* For controllability we focus on state evolution eq:

$$\dot{x} = A(t)x + B(t)u$$

w/ solution:

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

* we are looking @ what the input is capable of in terms of transfering from one state to another ($x_0 \to x_1$)

* Def 11.1 (Reachable subspace): Given $t_1 > t_0 \geq 0$ the reachable subspace $R[t_0, t_1]$ is the set of states x_1 that can be reached from the origin (i.e $x(t_0)=0$).

$$R[t_0, t_1] := \{ x_1 \in \mathbb{R}^n : \exists u(\cdot), \; x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \; \cancel{0 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau} \}$$

* Def 11.2 (Controllable subspace): Given $t_1 > t_0 \geq 0$ the controllable subspace $C[t_0, t_1]$ is the set of states that can be transferred to the origin (in finite time), i.e.

$$C[t_0, t_1] = \{ x_0 \in \mathbb{R}^n : \exists u(\cdot), \; 0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau) d\tau \}$$

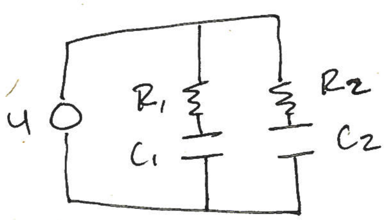
Note: $C(\cdot)$ & $D(\cdot)$ matrices play no role in these definitions — so only concerned w/ $\dot{x} = A(t)x + B(t)u$ eq.) or the pair $A(\cdot), B(\cdot)$

Note: $0 = \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) u(\tau) d\tau$

$\Rightarrow \quad -x_0 = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) u(\tau) d\tau$ (controllable Subspace)

$x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau$ (reachable Subspace)

Example: (Parralel RC Network)



$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

Solution:

$$x(t) = \begin{pmatrix} e^{-\frac{1}{R_1C_1}t} & 0 \\ 0 & e^{-1/R_2C_2 t} \end{pmatrix} x_0 + \int_0^t \begin{pmatrix} e^{-(t-\tau)/R_1C_1} & 0 \\ 0 & e^{-(t-\tau)/R_2C_2} \end{pmatrix} \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u(\tau) d\tau$$

Let $\frac{1}{R_1C_1} = \frac{1}{R_2C_2} = \omega$

$$x(t) = e^{-\omega t} x_0 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau$$

$\therefore$ If $x_0 = 0$ then the reachable subspace is $R[0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \alpha \in \mathbb{R} \right\}$

* We can transfer x_0 to the origin in finite time if

$$0 = e^{-\omega t} x_0 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau$$

only if $x_0 \in \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \mathcal{C}[0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$

→ this is ~~because~~ we chose the time constants to be the same. When they are different

$\frac{1}{R_1C_1} \neq \frac{1}{R_2C_2}$ then $R[t_0, t_1] = C[t_0, t_1] = \mathbb{R}^2$

~~Review~~:

~~- Given an $m \times n$ matrix W, the range or image is the set of vectors $y \in \mathbb{R}^m$ for which $y = Wx$ has a solution:~~

~~i.e. Im W =~~

- Range or Image = set of vectors $y \in \mathbb{R}^m$ for which $y = Wx$ has a solution.
- Im W = linear subspace of $\mathbb{R}^m$
- dimension of the subspace is called the rank.

Example:

$$W = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

Im W = Spanned by vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

rank Im A = 2 = # of linearly indep columns.

kernel or null set = set of vectors for which $Wx = 0$

dim of null space = nullity

Solve:

$Wx = 0$ [e-vectors live in null space of matrix $(SI - A)$]

$Wx \Rightarrow$
$x_1 - x_3 = 0$
$x_1 + 2x_2 - x_3 = 0$
$x_1 - x_3 = 0$

$x_2 = 0$
$x_1 = x_3$

$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ nullity = 1

Fundamental theorem of linear equations.

For every $\underline{m \times n}$ matrix W:

cols of W

$$\underbrace{\dim \ker W}_{\text{nullity } W} + \underbrace{\dim \operatorname{Im} W}_{\text{rank } W} = n$$

$\operatorname{Im} W = (\ker W^T)^\perp \quad \ker W = (\operatorname{Im} W^T)^\perp$

- orthogonal complement $(V)^\perp$ is the set of all vectors that are orthogonal to V.

$$V^\perp = \{ x \in \mathbb{R}^n : x^T z = 0, \ \forall z \in V \}$$

Reachability + Controllability Grammians :

$$W_R(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)\,d\tau$$

→ Allows us to compute the reachable subspace.

Thm 11.2 : $R[t_0,t_1] = \text{Im } W_R(t_0,t_1)$

And w/ $x_1 = W_R(t_0,t_1)\eta \in \text{Im } W_R(t_0,t_1)$
the control : $\underline{u(t) = B(t)^T\Phi(t_1,t)^T\eta}$
can be used to transfer the state from $x(t_0)=0$
to $x(t_1)=x_1$.

Proof :

① Show that $x_1 \in \text{Im } W_R(t_0,t_1) \Rightarrow x_1 \in R[t_0,t_1]$
(then will show opposite)
- when $x \in \text{Im } W_R(t_0,t_1)$ then $\eta_1 \in \mathbb{R}^n$ exists
s.t. $x_1 = W_R(t_0,t_1)\eta$
- Shows that input $u(t) = B(t)^T\Phi(t_1,t)^T\eta_1$
transfers state from $x(t_0)=0$ to $x(t_1)=x_1$
- $$x(t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)\underbrace{\left[B(t)^T\Phi(t_1,t)^T\eta_1\right]}_{u(\tau)}d\tau$$

$$= W_R(t_0,t_1)\eta_1 = x_1$$

② Show that $x_1 \in R[t_0, t_1] \Rightarrow x_1 \in \operatorname{Im} W_R(t_0, t_1)$

If x_1 is in the reachable subspace then

$$x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau \qquad \text{(definition)}$$

- Going to show that

$x_1^T \eta_1 = 0$ when $\eta_1 \in \ker W_R(t_0, t_1)$

(Remember $\operatorname{Im} W_r(t_0, t_1) = (\ker W_R(t_0, t_1))^\perp$

→ Pick an arbitrary vector ~~η~~ $\eta_1 \in \ker W_R(t_0, t_1)$

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \Phi(t_1, \tau)^T \eta_1\, d\tau$$

Since $\eta_1 \in \ker W_R(t_0, t_1)$

$$\eta_1^T W_R(t_0, t_1) \eta_1 = \int_{t_0}^{t_1} \eta_1^T \Phi(t_1, \tau) B(\tau) B(\tau)^T \Phi(t_1, \tau)^T \eta_1\, d\tau$$

$$= \int_{t_0}^{t_1} \| B(\tau)^T \Phi(t_1, \tau)^T \eta_1 \|^2 d\tau = 0$$

$$\Rightarrow B(\tau)^T \Phi(t_1, \tau)^T \eta_1 = 0 \quad \forall \tau \in [t_0, t_1]$$

→ And $x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T \underbrace{B(\tau)^T \Phi(t_1, \tau)^T \eta_1}_{=0}\, d\tau = 0$

$\therefore\ x_1^T \in$ ~~Im~~ $(\ker W_R(t_0, t_1))^\perp = \operatorname{Im} W_R(t_0, t_1)$

Controllability Gramnian : Allows us to compute the controllable subspace

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B(\tau)^T \Phi(t_0, \tau)^T d\tau$$

$$\mathcal{C}[t_0, t_1] = \operatorname{Im} W_c(t_0, t_1)$$

w/ control: $u(t) = -B(t)^T \Phi(t_0, t)^T \eta_0$

lets transfer from $X(t_0) = X_0$ to $X(t_1) = 0$
(back to the origin)