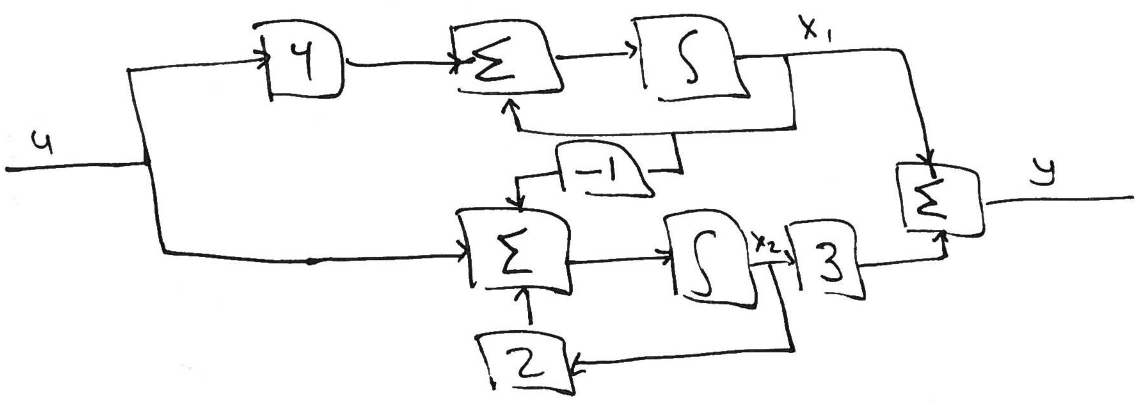
1.1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



2.1

$$T = sat(u) = \begin{cases} u & -1 \le u \le 1 \\ -1 & u < -1 \\ 1 & u > 1 \end{cases}$$

$$ml^2 \ddot{\theta} = mgl \sin\theta - b\dot{\theta} + T$$

$\theta = x_1$ $\quad \dot{x}_1 = x_2$

$\dot{\theta} = x_2$ $\quad \dot{x}_2 = \frac{1}{ml^2}(mgl \sin x_1 - bx_2 + T)$

a) $\theta = 0 \Rightarrow x_1 = 0$

$\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow \frac{1}{ml^2}(mgl\sin(0) - 0 + T) = 0$

$T = 0$

equilibrium point:

$x_1 = 0, \ x_2 = 0, \ u = 0$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}\cos x_1 & -\frac{b}{m\ell^2} \end{bmatrix}_{\substack{x_1=0 \\ x_2=0 \\ u=0}} = \begin{bmatrix} 0 & 1 \\ g/\ell & -b/m\ell^2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}_{\substack{x_1=0 \\ x_2=0 \\ u=0}} = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}_{\substack{x_1=0 \\ x_2=0 \\ u=0}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

b) linearize around $\theta = \pi = x_1$

$\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow \frac{1}{m\ell^2}\left(mg\ell \cancelto{0}{\sin(\pi)} - b\cancelto{0}{x_2} + T\right) \Rightarrow T = \text{sat}(u) = 0$

Eq. Point: $x_1 = 0,\ x_2 = 0,\ u = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -g/\ell & -b/m\ell^2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

c) linearize around $\theta = \frac{\pi}{4}$

$\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow \frac{1}{m\ell^2}\left(mg\ell\left(\overset{\frac{\sqrt{2}}{2}}{\downarrow}\frac{\sqrt{2}}{2}\right) + T\right) = 0 \Rightarrow \text{sat}(u) = -\frac{mg\ell\sqrt{2}}{2}$

$\Rightarrow \frac{mg\ell\sqrt{2}}{2} \in [-1, 1]$

$$A = \begin{bmatrix} 0 & 1 \\ +\frac{g}{\ell}\frac{\sqrt{2}}{2} & -\frac{b}{m\ell^2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

d) $b = \frac{1}{2}$, $mg\ell = 1/4$ Compute T needed

for fall from $\theta(0) = 0$ @ $\dot\theta(t) = 1$ $\forall t \geq 0$

$x_1(0) = 0$ $\qquad \dot x_1 = 1 \Rightarrow x_1 = t$

$x_2(t) = 1$ $\qquad \dot x_2 = 0$

$$\dot x_2 = 0 \Rightarrow \frac{1}{m\ell^2}\left(\frac{1}{4}\sin x_1 - \frac{1}{2}\overset{1}{x_2} + T\right) = 0$$

$$\Rightarrow \frac{1}{4}\sin x_1 - 1/2 + T = 0$$

$$\Rightarrow T(t) = \frac{1}{2} - \frac{1}{4}\sin t \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall t \geq 0$$

- least when $\sin t = 1$
  $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
- most when $\sin t = -1$
  $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}\cos t & \frac{-b}{m\ell^2} \end{bmatrix}, \quad B = \begin{pmatrix} 0 \\ 1/m\ell^2 \end{pmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- this is a function of t

2.2 – Local linearization & trajectory

a)

$u = [v \ \ w]^T$

$\dot{\theta} = w$

$\dot{p}_x = v\cos\theta$

$\dot{p}_y = v\sin\theta$

$x_1 = p_x\cos\theta + (p_y - 1)\sin\theta$

$$\dot{x}_1 = \dot{p}_x\cos\theta + -p_x\sin\theta\,\dot{\theta} + (\dot{p}_y\sin\theta) + (p_y - 1)\cos\theta\,\dot{\theta}$$

$$= (v\cos^2\theta) - p_x\sin\theta\, w + v\sin^2\theta + (p_y - 1)\cos\theta\, w$$

$$= -p_x\sin\theta\, w + v + (p_y - 1)\cos\theta\, w = \underline{v + x_2 w}$$

$x_2 = -p_x\sin\theta + (p_y - 1)\cos\theta$

$$\dot{x}_2 = -\dot{p}_x\sin\theta - p_x\cos\theta\,\dot{\theta} + \dot{p}_y\cos\theta + -(p_y - 1)\sin\theta\,\dot{\theta}$$

$$= -v\cancel{\sin\theta\cos\theta} - p_x\cos\theta\, w + v\cancel{\sin\theta\cos\theta} - (p_y - 1)\sin\theta\,\dot{\theta}$$

$$= -(p_x\cos\theta\, w + (p_y - 1)\sin\theta\, w) = \underline{-x_1 w}$$

$x_3 = \theta$

$\dot{x}_3 = \dot{\theta} = w$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} v + x_2 w \\ -x_1 w \\ w \end{bmatrix}$$

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) linearize @ $x^{eq}=0,\ u^{eq}=0$

$$A = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \cancel{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$w=0$
$x_1=0$
$x_2=0$
$\vdots$
etc.

$$B = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c) Show $\left.\begin{aligned} w(t) &= v(t) = 1 \\ p_x(t) &= \sin t \\ p_y(t) &= 1-\cos t \\ \theta(t) &= t \end{aligned}\right\}$ Show this is a solution to the system

$$x(t) = \begin{bmatrix} p_x\cos\theta + (p_y-1)\sin\theta \\ -p_x\sin\theta + (p_y-1)\cos\theta \\ \theta \end{bmatrix} = \begin{bmatrix} \sin t\cos t + (1-\cos t-1)\sin t \\ -\sin^2 t + (1-\cos t-1)\cos t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$$

$$\dot{x}(t) = \frac{d}{dt}\begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From part a)

$$\dot{x} = \begin{bmatrix} v + x_2 \omega \\ -x_1 \omega \\ \omega \end{bmatrix} \overset{\text{plugging in values}}{=} \begin{pmatrix} 1 + x_2 \\ -x_1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 + -p_x \sin\theta + (p_y - 1)\cos\theta \\ -p_x \cancel{\sin\theta}\,\cos\theta - (p_y - 1)\cancel{\cos\theta}\,\sin\theta \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \sin^2 t + (1 - \cos t - 1)\cos t \\ -\sin t \cos t - (\cancel{1} - \cos t - \cancel{1})\sin t \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ the parameters solve the system.

d) local linearization @ this system

$$A = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Bigg|_{\omega = 1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}_{\substack{x^{eq} \\ u^{eq}}} = \begin{bmatrix} 1 & -p_x \sin\theta + (p_y - 1)\cos\theta \\ 0 & -p_x \cos\theta - (p_y - 1)\sin\theta \\ 0 & 1 \end{bmatrix}_{\substack{x^{eq} \\ u^{eq}}} = \begin{pmatrix} 1 & -\sin^2 t - \cos^2 t \\ 0 & -\sin t \cos t + \sin t \cos t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ LTI system

4.3

— Given a TF $\hat{G}(s)$ let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be a realization for its transpose $\bar{G}(s) = \hat{G}(s)^T$

$A = \bar{A}^T$, $B = \bar{C}^T$, $C = \bar{B}^T$, $D = \bar{D}^T$

$\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$

$$\hat{G}(s) = \bar{G}(s)^T = (\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D})^T$$

$$= \bar{D}^T + \bar{B}^T((sI - \bar{A})^{-1})^T \bar{C}^T$$

$$= \bar{D}^T + \bar{B}^T(sI - \bar{A}^T)^{-1}\bar{C}^T$$

$$= C(sI - A)^{-1}B + D$$

4.5 — show zero-state equivalence, but are not algebraically equivalent.

a) $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} s-1 & 0 \\ 0 & s-1 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{(s-1)}{(s-1)^2} = \frac{1}{(s-1)}$$

E-values $= \lambda = 1$ (duplicates)

$\dot{\bar{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\quad \hat{G}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}$

$$= \frac{1}{(s-1)(s-2)} \begin{bmatrix} (s-2) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s-1)}$$

E-values $= \lambda = 1, 2$

$\longrightarrow$ E-values not preserved so aren't algebraically equivalent.

b)

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\hat{G}(s) = C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s-1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(s-1)^2} = \frac{1}{(s-1)}$$

E-values $= \lambda = 1$ (duplicate)

$$\dot{\bar{x}} = \bar{x} + u \qquad [1](s-1)^{-1}[1] = \frac{1}{s-1}$$

$$y = \bar{x}$$

E-values $\Rightarrow \lambda = 1$ (not duplicate)

$\rightarrow$ TF are same $\Rightarrow$ zero-state equivalent

$\rightarrow$ E-values are not preserved, so not algebraically equivalent.

$\rightarrow$ Also, similarity matrix T must be sq. to be invertible. Therfore can't convert from one dimension to another.

5.2

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix} x$$

a) $\dot{x}_1 = t x_2$

$\dot{x}_2 = 2x_2 \longrightarrow \dot{x}_2 - 2x_2 = 0$ $\qquad \frac{d}{dt}(e^{-2t}x_2) = -2e^{-2t}x_2 + e^{-2t}\dot{x}_2$

multiply by e^{-2t}

$\longrightarrow e^{-2t}(\dot{x}_2 - 2x_2) = 0$

$= \frac{d}{dt}(e^{-2t}x_2) = 0$

$= \int_{t_0}^{t} \frac{d}{dt}(e^{-2t}x_2)\,dt = 0$

$= e^{-2t}x_2 \Big|_{t_0}^{t} = e^{-2t}x_2(t) - e^{-2t_0}x_2(t_0) = 0$

$e^{-2t}x_2(t) = e^{-2t_0}x_2(t_0)$

$$\boxed{x_2(t) = e^{2(t-t_0)}x_2(t_0)}$$

$\dot{x}_1 = t x_2 = t\left(e^{2(t-t_0)}x_2(t_0)\right)$

$\longrightarrow$ Integrate both sides

$\int_{t_0}^{t} \frac{dx_1}{dt}\,dt = x_2(t_0)\int_{t_0}^{t} t e^{2(t-t_0)}\,dt$

$\Rightarrow x_1(t) - x_1(t_0) = x_2(t_0)\left[\frac{t}{2}e^{2(t-t_0)} - \frac{1}{4}e^{2(t-t_0)}\right]_{t_0}^{t}$

$= x_2(t_0)\left[\frac{t}{2}e^{2(t-t_0)} - \frac{1}{4}e^{2(t-t_0)} - \frac{t_0}{2} + \frac{1}{4}\right]$

$x_1(t) = x_1(t_0) + \frac{x_2(t_0)}{4}\left(1 - 2t_0 - (1-2t)e^{2(t-t_0)}\right)$

$$x_2(t) = e^{2(t-t_0)}\, x_2(t_0)$$

$$x_1(t) = x_1(t_0) + \frac{1}{4}\left(1 - 2t_0 - (1-2t)e^{2(t-t_0)}\right) x_2(t_0)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{4}\left(1-2t_0 - (1-2t)e^{2(t-t_0)}\right) \\ 0 & e^{2(t-t_0)} \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

But $x = \Phi(t, t_0)\, x(t_0)$ So —

$$\Phi(t, t_0) = \begin{bmatrix} 1 & \frac{1}{4}\left(1-2t_0 - (1-2t)e^{2(t-t_0)}\right) \\ 0 & e^{2(t-t_0)} \end{bmatrix}$$

b) Compute the system output to constant input $u(t) = 1$, $\forall t \geq 0$

$$y(t) = C(t)\Phi(t,t_0)\, x_0 + \int C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Phi(t,t_0) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} + \underbrace{\int_{t_0}^{t} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{4}\left(1-2\tau-(1-2t)e^{2(t-\tau)}\right) \\ 0 & e^{2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} d\tau}_{\text{Solve the integral}}$$

$$= \int_{t_0=0}^{t} \frac{\tau}{4}\left(1-2\tau-(1-2t)e^{2(t-\tau)}\right) d\tau = \frac{1}{4}\left(\frac{\tau^2}{2} - \frac{2\tau^3}{3}\right] + \frac{1}{16}\left[-2t(1+2\tau) + (1+2\tau)\right]e^{2(t-\tau)} \Bigg|_0^t$$

$$= \frac{1}{16}\left[\frac{4t^2}{2} - \frac{8}{3}t^3 - 2t(1+2t) + (1+2t) + (-2t+1)e^{2t}\right]$$

$$= \frac{1}{16}\left[\frac{4t^2}{2} - \frac{8}{3}t^3 - \cancel{2t} - 4t^2 + 1 + \cancel{2t} + (1-2t)e^{2t}\right]$$

$$= \frac{1}{16}\left(-2t^2 - \frac{8}{3}t^3 + 1 \mp (1-2t)e^{2t}\right)$$

$$= \frac{1}{48}\left(-6t^2 - 8t^3 + 3 \mp 3(1-2t)e^{2t}\right)$$

Add integral solution to I.C. part of y —

$$y = x_1(0) + \frac{1}{4}\left(1 - 2t_0 - (1-2t)e^{2(t-t_0)}\right)x_2(0)$$

$$+ \frac{1}{48}\left(-6t^2 - 8t^3 + 3 - 3(1-2t)e^{2t}\right)$$

$$= \frac{1}{48}\left[48x_1(0) + 12\left(1 - \overset{0}{\cancel{2t_0}} - (1-2t)e^{2t}\right)x_2(0) - 6t^2 - 8t^3 + 3 - 3(1-2t)e^{2t}\right]$$

$$= \frac{1}{48}\left[48x_1(0) + 12\left(1 - (1-2t)e^{2t}\right)x_2(0) - 6t^2 - 8t^3 + 3 - 3(1-2t)e^{2t}\right]$$