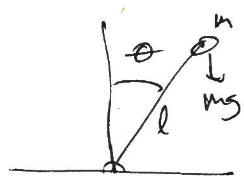
Objectives:
1. local linearization @ eq. point
2. Discrete time case
3. Feedback linearization (?)

- Most linear equations are only approximations for nonlinear ones.

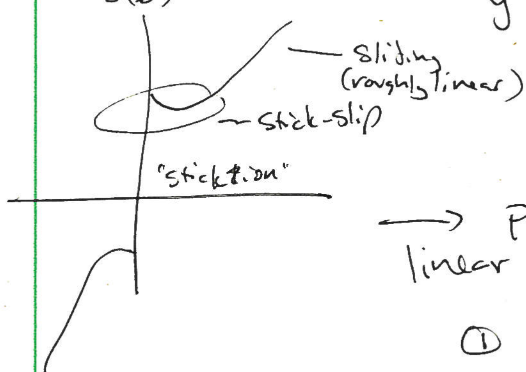
- Much easier to deal w/ linear systems — larger set of generalized tools we can apply.



$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + \tau$$

→ nonlinear eq. represented by:

$$\left.\begin{aligned} \dot{x} &= f(t,x,u) \\ y &= g(t,x,u) \end{aligned}\right\} \text{nonlinear state-space equations}$$



→ Previously we made the system linear by —

① Assuming θ = "small"

② Assuming always in the linear region for friction $b(\dot{\theta}) = b$

→ Want to develop a general method to turn any nonlinear diff eq. to a linear one.

Eq. Point of nonlinear system:

$$\dot{x} = f(t,x,u) \Rightarrow \dot{x} = 0 \Rightarrow f(x^{eq}, u^{eq}) = 0 \quad \forall t$$

$$\Rightarrow u(t) = u^{eq}, \quad x(t) = x^{eq}$$

(constants)

$$y(t) = g(t,x,u) \Rightarrow y = y^{eq} := g(x^{eq}, u^{eq})$$

→ We will perturb the system just slightly from this eq. point, then do a multi-variable taylor series expansion on the difference.

Reminder: Taylor Series

eq. point

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

where $x = a + \delta x$ + a is an eq. point

$\sin x$ around eq. point $x = 0$ (x is small)

$$f(x)\Big|_{x=0} \Rightarrow \sin(0) + \frac{\cos(0)}{1!}(x-0) + \frac{-\sin(0)}{2!}(x-0)^2 + \dots$$

$$= 0 + x - 0 - \frac{x^3}{3!} + \dots$$

$$= x + \text{remainder}$$

$\sin x \approx x$

Multi-variable Taylor exp of $h(x,u)$ (generic function) around an eq. point (x^{eq}, u^{eq})

evaluated @ eq. points

$$h(x,u) = h(x^{eq}, u^{eq}) + \frac{\partial h}{\partial x}(x^{eq}, u^{eq})\,\delta x + \frac{\partial h}{\partial u}(x^{eq}, u^{eq})\,\delta u$$

$= h(x^{eq}+\delta x, u^{eq}+\delta u) =$

$$+ \frac{1}{2!}\left(\frac{\partial^2 h}{\partial x^2}(x^{eq},u^{eq})\right)\delta x^2 + \frac{1}{2!}\left(\frac{\partial^2 h}{\partial u^2}(x^{eq},u^{eq})\right)\delta u^2 + \frac{1}{2!}\left(\frac{\partial^2 h}{\partial u\,\partial x}(x^{eq},u^{eq})\right)\delta u\,\delta x$$

$$+ \dots$$

$$= h(x^{eq},u^{eq}) + \frac{\partial h}{\partial x}(x^{eq},u^{eq})\,\delta x + \frac{\partial h}{\partial u}(x^{eq},u^{eq})\,\delta u + O(\|\delta x\|^2) + O(\|\delta u\|^2)$$

Let input : $u(t) = u^{eq} + \underbrace{\delta u(t)}_{\text{close, but not equal to } u^{eq}}$

$$x(t) = x^{eq} + \delta x(t)$$

$$\delta x(t) = x(t) - x^{eq}$$

$$\delta y(t) = y(t) - y^{eq}$$ ← If input signal + states are slightly perturbed from equilibrium – so will the output.

$$\delta y = g(x,u) - y^{eq} = g(x^{eq} + \delta x, u^{eq} + \delta u) - g(x^{eq}, u^{eq})$$

$$g(x^{eq} + \delta x, u^{eq} + \delta u) = g(x^{eq}, u^{eq}) + \frac{\partial g}{\partial x}(x^{eq}, u^{eq})\delta x + \frac{\partial g}{\partial u}(x^{eq}, u^{eq})\delta u$$
$$+ O(\|\delta x\|^2) + O(\|\delta u\|)^2$$

~~Similary~~

$$\delta y = \frac{\partial g}{\partial x}(x^{eq}, u^{eq})\delta x + \frac{\partial g}{\partial u}(x^{eq}, u^{eq})\delta u$$

→ Similarly: $\dot{\delta x} = \dot{x} = f(x,u) = f(x^{eq} + \delta x, u^{eq} + \delta u)$ ⟶ expand out $f(\cdot)$ exactly the same.

$$\dot{\delta x} = \underbrace{\frac{\partial f}{\partial x}(x^{eq}, u^{eq})}_{A}\delta x + \underbrace{\frac{\partial f}{\partial u}(x^{eq}, u^{eq})}_{B}\delta u$$

$$A = \frac{\partial f}{\partial x}(x^{eq}, u^{eq}) \qquad C = \frac{\partial g}{\partial x}(x^{eq}, u^{eq})$$

$$B = \frac{\partial f}{\partial u}(x^{eq}, u^{eq}) \qquad D = \frac{\partial g}{\partial u}(x^{eq}, u^{eq})$$

$$\begin{aligned} \dot{\delta x} &= A\,\delta x + B\,\delta u \\ \delta y &= C\,\delta x + D\,\delta u \end{aligned} \quad \xRightarrow{\text{change of variables}} \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Nonlinear equation for pendulum:

$$\dot{x}_1 = x_2 = f_1 \qquad y = x_1$$

$$\dot{x}_2 = \frac{g}{\ell}\sin x_1 - \frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}u = f_2$$

$$f(x,u) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}(x^{eq}, u^{eq}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

(Evaluated @ eq. point x^{eq}, u^{eq})

$$= \begin{bmatrix} 0 & 1 \\ +\frac{g}{\ell}\cos x_1 & -\frac{b}{m\ell^2} \end{bmatrix}_{\text{eq. point}}$$

– if eq. point is $x_1 = 0, x_2 = 0$ $\Longrightarrow$

$$\begin{bmatrix} 0 & 1 \\ +\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}$$

→ same as before

$$B = \frac{\partial f}{\partial u}(x^{eq}, u^{eq}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

$$C = \frac{\partial g}{\partial x}(x^{eq}, u^{eq}) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = \frac{\partial g}{\partial u}(x^{eq}, u^{eq}) = \begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

→ Generalized way of turning nonlinear systems into linear ones.

Consider the system:

$$\cancel{\dot{x}_1 = -x_1 + \frac{2x_2}{(1+x_2^2)}}$$

$$\cancel{\dot{x}_2 = -x_2}$$

$$\dot{x}_1 = x_2 \qquad\qquad y = x_1$$

$$\dot{x}_2 = \tau - x_2 - x_1^2 - 2x_1x_2 \qquad\qquad u = \tau$$

eq. point $\tau = u = 0, \quad x_2 = 0, \ x_1 = 0$

– Put into state space form — $\dot{x} = Ax + Bu$

$y = Cx + Du$

$$A = \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -2x_1 - 2x_2 & -1 - 2x_1 \end{bmatrix}_{\substack{x_1 = 0 \\ x_2 = 0}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

(Sanner #5)

Note:

1. There is an unlimited # of linear approximations to the original NL dynamics
=> one for every eq. point.

2. Different linearized models can vary tremendously in their properties
(some stable, some unstable, freq., damping of modes, etc.)

3. "Instability" in the linearized model doesn't neccesarily mean "physical" instability (divergence)
→ remainder $O(\|\delta x\|^2) + O(\|\delta u\|^2)$
=> $r(t)$ becomes non-negligable. the nonlinear higher order terms dominates the dynamics

4. For a linearized system to ~~be~~ stay representative of the nonlinear system then we need to ~~be~~ stay "close enough" the eq. point.
Will talk more @ what that means to be "close enough" in a subsequent class.

Discrete time

linear State model: (Difference equation rather than a differential eq.)

$$x(t+1) = A(t)x(t) + B(t)u(t) \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k \qquad t \in \mathbb{N} = \text{integers or "counters"}$$

$$y(t) = C(t)x(t) + D(t)u(t) \qquad y \in \mathbb{R}^m$$

For LTI (can drop the specific time step)

$$x^+ = Ax + Bu$$

$$y = Cx + Du$$

Given $\dot{x} = Ax + Bu$

$y = Cx + Du$

use $\dot{x} \cong \dfrac{x(k+1) - x(k)}{T}$

$$\dot{x} \cong \frac{x(k+1) - x(k)}{T} = Ax(k) + Bu(k)$$

$$\Rightarrow x(k+1) = \underbrace{(I + AT)}_{A_d} x(k) + \underbrace{TB}_{B_d} u(k)$$

$$y = \underbrace{C}_{C_d} x(k) + \underbrace{D}_{D_d} u(k)$$

Eq. Point: Continuous $\dot{x} = f(x^{eq}, u^{eq}) = 0$

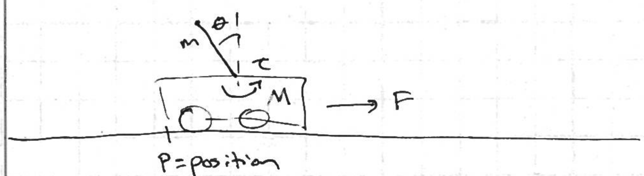
Discrete $x^+ = x^{eq}$ or $x(t+1) = x(t)$

$\rightarrow$ not changing over time.

$$\delta x^+ = A\,\delta x + B\,\delta u \quad \text{w/} \quad A := \frac{\partial f(x^{eq}, u^{eq})}{\partial x} \qquad B := \frac{\partial f(x^{eq}, u^{eq})}{\partial u}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C := \frac{\partial g(x^{eq}, u^{eq})}{\partial x} \qquad D := \frac{\partial g(x^{eq}, u^{eq})}{\partial u}$$

Example: Pendulum on a cart



$$(M+m)\ddot{p} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = F$$

$$(m\ell\cos\theta)\ddot{p} + m\ell^2\ddot{\theta} - mg\ell\sin\theta = \tau$$

$\Rightarrow$ $x_1 = p$, $x_3 = \theta$, $x_2 = \dot{p}$, $x_4 = \dot{\theta}$ — 4 state variables

→ 2 outputs position of the cart & orientation of pendulum

$$y = \begin{bmatrix} p \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

→ 2 control forces $u_1 = F$
$u_2 = \tau$

→ homework problem (?)