is no $x \neq 0$ s.t. $C^T x = 0$

$\Rightarrow$ rank $C = n$ $\Rightarrow$ Controllable

Thm 12.3 (Popov - Belevitch - Hautus - PBH test for Controllability) The LTI system ~~The LTI System (A,B - LTI)~~ (A, B) is controllable

iff $\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}$

Proof :

generic matrix

$$\operatorname{rank} W + \dim \ker W^T = n$$

$$\Rightarrow \dim \ker \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = n - \operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

So $\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \Rightarrow \dim \ker \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = 0$

$$\Rightarrow \ker \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = \{ x \in \mathbb{R}^n : A^T x = \lambda x + B^T x = 0 \} = \{0\}$$

$\Rightarrow$ No e-vectors of A^T are in $\ker B^T$

$\Rightarrow$ (A, B) is controllable

Example: PBH test for controllability

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Controllability matrix

$$C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad \left.\begin{matrix} \text{rank}(C) = 1 \\ \det(C) = 0 \end{matrix}\right\} \text{ not controllable}$$

PBH test:

$$\det(sI - A) = s^2 + 5s + 4 \qquad \lambda = -1, -4$$
$$= (s+4)(s+1)$$

For $\lambda = -1$

$$\text{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$$

$$\begin{bmatrix} A - (-1)I & B \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} -2 & 2 \\ 1 & -1 \end{matrix}}_{A+I} & \underbrace{\begin{matrix} 1 \\ 1 \end{matrix}}_{B} \end{bmatrix} \qquad \text{rank}([A+I \;\; B]) = 2$$

For $\lambda = -4$

$$\begin{bmatrix} A - (-4)I & B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \text{rank}([A+4I \;\; B]) = 1$$

$\longrightarrow$ The system is not controllable & -4 is the uncontrollable e-value.

Lyapunov test for controllability:

- Consider the LTI system $\dot{x} = Ax + Bu$

Thm 12.4: Assume A is a stability matrix (All e-values in LHP or Hurwitz). Then (A,B) is controllable iff $\exists$ a unique pos-def solution to $AW + WA^T = -BB^T$, w/ unique solution

$$W = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} \, d\tau = \lim_{(t_1 - t_0)\to\infty} W_R(t_0, t_1)$$

→ Note - even though the Gramian has limits to infinity it still provides info about finite time.

Proof: (1) Assume $AW + WA^T = -BB^T$ has pos-def soln W. Then show (A,B) is controllable.

Suppose there exists a unique positive-def soln to $AW + WA^T = -BB^T$. We will use the e-vector test by letting $x \neq 0$ be an e-vector of A^T. We need to show that $B^T x \neq 0$

$$x^* (AW + WA^T) x = -x^* BB^T x = -\| B^T x \|^2$$

x^* ← complex conjugate transpose

But, $x^*(AW + WA^T)x = (A^T\bar{x})^T W x + x^* W A^T x$

$= \bar{\lambda} x^* W x + \lambda x^* W x$

$= 2\,\mathrm{Re}\{\lambda\}\, x^* W x$

< 0 since $W > 0$ + A is a Stability matrix ($\mathrm{Re}(\lambda) < 0$)

$\therefore \|B^T x\| \neq 0 \Rightarrow (A,B)$ is controllable.

(Every e-value of A^T is not in the ker of B^T)

② Assume controllability $\Rightarrow AW + WA^T = -BB^T$ has pos.-def solution W.

Let $BB^T = Q$ + $A^T = \bar{A}$

$\Longrightarrow \bar{A}^T W + W\bar{A} = -Q$ (same format as Lyap eq. $A^T P + PA = -Q$ w/ sol. $P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$)

$$W = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\, d\tau \quad \text{is a solution of}$$

$$AW + WA^T = -BB^T$$

But $\Rightarrow BB^T$ is not guaranteed to be pos-def.
(only showed $W \geq 0$)

But $x^T W x = x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\, d\tau \right) x$

$\geq x^T \left(\int_0^1 e^{A\tau} BB^T e^{A^T\tau}\, d\tau \right) = x^T W_R(0,1)\, x > 0$

$\rightarrow$ Because (A,B) is controllable

$\Rightarrow W > 0$

- we can also show that <u>if</u> $\exists$ a pos-def solution P to $AP + PA^T = -BB^T$ & (A,B) is controllable,

<u>Then</u> A is a stability matrix

Proof — Let (λ, x) be an e-pair of A^T

Then (A,B) = controllable $\Rightarrow$ $B^T x \neq 0$

So $$x^*(AP + PA^T)x = -x^* BB^T x$$

$$\underbrace{2\,\mathrm{Re}\{\lambda\}\; x^* P x}_{\substack{>0 \\ \text{by assumption}}} = \underbrace{-\|B^T x\|^2}_{\substack{\text{non-zero by} \\ \text{controllability}}}$$

$\therefore$ $\mathrm{Re}\{\lambda\} < 0$

$\Rightarrow$ A^T = stability matrix

$\Rightarrow$ A = stability matrix (a matrix & its transpose have same set of e-values)

<u>Feedback Stabilization:</u>

If (A,B) is controllable

Then $(-\mu I - A, B)$ is controllable for every $\mu \in \mathbb{R}$

<u>Proof</u>: Since $A^T x = \lambda x \iff \cancel{(-\mu I + A)^T x = -\mu x - \mu}$

$$(-\mu I - A)^T x = -\mu x - \lambda x$$

$$= -(\mu + \lambda) x$$

$\Rightarrow$ This implies that A^T & $(-\mu I - A)^T$ have the same e-vector, and $B^T x \neq 0$

$\therefore$ We can choose μ to make $(-\mu I - A)$ a stability matrix.

$\longrightarrow$ w/ (A,B) = controllable & sufficiently large μ

Then $(-\mu I - A) W + W(-\mu I - A)^T = BB^T$

$$\Longrightarrow \quad AW + WA^T - BB^T = -2\mu W$$

Let $P = W^{-1} > 0$ & pre/post multiply by P

$$PA + A^T P - PBB^T P = -2\mu P$$

$$\Rightarrow \quad P(A - \tfrac{1}{2}BB^T P) + (A - \tfrac{1}{2}BB^T P)^T P = -2\mu P$$

$$\Rightarrow \quad P(A - BK) + (A - BK)^T P = -2\mu P$$

$$\text{w/ } K = \tfrac{1}{2}B^T P$$

$P > 0 \Rightarrow 2\mu P > 0$

$\therefore$ $(A - BK)$ is a stability matrix & $u = -Kx$ asymptotically stabilizes the system

Thm. 12.6

When (A,B) - controllable.

For every $\mu > 0$, it is possible to find a controller $u = -kx$ that places all of the e-values of $\dot{x} = (A - Bk)x$ on $\mathrm{Re}\{s\} \leq -\mu$

Note: Controllability is uneffected by state feedback.