Motivation:

- Given an LTI System $\dot{x} = Ax + Bu$
  $y = Cx + Du$

* we learned that if (A,B) - stabilizable then the feedback control will asymptotically stabilize the CLTI system. $(A-BK)$ is a stability matrix.

* But, now assume we can only measure the output y (rather than the entire state).

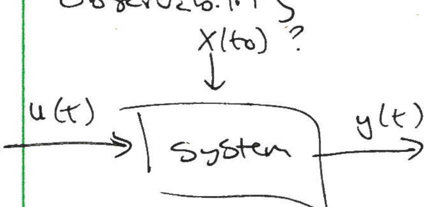
* Then we can't implement state feedback unless we can reconstruct the state of the system based on measured output.

Two Definitions:

① Observability — determining $x(t_0)$ from future inputs + outputs $u(t)$ + $y(t)$, $t \in [t_0, t_1]$.

② Constructability — determining $x(t_1)$ from past inputs + outputs $u(t)$ + $y(t)$, $t \in [t_0, t_1]$

Observability



* Can figure out what $x(t_0)$ equals (eventually) by watching the inputs/outputs.

Given an LTV Solution — (know solution = variation of constants formula)

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)$$

can we solve for this?

If C matrix was invertible could simple solve for X

$$y = Cx + Du \longrightarrow x = C^{-1}(y - Du)$$

→ Usually C matrix is not invertible $(m<n)$

→ put everything we know on the left hand side —

$$y - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau - D(t)u(t) = \underset{\substack{\uparrow \\ \text{generally} \\ \text{not invertible}}}{C(t)}\Phi(t,t_0)x_0$$

$$(C(t)\Phi(t,t_0))^T C(t)\Phi(t,t_0) = \underbrace{\Phi(t,t_0)^T C^T(t) \overset{(m\times n)}{C(t)} \overset{(n\times n)}{\Phi(t,t_0)}}$$

— sq. matrix, but does not have full rank, so not invertible

— (rank at most $m < n$)

— left-inverse will not be invertible.

— But, hopefully over time we will sweep out a large enough area to span $\mathbb{R}^n$ space.

~~$(C(t)\Phi(t,t_0))^T C$~~

$$\int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)\left[y - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau - D(t)u(t)\right] dt$$

$$= \underbrace{\int_{t_0}^{t_1} \Phi^T(t,t_0)\, C^T(t)C(t)\Phi(t,t_0)\,dt}\, x_0$$

— observability Grammian —
If this is pos-definite (or equivalently the rank$(\mathrm{Im}(W_0)) = n$) then will be invertible & can solve for x_0.

Corrallary 15.1: The system is observable iff rank$(W_0[t_0,t_1]) = n$

Definition: The $\ker(C(t)\Phi(t,t_0))$, i.e. all states $x_0 \in \mathbb{R}^n$, s.t. $C(t)\Phi(t,t_0)x_0 = 0 \quad \forall t \in [t_0, t_1]$ is called the <u>unobservable subspace</u> $\mathcal{UO}[t_0,t_1]$

Definition: The system is observable only if $\ker(C(t)\Phi(t,t_0)) = \{\phi\}$, i.e. $\mathcal{UO}[t_0,t_1] = \{0\}$

~~Corollary 15.1: The system is observable iff~~
~~$\text{rank}(W_0[t_0,t_1]) = n$~~
~~Proof: Since $W_0 = W_0^T$ we have that~~
~~$\ker(W_0) = \ker(W_0^T)$ we also know that~~
~~$\ker(W_0^T) = \ker((C(t)\Phi(t,t_0))^T(C(t)\Phi(t$~~

Thm 15.1: Given $t_1 > t_0 \geq 0$ $\quad \mathcal{UO}[t_0,t_1] = \ker W_0(t_0,t_1)$
w/ $W_0(t_0,t_1) = \int_{t_0}^{t_1} \Phi(\tau,t_0)^T C(\tau)^T C(\tau)\Phi(\tau,t_0)\,d\tau$.

Proof - for $x_0 \in \mathbb{R}^n$

$$x_0^T W_0(t_0,t_1)x_0 = \int_{t_0}^{t_1} x_0^T \Phi(\tau,t_0)^T C(\tau)^T C(\tau)\Phi(\tau,t_0)x_0\,d\tau$$

$$= \int_{t_0}^{t_1} \|C(\tau)\Phi(\tau,t_0)x_0\|^2\,d\tau$$

$\Rightarrow x_0 \in \ker W_0(t_0,t_1) \Rightarrow C(\tau)\Phi(\tau,t_0)x_0 = 0 \quad \forall \tau \in [t_0,t_1]$

Conversely -

$x_0 \in \mathcal{UO}[t_0,t_1] \Rightarrow C(\tau)\Phi(\tau,t_0)x_0 = 0 \quad \forall \tau \in [t_0,t_1]$
$\Rightarrow x_0 \in \ker W_0(t_0,t_1)$

(because W_0 is pos semi-definite, so $x^T W_0 x = 0 \Rightarrow W_0 x = 0$)

$\Rightarrow$ system is observable iff rank $W_0(t_0,t_1) = n$ (~~i.e. ker W_0,~~
(i.e. nullity $(\ker W) = 0$) $\quad$ Grammian matrix is nonsingular

Constructability ~~Gramian~~ Gramian :

$$W_{cn}(t_0,t_1) = \int_{t_0}^{t_1} \Phi(\tau,t_1)^T C(\tau)^T C(\tau)\Phi(\tau,t_1)\,d\tau$$

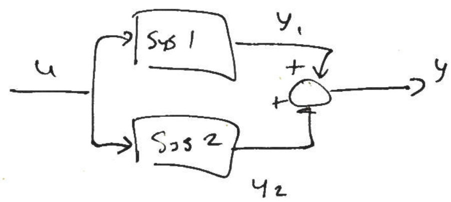
→ Can construct the same way using —

$$y(t) = C(t)\Phi(t,t_1)x_1 + \int_{t_1}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)$$

$$\forall t \in [t_0,t_1]$$

→ "constructing" the future state based upon past states from t_0 to t_1.

Example: Parallel Interconnection



$\dot{x}_1 = A_1 x_1 + B_1 u$

$y_1 = C_1 x_1$

$\dot{x}_2 = A_2 x_2 + B_2 u$

$y_2 = C_2 x_2$

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad \text{w/} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = [C_1 \;\; C_2]\, x$$

$$y(t) = \cancel{C_1 e^{A_1 t} + C_2 e^{A_2 t}}$$

$$C_1 e^{A_1 t} x_1(0) + C_2 e^{A_2 t} x_2(0) + \int_0^t \left(C_1 e^{A_1(t-\tau)} B_1 + C_2 e^{A_2(t-\tau)} B_2 \right) u(\tau)\, d\tau$$

When $A_1 = A_2 = A$ + $C_1 = C_2 = C$

$$y(t) = Ce^{At} \underbrace{(x_1(0) + x_2(0))} + \int_0^t C e^{A(t-\tau)} (B_1 + B_2) u(\tau)\, d\tau$$

- Even knowing the input + output of the system we won't be able to ~~fi~~ figure out what the initial states will be.
- only can see the combined initial states.

Duality : For an LTI System : $\dot{x} = Ax + Bu$

$y = Cx + Du$

$$W_o(t_0,t_1) = \int_{t_0}^{t_1} e^{A^T(\tau - t_0)} C^T C e^{A(\tau - t_0)} d\tau = \int_0^{t_1 - t_0} e^{A^T t} C^T C e^{At} dt$$

Recall the ~~controllability~~ reachability Gramian —

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} e^{A(\tau - t_0)} BB^T e^{A^T(\tau - t_0)} d\tau = \int_0^{t_1 - t_0} e^{At} BB^T e^{A^T t} dt$$

where we have —

(A,B) is ~~controllable~~ reachable/controllable (LTI) $\Leftrightarrow \text{rank}(W_R) = n$

&

(A,C) is observable $\Leftrightarrow \text{rank}(W_o) = n$

Note that if A & B in W_R is replaced w/ A^T, C^T, then the result is W_o, i.e.

$$W_R(A,B) = W_o(A^T, B^T)$$

& $$W_o(A,C) = W_R(A^T, C^T)$$

Thm 15.5 (Duality for LTI systems)

(A,B) - reachable $\Leftrightarrow$ (A^T,B^T) - observable

(A,C) - observable $\Leftrightarrow$ (A^T,C^T) - reachable

Thm 15.6

(A,B) - controllable $\Leftrightarrow$ (A^T,B^T) - constructable

(A,C) - constructable $\Leftrightarrow$ (A^T,C^T) - ~~reachable~~ controllable

- Using the duality result we can find tests for observability.

$$\mathcal{C}(A,B) = [B \;\; AB \; \cdots \; A^{n-1}B]$$

Then $\mathcal{C}(A^T, C^T) = [C^T \;\; A^TC^T \; \cdots \; (A^T)^{n-1}C^T]$

And $\mathcal{C}^T(A^T, C^T) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Define $\mathcal{O}(A,C) \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ = the observability matrix.

$(\operatorname{rank}(\mathcal{C}) = \operatorname{rank}(\mathcal{C}^T))$

Thm 15.7: (A,C) - observable iff $\operatorname{rank} \mathcal{O}(A,C) = n$

Alternate Proof:

Let $\dot{x} = Ax$, $y = Cx$ $\quad x(t_0) = x_0$

Then $y(t) = Ce^{At}x_0$

Also, $\dot{y}(t) = CAe^{At}x_0$

$\ddot{y}(t) = CA^2e^{At}x_0$

$\vdots$

$y^{(n-1)}(t) = CA^{n-1}e^{At}x_0$

$\therefore$ evaluating at time $t=0$

$$\begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = \mathcal{O}(A,C)\,x_0$$

So if (A,C) - observable Then $x_0 = \mathcal{O}^{-1}(A,C)\begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix}$

Tests for Observability: (LTI Systems)

① rank $\mathcal{O} = n$

② No E-vector of A is in ker C (e-vector test)

③ PBH test

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}$$

④ Lyapunov Test — If unique pos-definite solution to W of $A^T W + WA = -C^T C$

w/ unique solution

$$W = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau}\, d\tau = \lim_{t_1 - t_0 \to \infty} W_o(t_0, t_1)$$