Objectives:

1. Understand state space representation of linear systems
2. Familiar w/ notation + definitions
3. ~~Remember~~ Review Laplace transforms
4. Block diagrams

* There are three common mathematical tools for representing dynamic systems.

<u>Def</u>: <u>Dynamic System</u> — a system that moves or has memory, or consumes/generates energy.

- For mechanical systems: anything w/ position, velocity, angles, etc.
- For computer systems: anything w/ memory

Methods for representing dynamic systems:

1. Differential equation (partial and ordinary)
2. Transfer functions (limited to <u>LTI</u>)
3. State space

State space models are the most useful

- Apply to the largest class of problems
- Easy to account for complex interactions
- Standard computation methods for solving
- Powerful mathematical tools for analysis + design

– Continuous state space equation for <u>nonlinear</u> systems:

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u)$$

$u \in \mathbb{R}^m$ = input

$x \in \mathbb{R}^n$ = state variables

$y \in \mathbb{R}^p$ = output

$$\dot{x} = \frac{dx}{dt}$$

– Continuous state-space equations for linear time-varying (LTV) systems

$$\underset{(n \times 1)}{\dot{x}(t)} = \underset{(n \times n)}{A(t)}\,\underset{(n \times 1)}{x(t)} + \underset{(n \times m)}{B(t)}\,\underset{(m \times 1)}{u(t)}$$

$$\underset{(p \times 1)}{y(t)} = \underset{(p \times n)}{C(t)}\,\underset{(n \times 1)}{x(t)} + \underset{(p \times m)}{D(t)}\,\underset{(m \times 1)}{u(t)}$$

– If A, B, C, D ~~do not depent or vary w/ time~~ are constant then w/ have a linear time-invariant (LTI) system

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

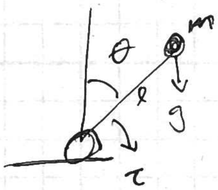
– Note: the signals u, x, y will always be depentent on time (or we will assume they are) – for convenience will drop their (t) depentence, but it is always implied.

– Note: Generally there is more that can be done w/ LTI than LTV systems.

SISO: Single input / Single output $\longrightarrow$ $K = 1$ + $p = 1$

MIMO: multiple Input / multiple output $\longrightarrow$ $K > 1$, $p > 1$

Example:



Newton's law: $\underbrace{ml^2\ddot{\theta}}$ $= \underbrace{mgl\sin\theta}_{\text{gravity}} - \underbrace{b\dot{\theta}}_{\text{friction}} + \underbrace{\tau}_{\text{torque/ external force}}$ (★)
(for rotation)

- Equation (★) is an ODE representation of the system.
- We can create several different state space representations

Select: $x_1 := \theta$ $\qquad u := \tau$

$x_2 := \dot{\theta}$ $\qquad y := \theta$

Then: $\dot{x}_1 = \dot{\theta} = x_2$

$$\dot{x}_2 = \ddot{\theta} = \frac{1}{ml^2}\left(mgl\sin\theta - b\dot{\theta} + \tau\right)$$

$$= \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{u}{ml^2}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u \end{bmatrix}}_{f(x,u)}$$

$$y = \underbrace{x_1}_{h(x,u)}$$

→ use the small angle approximation $\sin\theta \simeq \theta$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}_{B} u \qquad \dot{x} = Ax + Bu$$

$$\underbrace{y}_{y} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{0}_{D} \cdot u \qquad y = Cx + Du$$

- Or we could have choosen ——

$$x_1 = \theta + \dot\theta \implies \theta = x_1 - \dot\theta = x_1 - 2x_2$$

$$x_2 = \tfrac{1}{2}\dot\theta \implies \dot\theta = 2x_2$$

$$y = \theta$$

$$u = \tau$$

Then:

$$\dot x_1 = \dot\theta + \ddot\theta = 2x_2 + \frac{1}{m\ell^2}\left(mg\ell\cancel{\sin}\,\theta - b\dot\theta + \tau\right)$$

$$= 2x_2 + \frac{g}{\ell}\underbrace{\theta}_{x_1 - 2x_2} - \frac{b}{m\ell^2}\underbrace{\dot\theta}_{(2x_2)} + \frac{\tau}{m\ell^2}$$

$$= \cancel{2\left(1 - \frac{g}{\ell}\right)x_2}\; 2x_2 + \frac{g}{\ell}x_1 - 2\frac{g}{\ell}x_2 - 2\frac{b}{m\ell^2}x_2 + \frac{u}{m\ell^2}$$

$$= 2\left(1 - \frac{g}{\ell} - \frac{b}{m\ell^2}\right)x_2 + \frac{g}{\ell}x_1 + \frac{1}{m\ell^2}u \qquad \text{(group terms)}$$

$$\dot x_2 = \frac{1}{2}\ddot\theta = \frac{1}{2m\ell^2}\left(mg\ell\theta - b\dot\theta + \tau\right)$$

$$= \frac{1}{2m\ell^2}\left(mg\ell(x_1 - 2x_2) - 2bx_2 + u\right)$$

$$= \frac{g}{2\ell}x_1 - \frac{g}{\ell}x_2 - \frac{b}{m\ell^2}x_2 + \frac{1}{2m\ell^2}u$$

$$= \frac{g}{2\ell}x_1 - \left(\frac{g}{\ell} + \frac{b}{m\ell^2}\right)x_2 + \frac{1}{2m\ell^2}u$$

$\therefore$

$$\begin{bmatrix}\dot x_1 \\ \dot x_2\end{bmatrix} = \begin{bmatrix} g/\ell & 2(1 - g/\ell - b/m\ell^2) \\ \frac{g}{2\ell} & -\left(\frac{g}{\ell} + \frac{b}{m\ell^2}\right)\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix} 1/m\ell^2 \\ \frac{1}{2m\ell^2}\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 0\cdot u$$

$\longrightarrow$ Different state space models for the same system.

LaPlace Transform Review —

$$\mathcal{L}\left[x(t)\right] = \hat{X}(s) = \int_0^\infty e^{-st}\, x(t)\, dt \qquad s \in \mathbb{C}$$

$x(t)$ = continuous signal
$t \geq 0$

$$\mathcal{L}\left[\dot{x}(t)\right] = s\hat{X}(s) - x(0) \qquad s \in \mathbb{C}$$

Proof : $\frac{d}{dt} e^{-st} x(t) = e^{-st}\dot{x}(t) - s e^{-st} x(t)$

$$\Rightarrow \int_0^\infty \frac{d}{dt} e^{-st} x(t)\, dt = \int_0^\infty e^{-st}\dot{x}(t)\, dt - \int_0^\infty s e^{-st} x(t)\, dt$$

$$\int_0^\infty d\left(e^{-st} x(t)\right) \qquad\qquad \mathcal{L}\left[\dot{x}(t)\right] - s\int_0^\infty e^{-st} x(t)\, dt$$

$$\lim_{t\to\infty} e^{-st} x(t) - e^{-s(0)} x(0) \qquad\qquad \mathcal{L}\left[\dot{x}(t)\right] - s\hat{X}(s)$$

$$0 - x(0)$$

$$\therefore \mathcal{L}\left[\dot{x}(t)\right] = s\hat{X}(s) - x(0)$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\left[(x * y)\right] = \mathcal{L}\left[\int_0^t x(\tau)y(t-\tau)\, d\tau\right] = \hat{X}(s)\hat{y}(s)$$

$$\mathcal{L}\left[\ddot{x}(t)\right] = s^2\hat{X}(s) - s x(0) - \dot{x}(0)$$