Method #1 : LaPlace Transform

Note: when you take the LaPlace transform of a vector you take the LaPlace of each component.

$$\mathcal{L}\{x(t)\} = \hat{x}(s) = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \mathcal{L}\{x_2(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} = \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \vdots \\ \hat{x}_n(s) \end{bmatrix}$$

$x \in \mathbb{R}^n$

If A is constant —

$$\mathcal{L}\{Ax(t)\} = A\mathcal{L}[x(t)] = A\hat{x}(s)$$

$$\mathcal{L}\{\dot{x}(t)\} = \begin{bmatrix} \mathcal{L}\{\dot{x}_1(t)\} \\ \mathcal{L}\{\dot{x}_2(t)\} \\ \vdots \\ \mathcal{L}\{\dot{x}_n(t)\} \end{bmatrix} = \begin{bmatrix} s\hat{x}_1(s) - x_1(0) \\ s\hat{x}_2(s) - x_2(0) \\ \vdots \\ s\hat{x}_n(s) - x_n(0) \end{bmatrix} = s\hat{x}(s) - x_0$$

$\hat{x}, x_0 \in \mathbb{R}^n$

$$= sI\hat{x}(s) - x_0$$

$I = n \times n$ identity matrix

Remember if :

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x_0$$

let $t_0 = 0$ (time invariant so we can to this)

Then soln :

$$x(t) = \Phi(t,0)x_0 = e^{At}x_0$$

Taking the Laplace transform of $\dot{x} = Ax$

$$\mathcal{L}[\dot{x}(t)] = s\hat{x}(s) - x_0 = A\hat{x}(s)$$

$$(sI - A)\hat{x}(s) = x_0 \quad \text{(gathering terms)}$$

$$\Rightarrow \hat{x}(s) = (sI-A)^{-1}x_0$$

$$\Rightarrow$$

$$x(t) = \underline{\mathcal{L}^{-1}\{(sI-A)^{-1}\}}\,x_0 \quad \text{(take the inverse Laplace)}$$

But, $x(t) = \Phi(t,0)x_0 = \underline{e^{At}}\,x_0$

$$\therefore \quad e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

Example:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \qquad T^{-1} = \frac{\operatorname{adj}(T)}{\det(T)}$$

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}\right\}$$

~~P~~ use partial fraction expansion to separate denominators —

Example: a_{11} element

$$\frac{s+3}{(s+2)(s+1)} = \frac{a}{(s+2)} + \frac{b}{(s+1)} = \frac{a(s+1)+b(s+2)}{(s+2)(s+1)} = \frac{s+3}{(s+2)(s+1)}$$

$$\Rightarrow a(s+1)+b(s+2) = s+3$$

$$(a+b)s + (a+2b) = s+3 \quad \Rightarrow \quad \left.\begin{matrix} a+b=1 \\ a+2b=3 \end{matrix}\right\} \quad \begin{matrix} b=2 \\ a=-1 \end{matrix}$$

$$\mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}\right\}$$

* this is in a format that we can easily take the inverse Laplace.

* Take the inverse Laplace of each element.

$$= \begin{bmatrix} 2e^{-t} - 1e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Note: The roots of $(sI-A) \Rightarrow \det(sI-A)=0$ are the same as the e-values of A.

- These are the modes of our system — if we push the system with a given input u then it is going to move as a linear combination of the eigenvalues of A.

Method # 2: Cayley - Hamilton:

Cayley - Hamilton Theorem: Given a square matrix $A \in \mathbb{R}^{n \times n}$.

The matrix A satisfies its own characteristic equation.

where characteristic equation =>

$$P_A(s) = \det(sI - A) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

$$P_A(A) = 0$$

i.e. you can plug A in for s and you will get zero.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \cancel{w/\ p(s) = s^2 + 2s + 5} \quad + \ \cancel{A \cdot A} \ A^2 = A \cdot A = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$\cancel{A^2 + 2A + 5 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}}$$

$$p(s) = \det(sI - A) = s^2 - 5s - 2$$

- Plug in for A

$$p(A) = A^2 - 5A - 2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

How does this help us?

Implication: Powers of A, A^k for $k \geq n$ can be written as linear combinations of the powers A^i, $0 \leq i \leq n-1$

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_{n-1} A + a_n = 0$$

$$\Rightarrow \quad A^n = \frac{1}{a_0}\left(-a_1 A^{n-1} - a_2 A^{n-2} + \dots + -a_{n-1} A - a_n\right) = 0$$

- And every higher power ~~can be~~ $(k \geq n)$ can be written as a linear combination of lower powers.

$$\Rightarrow A^{n+1} = A A^n = \frac{A}{a_0}\left(-a_1 A^{n-1} - a_2 A^{n-2} + \dots + -a_{n-1} A - a_n\right) = 0$$

$$= -\frac{a_1}{a_0} A^n - \frac{a_2}{a_0} A^{n-1} + \dots - \frac{a_{n-1}}{a_n} A^2 - a_n A = 0$$

plug in our solution to A^n here.

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k \Rightarrow \sum_{k=0}^{n-1} \alpha_k(t) A^k$$

no longer an infinite sum.

- Combination of the t values multiplied by the combination of lower power coefficients.

- How do we find these new coefficients?
- Use the fact that $p_A(\lambda_i) = 0$ if $\lambda_i \in \text{eig}(A)$, then if we have n distinct e-values we will have n linearly independent equations and can solve for the coefficients.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \det(\lambda I - A) = 0$$

$$\Rightarrow \cancel{s(s+1)+2 = s^2+s+2}$$

$$\cancel{\lambda(\lambda+3)+2} = \cancel{\lambda^2+\lambda+2=0}$$

$$\cancel{\det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} =}$$

$$\cancel{\lambda = -2, -1}$$

$$= \cancel{\lambda^2 + 3\lambda + 2}$$

$$\lambda(\lambda+3)+2 = \lambda^2+3\lambda+2 = 0$$

$$(\lambda+2)(\lambda+1) = 0 \quad \boxed{\lambda = -2, -1} \qquad n=2$$

$$e^{\lambda_1 t} = \alpha_0 + \alpha_1 \lambda_1 \Rightarrow e^{-2t} = \alpha_0 + \alpha_1(-2)$$

$$e^{\lambda_2 t} = \alpha_0 + \alpha_1 \lambda_2 \Rightarrow e^{-t} = \alpha_0 + \alpha_1(-1)$$

- Put into a matrix equation:

$$\begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = \cancel{\frac{\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}}{(-2+1)}} \cancel{\begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix}} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-2t} \\ \underbrace{\cancel{2e^{-t} + e^{-2t}}}_{2e^{-t} - e^{-2t}} \end{bmatrix}$$

$$\cancel{e^{At} = \alpha_1 A + \alpha_0 I = (2e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}}{(-2+1)} \begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-2t} \\ 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$e^{At} = \alpha_1 A + \alpha_0 I = (e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

* General way for finding the coefficients —

$$\underbrace{\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vantermonde matrix}} \begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

→ Solve for coefficients by inverting the Vantermonde matrix.

→ Only works when we have n distinct eigenvalues (i.e. none repeated)

Method #3: Eigenvalue / EigenVector ~~ta~~

Review of e-vectors:

$$AV_k = \lambda_k V_k \ , \quad V_k \neq 0 \quad V_k \in \mathbb{R}^n$$

$$(\lambda_k I - A) V_k = 0$$ * So, V_k lives in the null space of the matrix $(\lambda_k I - A)$

- Create a matrix of the eigenvectors

$$V = [v_1 \ v_2 \ v_3 \ \dots \ v_n]$$

$$AV = [Av_1 \ Av_2 \ Av_3 \ \dots \ Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \ \cdots \ \lambda_n v_n] = V\Lambda$$

w/ $$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \phi \\ & \phi & \ddots & \\ & & & \lambda_n \end{bmatrix} = \operatorname{diag}\{\lambda_k\}_{k=1}^{n}$$

- If the e-values are distinct (non-repeatable) then the corresponding e-vectors are linearly indepentent.
- And V is nonsingular, $\det V \neq 0$, V^{-1} exists.

$$\Rightarrow AV = V\Lambda \Rightarrow \Lambda = V^{-1}AV$$

$$A = V\Lambda V^{-1}$$ (use this in the matrix exp. ~~form~~ formula)

$$e^{At} = I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{6}t^3A^3 + \dots$$

$$= \underbrace{I}_{VV^{-1}} + t(V\Lambda V^{-1}) + \tfrac{1}{2}t^2(V\Lambda V^{-1})(V\Lambda V^{-1}) + \tfrac{1}{6}t^3(V\Lambda V^{-1})(V\Lambda V^{-1})(V\Lambda V^{-1}) + \dots$$

$$= V\underbrace{(I + t\Lambda + \tfrac{1}{2}t^2\Lambda^2 + \tfrac{1}{6}t^3\Lambda^3 + \dots)}_{= e^{\Lambda t} \text{ (by definition)}}V^{-1}$$

- And since this is a diagonal matrix, can easily write out what it is.

$$\therefore e^{At} = V e^{\Lambda t} V^{-1}$$

V = matrix of linearly independent e-vectors

Λ = diagonal matrix of e-values.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$\det(sI - A) = \lambda^2 + 3\lambda + 2 \Rightarrow \lambda = -2, -1$ (computed as before)

$$\therefore \quad \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

Compute e-vector associated w/ $\lambda_1 = -2$

$$(\lambda_k I - A)v_k = 0 \Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -2v_{11} - v_{12} = 0 \\ 2v_{11} + v_{12} = 0 \end{matrix} \Rightarrow \underbrace{2v_{11}}_{\text{choose } v_{11}=1} = -v_{12} \qquad v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Compute e-vector for $\lambda_2 = -1$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_{21} + v_{22} = 0 \qquad v_{21} = -v_{22}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \quad V = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \quad \text{w/ } V^{-1} = \frac{\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}}{(-1+2)} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad e^{At} = V e^{\Lambda t} V^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Properties of the matrix exponential: (Recap)

#1 $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$

#2 $e^{A \cdot 0} = I$

#3 $e^{At} e^{A\tau} = e^{A(t+\tau)}$

#4 For every $t \in \mathbb{R}$, e^{At} is nonsingular (i.e. has an inverse)

$(e^{At})^{-1} = e^{-At}$ * ~~to~~ to invert simply put through a negative sign.

Discrete time:

Recall $\Phi(t, t_0) = \begin{cases} I & t = t_0 \\ A(t-1)A(t-2)\cdots A(t_0+1)A(t_0) & t > t_0 \end{cases}$

Define $A^0 = I$, then

$\Phi(t, t_0) = A^{t-t_0}$

Methods for computing A^t:

#1 z-transform —

$A^t = \mathcal{Z}_1^{-1}\{z(zI - A)^{-1}\}$

#2 e-value / e-vectors —

$A^t = \cancel{P\Lambda^t}\ V\Lambda^t V^{-1}$

#3 Cayley-Hamilton
(for $t \geq n$)