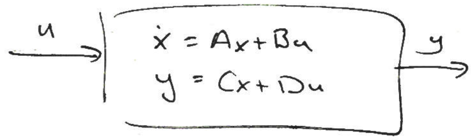
State Estimation:



– what if we want to stabilize the system using state feedback?

$$u = -Kx$$

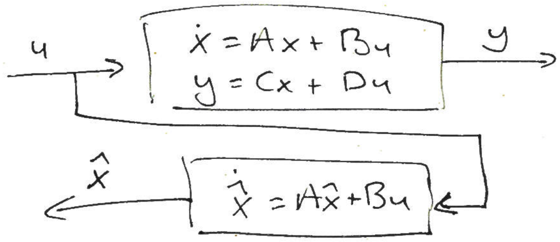
– So $\quad \dot{x} = (A - BK)x$

→ We can't implement this when only the output y is measured.

→ But, if (A, C) is detectable we can estimate $x(t)$ asymptotically.

→ Simplest state estimator is to copy the original system. Assumes (A, B, C, D) are known.

$$\dot{\hat{x}} = A\hat{x} + Bu$$



Define: $\quad e = \hat{x} - x \quad$ to be the state estimation error.

Then $\quad \dot{e} = A\hat{x} + Bu - (Ax + Bu) = A(\hat{x} - x) = Ae$

w/ $\quad e(0) = \hat{x}(0) - x(0)$

→

$$\dot{e} = Ae$$

$$e(t) = e^{At} e(0)$$

← If A is a stability matrix this will go to zero exp. fast.

→ If A is not a stability matrix then the estimation error diverges.

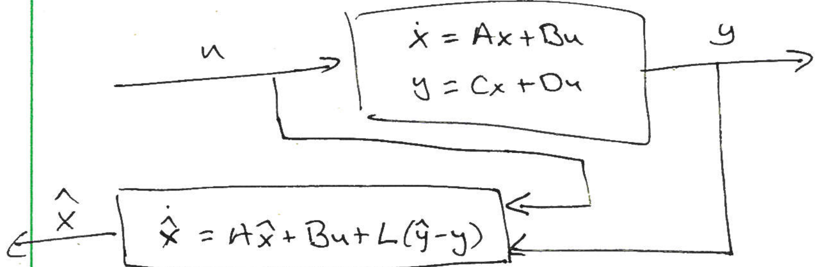
→ But, if we use a closed loop estimator we can still create an asymptotically correct state estimate – even if A is not a stability matrix.

→ Have the estimator correct based upon its output (don't *just* use system inputs)

$y \in \mathbb{R}^m$
$x \in \mathbb{R}^n$

Let: $\dot{\hat{x}} = A\hat{x} + Bu - L\underbrace{(\hat{y} - y)}_{\text{innovation}}$ $\qquad L \in \mathbb{R}^{n \times m}$ ← output injection gain

w/ $\hat{y} = C\hat{x} + Du$



Again, let $\dot{e} = \dot{\hat{x}} - \dot{x} = \underbrace{A\hat{x} + Bu - L(\hat{y} - y)}_{\dot{\hat{x}}} - \underbrace{(Ax + Bu)}_{\dot{x}}$

$$= A(\hat{x} - x) - L(\hat{y} - y)$$

$$= Ae - L(C\hat{x} + \cancel{D}u - Cx + \cancel{D}u)$$

$$= (A - LC)e$$

$$e(t) = e^{(A-LC)t} e(0)$$

Thm 16.7 — If the output injection gain $L \in \mathbb{R}^{n \times m}$ makes $(A - LC)$ a stability matrix, then $e \to 0$ exp. fast for every input signal u.

E-Value Assignment by Output Injection:

→ Can use "duality" to get same results from e-value assignment.

Thm. 16.8 – When pair (A, C) is detectable it is always possible to find a matrix gain $L \in \mathbb{R}^{n \times m}$ s.t. $A - LC$ is a stability matrix.

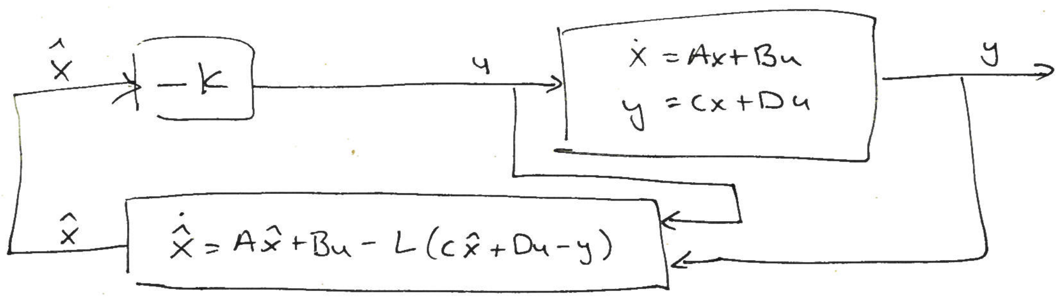
→ i.e. can push poles into the LHP

Thm 16.9 – When pair (A, C) is observable. Given any set of "legal" (i.e complex conj. pairs) $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$ there exists a state feedback matrix $L \in \mathbb{R}^{n \times m}$ s.t. $(A - LC)$ has e-values equal to λ_i.

→ i.e. Can put the poles anywhere (assuming "legal").

Separation Principle:

- Suppose we have the following feedback system:



Is the closed-loop system stable?

We can write the closed-loop system as —

$$\dot{x} = Ax - BK\hat{x}$$
$$\dot{\hat{x}} = (A - LC)\hat{x} - BK\hat{x} + LCx$$

or
$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Alternatively, lets make a change of variables —

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \qquad e = \hat{x} - x$$

Then —

$$\dot{x} = Ax - BKe - BKx = (A - BK)x - BKe$$

$$\dot{e} = \dot{\hat{x}} - \dot{x} = A\hat{x} - LC\hat{x} - \cancel{BK\hat{x}} + LCx - Ax + \cancel{BK\hat{x}}$$
$$= A(\hat{x} - x) - LC(\hat{x} - x) = (A - LC)e$$

or
$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{\text{e-values} \,=\, \text{eig}(A-BK) \,\cup\, \text{eig}(A-LC)} \begin{bmatrix} x \\ e \end{bmatrix}$$

⇒ The state feedback gain K can be designed by placing $(A-BK)$ at arbitrary locations.

⇒ And the output injection gain L can be designed ~~by~~ separately by placing $(A-LC)$ at arbitrary locations.
(Separation Principle)

→ In practice this doesn't work extremely well, (will see why later in semester).

→ A simple explanation —

The closed-loop e-values are at :

$$\mathrm{eig}(A-BK) \cup \mathrm{eig}(A-LC)$$

But, e-values of the observer-based controller (can be written as)

$$u \leftarrow \boxed{\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu - L(\hat{y}-y)\\ u &= -K\hat{x}\end{aligned}} \leftarrow y$$

or

$$u \leftarrow \boxed{\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu - LC\hat{x} - LDu + Ly\\ u &= -K\hat{x}\end{aligned}} \leftarrow y$$

$$u \leftarrow \boxed{\begin{aligned}\dot{\hat{x}} &= (A-BK-LC+LDK)\hat{x} + Ly\\ u &= -K\hat{x}\end{aligned}} \leftarrow y$$

Note: This is a state space system w/ the

"A-matrix" $= (A-BK-LC+LDK)$

"B-matrix" $= L$

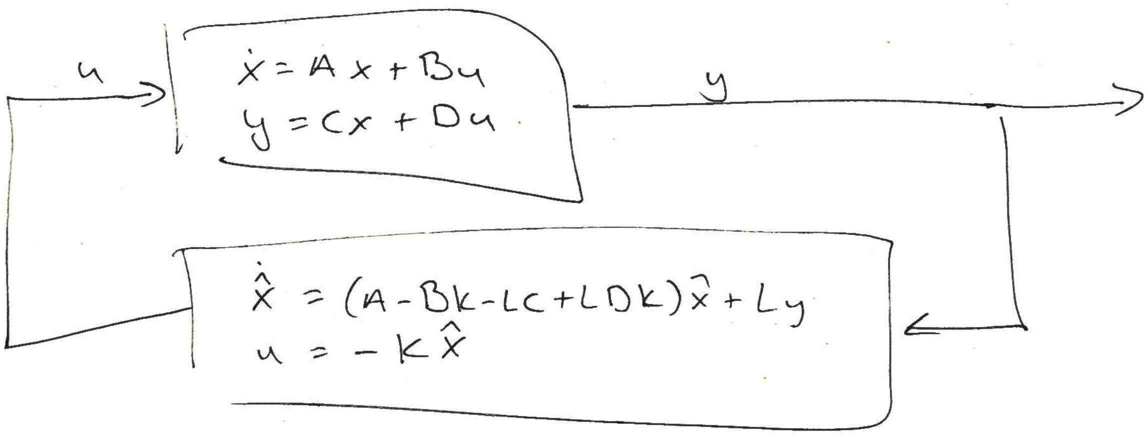
"C-matrix" $= -K$

"D-matrix" $= 0$

$\therefore$ The e-values of controller are −

$\text{eig}(A - BK - LC + LDK)$

→ which are not related to $\text{eig}(A-BK) \cup \text{eig}(A-LC)$ and may be unstable.



open-loop poles : $\text{eig}(A) \cup \text{eig}(A - BK - LC + LDK)$

closed-loop poles : $\text{eig}(A-BK) \cup \text{eig}(A-LC)$

poles of the plant : $\text{eig}(A)$

poles of controller : $\text{eig}(A - BK - LC + LDK)$