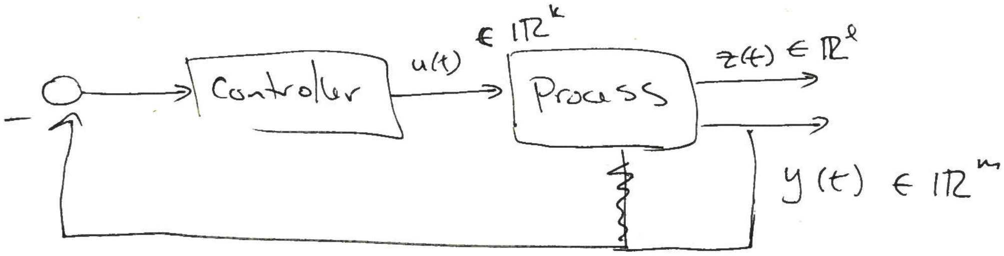
Deterministic LQR



$z(t)$: signals we would like to make as small as possible

$y(t)$: signals that can be measured (and controlled) (from sensors)

~~✱~~ Sometimes $z(t) = y(t)$ or $z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$

Process:

$\dot{x} = Ax + Bu$ — state equation

$y = Cx$ — measured output

$z = Gx + Hu$ — controlled output (things to make small)

Example: to make $\begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ small, let

$$z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} Cx \\ C\dot{x} \end{bmatrix} = \begin{bmatrix} Cx \\ CAx + CBu \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \end{bmatrix}}_{G} x + \underbrace{\begin{bmatrix} 0 \\ CB \end{bmatrix}}_{H} u$$

* LQR problem is to find the control input $u(t)$, $t \in [0,\infty]$ that minimizes

$$J_{LQR} = \int_0^\infty \Big(\underbrace{\|z(t)\|^2}_{\text{Energy of the controlled output}} + \rho \underbrace{\|u(t)\|^2}_{\text{Energy of the controlled signal}} \Big) dt$$

ρ: determines trade-off between increasing control to minimize $z(t)$ + vice-versa

ρ - large $\Rightarrow$ $\int_0^\infty \|u(t)\| dt$ is small @ expense of $z(t)$

ρ - small $\Rightarrow$ $z(t)$ is small @ expense of large input

More generally we have —

$$J_{LQR} = \int_0^\infty z^T(t)\bar{Q}z(t) + \rho u^T(t)\bar{R}u(t)$$

where $\bar{Q} \in \mathbb{R}^{l\times l}$, $\bar{R} \in \mathbb{R}^{k\times k}$ are symmetric, pos-def

$\rho \in \mathbb{R}$, $\rho > 0$

~~Since~~
Since $z = Gx + Hu$ we get

$$J_{LQR} = \int_0^\infty \left((Gx+Hu)^T \bar{Q} (Gx+Hu) + \rho u^T \bar{R} u \right) dt$$

$$= \int_0^\infty \left(x^T G^T \bar{Q} G x + u^T(\rho \bar{R} + H^T \bar{Q} H)u + 2x^T G^T \bar{Q} H u \right) dt$$

Let: $Q \triangleq G^T \bar{Q} G$, $R \triangleq \rho \bar{R} + H^T \bar{Q} H$, $N \triangleq G^T \bar{Q} H$

$$J_{LQR} = \int_0^\infty \left(x^T Q x + u^T R u + 2x^T N u \right) dt$$

Feedback Invariants :

Given an LTI system : $\dot{x} = Ax + Bu$

function of functions

* The functional $H(x(t), u(t))$ is feedback invariant

should be dots here, not yet specific to t

if it only depends on x_0 and not $u(t)$, $t \geq 0$

Example :

$$H(x(t); u(t)) = -\int_0^\infty \left((Ax+Bu)^T P x + x^T P (Ax+Bu)\right) dt$$

(P = symmetric) is feedback invariant

if $\lim_{t \to \infty} x(t) = 0$

Proof : $$H(x(\cdot); u(\cdot)) = -\int_0^\infty \frac{d}{dt}\left(x^T P x\right) dt$$

$$= \underbrace{\lim_{t \to \infty} x^T P x}_{\text{require this to go to zero.}} + x_0^T P x_0$$

$$= x_0^T P x_0$$

Feedback Invariants in Optimal Control :

$$J_{LQR} = \int_0^\infty \left(x^T Q x + u^T R u + 2x^T N u\right) dt$$

(Add in then subtract the feedback invariant)

$$J_{LQR} = H(x(t); u(t)) + \int_0^\infty \Big(x^T Q x + u^T R u + 2x^T N u + (Ax+Bu)^T P x + x^T P (Ax+Bu)\Big) dt$$

(group terms)

$$= H(x(t); u(t)) + \int_0^\infty \Big[x^T (A^T P + PA + Q) x + \underbrace{u^T R u + 2u^T (B^T P + N^T) x}_{\text{Compute the sq.}}\Big] dt$$

$$(u+Kx)^T R (u+Kx) = u^T R u + u^T R K x + x^T K^T R u + x^T K^T R K x$$
$$= u^T R u + 2u^T R K x + x^T K R K x$$

(comparing terms) $\quad RK = B^T P + N^T$
$$K = R^{-1}(B^T P + N^T)$$

$$\therefore \quad J_{LQR} = H(x(t); u(t)) + \int_0^\infty \Big[(u+Kx)^T R (u+Kx) + x^T (A^T P + PA + Q - K^T R K) x \Big] dt$$

$\rightarrow$ * Since $H(x(t); u(t))$ is a feedback invariant for any symmetric P, select P as the symmetric solution of

$$A^T P + PA + Q - K^T R K = 0 \qquad (\star)$$

to get –

$$J_{LQR} = H(x(t); u(t)) + \int_0^\infty \overbrace{(u+Kx)^T R (u+Kx)}^{\text{Quadratic - lowest it can be is zero}} dt$$

Note: if $R > 0$ then $\min_{w \in \mathbb{R}^k} (w+Kx)^T R (w+Kx) = 0$

And $u^*(t) = \arg\min_{w \in \mathbb{R}^k} (w+Kx)^T R (w+Kx)$
$$= -Kx(t)$$
$$\boxed{u^*(t) = R^{-1}(B^T P + N^T)\, x(t)}$$

$\longrightarrow$ plug K into $(\star)$ $\qquad$ w/ closed loop system: $\dot{x} = (A - BR^{-1}(B^T P + N^T))x$

$$A^T P + PA + Q - (R^{-1}(B^T P + N^T))^T R (R^{-1}(B^T P + N^T)) = 0$$
$$\boxed{A^T P + PA + Q - (PB + N) R^{-1} (B^T P + N^T) = 0}$$

* Algebraic Riccati Equation

$$\therefore \quad \underset{(\text{minimum})}{J_{LQR}} = \int_0^\infty (x^T Q x + u^T R u + 2x^T N u)\,dt = X^T(0) P X(0)$$

Discrete Time:

$$X_{k+1} = A X_k + B u_k \qquad X_0 \text{ - initial condition}$$

$$y_k = C X_k$$

$$z_k = G X_k + H u_k$$

$$J_{LQR} = \sum_{j=0}^{\infty} X_j^T Q x_j + u_j^T R u_j + 2 x_j^T N u_j$$

Use discrete feedback invariant —

$$H(x(t); u(t)) = -\sum_{j=0}^{\infty} \left\{ (Ax_j + Bu_j)^T P (Ax_j + Bu_j) - x_j^T P x_j \right\}$$

$$= -\sum_{j=0}^{\infty} \left(X_{j+1}^T P X_{j+1} - x_j^T P x_j \right)$$

$$= -\Big\{ (x_1^T P x_1 - X_0^T P X_0)$$

$$+ (X_2^T P X_2 - X_1^T P X_1)$$

$$+ (X_3^T P X_3 - X_2^T P X_2)$$

$$\vdots$$

$$+ (\qquad\qquad) \Big\}$$

$$= X_0^T P X - \lim_{j \to \infty} X_j^T P X_j$$

$\therefore$ If $x_j \to 0$ then $H(x(t); u(t))$ is a feedback invariant for any P.

(Add + subtract feedback invariant)

$$J_{LQR} = H(x(t);u(t)) + \sum_{j=0}^{\infty} \{ x_j^T Q x_j + u_j^T R u_j + 2x_j^T N u_j$$
$$- x_j^T A^T P A x_j - u_j^T B^T P B u_j$$
$$- 2u_j^T B^T P A x_j - x_j^T P x_j \}$$

$$= H(x(t);u(t)) + \sum_{j=0}^{\infty} \{ u_j^T (R - B^T P B) u_j + 2u_j^T (B^T P A - N^T) x_j$$
$$- x_j^T (A^T P A + P - Q) x_j \}$$

(complete the square on 1st 2 terms)

$$(u_j + K x_j)^T \bar{R} (u_j + K x_j) = u_j^T \bar{R} u_j + 2u_j^T \bar{R} K x_j + x_j^T K^T \bar{R} K x_j$$
w/ $\bar{R} = (R - B^T P B)$

And $\bar{R} K = B^T P A - N^T$

$$\Longrightarrow \boxed{K = (R - B^T P B)^{-1} (B^T P A - N)}$$

$$J_{LQR} = H(x(t);u(t)) + \sum_{j=0}^{\infty} \{ (u_j + K x_j)^T (R - B^T P B)(u_j + K x_j)$$
$$- x_j^T \underbrace{(A^T P A + P - Q - K^T (R - B^T P B) K)}_{=0} x_j \}$$

If we select P as soln to:

$$A^T P A + P - Q - (A^T P B - N)(R - B^T P B)^{-1} (B^T P A - N^T)$$

– plugged in for K + $^{-1}$ ~~all the~~ pulled through the transpose (R, P symmetric)

$$\Longrightarrow J_{LQR} = H(x(t);u(t)) + \sum_{j=0}^{\infty} (u_j + K x_j)^T (R - B^T P B)(u + K x_j)$$

if $R - B^T P B > 0$ Then $u_j^* = -K x_j$

Bryson's Rule:

For $J_{LQR} = \int_0^\infty (z^T \bar{Q} z + \rho u^T \bar{R} u) \, dt$

Bryson's rule is to select $\bar{Q}$ + $\bar{R}$ as diag. matrices

w/ $$\bar{Q}_{ii} = \frac{1}{\text{max acceptable value of } z_i^2}$$

$$\bar{R}_{jj} = \frac{1}{\text{max acceptable value of } u_j^2}$$

$\longrightarrow$ this normalizes units

Example:

$$J = \int_0^\infty u^2 \, dt \quad w/ \quad \dot{x} = x + u$$

optimal solution $u^* = 0 = R^{-1}(B^T P + N^T)\, x(t)$

In this case $B = 1$, $R = 1$, $N^T = 0 \Rightarrow P = 0$

And closed-loop dynamics: $\dot{x} = x$ are unstable

$$\dot{x} = [1]\, x \Rightarrow x = e^{At} x_0$$
$$= x = e^{t} x_0$$

Example #2:

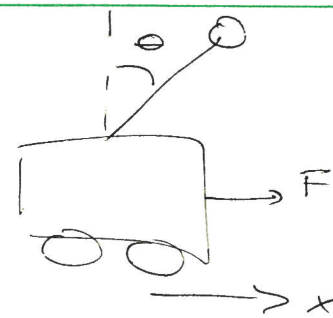
$$J = \int_0^\infty u^2 \, dt \quad w/ \quad \dot{x} = -x + u$$

→ Same result, but this time closed-loop system is stable $x = e^{-t} x_0$

→ Note: in both cases $P = 0$ and is not strictly pos ~~semi~~-definite.

Bryson's Rule :

Inverted Pendulum on cart :



State vector: $x = [x \ \dot{x} \ \theta \ \dot{\theta}]^T$

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$ w/ input = Force to move cart.

$$Q = \begin{pmatrix} 3.28 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 28.65 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1 foot = .3048 meters

$\frac{1}{\max x} = 3.28$

$2^\circ = 0.035 \text{ rad} = \max \theta$

$\frac{1}{\max \theta} = 28.65$

Q has large penalty on x_1 + x_3 states

→ If those states are large then the Q is going to amplify them.

→ So to minimize cost the u input will need to minimize these two states.

$$R = \left[\frac{1}{\text{Max Force desired}} \right]$$