23.3

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = (A - LC - BK)\hat{x} + Ly$$

$$u = -K\hat{x}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = \underbrace{(Ax + Bu + \bar{B}d)}_{\dot{x}} - ((A - LC \overbrace{- BK)\hat{x}}^{Bu} + Ly)$$

$$= Ax - A\hat{x} + LC\hat{x} - Ly + \bar{B}d$$

$$y = Cx + n$$

$$\dot{e} = \quad = (A - LC)e - Ln + \bar{B}d$$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix}$$

$$u = -K\hat{x} = -K(x - e)$$

$$\dot{x} = (Ax + Bu + \bar{B}d) = Ax - BKx + BKe + \bar{B}d$$

$$z = Gx + Hu = Gx + H(-K(x - e)) = Gx - HKx + HKe$$

$$z = \begin{bmatrix} G - HK & +HK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

→ Since $(A - BK)$ + $(A - LC)$ are Hurwitz, the matrix

$$A = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$ will also be Hurwitz. Made up of diag. elements.

23.4 —

$$\begin{bmatrix} -A & B \\ -G & H \end{bmatrix} \begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \qquad (23.17)$$

$$\begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = P(0)^T (P(0) P(0)^T)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

w/ $P(s) = \begin{bmatrix} sI - A & B \\ -G & H \end{bmatrix} \Rightarrow P(0) = \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}$

$\rightarrow$ plug in $P(0)^T (P(0) P(0)^T)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$ for sol. $\begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix}$ to 23.17

$$\begin{bmatrix} -A & B \\ -G & H \end{bmatrix} = P(0) \Rightarrow \underbrace{(P(0) P(0)^T)(P(0) P(0)^T)^{-1}}_{= I} \begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

∴ $P(0)^T (P(0) P(0)^T)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$ is a solution.

23.5

$$\dot{\bar{x}} = (A - LC - BK)\bar{x} - L(y - Cx_{eq}) \qquad u = K\bar{x} + u_{eq}$$

$$\Rightarrow \quad \dot{\hat{x}} = (A - LC)\hat{x} - BK\hat{x} + BKx_{eq} + Bu_{eq} + Ly$$

$$u = -K(\hat{x} - x_{eq}) + u_{eq}$$

$$e = x - \hat{x}$$

$$\tilde{x} = x - x_{eq}$$

$$\dot{\tilde{x}} = A(x - x_{eq}) + B(\underset{\substack{\uparrow \\ \text{plug-in for } u}}{u} - u_{eq}) + Ax_{eq} + Bu_{eq}$$

$$= A(x - x_{eq}) + B(-K\underset{\substack{\uparrow \\ \hat{x} = x - e}}{\hat{x}} + Kx_{eq} + \cancel{u_{eq}} - \cancel{u_{eq}}) + Ax_{eq} + Bu_{eq}$$

$$= \cancel{Ax} \; A\tilde{x} + B(-Kx + Ke + Kx_{eq}) + Ax_{eq} + Bu_{eq}$$

$$= (A - BK)\tilde{x} + BKe + \underbrace{Ax_{eq} + Bu_{eq}}_{=0 = \dot{x}_{eq}}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + B\underset{\substack{\uparrow \\ \text{plug-in for } u}}{u} - (A - LC - BK)\hat{x} - BKx_{eq} - Bu_{eq} - Ly$$

$$= Ax + B(-K\hat{x} + K\cancel{x_{eq}} + \cancel{u_{eq}}) - \cancel{Ax} \; (A - LC - BK)\hat{x} - B\cancel{K}x_{eq} - B\cancel{u_{eq}} - Ly$$

$$= Ax - \cancel{BK}\hat{x} - A\hat{x} + LC\hat{x} + \cancel{BK}\hat{x} - Ly$$

$$= (A - LC)e$$

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ e \end{bmatrix}$$

→ Again $(A - BK)$ is designed to be Hurwitz, as is $(A - LC)$. So the combined A matrix will be made up of those poles (since diagonal) & will also be Hurwitz.

23.6 Show that For single controlled output ($\ell = 1$) we can take $u_{eq} = 0$ in (23.17) when matrix A has a e-value @ origin and its mode is observable through z.

Let (λ, v) – e-pair of A $\longrightarrow$ w/ $Gv \neq 0$

(23.17) $$\begin{bmatrix} -A & B \\ -G & H \end{bmatrix} \begin{bmatrix} -x_{eq} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$\left.\begin{aligned} A x_{eq} &= 0 \\ + G x_{eq} &= r \end{aligned}\right] \longrightarrow \text{let } x_{eq} = \frac{r}{Gv} v$$

e-vector

$$\text{Then } A\left(\frac{r}{Gv} v\right) = \frac{r}{G} Av = 0$$

~~let $x_{eq} = \frac{}{(Gv)}$ $(1\times n)(n\times 1)$~~

$\underset{(1\times n)(n\times 1)}{Gv} = (1\times 1)$ because $\ell = 1$ + r = single output

~~x_e~~ $A x_{eq} = 0 \longrightarrow A \frac{rv}{Gv} = \underbrace{\frac{r}{Gv}}_{\text{Scalars}} Av = 0$

$$G x_{eq} = G\left(\frac{r}{Gv}\right) v = r$$

```
A = [2 0 0; 0 -1 0; 0 0 -1];
B = [1 0; 1 0; 0 1];
C = [1 0 2; 0 -1 0];
D = [1 0; 1 0];

% this is a minimal realization, so invariant zeros should equal
% transmission zeros.  Meaning that tzero will give the right answer to
% both.
rank(ctrb(A,B))
rank(obsv(A,C))

msys = minreal(ss(A,B,C,D)) % will have an invariant/transmission zero
                            % at s=0 and s=2

tzero(ss(A,B,C,D))

% look at the Rosenbrock matrix and the TF to verify all of this.
syms s
P = [(s*eye(3)-A) B; -C D]
G = C*inv(s*eye(3)-A)*B+D % poles will be s=2, w/ multiplicity 1, and
                          % s=-1, w/ multiplicity 2


det(P)  % as expected -- zeros at 0, 2.


det(G)  % shows we will also have a zero at s=0, it is also clear from the
        % TF that you will lose rank as s->infinity.  What is not as clear
        % is that there is also a zero at 2, but we know because this is a
        % minimal realization that there will be one.

% To show the zero at 2:
% Let
u_s = [ -2*(s-2); (s+1)*(s-1)]
H=simplify(G*u_s)
% take the limit as s->2 --> H=[0;0]
subs(H,s,2)
```

```
syms s
G = C*inv(s*eye(3)-A)*B+D
P = [(s*eye(3)-A) B; -C D]

P1=subs(P,s,zi)
xu0= null(P1)



x0=-eval(xu0(1:3)) % null command returns something other than double,
u0=eval(xu0(4:5))  % this needs to be fixed for lsim.

% set the input
t = linspace(0,5);
u = exp(zi*t);
in = (u0*u);

% Note that this exponentially diverges from zero at about 10 seconds.
% This is mathmatical inaccuracies since we can show that by computing y
% directly we get identically zero.
[y,x] = lsim(ss(A,B,C,0),in,t,x0);
plot(t,y)
axis([0 5 -1 1])
xlabel('time, seconds')
title('response to x_{0} and u(t)=u_{0}e^{t}')

% a second way of showing will get zero output...
syms t tao
u=u0*exp(zi*tao);
x_t = expm(A*t)*x0+int(expm(A*(t-tao))*B*u,tao,0,t);
y_t = C*expm(A*t)*x0+int(C*expm(A*(t-tao))*B*u,tao,0,t)
simplify(y_t)


ans =

     3
```

```
   x1   1   0
   x2   1   0
   x3   0   1

  C =
       x1  x2  x3
   y1   1   0   2
   y2   0  -1   0

  D =
       u1  u2
   y1   1   0
   y2   1   0

Continuous-time state-space model.


ans =

   -0.0000
    2.0000


P =

[ s - 2,     0,     0, 1, 0]
[     0, s + 1,     0, 1, 0]
[     0,     0, s + 1, 0, 1]
[    -1,     0,    -2, 1, 0]
[     0,     1,     0, 1, 0]


G =

[ 1/(s - 2) + 1, 2/(s + 1)]
[ 1 - 1/(s + 1),         0]


ans =
```

```
 (2*s - 4)*(1/(s + 1) - 1)


ans =

 0
 0


zi =

    1.0000


G =

[ 2/(s + 1) - 2/(s + 2), 4/(s + 2) - 2/(s + 1)]
[ 2/(s + 1) - 4/(s + 2), 2/(s + 2) - 2/(s + 1)]


P =

[ s + 1,     0,     0,  2, -2]
[     0, s + 2,     0, -2,  4]
[     0,     0, s + 2, -4,  2]
[    -1,    -1,     0,  0,  0]
[    -1,     0,    -1,  0,  0]


P1 =

[  2,  0,  0,  2, -2]
[  0,  3,  0, -2,  4]
[  0,  0,  3, -4,  2]
[ -1, -1,  0,  0,  0]
[ -1,  0, -1,  0,  0]


xu0 =
```

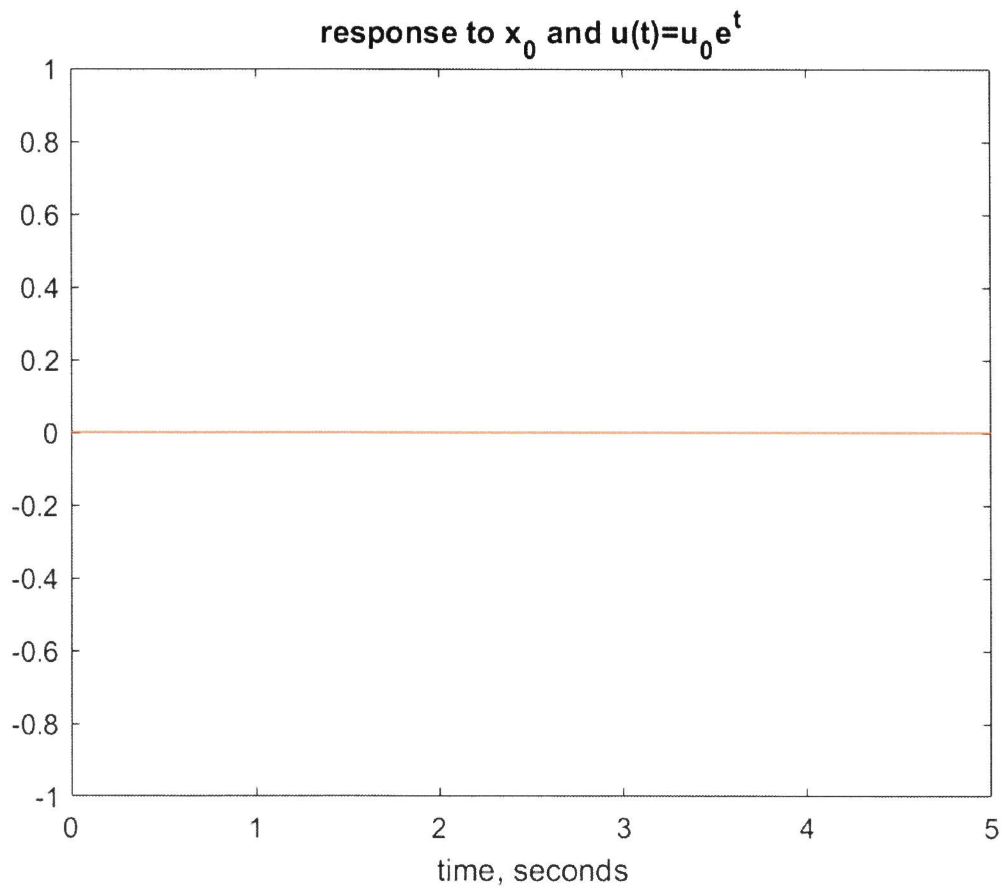
```
     1


y_t =

 2*exp(-2*t) - 2*exp(-t) + 2*exp(-2*t)*(exp(t) - 1)
 2*exp(-2*t) - 2*exp(-t) + 2*exp(-2*t)*(exp(t) - 1)


ans =

 0
 0
```



Problem # 3

$\hat{x}(t) = e^{\alpha t} x(t)$

$\hat{u}(t) = e^{\alpha t} u(t)$

$\dot{\hat{x}} = (A + \alpha I)\hat{x} + B\hat{u}$

$$\dot{\hat{x}} = \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t)$$
$$= \alpha e^{\alpha t} x(t) + e^{\alpha t}(Ax + Bu)$$
$$= (A + \alpha I)\hat{x}(t) + B\hat{u}(t)$$

$$J_{LQR} = \int_0^\infty (\hat{x}^T Q \hat{x} + \hat{u}^T R \hat{u})\,dt \qquad N = 0$$

$$H(\hat{x};\hat{u}) = -\int_0^\infty \frac{d}{dt}(\hat{x}^T P \hat{x})\,dt$$

$$= -\int_0^\infty (\dot{\hat{x}}^T P \hat{x} + \hat{x}^T P \dot{\hat{x}})\,dt = \int ((A+\alpha I)\hat{x} + Bu)P\hat{x} + \hat{x}^T P((A+I\alpha)\hat{x} + B\hat{u}))\,dt$$

$$J_{LQR} = H(\hat{x};\hat{u}) + \int \hat{x}^T Q \hat{x} + \hat{u}^T R \hat{u} + ((A+I\alpha)\hat{x} + B\hat{u})^T P \hat{x} + \hat{x}^T P((A+\alpha I)\hat{x} + B\hat{u}))\,dt$$

$$= H(\hat{x};\hat{u}) + \int_0^\infty (\hat{x}^T \underbrace{(Q + (\alpha I + A^T)P + P(\alpha I + A))}\hat{x} + \hat{u}^T R \hat{u} + 2\hat{u}^T B^T P \hat{x})\,dt$$

→ complete the sq.

$$= H(\hat{x};\hat{u}) + \int_0^\infty \Big(\underbrace{\hat{x}^T(\quad)\hat{x} - \hat{x}^T P B R^{-1} B^T P \hat{x}}_{\text{- set this equal to zero}} + (\hat{u}^T + \hat{x}^T K^T) R (u + K\hat{x})\Big)\,dt$$

$$Q + (A + \alpha I)^T P + P(A + \alpha I) - PBR^{-1}B^T P = 0 \qquad \text{(ARE)}$$

w/ $\hat{A} = (A + I\alpha)$

Control law: $\hat{u}^T = -K\hat{x} \implies$ ~~$u(t)$~~ $e^{\alpha t} u(t) = -K e^{\alpha t} x(t)$

w/ $K = R^{-1}B^T P$

$\boxed{u(t) = -K x(t)}$

→ Same control law

b) Minimum possible value: $\hat{x}_0^T P \hat{x} = e^{\alpha(0)} x^T(0) P e^{\alpha(0)} x(0) = x^T(0) P x(0)$

w/ P solution to modified ARE eq. which will depend on alpha

c) The $(A+\alpha I)$ has the effect of shifting poles further to LHP. All e-values w/ have real parts less than $-\alpha$ (instead of less than 0).

#4 Show

$$\det\begin{bmatrix} z_i - A & B \\ -C & D \end{bmatrix} = \det(Iz_i - A)\,\det(G(z_i))$$

$$\det\begin{bmatrix} z_i - A & B \\ -C & D \end{bmatrix} = \det\left(\begin{bmatrix} z_iI - A & B \\ -C & D \end{bmatrix}\underbrace{\begin{bmatrix} I & -(z_iI - A)^{-1}B \\ 0 & I \end{bmatrix}}_{\substack{\text{det of this matrix} \\ = \det(I) = 1}}\right)$$

$$= \det\left(\begin{bmatrix} z_iI - A & 0 \\ -C & C(z_i - A)^{-1}B + D \end{bmatrix}\right)$$

$\longrightarrow$ equal to det of diag because of upper-right zero

$$= \det(z_iI - A)\det(C(z_i - A)^{-1}B + D)$$

$$= \det(z_iI - A)\det(G(z_i))$$