Bounded Input/Bounded Output :

LTV System : $\dot{x} = A(t)x + B(t)u$
$y = C(t)x + D(t)u$

when $x(0) = 0$ — Soln :

$$y_f(t) = \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Def : the system is BIBO stable if there exists $0 < g < \infty$, s.t for every input $u(t)$, the forced response satisfies :

$$\sup_{t\in[0,\infty)} \|y_f(t)\| \le g \sup_{t\in[0,\infty)} \|u(t)\|$$

Supremum = least upper bound. upper value for those included in the set.

* Max is largest # w/ in the set. Sup bounds the set. (may not or may be part of the set)

Note : In the definition the I.C. are assumed to be zero. If they are not zero, then still only use forced response.

Conditions for BIBO Stability : (equivalent statements) (9.2)

1. The CLTV system is uniformly BIBO stable
2. Every entry of $D(t)$ is uniformly bounded and

$$\sup_{t\ge 0} \int_0^t |g_{ij}(t,\tau)|d\tau < \infty$$

for every entry $\underbrace{g_{ij}(t,\tau)}_{\text{Impulse Response of the system}}$ of $C(t)\Phi(t,\tau)B(\tau)$

Proof :

(2) $\Rightarrow$ (1)

$$\|y_f(t)\| \leq \int_0^t \|C(t)\Phi(t,\tau)B(\tau)\| \, \|u(\tau)\| \, d\tau + \|D(t)\| \, \|u(t)\|$$

~~let~~ $\cancel{\mu := \sup_{t\in[0,\infty)} \|u(t)\|}$ $\quad \cancel{\delta := \sup_{t[0,\infty)} \|D(t)\|}$

$$\|y_f(t)\| \leq \underbrace{\left(\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\| \, d\tau + \sup_{t\geq 0} \|D(t)\| \right)}_{g} \sup_{t\geq 0} \|u(t)\|$$

For g to be finite we need:

1) $\sup_{t\geq 0} \|D(t)\| < \infty$, i.e. every $D(t)$ is uniformly bounded.

2) $\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\| \, d\tau < \infty$

But $\|P\| \leq \sum_{i,j} |p_{ij}|$, why? $\qquad \|A+B\| \leq \|A\| + \|B\|$

$$\|P\| = \left\| \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & & & \\ p_{m1} & \cdots & \cdots & p_{mn} \end{pmatrix} \right\| = \left\| \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \begin{pmatrix} 0 & p_{12} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \right.$$

$$\left. + \cdots \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & p_{mn} \end{pmatrix} \right\| \leq \sum \left\| \begin{pmatrix} 0 & & 0 \\ \vdots & p_{ij} & \vdots \\ 0 & & 0 \end{pmatrix} \right\|$$

$$= \sum |p_{ij}|$$

$$\therefore \; g = \sup_{t\geq 0} \int_0^t \|C(t)\Phi(t,\tau)B(\tau)\| \, d\tau + \sup_{t\geq 0} \|D(t)\|$$

$$\leq \sup_{t\geq 0} \sum_{ij} \int_0^t |g_{ij}(t,\tau)| \, d\tau + \sup_{t\geq 0} D(t) < \infty$$

(1) $\Rightarrow$ (2) (Prove by implication, i.e. $\neg(2) \Rightarrow \neg(1)$, 2nd statement false $\Rightarrow$ 1st statement is false)

* Suppose at least one element of $D(t)$ is unbounded.

let $d_{ij}(t)$ = the unbounded element

– Pick an arbitrary time + consider the step input

$$u_T = \begin{cases} 0 & 0 \le \tau < T \\ e_j & \tau \ge T \end{cases} \qquad \forall \tau \ge 0$$

$e_j \in \mathbb{R}^k$ is the jth vector in the cannonical basis of $\mathbb{R}^k$

Note that $\sup_{t \ge 0} \|u_T(t)\| = 1$

And @ time T we have —

$$y_f(T) = \int_0^T C(T)\Phi(T,\tau)\overset{B(\tau)}{\wedge} u(\tau)\,d\tau + D(T)u_t(T) = D(T)u_T(T)$$

$$\Rightarrow \sup_{t \in [0,\infty)} \|y_f(t)\| \ge \|y_f(T)\| = \|D(T)u_T(T)\|$$

$$= \|D(T)e_j\| \cancel{=} \ge |d_{ij}(T)|$$

$\longrightarrow$ the norm must be larger or equal to a single entry of the jth col of D.

$\rightarrow$ Since at least one element of d_{ij} is unbounded, we can make $\sup_{t \in [0,\infty)} \|y_f(t)\|$ arbitrarily large using a bounded input. $\Rightarrow$ Not BIBO stable.

* Now suppose (2) is false because $\sup_{t \ge 0} \int |g_{ij}(t)|dt$ is unbounded. ($D(t)$ is bounded)

~~For some i, j~~

~~let $u_T(t) = \begin{cases} +e_j \\ -e_j \end{cases}$~~

- Again pick an arbitrary T and consider the switching function.

$$u_T(\tau) = \begin{cases} e_j & g_{ij}(t,\tau) \geq 0 \\ -e_j & g_{ij}(t,\tau) < 0 \end{cases} \qquad \tau \geq 0$$

$$y_f(t) = \begin{pmatrix} \int_0^t |g_{ij}(t,\tau)|\,d\tau \\ \vdots \\ \int_0^t |g_{nj}(t,\tau)|\,d\tau \end{pmatrix} \pm \begin{pmatrix} d_{ij}(t) \\ \vdots \\ d_{nj}(t) \end{pmatrix}$$

Note that the output.

$$\sup_{t \in [0,\infty)} \|y_f(t)\| \geq \|y_f(T)\| \geq \underbrace{\int_0^t |g_{ij}(T,\tau)|\,d\tau}\pm d_{ij}(t)$$

— choose this to be arbitrarily large (unbounded)

$\Rightarrow \sup_{t \in [0,\infty)} \|y_f(t)\|$ is not bounded even though the input is bounded.

$\therefore$ (9.2) must hold for a system to be BIBO stable.

(1) $\Rightarrow$ (2)

For LTI Systems : $\dot{x} = Ax + Bu$
$y = Cx + Du$

(9.2) $C\Phi(t,\tau)B \Rightarrow Ce^{A(t-\tau)}B$

$$\Rightarrow \sup_{t \geq 0} \int_0^t |\bar{g}_{ij}(t-\tau)| d\tau < \infty$$

– LTI can look @ time difference of systems.

$p = t - \tau$

$$\Leftrightarrow \int_0^\infty |\bar{g}_{ij}(p)| dp < \infty$$

Freq. Domain Characteristics :

(Theorem 9.3) For LTI systems $\dot{x} = Ax + Bu$, $y = Cx + Du$

where TF $\hat{G}(s) = \mathcal{L}\{C(sI-A)^{-1}B\}$ (leaving off D matrix, but D is constant & won't change the poles) → TF of impulse response.

The following statements are equivalent —

1) The system is uniformly BIBO stable
2) Every pole of every entry in $\hat{G}(s)$ has a strictly negative real part.

Proof — We need to show that :

$$\int_0^\infty |g_{ij}(p)| dp < \infty \Longleftrightarrow$$ the poles of $\hat{g}_{ij}(s)$ are in the left hand plane.

→ We know that all the elements of $\hat{G}(s)$ are strictly proper (will be true even if we included the D matrix because D will be constant & wont change the poles).

Strictly Proper rational = fewer zeros than poles.

$$\Rightarrow \quad \hat{g}_{ij}(s) = \frac{a_0 s^q + a_1 s^{q-1} + \cdots + a_{q-1} s + a_q}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}} \qquad \sum m_k > q$$

$$= \frac{a_{11}}{s-\lambda_1} + \frac{a_{12}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s-\lambda_1)^{m_1}} + \cdots$$

$$+ \frac{a_{k1}}{(s-\lambda_k)} + \frac{a_{k2}}{(s-\lambda_k)^2} + \cdots + \frac{a_{km_k}}{(s-\lambda_k)^{m_k}}$$

The inverse Laplace transform is

$$g_{ij}(t) = a_{11} e^{\lambda_1 t} + a_{12} t e^{\lambda_1 t} + \cdots + a_{1m_1} t^{m_1 - 1} e^{\lambda_1 t} + \cdots$$

$$+ a_{k1} e^{\lambda_k t} + \cdots + \cdots + a_{km_k} t^{m_k - 1} e^{\lambda_k t}$$

for each term we have

$$\int_0^\infty a_{ij} t^{j-1} e^{\lambda_i t} dt < \infty \text{ iff } Re\{\lambda_i\} < 0$$

$\rightarrow$ If one of these has a positive or <u>even</u> a zero pole then the terms don't converge to zero and $|g_{ij}(t)|$ will not converge to zero because the integral will keep growing. $\longrightarrow$ System not BIBO stable.

BIBO vs. Lyapunov Stability: (CLTI systems)

$$\left[\begin{array}{c}\text{exp stability} \\ \text{in the sense of} \\ \text{Lyapunov}\end{array}\right] \Longrightarrow \left[\text{BIBO stability}\right]$$

Proof —

Exp stability $\Longrightarrow$ All e-values of A are in ~~RHP~~ LHP

$\Longrightarrow$ all poles of $C(sI-A)^{-1}B$ are in RHP

$\Longrightarrow$ BIBO stability

$$\left[\text{BIBO stability}\right] \not\Longrightarrow \left[\begin{array}{c}\text{exp stability} \\ \text{in the sense of} \\ \text{Lyapunov}\end{array}\right]$$

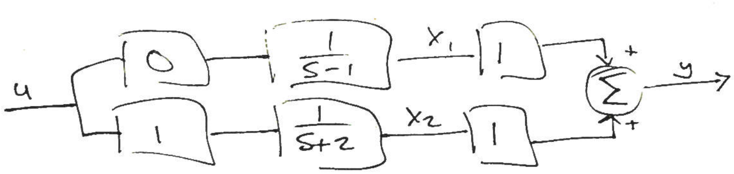
— this can happen when Ce^{At} or $e^{At}B$ cancel terms in e^{At} that are not converging exp. fast.

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

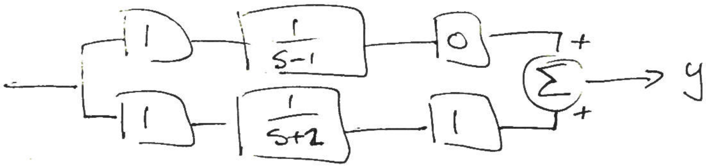
$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \longrightarrow \text{not exp. stable because of } e^{t}$$

But $$Ce^{At}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-2t}$$

2- situations



$$G(s) = \frac{1\cdot 1}{s+2} + \underbrace{\frac{1\cdot 0}{s-1}}_{\text{This is unstable, but mode is not effected by the input.}} = \frac{1}{s+2} = \frac{Y(s)}{U(s)}$$



$\dot{x}_1 = x_1 + u$

$\dot{x}_2 = -2x_2 + u$

$y = x_2$

$G(s) = \frac{1}{s+2}$ $\longrightarrow$ Same TF, but the output is not "seeing" the "run-away" mode. The output sensor does not measure this.

Two problems:

① Input does not effect all the modes $\Rightarrow$ this system lacks "controllability"

② Sensors do not see all of the modes $\Rightarrow$ This system lacks "observability"

$\longrightarrow$ System must be completely controllable + observable in order for BIBO stability to imply internal stability.

Discrete Time:

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

Forced response:

$$y_f(t) = \sum_{\tau=0}^{t-1} C(t)\underbrace{\Phi(t,\tau+1)}_{\text{discrete state transition matrix}}B(\tau)u(\tau)d\tau \qquad \forall t \geq 0$$

Definition of BIBO stability is almost identical to cont. case

$$\sup_{t\in\mathbb{N}} \|y_f(t)\| \leq g \sup_{t\in\mathbb{N}} \|u(t)\| \quad w/\ g>0$$

<u>Thm 9.5</u> the following 2 statements are equivalent

1. The DLTV system is BIBO stable
2. Every entry of $D(t)$ is uniformly bounded and
$$\sup_{t\geq 0} \sum_{\tau=0}^{t-1} |g_{ij}(t,\tau)| < \infty$$
where $G(t,\tau) = C\,\Phi(t,\tau)B(\tau)$

<u>Thm 9.6</u> The following three statements are equivalent

1) The time-invariant system
$$x^+ = Ax + Bu, \quad y = Cx + Du$$
is BIBO stable

2) Every entry of $G(p) = CA^pB$ satisfies
$$\sum_{p=1}^{\infty} |g_{ij}(p)| < \infty$$

3) Every pole of $\hat{G}(z) = \mathcal{Z}\{CA^pB\}$ has magnitude in the open unit circle.