16.1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{e-values} = (1, 0, -1)$$

Detectability need to check the ~~TR~~ pos. or zero e-values of A:

$\lambda = 0$

$$(sI - A) = -A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-A v_1 = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \phi \qquad \begin{matrix} -x_1 = 0 \\ x_3 = 0 \end{matrix}$$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$(I - A)v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \phi \Rightarrow \begin{matrix} x_2 = 0 \\ 2x_3 = 0 \end{matrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ check that these vectors don't reside in ker C

$$[c_1 \; c_2 \; c_3]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow c_2 \neq 0 \qquad [c_1 \; c_2 \; c_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow c_1 \neq 0$$

b) The entries in C_i which correspond to non-neg. e-values (neg. entries of diagonal A matrix) must be non-zero.

17.1 Is the realization minimal? i.e is it both controllable & observable?

~~→ Already in controller cannonical form so is controllable~~

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad \text{rank}(\mathcal{C}) = 2 \Rightarrow \text{controllable}$$

$$\mathcal{O} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{rank}(\mathcal{O}) = 1 \Rightarrow \text{not observable}$$

⇒ not a minimal realization

→ observer decomposition

$$\ker(\mathcal{O}^T) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -x_1 + x_2 = 0 \\ x_1 = x_2 \end{matrix} \qquad = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{unobs. space}$$

$$\text{Im}(\mathcal{O}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \bar{B} = TB = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \bar{D} = D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T$$

$$\left(A_o = -1, \quad B_o = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad C_o = 1, \quad D_o = \begin{bmatrix} 2 & 1 \end{bmatrix} \right)$$

17.2 Repeated e-values

a) $A = \begin{bmatrix} \lambda_1 & & \phi \\ & \lambda_2 & \\ \phi & & \ddots \\ & & & \lambda_n \end{bmatrix}$ $\quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ $\quad C = [c_1 \cdots c_n]$

$$\mathcal{C} = [B \;\; AB \;\; A^2B \cdots] = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ \vdots & \lambda_2 b_2 & \lambda_2^2 b_2 & & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & & \lambda_n^{n-1} b_n \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ \lambda_1 c_1 & \lambda_2 c_2 & \cdots & \lambda_n c_n \\ \vdots & & & \\ \lambda_1^{n-1} c_1 & & \cdots & \lambda_n^{n-1} c_n \end{bmatrix}$$

b) If there are repeated e-values

$(A, B) \rightarrow \mathcal{C} = \begin{bmatrix} \cancel{b_1 \;\; \lambda_1 b_1 \;\; \lambda_1^2 b_1 \cdots \lambda_1^{n-1} b_1} \\ \vdots \\ b_l \;\; \lambda_i b_l \;\; \lambda_i^2 b_l \cdots \lambda_i^{n-1} b_l \\ \cancel{b_m \;\; \lambda_i b_m \;\; \lambda_i^2 b_m} \cdots \lambda_i^{n-1} b_m \\ \vdots \end{bmatrix}$ $\quad$ (i = repeated e-value for entry l & m)

λ_1

$$\mathcal{C} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ \vdots & & & & \\ b_l & \lambda_i b_l & \lambda_i^2 b_l & \cdots & \lambda_i^{n-1} b_l \\ b_m & \lambda_i b_m & \lambda_i^2 b_m & \cdots & \lambda_i^{n-1} b_m \\ \vdots & & & & \\ b_n & \lambda_n b_n & & \cdots & \lambda_n^{n-1} b_n \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \cdots & & \lambda_1^{n-1} b_1 \\ b_l [1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1}] \\ b_m [1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1}] \\ \vdots & & & & \\ b_n & \lambda_n b_n & & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

these rows are linearly dependent

$$\mathcal{O} = \begin{bmatrix} c_1 & \cdots & c_l \begin{bmatrix} 1 \\ \lambda_i \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix} & c_m \begin{bmatrix} 1 \\ \lambda_i \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix} & \cdots & c_n \\ \lambda_1 c_1 & & & & & \lambda_n c_n \\ \vdots & & & & & \vdots \\ \lambda_1^{n-1} c_1 & & & & & \lambda_n^{n-1} c_n \end{bmatrix}$$

c) No - because ~~a~~ we just looked @ general diagonizable form + found it didn't work.

d) Yes - if in block Jordan form this will work -

i.e.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\longrightarrow$ controllable + observable, so this is a minimal realization.

21.1

$P^TA + A^TP =$ symmetric

- Verify that for every matrix P, the following matrix is symmetric:

$$\begin{bmatrix} -P^T & I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} -P^T & I \end{bmatrix} \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix}$$

$$= \begin{bmatrix} -P^TA + P^TBR^{-1}N - Q + NR^{-1}N^T & PBR^{-1}B^T - (A - BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix}$$

$$= -P^TA + P^TBR^{-1}N - Q + NR^{-1}N^T + PBR^{-1}B^TP - (A - BR^{-1}N^T)^TP$$

$$= -P^T\underbrace{(A - BR^{-1}N)}_{\bar{A}} - (A - BR^{-1}N^T)^TP - Q + NR^{-1}N^T + PBR^{-1}B^TP$$

$$= -\underbrace{(P^T\bar{A} + \bar{A}^TP)}_{\text{Symmetric because adding transpose}} \underbrace{- Q + NR^{-1}N^T + PBR^{-1}B^TP}_{\text{- Symmetric because Q \& R are symmetric.}}$$

$T + T^T =$ symmetric

21.4 Show detectability of (A, G) is eq. to detectability of (A, Q)
w/ $Q = G^T G$

Note $\ker G = \ker G^T G$

Assume (λ, x) - e-pair of A w/ λ having pos or zero real part.

(A, G) - detectable $\Rightarrow$ $Gx \neq 0$

If $Gx \neq 0$ then $Qx = G^T G x \neq 0$

Because $\ker G = \ker G^T G$

$\therefore$ ~~AB,~~ x is also not in the $\ker Q$ and (A, Q) - detectable.

```
% HW4_LQR

% put the ODE in state space form
A = [0 1 0; 0 0 1; -2.1 -.65 -.1];
B = [0; 0; -2];


% use matlab's lqr -- this is designed to input x'Qx + u'Ru + 2x'Nu so need
% to get our system into that form:

C = [1 0 0];
Q = C'*C;
% verify that [A,Q]-detectable
O = obsv(A,Q)
rank(O)

% verify [A,B]-controllable
C =ctrb(A,B);
rank(C)

% find the feedback gains

% pick an R
R = [1];
[K,S,E] = lqr(A,B,Q,R);

% state feedback u=-Kx: given by k1, k2, k3=
-K    = [.4  1.5874  1.2109]

% minimum value given by: x0'Px0 -- S is solution to ARE from Matlab
x0=[1; 1; 1];
x0'*S*x0

% equals 10.77
```

```
% A = [0 1; -1 -2]; B = [0; 1]; C = [0 1];
A = [0 1; -2.25 -6]; B = [0; 1]; C = [0 1];
x0 = [1; 1];

% P = [-.25 -.5];
P = [-.85 -.75];


% A = [0 1 0; 0 0 1; -.1 -.65 -2.1];
% B = [0; 0; -2];
% C = [1 0 0];
% x0 = [1; 1; 1];


% P = [-.25 -.5-i -.5+i];


L = place(A',C',P)


syms tao t
phi2 = expm(A*(t-tao));
phi = expm(A*(t));
u=3+.5*sin(.75*t);






u2 = 3 + .5*sin(.75*tao)
x_int = phi2*B*u2;
x2 = int(x_int,tao,0,t);

t1=[0:50];
x = phi*x0+x2;

x_real = subs(x,t1)

figure;plot(t1,x_real)
hold on;

% look at the error
x_hat_0 = [0; 0];
e_0=x_hat_0-x0;
AL=expm((A-L.'*C)*t);
e =AL*e_0;
e_real = subs(e,t1);

x_hat = e_real+x_real;
plot(t1,x_hat)
```