Controllibility Matrix (LTI):

For LTI system: $\dot{x} = Ax + Bu$

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

→ Reachability Grammian

τ

$$W_R(t_0,t_1) = \int_{t_0}^{t_1} \Phi(\underline{t_1},\tau)B(\tau)B^T(\tau)\Phi^T(\underline{t_1},\tau)\,d\tau$$

$$= \int_{\tau=t_0}^{t_1} e^{A(t_1-\tau)} BB^T e^{A^T(t_1-\tau)}\,d\tau$$

$$\sigma = t_1 - \tau \Rightarrow d\sigma = -d\tau$$

$$= \int_{t_1-t_0}^{0} e^{A\sigma}BB^Te^{A^T\sigma}(-d\sigma)$$

$$= \int_0^{t_1-t_0} e^{A\sigma}BB^Te^{A^T\sigma}\,d\sigma$$

→ Controllability Grammian

$$W_c(t_0,t_1) = \int_{\tau=t_0}^{t_1} \Phi(\underline{t_0},\tau)BB^T\Phi^T(\underline{t_0},\tau)\,d\tau$$

$$= \int_{\tau=t_0}^{t_1} e^{A(t_0-\tau)}BB^Te^{A^T(t_0-\tau)}$$

$$= \int_{t_0}^{t_1} e^{-A(\tau-t_0)}BB^Te^{-A^T(\tau-t_0)}\,d\tau$$

$$\sigma = \tau - t_0 \Rightarrow d\sigma = d\tau$$

$$= \int_{\sigma=0}^{t_1-t_0} e^{-A\sigma}BB^Te^{-A^T\sigma}\,d\sigma$$

Define the Controllability Matrix:

$u \in \mathbb{R}^k$

$x \in \mathbb{R}^n$

$$C := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times kn}$$

Thm 11.5 : for $t_1 > t_0 \geq 0$

$$R[t_0, t_1] = Im(W_R(t_0,t_1)) = Im(C) = Im(W_c(t_0,t_1))$$
$$= C[t_0, t_1]$$

→ controllability matrix provides a way to compute reachable + controllable subspaces for LTI systems.

→ Controllable + reachable subsystems are the same for CLTI systems. (If you can reach a state from the origin then you will be able to ~~go back fr~~ control back from that state ~~to~~ back to the origin).

(A) Proof that → $Im(W_R(t_0,t_1)) = Im(C)$

Assume $x_1 \in Im(W_R(t_0,t_1))$

$x_1 \in R[t_0, t_1]$

$$\exists\, u(t) \text{ s.t. } x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

By the ~~Cayle~~ Cayley-Hamilton theorem (lecture #6)

$$\cancel{e^{At} = \sum_{i=0}^{n-1} e^{A(t_1-\tau)} Bu(\tau)d\tau} \qquad \cancel{\sum_{i=0}^{n-1} e^{A(t_1-\tau)} Bu(\tau)d\tau}$$

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i \qquad \forall t \in \mathbb{R}$$

sub into

$$\cancel{x_1 = \sum_{i=0}^{n-1} A^i B \left(\int_{t_0}^{t_1} \alpha_i(t_1-\tau)u(\tau)d\tau\right)}$$

$$x_1 = \sum_{i=0}^{n-1} \int_{t_0}^{t_1} \alpha_i(t_1-\tau) A^i B u(\tau) d\tau$$

$$= \underbrace{[B \;\; AB \;\; \cdots \;\; A^{n-1}B]}_{C = \text{controllability matrix}} \begin{bmatrix} \int_{t_0}^{t_1} \alpha_0(t_1-\tau) u(\tau) d\tau \\ \int_{t_0}^{t_1} \alpha_1(t_1-\tau) u(\tau) d\tau \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(t_1-\tau) u(\tau) d\tau \end{bmatrix}$$

$\Rightarrow x_1 \in Im(C)$

(B) Prove that — $x_1 \in \text{Im}(C) \Rightarrow x_1 \in \text{Im}(W_R(t_0,t_1))$

Assume $x_1 \in \text{Im}\, C$

Then $x_1 = Cv$ for a vector $v \in \mathbb{R}^{kn}$

If $x_1 \in \text{Im}\, W_R(t_0,t_1) = (\ker W_R(t_0,t_1))^{\perp}$

i.e. $\eta_1^T x_1 = \eta_1^T C v = 0 \quad \forall \eta_1 \in \ker W_R(t_0,t_1)$

[i.e if η_1 is in the $\ker W_R(t_0,t_1)$ then it will be orthogonal to anything in the $\text{Im} W_R(t_0,t_1)$ so if it is orthogonal to any vector $x_1 \in \text{Im}\, C$ then $x_1 \in \text{Im}\, W_R(t_0,t_1)$]

From Thm 11.2 $\quad \eta_1 \in \ker W_R(t_0,t_1)$

$$\Rightarrow \eta_1^T e^{A(t_1-\tau)} B = 0 \quad \forall \tau \in [t_0,t_1]$$

when $\tau = t_1 \Rightarrow B = 0$

Differentiate both sides —

$$-\eta_1^T A e^{A(t_1-\tau)} B = 0$$

when $\tau = t_1 \Rightarrow AB = 0$

Differentiate again —

$$\eta_1^T A^2 e^{A(t_1-\tau)} B = 0$$

when $\tau = t_1 \Rightarrow A^2 B = 0$

Differentiate k times $\longrightarrow \eta_1^T (-1)^k A^k e^{A(t_1-\tau)} B = 0$

$$(A^k) B = 0$$

$\Rightarrow \eta_1^T C = 0$ ~~Q.E.D~~.

Because the terms all equal zero : $\eta_1^T [B \;\; AB \;\cdots\; A^k B] = 0$

→ Proof for controllable subspace is analogous.

- For CTI systems: The system is both reachable + controllable if $\text{rank}([B \ AB \cdots A^{n-1}B]) = n$

Example: Parrallel RC circuit —

$x \in \mathbb{R}^2$

$u \in \mathbb{R}^1$

$C = 2 \times 2$ matrix

$$C = \underset{n \times nk}{[B \quad AB]}$$

$$= \Bigg[\underbrace{\begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -\frac{1}{R_1^2 C_1^2} \\ -\frac{1}{R_2^2 C_2^2} \end{bmatrix}}_{AB} \Bigg]$$

$$B = \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{bmatrix} \quad A = \begin{pmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{pmatrix}$$

If: $\frac{1}{R_1 C_1} = \frac{1}{R_2 C_2} = w$

$$C = \begin{pmatrix} w & -w^2 \\ w & -w^2 \end{pmatrix} \qquad \text{rank}(C) = 1 \neq \text{not controllable}$$

More Examples :

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u$$

$u \in \mathbb{R}^2$
$x \in \mathbb{R}^3$

$C = 3 \times 6$ matrix

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -1 & 9 & 3 \end{bmatrix}$$

rank $C = 2 \neq$ not controllable

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$\longrightarrow$ in controller cannonical form.

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

rank $C = 2 \Rightarrow$ controllable