Observable Decomposition :

Thm 16.2 — For every LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$

There is a similarity transformation T, s.t.

$$T^{-1}AT = \begin{pmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{pmatrix}, \quad T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}, \quad CT = \begin{pmatrix} C_o & 0 \end{pmatrix}$$

where (A_o, C_o) - observable

Proof : By duality since for (A^T, C^T, B^T) there exists a T s.t

$$T^{-1}A^T T = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix}, \quad T^{-1}C^T = \begin{pmatrix} C_o \\ 0 \end{pmatrix}, \quad B^T T = \begin{bmatrix} B_o & B_{\bar{o}} \end{bmatrix}$$

Note that the SVD of $\underset{(mn \times n)}{\mathcal{O}(A,C)} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$

v_1 spans the observable subspace and

v_2 spans the unobservable subspace.

The transformed system can be written as

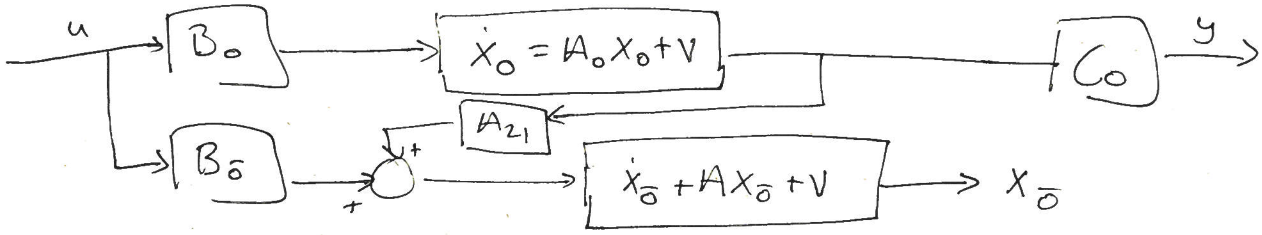
$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du$$

or

$$\dot{x}_o = A_o x_o + B_o u, \quad y = C_o x_o \quad \text{no } x_{\bar{o}}$$

$$\dot{x}_{\bar{o}} = A_{12} x_o + A_{\bar{o}} x_{\bar{o}} + B_{\bar{o}} u$$



$x_{\bar{o}}$ is unobservable

no contribution to the output

Kalman Decomposition:

We saw that $\exists$ a similarity transform $T_c = [V_c \ V_{\bar{c}}]$

$$\begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} = T_c^{-1} x$$

s.t.
$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = [C_c \ C_{\bar{c}}] \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + Du$$

Also $\exists$ a similarity transformation $T_o = [V_o \ V_{\bar{o}}]$

$\S$ $$\begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} = T_o^{-1} x$$

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = [C_o \ 0] \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du$$

Now, using T_c + $T_{\bar{o}}$, find

$$T = [V_{co} \ V_{c\bar{o}} \ V_{\bar{c}o} \ V_{\bar{c}\bar{o}}]$$

span $V_{co} = \mathrm{Im}(\mathcal{C}) \cap \mathrm{Im}(\mathcal{O}^T)$

span $V_{c\bar{o}} = \mathrm{Im}(\mathcal{C}) \cap \ker(\mathcal{O})$

span $V_{\bar{c}o} = \ker(\mathcal{C}^T) \cap \mathrm{Im}(\mathcal{O}^T)$

span $V_{\bar{c}\bar{o}} = \ker(\mathcal{C}^T) \cap \ker(\mathcal{O})$

and let $\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = T^{-1}x$

Then

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{xo} & 0 \\ A_{cx} & A_{c\bar{o}} & A_{xx} & A_{x\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}x} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

where —

1) $\left(\begin{bmatrix} A_{co} & 0 \\ A_{cx} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix} \right)$ is controllable

2) $\left(\begin{bmatrix} A_{co} & A_{xo} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix} \right)$ is observable

3) (A_{co}, B_{co}, C_{co}) is both observable & controllable

4) $C(sI-A)^{-1}B + D = C_{co}(sI - A_{co})^{-1}B_{co} + D$

→ TF reduces to just to the controllable & observable parts of the system.

Proof of 4:

$$C(sI-A)^{-1}B = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} sI-A_{co} & 0 & -A_{xo} & 0 \\ -A_{cx} & sI-A_{c\bar{o}} & -A_{xx} & -A_{x\bar{o}} \\ 0 & 0 & sI-A_{\bar{c}o} & 0 \\ 0 & 0 & -A_{\bar{c}x} & sI-A_{\bar{c}\bar{o}} \end{bmatrix}^{-1} \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{co} G_{11} + C_{\bar{c}o} G_{31} \;\middle|\; C_{co} G_{12} + C_{\bar{c}o} G_{32} \;\middle|\; * \;\middle|\; * \end{bmatrix} \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$= C_{co} G_{11} B_{co} + C_{\bar{c}o} G_{31} B_{co} + C_{co} G_{12} B_{c\bar{o}} + C_{\bar{c}o} G_{32} B_{c\bar{o}}$$

From the formula $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$ $\leftarrow$ cofactors transpose,

$G_{31} = 0$

$G_{12} = 0$ $\quad$ + $\quad G_{11} = (sI - A_{co})^{-1}$

$G_{32} = 0$

$$= C_{co} (sI - A_{co})^{-1} B_{co}$$

16.3 Detectability:

Any LTI system is algebraically equivalent to —

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u \qquad \begin{matrix} x_o \in \mathbb{R}^{\bar{n}} \\ x_{\bar{o}} \in \mathbb{R}^{n-\bar{n}} \end{matrix}$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du$$

<u>Def 16.1</u> (A,C) is detectable if $n=\bar{n}$ or $A_{\bar{o}}$ is a stability matrix.

Basic idea: x_o is observable, and the unobservable states $x_{\bar{o}}$ converge to zero asymptotically.

<u>Detectability Tests</u> — All results follow from duality to stabilizability.

<u>Eigenvector Test</u>: An (LTI) is detectable iff every unstable e-vector of A is not in the kernel of C.

<u>PBH Test</u>: (A,C) - detectable iff $\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$

$\forall$ unstable eigenvalue.

<u>Lyapunov Test</u>: (A,C) - detectable iff $\exists$ a pos-def P s.t. $A^T P + PA - C^T C < 0$