- use LaPlace transform to find the TF (or relationship between input u & output y) for statespace system

Objectives

1. TF for LTI Systems
2. Computing state space from TF
3. Realizations + zero-state equivalent
4. Similarity transforms

Review: linearization, — take a generic NL system & linearize @ an eq. point using multi-variable taylor series.

(LTI system)

$$\dot{x} = Ax + Bu \qquad \xrightarrow{\text{LaPlace Transform}} \qquad s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$

$$y = Cx + Du \qquad\qquad \hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

$$\xrightarrow{\text{solve for } \hat{x}(s)} \qquad (sI - A)\hat{x}(s) = B\hat{u}(s) + x(0)$$

$$\hat{x}(s) = (sI-A)^{-1}B\hat{u}(s) + \cancel{x(0)} \; (sI-A)^{-1}x(0)$$

$$\xrightarrow{\text{plug into } \hat{y}(s)} \qquad \hat{y}(s) = \underbrace{\left(C(sI-A)^{-1}B + D\right)}_{\hat{G}(s) \text{ transfer function}}\hat{u}(s) + \underbrace{C(sI-A)^{-1}x(0)}_{\hat{\psi}(s)}$$

$\hat{\psi}(s)$ = depends on initial conditions

Definition:

$$\left.\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\}$$ is said to be a realization of $\hat{G}(s)$ if

$$\hat{G}(s) = \left(C(sI-A)^{-1}B + D\right)$$

Definition: Two state-space systems are said to be "zero-state equivalent" if they realize the same transfer function.

(same zero-state response, but could have a different initial condition response)

Example:

$$\underset{p\times n}{C}\,(\underset{n\times n}{sI-A})^{-1}\,\underset{n\times m}{B} + \underset{p\times m}{D} = \underset{p\times m}{\hat{G}(s)}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 5 \end{bmatrix} x$$

$$\hat{G}(s) = \begin{bmatrix} 4 & 5 \end{bmatrix}\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 \end{bmatrix}\underbrace{\begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}}_{T}{}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: matrix Inverse

$$T^{-1} = \frac{\operatorname{adj}(T)}{\det(T)}$$ Adj(T) = transpose of the cofactors of T

if $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\operatorname{adj}(T) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(T) = ad - bc$$

$$\Rightarrow T^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$T = \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix} \qquad \det(T) = s(s+2)+3 = s^2+2s+3$$

$$\operatorname{adj}(T) = \begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix}$$

$$\hat{G}(s) = \frac{1}{s^2+2s+3}\begin{bmatrix} 4 & 5 \end{bmatrix}\begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+2s+3}\begin{bmatrix} 4s+8-15 & 5s+4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\hat{G}(s) = \frac{5s+4}{s^2+2s+3}}$$

Let

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -6 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1/2 \end{bmatrix} x$$

$$\det(sI - A) = (s+3)(s-1) + 6$$
$$= s^2 + 3s - s - 3 + 6$$
$$= s^2 + 2s + 3$$

$$\hat{G}(s) = \begin{bmatrix} 4 & 1/2 \end{bmatrix} \begin{bmatrix} s+3 & -1 \\ 6 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+3} \begin{bmatrix} 4 & 1/2 \end{bmatrix} \begin{bmatrix} \cancel{s+3}\; s-1 & 1 \\ -6 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s^2+2s+3} \begin{bmatrix} 4 & 1/2 \end{bmatrix} \begin{bmatrix} s+1 \\ 2s+6-6 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+3} \, 4s+4+s = \frac{5s+4}{s^2+2s+3}$$

∴ The two systems are zero state equivalent.

→ The two systems do not have the same initial condition responses.

Initial Condition Responses: $C(sI-A)^{-1} x_0 = \hat{y}(s)$

1st system:

$$\hat{y}(s) = \begin{bmatrix} 4 & 5 \end{bmatrix} \left(\frac{1}{s^2+2s+3} \begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix} \right) x_0 = \frac{1}{s^2+2s+3} \begin{bmatrix} 4s-7 & 29 \end{bmatrix} x_0$$

2nd system:

$$\hat{y}(s) = \begin{bmatrix} 4 & 1/2 \end{bmatrix} \left(\frac{1}{s^2+2s+3} \begin{bmatrix} s-1 & 1 \\ -6 & s+3 \end{bmatrix} \right) = \frac{1}{s^2+2s+3} \begin{bmatrix} 4s-7 & \frac{s}{2} + \frac{11}{2} \end{bmatrix} x_0$$

Note that since:

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \frac{C\,\mathrm{adj}(sI-A)\,B}{\det(sI-A)} + D$$

$$= \frac{C\,\mathrm{adj}(sI-A)\,B + \det(sI-A)\,D}{\det(sI-A)}$$

$AV = \lambda v \Rightarrow (\lambda I - A)v = 0$

$v \neq 0$

Note: $\det(sI-A)$ is a polynomial called the characteristic polynomial of A (roots are the eigenvalues of A)

Also: since we cancel one row and one col to compute $\mathrm{adj}(sI-A)$, each element of $\mathrm{adj}(sI-A)$ is a polynomial of degree $n-1$ or lower.

$\therefore$ $C\,\mathrm{adj}(sI-A)\,B$ is a matrix of polynomials w/ degree at most of $n-1$

And

$\det(sI-A)\,D$ is a matrix of polynomials w/ degree at most of n

If $D=0$ Then we say that $\hat{G}(s)$ is a "strictly proper rational function"

rational because ratio of polynomials

If $D \neq 0$ then we say that $\hat{G}(s)$ is a "proper rational function"

Thm 4.3: A transfer function $\hat{G}(s)$ can be realized by an LTI state space system iff $\hat{G}(s)$ is a proper rational function.

- Given a state space realization we have seen how to obtain the transfer matrix

Thm 4.3 says we can also go the other way

Step #1: Break up the $m \times k$ $\hat{G}(s)$ matrix to a strictly proper matrix:

$$\hat{G}(s) = \hat{G}_{sp}(s) + D \qquad \text{where } D = \lim_{s\to\infty} \hat{G}(s)$$

$$\text{and } \hat{G}_{sp}(s) = C(sI-A)^{-1}B$$

Example:

$$\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{(s+1)}{(s+2)^2} \end{bmatrix}$$

$$D = \lim_{s\to\infty} \hat{G}(s) = \lim_{s\to\infty} \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{(s+1)}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} \frac{4-10/s}{2+1/s} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\cancel{\hat{G}_{11} = 2}$

$$\hat{G}_{11} + 2 = \frac{4s-10}{2s+1} \qquad \hat{G}_{11} = \frac{4s-10}{2s+1} - \frac{2(2s+1)}{2s+1} = \frac{-12}{2s+1}$$

$$\hat{G}(s) = \underbrace{\begin{bmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{(s+1)}{(s+2)^2} \end{bmatrix}}_{C(sI-A)^{-1}B} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Step #2: Find the monic least common denominator for $\hat{G}_{sp}(s)$

leading coefficient ~~equal~~ equal 1.

$$d(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$$

Example

$$d(s) = (2s+1)(s+2)^2 = (2s+1)(s^2+4s+4) = 2s^3 + 8s^2 + 8s + s^2 + 4s + 4$$
$$= 2s^3 + 9s^2 + 12s + 4$$

not monic so divide by 2

$$d(s) = s^3 + \frac{9}{2}s^2 + 6s + 2, \quad \alpha_1 = \frac{9}{2}, \quad \alpha_2 = 6, \quad \alpha_3 = 2$$

Step #3: Expand $\hat{G}_{sp}(s)$ as

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_n\right]$$

Example:

$$\hat{G}_{sp} = \begin{bmatrix} \frac{-12(s+2)^2(\frac{1}{2})}{(2s+1)(s+2)^2} & \frac{3(2s+1)(s+2)(\frac{1}{2})}{(2s+1)(s+2)^2} \\ \frac{(s+2)(\frac{1}{2})}{(2s+1)(s+2)^2} & \frac{(s+1)(2s+1)(\frac{1}{2})}{(2s+1)(s+2)^2} \end{bmatrix} = \frac{\begin{bmatrix} -6(s^2+4s+4) & \frac{3}{2}(2s^2+5s+2) \\ \frac{s+2}{2} & \frac{1}{2}(2s^2+3s+1) \end{bmatrix}}{(s^3+\frac{9}{2}s^2+6s+2)}$$

$$\cancel{\text{Step \#4}} = \frac{1}{(s^3+\frac{9}{2}s^2+6s+2)}\left(\underbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}_{N_1} s^2 + \underbrace{\begin{bmatrix} -24 & 15/2 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}}_{N_2} s + \underbrace{\begin{bmatrix} -24 & 3 \\ 1 & 1/2 \end{bmatrix}}_{N_3}\right)$$

Step #4: Choose the ~~for~~ state space form:

$$A = \begin{bmatrix} -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & \cdots & -\alpha_{n-1} I_{k\times k} & -\alpha_n I_{k\times k} \\ I_{k\times k} & 0 & & \phi & \\ 0_{k\times k} & I_{k\times k} & \ddots & & \\ & \phi & & I_{k\times k} & 0_{k\times k} \end{bmatrix}_{(nk\times nk)} \quad B = \begin{bmatrix} I_{k\times k} \\ 0_{k\times k} \\ \vdots \\ 0_{k\times k} \end{bmatrix}_{(nk\times k)}$$

$$C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \end{bmatrix}_{(m\times nk)}$$

k = u dimension
m = y dim.

→ this is the controllable cannonical form that we will talk @ in further class.

→ choose the coefficients that fit this form

$$A = \begin{bmatrix} -\frac{9}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix} & -6\begin{bmatrix}1&0\\0&1\end{bmatrix} & -3\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ \begin{bmatrix}1&0\\0&1\end{bmatrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix} \\ \begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix} \end{bmatrix} = \begin{bmatrix} -9/2 & 0 & -6 & 0 & -3 & 0 \\ 0 & -9/2 & 0 & -6 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} \underbrace{\begin{matrix}-6 & 3 \\ 0 & 1\end{matrix}}_{N_1} & \underbrace{\begin{matrix}-24 & 15/2 \\ 1/2 & 3/2\end{matrix}}_{N_2} & \underbrace{\begin{matrix}-24 & 3 \\ 1 & 1/2\end{matrix}}_{N_3} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Equivalent State Space Systems: (or Similarity transform)

- Suppose T is a nonsingular matrix
- Define $\bar{x} := Tx$

Then:

$$\dot{\bar{x}} = T\dot{x} = TAx + TBu \implies \dot{\bar{x}} = TAT^{-1}\bar{x} + TBu$$

$$\uparrow$$
$$x = T^{-1}\bar{x}$$

$$y = CT^{-1}\bar{x} + Du$$

Let $\bar{A} := TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$

Then

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = \bar{C}\bar{x} + \bar{D}u$$

→ we say the two systems

$$\dot{x} = Ax + Bu \qquad \text{and} \qquad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = Cx + Du \qquad \qquad \bar{y} = \bar{C}x + \bar{D}u$$

are algebraically equivalent.

~~Pf.~~ Property of algebraically equivalent systems.

P1: For every input signal, both systems have the same set of outputs y. zero-state response always the same $(\hat{G}(s))$.

kind of saying the same thing

(But w/ same initial conditions won't usually get same output).
↳ need initial conditions related by $\bar{x}(0) = Tx(0)$

P2: Both are zero-state equivalent. Have the same TF.

Note: zero-state equivalence does not ~~at~~ automatically imply algebraic equivalence. i.e. Two systems can have the same TF, but not be related through a similarity transform.

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- use $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to perform a similarity transform and verify that the new realization produces the same ~~TF~~ TF.

$$\bar{x} = Tx$$

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu$$
$$y = CT^{-1}\bar{x} + Du$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}$$

$$\bar{B} = TB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

~~old TF~~ TF of x system:

$$C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} = \boxed{1}$$

TF of $\bar{x}$ system:

$$C(sI-A)^{-1}B = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} s+2 & 0 \\ 2 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} s+1 & 0 \\ -2 & s+2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s+3 & -s-2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = s+3-s-2$$
$$= \boxed{1}$$

* Note: Eigenvalues are uneffected by a similarity transform. (they are the poles of TF which are same)