The system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k$$

$$y = Cx + Du \qquad y \in \mathbb{R}^m$$

is a realization of $\hat{G}(s)$ if:

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

w/ n = order of the realization = size of state space vector.

Def 17.1: A realization is <u>minimal</u> if there is no other realization of smaller order.

Thm 17.1: Every minimal realization must be both controllable + observable.

Proof — (by contradiction) Assume a realization is either not controllable or not observable, Then by Kalman ~~decocom~~ decomposition theorem we could find another, <u>smaller</u> order realization that realizes the same TF.

* Controllability + observability are not only neccesary, but also sufficient for minimality.

Markov Parameters:

~~Recall that~~

Remember: $(sI-A)^{-1} = \mathcal{L}\{e^{At}\} = \mathcal{L}\left\{\sum_{i=0}^{\infty} \frac{t^i}{i!} A^i\right\}$

$$= \sum_{i=0}^{\infty} \mathcal{L}\left\{\frac{t^i}{i!}\right\} A^i$$

$$= \sum_{i=0}^{\infty} s^{-(i+1)} A^i$$

$$\therefore \quad \hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \sum_{i=0}^{\infty} s^{-(i+1)} CA^iB + D$$

The matrices $D, CB, CAB, CA^2B, \dots$

are called the Markov parameters.

Note also that the impulse response —

$$G(t) = \mathcal{L}^{-1}\{\hat{G}(s)\} = \mathcal{L}^{-1}\{C(sI-A)^{-1}B + D\} = Ce^{At}B + D\delta(t)$$

Taking the derivative gives

$$\frac{d^i}{dt^i} G(t) = CA^ie^{At}B \qquad \forall i \geq 1,\ t \geq 0$$

Evaluating @ $t=0$ gives

$$\left.\frac{d^i}{dt^i} G(t)\right|_{t=0} \quad CA^iB \qquad i \geq 1$$

Note that if $D=0$ then $G(0) = CB$

* So the Markov parameters may also be recovered from the impulse response & its derivatives.

Thm 17.2 —

Two realizations:

$$\dot{x} = Ax + Bu \qquad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = Cx + Du \qquad y = \bar{C}\bar{x} + \bar{D}u$$

are zero state equivalent (respond equivalently to inputs when $x(t_0) = x_0 = 0$) iff they have the same Markov parameters, i.e.

$$D = \bar{D}, \quad CA^iB = \bar{C}\bar{A}^i\bar{B}, \quad \forall i \geq 0$$

Note also that

$$\mathcal{C} = [B \ \ AB \ \cdots \ A^{n-1}B], \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$\mathcal{O}\mathcal{C} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & & & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix}$$

Are the Markov parameters

Thm 17.3 : A realization is minimal iff it is both controllable + observable.

Proof : (Assume theorem 17.1 already proven — a minimal realization is both controllable + observable).

By contradiction show {controllable + observable} $\Rightarrow$ minimal

Assume : $\dot{x} = Ax + Bu$, $y = Cx + Du$ is a controllable/observable realization of $\hat{G}(s)$

But, is <u>not</u> <u>minimal</u>.

Then $\exists$ another realization

$$\left.\begin{array}{l} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{array}\right\} \text{ w/ TF } \hat{G}(s) \text{ where } \bar{n} < n$$

Since (A, B, C) - controllable + observable

Then $\text{rank}\left(\underset{(nm\times n)}{\mathcal{O}}\underset{(n\times nk)}{\mathcal{C}}\right) = n$

And $\text{rank}(\bar{\mathcal{O}}\bar{\mathcal{C}}) = \bar{n}$

But since both have the same Markov parameters

$$\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$$

And we have a contradiction

$\therefore$ (A, B, C, D) - must be minimal

Thm 17.4 – All minimal realizations of a TF are algebraically equivalent.

↳ there exists a similarity transformation T that can convert between the two.

Def: Pseudoinverse – If M is full col rank then $M^T M$ is nonsingular, and $M^\ell = (M^T M)^{-1} M^T$ is the "left inverse" of M. i.e $M^\ell M = I$

– If N is full row rank NN^T is nonsingular and $N^r = N^T (NN^T)^{-1}$ is the "right inverse" w/ $NN^r = I$

Proof of thm 17.4 :

Assume two minimal realizations of the same TF:

$$\dot{x} = Ax + Bu \qquad \dot{\bar{x}} = \bar{A}x + \bar{B}u$$
$$y = Cx + Du \qquad y = \bar{C}x + \bar{D}u$$

Both are controllable + observable w/

$$\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$$

Define the transformation:

$$T = \bar{\mathcal{C}}\, \mathcal{C}^T (\mathcal{C}\mathcal{C}^T)^{-1}$$

Claim: $T^{-1} = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\bar{\mathcal{O}}$

Proof: $T^{-1}T = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\bar{\mathcal{O}}\bar{\mathcal{C}}\,\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}$ $\quad (\bar{\mathcal{O}}\bar{\mathcal{C}} = \mathcal{O}\mathcal{C})$

$= (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\mathcal{O}\mathcal{C}\,\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}$

$= I$

Now note that —

$$\bar{\mathcal{O}} T = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} T = \bar{\mathcal{O}} \bar{\mathcal{C}} \mathcal{C}^T (\mathcal{C}\mathcal{C}^T)^{-1}$$

$$= \cancel{\mathcal{O}\mathcal{C}}$$

$$= \mathcal{O}\mathcal{C}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1} = \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{C}T \\ \bar{C}\bar{A}T \\ \vdots \\ \bar{C}\bar{A}^{n-1}T \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\therefore \quad \bar{C}T = C \qquad$ ($y = Cx + Du$, C not cotrollability $\mathcal{C}$)

Also, $$T^{-1}\bar{\mathcal{C}} = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\bar{\mathcal{O}}\bar{\mathcal{C}}$$

$$= \underbrace{(\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\mathcal{O}}_{I}\mathcal{C}$$

$$= \mathcal{C}$$

$$\therefore \quad T^{-1}[\bar{B} \ \ \bar{A}\bar{B} \cdots \bar{A}^{n-1}\bar{B}] = [B \ \ AB \cdots A^{n-1}B]$$

$$\Rightarrow \quad T^{-1}\bar{B} = B$$

Also, since the Markov parameters are equivalent

$$\underbrace{\mathcal{O}A\mathcal{C}}_{\text{All the Markov params}} = \bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}$$

All the Markov params $= \begin{bmatrix} CAB & CA^2B & \cdots & CA^nB \\ CA^2B & & & \\ \vdots & & & \\ CA^nB & & \cdots & CA^{2n-1}B \end{bmatrix}$

But, $\Theta = \bar{\Theta} T$ & $\mathcal{C} = T^{-1}\bar{\mathcal{C}}$

$\Rightarrow \Theta A \mathcal{C} = \bar{\Theta} T A T^{-1} \bar{\mathcal{C}} = \bar{\Theta} \bar{A} \bar{\mathcal{C}}$

$\therefore \quad \bar{A} = T A T^{-1}$