Jordan Form

Since $\det(sI-A) = p$ is a n^{th} order polynomial by the fundamental theorem of algebra there are n roots, or eigenvalues, some of which may be repeated.

Let $\det(sI-A) = (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_p)^{m_p} = 0$

Def — The algebraic multiplicity of λ_i is m_i

Def — the geometric multiplicity of λ_i is

$$q_i = \dim\{N(\lambda_i I - A)\} \longrightarrow \text{dimension of the nullspace of } N(\lambda_i I - A)$$

Example: Double Integrator

$\dot y_1 = u$
$\dot y_2 = y_1$
$y = y_2$

$u \longrightarrow \boxed{\int} \xrightarrow{y_1} \boxed{\int} \xrightarrow{y_2}$ ~~$x_1 = y_1$~~ ~~$\dot x_1 = y_1$~~ ~~$x_2 = y_1$~~ ~~$\dot x_2 =$~~

$\boxed{x_1 = y_2}$ $\quad \dot x_1 = \dot y_2 = y_1 = x_2$

$\boxed{x_2 = y_1 \quad \dot x_1 = y_1}$

$\dot x_2 = \dot y_1 = u$

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$y = x_1$

$$\det(sI-A) = \det\begin{bmatrix}s & -1\\ 0 & s\end{bmatrix} = (s-0)^2$$

$\Longrightarrow \lambda_1 = 0$ has algebraic multiplicity $m_1 = 2$

$N\{(0I-A)\} = N\left\{\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\right\}$ or $\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \Longrightarrow -x_2 = 0$

$\Longrightarrow$ Null space spanned by $\begin{pmatrix}1\\ 0\end{pmatrix} \Longrightarrow \dim\{N(\lambda_1 I - A)\} = 1$

$\Longrightarrow$ geometric multiplicity is $q_i = 1$

$\longrightarrow$ In this case there are not two linearly independent eigenvectors and we can't diagonalize A with a similarity transformation.

- To diagonalize we instead find "generalized eigenvectors.

- Given (λ_1, v_1) s.t. $Av_1 = \lambda_1 v_1$, the generalize e-vectors are found by solving the following <u>Chain</u>

$$Av_2 = \lambda_1 v_2 + v_1$$
$$Av_3 = \lambda_1 v_3 + v_2$$
$$\vdots$$
$$Av_{m_i} = \lambda_1 v_{m_i} - v_{m_i - 1}$$

m_i = algebraic multiplicity

Example: Double Integrator

$$\lambda_1 = 0, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Find v_2 by solving : $Av_2 = \lambda_1 v_2 + v_1$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \underbrace{0}_{\lambda_1} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} + \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \Rightarrow v_{22} = v_{11} = 1$$

$\Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a generalized e-vector of A

Note — we can write the <u>chain</u> as:

$$A[v_1 \; v_2 \cdots v_{m_i}] = [v_1 \; v_2 \cdots v_{m_i}] \underbrace{\begin{bmatrix} \lambda_1 & 1 & & \phi \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ \phi & & & \lambda_1 \end{bmatrix}}_{\text{Jordan block}}$$

- It can be shown that there are $m_i - 1$ ~~linearized~~ linearly independent generalized eigenvectors associated w/ each eigenvalue.

- In our example $m_i = 2$ so we found 1 generalized e-vector.

If $q_i = 1$, then there is only one Jordan block w/ $m_i - 1$ generalized e-vectors generatable by v_i

However, if $1 < q_i < m_i$, then there are q_i linearly independent eigenvectors, each of which can be used to generate generalized eigenvectors, and there are several potentially different ~~block~~ possible Jordan blocks.

— For example — if $m_i = 4$ + $q_i = 2$ the possible Jordan blocks are —

$$\begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \text{ and } \lambda_i$$

or $$\begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}$$

or $$\lambda_i \text{ and } \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix}$$

Which one is it?

* To decide, generate the possible generalized eigenvector for each e-value and pick the linearly independent one.

Example:

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\det(sI - A) = (\lambda - 1)^4$ we have $\lambda_1 = 1$, $m_1 = 4$

$$q_1 = \dim\left(N \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = 2$$ — since there are two linearly independent rows.

So there are 2 linearly intependent eigenvectors:

$$(\lambda_1 I - A) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So $x_4 = 0$, $x_2 = x_3$

Eigenvectors are —

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \& \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Find possible generalized e-vectors —

→ those associated w/ v_1 :

$$Av_3 = 1 \cdot v_3 + v_1 \implies (\lambda_1 I - A) v_3 = v_1$$

$$\begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} x_3 &= 1 + x_2 \\ x_4 &= 0 \end{aligned}$$

Could use — $v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ which is linearly intependent of v_1 + v_2

Try $Av_4 = \lambda_1 v_4 + v_3 \implies \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ ← not possible

So try, $Av_4 = \lambda_1 v_4 + v_2 \implies \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ which is linearly intependent

→ So we have v_1 and we generated one e-vector from v_1 (v_3). We couldn't generate a 2nd generalized e-vector from v_1 (through v_3) so the Jordan block for this part is

$$\cancel{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}} \qquad A\,[v_1 \ \ v_3] = [v_1 \ \ v_3]\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

$$A v_1 = v_1 \lambda$$

$$A v_3 = v_1 + \lambda_1 v_3$$

Jordan Block $= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

→ w/ 2nd e-vector v_2 was also able to generate a generalized e-vector.

Jordan Block $= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

→ Full ~~block~~ Jordan block is

$$\underbrace{A\,[v_1 \ \ v_3 \ \ v_2 \ \ v_4]}_{P^{-1}} = \underbrace{[v_1 \ \ v_3 \ \ v_2 \ \ v_4]}_{P^{-1}}\underbrace{\begin{bmatrix} \phi & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{J_1}$$

$$J = PAP^{-1} \Rightarrow A = P^{-1} J P$$

$$A^2 = AA = P^{-1}JPP^{-1}JP = P^{-1}J^2P$$

$$A^k = \qquad\qquad = P^{-1}J^kP$$

$$\Rightarrow e^{At} = P^{-1} e^{Jt} P \quad \left(\text{because } e^{At} = \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k\right)$$

If $J = \begin{pmatrix} J_1 & & \phi \\ & \ddots & \\ \phi & & J_e \end{pmatrix}$ where J_i is a Jordan block,

Then $e^{At} = P^{-1} \begin{pmatrix} e^{J_1 t} & & \phi \\ & \ddots & \\ \phi & & e^{J_e t} \end{pmatrix} P$

$\rightarrow$ we need a formula for $e^{J_i t}$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}$$

$$e^{J_i t} = e^{\lambda_i t} \underbrace{\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{n_i-2} \\ 0 & 0 & \phi & t & \cdots & \frac{t^{n_i-3}}{n_i-3} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & t \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}}_{\text{call this } M \rightarrow \text{ for proof (next page)}}$$

Proof –

Show that $e^{J_i \cdot 0} = I$ and $\frac{d}{dt} e^{J_i t} = J_i e^{J_i t}$

$$e^{J_i \cdot 0} = e^0 \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \ddots & \\ 0 & \cdots & & \cdots & 1 \end{bmatrix} = I$$

$$\frac{d}{dt} e^{J_i t} = \frac{d}{dt}\left(e^{\lambda_i t} M\right) = \lambda_i e^{\lambda_i t} M(t) + e^{\lambda_i t} \frac{dM}{dt}$$

$$= \lambda_i e^{J_i t} + e^{\lambda_i t} \frac{dM}{dt}$$

$$\frac{dM}{dt} = \begin{bmatrix} 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} \\ & & \ddots & & & \\ & & & & \ddots & 1 \\ 0 & \cdots & & \cdots & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 1 \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} M$$

$$\Rightarrow \frac{d}{dt} e^{J_i t} = \lambda_i e^{J_i t} + e^{\lambda_i t} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 1 \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} M = \lambda_i e^{J_i t} + \left[\quad \right] \underbrace{e^{\lambda_i t} M}_{e^{J_i t}}$$

$$= \left(\lambda_i + \left[\quad \right]\right) e^{J_i t} = J_i e^{J_i t}$$