Controllable Decompositions:

What if the system is not controllable?

Note: the controllable subspace $\mathcal{C}$ of the system is A-invariant + contains Im B (HW problem)

$\Rightarrow$ Let rank $\mathcal{C} = \bar{n} < n$

Break up $\mathcal{C}$ s.t.

$$\underset{(n \times nk)}{\mathcal{C}} = [\underset{n\times\bar{n}}{u_1} \quad \underset{n\times(n-\bar{n})}{u_2}]$$

where $\text{span}(u_1) = \text{Im}\,\mathcal{C}$

$\text{span}(u_2) = \ker \mathcal{C}^T$

* span of the cols of a matrix is same as the range, image, or column space of the matrix.
* together u_1 + u_2 span all of $\mathbb{R}^n$ $\downarrow$ form a basis for all $\mathbb{R}^n$

$\longrightarrow$ Since Im $\mathcal{C}$ is A-invariant

$$Au_1 = u_1 A_c = [u_1 \quad u_2]\begin{bmatrix} A_c \\ 0 \end{bmatrix}$$

linear combination of the cols of u_1

$\longrightarrow$ Also, since $\mathcal{C} = [B \quad AB \ \cdots \ A^{n-1}B]$ we have $\text{span}(B) \leq \text{Im}\,\mathcal{C}$

$$= [I \quad A \ \cdots \ A^{n-1}]B$$

$\text{Im}\, B \subset \mathcal{C}$ $\longrightarrow$ The columns of B can be written as a linear combination of the cols of u_1

proper subset.

$$\Rightarrow B = u_1 B_c \quad \Rightarrow \quad B = [u_1 \quad u_2]\begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

~~Note:~~ $$A[u_1 \quad u_2] = [\underbrace{Au_1}_{u_1 A_c} \quad Au_2] = [u_1 \quad u_2]\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}$$

$$[u_1\ u_2]^{-1} A [u_1\ u_2] = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}$$

let $U = [u_1\ u_2]$ Then

$$U^{-1}AU = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \qquad U^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

Thm 13.2 — For any LTI system (A, B), there is a similarity transform T, s.t.

$$T^{-1}AT = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

s.t.

① The controlable subspace of the transformed system is

$$\operatorname{Im} \bar{\mathcal{C}} = \operatorname{Im} \begin{bmatrix} I_{\bar{n} \times \bar{n}} \\ 0 \end{bmatrix}$$

② (A_c, B_c) is controllable

<u>Proof:</u>

$$\begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix}\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = \begin{pmatrix} A_c^2 & A_cA_{12}+A_{12}A_u \\ 0 & A_u^2 \end{pmatrix}$$

this term keeps growing, but it doesn't matter.

① $$\bar{\mathcal{C}} = \left(\begin{bmatrix} B_c \\ 0 \end{bmatrix} \ \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} B_c \\ 0 \end{bmatrix} \cdots \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1}\begin{bmatrix} B_c \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{n-1}B_c \\ 0 & 0 & & 0 \end{bmatrix} \Big\} \text{ 1st } \bar{n} \text{ rows}$$

→ Since $\bar{\mathcal{C}}$ is rank $\bar{n}$ the first $\bar{n}$ rows are linearly independent.

$$\Longrightarrow \operatorname{Im}\{\bar{\mathcal{C}}\} = \operatorname{Im}\left\{ \begin{matrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{matrix} \right\}$$

(2) By Cayley-Hamilton theorem —

$$\operatorname{rank}\left[B_c \;\; A_cB_c \;\cdots\; A_c^{n-1}B_c\right] = \operatorname{rank}\left[B_c \;\; A_cB_c \;\cdots\; A_c^{\bar{n}-1}B_c\right] = \bar{n}$$

can write higher order terms as linear combination of lower order ones.

$\therefore$ (A_c, B_c) is controllable

<u>Block Diagram:</u>

$$\bar{x} = U^{-1}x = \begin{bmatrix} x_c \\ x_u \end{bmatrix}$$

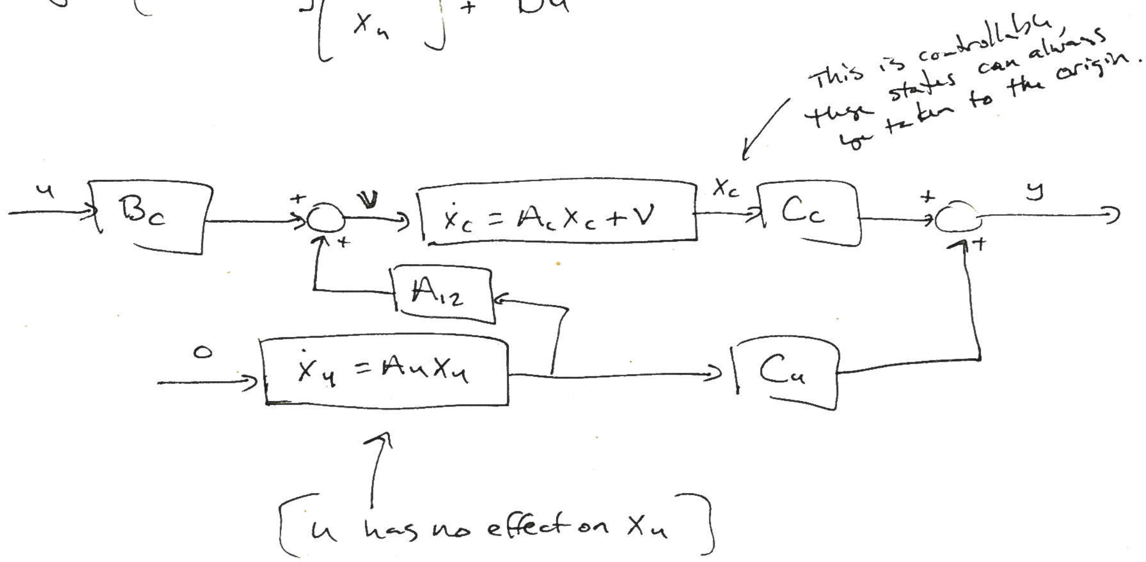
$$\begin{pmatrix} \dot{x}_c \\ \dot{x}_u \end{pmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix}u$$

or

$$\dot{x}_c = A_cx_c + A_{12}x_u + B_cu$$

$$\dot{x}_u = A_ux_u$$

$$y = \begin{bmatrix} C_c & C_u \end{bmatrix}\begin{bmatrix} x_c \\ x_u \end{bmatrix} + Du$$



Transfer Function :

- Transfer Functions are the same for all state space realizations.

- Compute TF from the transformed system —

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{pmatrix} sI-A_c & -A_{12} \\ 0 & sI-A_u \end{pmatrix}^{-1} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D$$

- Inverting an upper triangle matrix —

$$\hat{G}(s) = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{pmatrix} (sI-A_c)^{-1} & (\text{something}) \\ 0 & (sI-A_u)^{-1} \end{pmatrix} \begin{pmatrix} B_c \\ 0 \end{pmatrix}$$

$$= C_c(sI-A_c)^{-1}B_c + D$$

⟶ The TF of the system is equal to the TF of the controllable part.