Certainty Equivalence:

* LQR is state feedback:

$$u(t) = -Kx$$

+ requires full knowledge of the states —

* Instead use observer to estimate the states. Use estimate in control: Certainty Equivalence

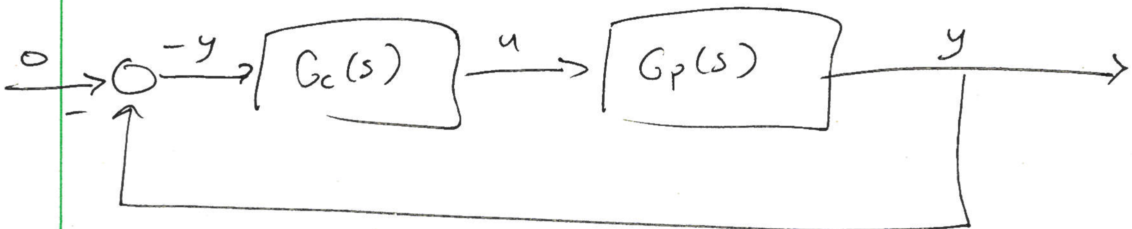
Model Based Compensator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \quad \leftarrow \text{model-based observer}$$

$$u = -K\hat{x} \quad \leftarrow \text{certainty eq. feedback}$$

Separation Principle: Closed loop e-values of system $\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix}$

$$\underbrace{\text{eig}(A-BK)}_{\substack{\text{Closed-loop} \\ \text{State feedback} \\ \text{poles}}} \cup \underbrace{\text{eig}(A-LC)}_{\text{Observer poles}}$$



$$\left.\begin{aligned} \dot{\hat{x}} &= (A - LC - BK)\hat{x} - L(-y) \\ u &= -Kx \end{aligned}\right\} \quad \begin{aligned} G_c(s) &= (-K)(sI - A + BK + LC)^{-1}(-L) \\ &= K(sI - A + LC + BK)^{-1}L \end{aligned}$$

- Can we find an optimal gain for L like we did for k?
- Used copy of system (A, B, C, D) matrices which are rarely exact.

Generally system closer to:

$$\dot{x} = Ax + Bu + d \leftarrow \text{external disturbances}$$

$$y = Cx + n \leftarrow \text{sensor errors}$$

w/ n & d random

<u>Error Dynamics</u>:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (Ax + Bu + d) - (A\hat{x} + Bu + L(Cx + n - C\hat{x}))$$

$$= Ax - A\hat{x} + LCx - LC\hat{x} + d - Ln$$

$$= (A + LC)e + \underbrace{d - Ln}_{\text{error no longer converges to zero.}} \leftarrow \text{randomness present in error}$$

- Can we squeeze down the error as small as possible?
- What L will minimize the effects due to randomness?

Expected value:

$$E\{h(x)\} = \int_{-\infty}^{\infty} h(x) f(x)\, dx = m_H$$

variance:

$$\operatorname{var}\{x\} = \sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 f(x)\, dx = E\{(x-m_x)^2\}$$

σ_x = standard deviation

Covariance Matrix: (vector)

$$V(t) = E\{z(t)\, z^T(t)\}$$

Correlation Matrix: (vector)

$$R(t,\tau) = E\{z(t)\, z^T(\tau)\}$$

Minimize the error variance:

$$\lim_{t\to\infty} \sum_{k=1}^{n} \text{var}\{e_k(t)\} = \lim_{t\to\infty} \sum_{k=1}^{n} \sigma_k^2(t)$$

For a normal dist:

$$|e_k(t)| \le 3\sigma_k \quad 99.7\% \text{ of the time}$$

- Quantify size of disturbances:

$\longleftarrow$ Delta function

$$E\{n(t)n^T(\tau)\} = N\,\delta(t-\tau) \qquad n \sim \mathcal{N}(0, N)$$

$$E\{d(t)d^T(\tau)\} = D\,\delta(t-\tau) \qquad d \sim \mathcal{N}(0, D)$$

$$N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \phi \\ & & \ddots & \\ \phi & & & N_m \end{bmatrix}$$

$\longrightarrow$ How big the noise is in the sensors

$$D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \phi \\ \phi & & \ddots & \\ & & & D_k \end{bmatrix}$$

$\longrightarrow$ How big disturbances are in the actual system

$$|y_i(t) - \bar{y}_i(t)| \le 3\sqrt{N_i} \quad 99.7\% \text{ of time}$$

$\uparrow$ sensor reading

$\uparrow$ actual output

Matrix Background:

Trace of a matrix:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$ $\longrightarrow$ sum of the diag elements of matrix.

Properties:

1. $tr\{AB\} = tr\{BA\}$ — if the dimensions are compatible, i.e $A \in \mathbb{R}^{n \times m}$ + $B \in \mathbb{R}^{m \times n}$

2. $tr\{A\} = \sum_{i=1}^{n} \lambda_i$ — If $A \in \mathbb{R}^{n \times n}$

3. If $P \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ then $x^T P x = tr\{P x x^T\}$

$$x^T A x = \text{scalar} \Rightarrow tr\{x^T A x\} \underset{\text{by \#1}}{=} tr\{A x x^T\}$$

Stochastic - Deterministic Dualism:

- Show that the two problems are dualistic:

$$V_d(x_0) = \int_0^\infty x^T Q x \, dt$$

$$\dot{x} = Ax; \quad x(0) = x_0$$

+ A-stable

$$V_s(x) = \lim_{t \to \infty} E\{x^T C C^T x\}$$

$$\dot{x} = Ax + \dot{\omega}$$

$$\cancel{dx = Ax\,dt + dw} \quad x(0) = x_0$$

A-stable +

$$\cancel{E\{dw\,dw^T\} = w(t)\,dt}$$

$$E\{\dot{\omega}(t)\dot{\omega}^T(\tau)\} = W\delta(t-\tau)$$

From LQR:

$$V_d(x_0) = x_0^T P x_0 = tr\{P x_0 x_0^T\}$$

Why?

$$V_d(x_0) = \int_0^\infty x^T(t) Q x(t)\, dt \qquad w/ \; x = e^{At}x_0$$

$$V_d(x_0) = \int_0^\infty x_0^T e^{A^T t} Q e^{At} x_0 \, dt$$

$$= x_0^T \underbrace{\left(\int_0^\infty e^{A^T t} Q e^{At}\, dt\right)}_{= P \quad (\text{Lecture } \#8)} x_0$$

Plug into the Lyap eq:

$$A^T P + PA = \int_0^\infty A^T e^{A^T t} Q e^{At}\, dt + \int_0^\infty e^{A^T t} Q e^{At} A\, dt$$

$$= \int_0^\infty \frac{d}{dt}\left[e^{A^T t} Q e^{At} \right] dt$$

$$= e^{A^T t} Q e^{At} \Big|_0^\infty = \underbrace{\lim_{t \to \infty} e^{A^T t} Q e^{At}}_{\substack{A\text{-stable} \\ \to 0 \text{ as } t \to \infty}} - e^{A \cdot 0} Q e^{A \cdot 0}$$

$$= -Q$$

$\therefore$ P satisfies $A^T P + PA = -Q$

$\Rightarrow$ solution to —

$$V_d(x_0) = \int_0^\infty x^T Q x \, dt \Rightarrow x_0^T P x_0 = tr\{ P x_0 x_0^T \}$$

when $\dot{x} = Ax$; $x(0) = x_0$ + A-stable

w/ P sol. to lyap eq.

$A^T P + PA = -Q$

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look @ the stochastic problem:

$$V_s(x_0) = \lim_{t \to \infty} E\{ x^T(t) C C^T x(t) \}$$

when $\dot{x} = Ax + \dot{w}$; $x(0) = x_0$
+ A-stable

$x(t)$ satisfies —

playing part of $u(\tau)$ function.

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} \dot{w}(\tau) d\tau$$

$$V_s(x_0) = \lim_{t \to \infty} E\Big\{ \underbrace{\left[x_0^T e^{A^T t} + \int_0^t \dot{w}^T e^{A^T(t-\tau)} d\tau \right]}_{x^T} C C^T \underbrace{\left[e^{At} x_0 + \int_0^t e^{A(t-\tau)} \dot{w}(\tau) d\tau \right]}_{x} \Big\}$$

$$= \lim_{t \to \infty} E \Big\{ \Big[\underbrace{x_0^T e^{A^T t} C C^T e^{At} x_0}_{A\text{-stable } = 0} + \underbrace{2 \int_0^t x_0^T \cancel{e^{A(t-\tau)}}\, e^{A^T t} C C^T e^{A(t-\tau)} \dot{w}(\tau) d\tau}_{= 0 \text{ because } \dot{w}(\tau) = \text{zero mean}}$$

$$+\int_0^t\int_0^t \dot{w}^T(\tau_1)\, e^{A^T(t-\tau_1)} C C^T e^{A(t-\tau_2)} \dot{w}(\tau_2)\, d\tau_1 d\tau_2 \Big\}$$

$\longrightarrow$ this term is a 1×1 value

$$V_s(x_0) = \lim_{t\to\infty} E\left\{ tr\left[\int_0^t\int_0^t e^{A^T(t-\tau_1)} C C^T e^{A(t-\tau_2)} \dot{w}(\tau_1)\dot{w}^T(\tau_2)\, d\tau_1 d\tau_2\right]\right\}$$

$$= \lim_{t\to\infty} tr\left[\int_0^t\int_0^t e^{A^T(t-\tau_1)} C C^T e^{A(t-\tau_2)} \underbrace{E\{\dot{w}(\tau_1)\dot{w}^T(\tau_2)\}}_{w\,\delta(\tau_1-\tau_2)} d\tau_1 d\tau_2\right]$$

$$= \lim_{t\to\infty} tr\left[\int_0^t \underbrace{e^{A^T(t-\tau)} C C^T}_{\substack{\text{move to end}\\ \text{by trace property}}} e^{A(t-\tau)}\, w\, d\tau\right]$$

$$= \lim_{t\to\infty} tr\left[\int_0^t e^{A(t-\tau)}\, w\, e^{A^T(t-\tau)} \underbrace{C C^T}_{\substack{\text{not function}\\ \text{of }\tau}} d\tau\right]$$

push the limit into the integral

$$= tr\left[\underbrace{\left(\int_0^\infty e^{A\tau}\, w\, e^{A^T\tau}\, d\tau\right)}_{\triangleq S} C C^T\right] \qquad w/\ AS + SA^T = -w$$

In Summary: if A-stable & $x(0) = x_0$

$$V_d(x_0) = \int_0^\infty x^T Q x \, dt$$

$$\dot{x} = Ax$$

is solved by

$$V_d(x_0) = \mathrm{tr}\{P x_0 x_0^T\}$$

where P satisfies

$$A^T P + PA = -Q$$

$$V_s(x_0) = \lim_{t \to \infty} E\{x^T(t) C C^T x(t)\}$$

$$\dot{x} = Ax + w$$

$$w \sim N(0, W)$$

is solved by

$$V_s(x_0) = \mathrm{tr}\{S C C^T\}$$

where S solves

$$AS + SA^T = -W$$

Dual Quantities

$$x_0 \longleftrightarrow C$$

$$Q \longleftrightarrow W$$

$$A \longleftrightarrow A^T$$

i.e. if we replace (x_0, Q, A) in solution of deterministic problem w/ (C, W, A^T) then we have solved the stochastic problem.

→ Use this to derive an optimal observer.