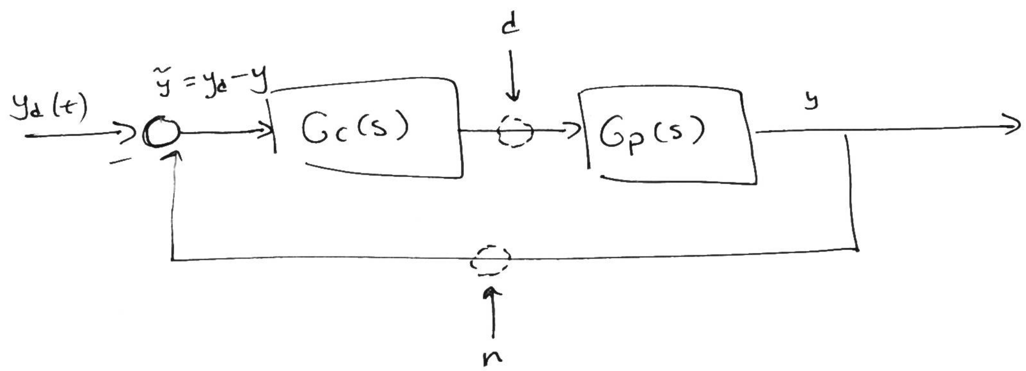
MBC for Tracking:

Can we track a wider class of problems?



$G_c(s)$ is model-based Compensator: (state-feedback + observer)

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} - L\tilde{y} \quad \text{w/ } \tilde{y} = y_d - y$$

$$u = -K\hat{x}$$

$\longrightarrow$ will $\tilde{y}(t) \to 0$ for a large class of $y_d(t)$ or at least $\lim_{t\to\infty} \|\tilde{y}(t)\| \leq \varepsilon \ll 1$

Closed-loop dynamics:

$$Y(s) = G_p(s) G_c(s) \tilde{Y}(s)$$

$$= G_p(s) G_c(s) \left(Y_D(s) - Y(s) \right)$$

$$\left[I + G_p(s) G_c(s) \right] Y(s) = G_p(s) G_c(s) Y_D(s)$$

$$Y(s) = \underbrace{\left[I + G_p(s) G_c(s) \right]^{-1} G_p(s) G_c(s)}_{T(s) \,=\, \text{closed-loop transfer function matrix}} Y_D$$

open-loop transfer function matrix

$$L(s) = G_p(s) G_c(s) \longrightarrow T(s) = \left[I + L(s) \right]^{-1} L(s)$$

Define a TF for the error:

$$\tilde{Y}(s) = Y_D(s) - \underbrace{Y(s)}_{Y(s) = T(s)Y_D(s)} = \left[I - T(s)\right] Y_D(s)$$

Let $S(s) = I - T(s)$ be the "sensitivity" TFM

$$S(s) = I - T(s) = \left[I + L(s)\right]^{-1}\left[(I + L(s)) - L(s)\right]$$

because $I = \left(I + L(s)\right]^{-1}\left[I + L(s)\right]$

$$= \left[I + L(s)\right]^{-1}$$

Note: $T(s) + S(s) = I$ for any s (algebraic constraint)

$S(s)$ = sensitivity TFM

$T(s)$ = complementary sensitivity TFM

} both have same denominator poles $\det\left[I + L(s)\right]$

$$\Rightarrow \tilde{Y}(s) = S(s)\, Y_D(s)$$

$$y_d \longrightarrow \boxed{S(s)} \xrightarrow{\tilde{y}\, = \text{ error}}$$

→ This shows closed-loop dynamics from what we want to track to the errors.

→ we have guaranteed stability w/ MBC (LQG = optimal), but how well will it track?

Steady state error: $e_{ss} = \lim_{t \to \infty} \tilde{Y}(s) = \lim_{s \to 0} s\tilde{Y}(s) = \lim_{s \to 0} s\left[S(s)\, Y_D(s)\right]$

Suppose:

$y_d(t) = y_0 \mathbb{1}(t)$ (step input w/ y_0 = constant vector)

could have potentially different levels for each output.

→ This is the min would like to track.

Then:

$$Y_D(s) = \frac{1}{s} y_0$$

$$e_{ss} = \lim_{s \to 0} s\left[S(0)\frac{1}{s}y_0\right] = S(0)\,y_0$$

→ would like $e_{ss} = 0$ (zero steady-state error). Can get this in two ways.

1. $S(0)$ has non-trivial nullspace that contains y_0.
2. $S(0)$ = identically zero.

→ would like to guarantee the second to get $S(0) = \phi$ (matrix) to ensure $e_{ss} = 0$ for all possible step inputs.

When will this happen?

Suppose: $L(s) = \frac{I}{s} L_1(s)$ where $L_1(0)$ = nonsingular

Then: $S(s) = \left[I + L(s)\right]^{-1} = \left[I + \frac{I}{s} L_1(s)\right]^{-1}$

$$= \left[\frac{1}{s}\left(sI + L_1(s)\right)\right]^{-1} = s\left[sI + L_1(s)\right]^{-1}$$

$$\lim_{s \to 0} S(s) = \lim_{s \to 0} s\left[L_1(0)\right]^{-1} = 0$$

→ i.e. if we can factor out a $\frac{1}{s}$ from open loop TF $L(s)$ then can guarantee constant ~~step~~ tracking.

Trivially could do this for any $L(s)$:

$$L(s) = \frac{1}{s+3} = \frac{1}{s}\underbrace{\left(\frac{s}{s+3}\right)}_{L_1(s)}$$

$L_1(s) \longrightarrow$ This is not invertible. Need $L_1(0)$ to be nonsingular.

* Let's consider a more general form of the same problem.

$$y_d(s) = \frac{a_y(s)}{b_y(s)}$$

where $b_y(s)$ contains only roots in the closed right half plane. (count marginally stable as being w/ unstable here)

* Under what conditions can we track this? When is $e_{ss} = 0$?
* Proceed as before + suppose:

$$L(s) = \frac{I}{b(s)} L_1(s)$$

with $(L_1(0) + b_y(0) I)$ – nonsingular

* Force out from open-loop TF a denominator that looks like denominator of $Y_D(s)$.

Then: $$S(s) = \left[\frac{1}{b_y(s)}\left(b_y(s)I + L_1(s)\right)\right]^{-1}$$

$$= b_y(s)\left[b_y(s)I + L_1(s)\right]^{-1}$$

And: $$e_{ss} = \lim_{s \to 0} s\, \overset{b_y(s)}{\cancel{b_y(s)}}\left[b_y(s)I + L_1(s)\right]^{-1} \frac{a_y(s)}{b_y(s)} = 0 \cdot \left[b_y(0)I + L_1(0)\right]^{-1} a_y(0) = 0$$

cancels w/ factored $b_y(s)$

$\therefore$ To track unstable $y_d(t)$ w/ $e_{ss} = 0$ we must be able to factor:

$$L(s) = \left(\frac{I}{b_y(s)}\right) L_1(s)$$

- where $b_y(s)$ contains the unstable (+ marginally stable) poles of $y_d(s)$ + $L_1(s)$ - nonsingular
- Note: Implies that the unstable poles of $y_d(s)$ are also poles of open-loop feedback system.

  i.e. $L(s)$ contains an "internal model" of the unstable parts of $y_d(t)$.
- If $L(s)$ has a pole @ origin can track a step w/ zero error, two poles = ramp, pole @ Im axis w/ specific freq = can track sin wave @ that freq.

~~$Ae^{-\sigma t}\sin(\omega t + \phi)$~~

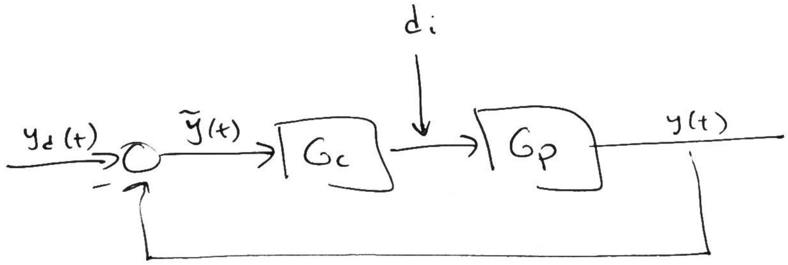
$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

or

$$= A e^{\sigma t}\sin(\omega t + \phi)$$

Additional feature of Internal model ~~#~~ principle:



d_i = deterministic input disturbance (not random).

How does error depend on input disturbance?

$$Y(s) = G_p(s)\left(U(s) + d_i\right)$$

$$G_c(s)\tilde{Y}(s) = G_c(s)(Y_D - Y)$$

$$= G_p(s)\left(G_c(s)Y_D - G_c Y + d_i\right)$$

$$\left[I + G_p(s)G_c(s)\right]Y(s) = G_p(s)G_c(s)Y_D + G_p(s)d_i$$

$$Y(s) = \left[I + G_p(s)G_c(s)\right]^{-1}G_p(s)G_c(s)Y_D + \left[I + G_p(s)G_c(s)\right]^{-1}G_p(s)d_i$$

$$= \left[I + L(s)\right]^{-1}L(s)Y_D + \left[I + L(s)\right]^{-1}G_p(s)d_i$$

$$\left[Y(s) = T(s)Y_D + S(s)G_p(s)d_i\right]$$

$$\tilde{Y}(s) = Y_D - Y = \left[I - T(s)\right]Y_D - S(s)G_p(s)d_i$$

$$= \underbrace{S(s)Y_D}_{\text{Same error as before}} - \underbrace{S(s)G_p(s)d_i}_{\text{tracking error due to disturbance.}}$$

(look @ separately = linear system)

$$\left[E(s)\right]_{d_i} = -S(s)G_p(s)d_i(s)$$

Take same approach:

Suppose: $d_i(s) = \dfrac{a_d(s)}{b_d(s)}$ w/ $b_d(s)$ containing the unstable poles of $d_i(s)$

Factor: (Assumption) $L(s) = \left(\dfrac{I}{b_d(s)}\right) L_1(s)$ w/ $\left(b_d(0) + L_1(0)\right)$ nonsingular

$$e_{ss}\big|_{d_i} = \lim_{s\to 0} s\, S(s)\, G_p(s) \frac{a_d(s)}{b_d(s)}$$

$$= \lim_{s\to 0} s \left[b_d(s) I + L_1(s) \right]^{-1} G_p(s)\, b_d(s) \overbrace{\left(\frac{a_d(s)}{b_d(s)}\right)}^{d_i(s)}$$

- If $G_p(s)$ has poles $b_d(s)$ then will cancel here, & nothing left to cancel the $b_d(s)$ from disturbance
- Result may or may not be zero.

→ Need to be able to factor unstable poles out of the compensator. (Even if can factor out of poles, from plant fact that can cancel out of the compensator will fix things).

$$e_{ss}\big|_{d_i} = 0 \quad \text{if} \quad G_c(s) = \frac{I}{b_d(s)} G_i(s)$$

→ Unstable model in compensator, ~~but not~~ (for disturbances)

→ unstable model can be in either compensator or plant when dealing w/ y_d.

Example:

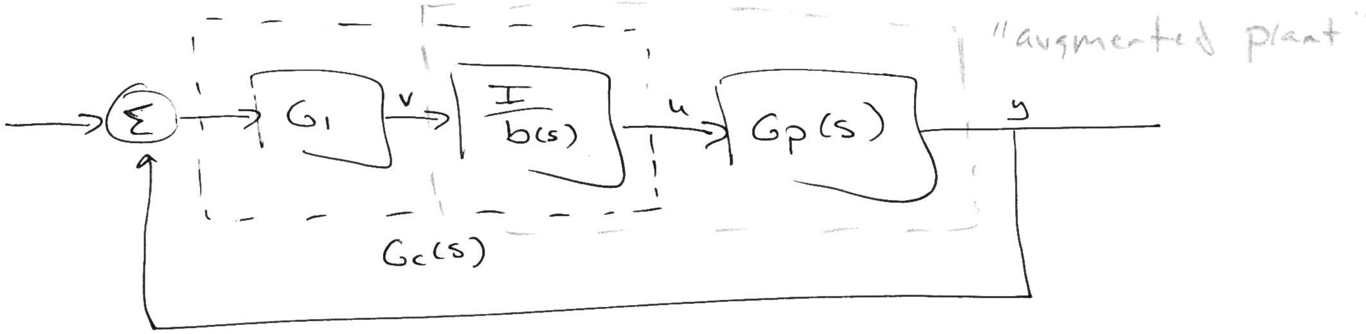
$$A = \begin{bmatrix} 0 & 1 \\ -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

→ Find the augmented plant system + the compensator dynamics. (Assume generic K_a + L_a)

Augmented Plant:

$$A_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -5 \end{bmatrix} \quad B_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C_a = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

* Generally, don't have ~~poles~~ unstable poles in right places to give perfect tracking or disturbance rejection.
  (Force compensator to have poles in the right places)



* Design G_1 based upon augmented plant (to guarantee stability). ~~Absorb~~
* then "absorb" these dynamics back into the compensator.

State model perspective:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Let $[A_b, B_b, C_b]$ be a realization of this TF.

$$\dot{\xi} = \cancel{A_b\xi}\ A_b\xi + B_b v$$
$$u = C_b\xi$$

"Augmented Plant"

Define: $X_a = \begin{bmatrix} \xi \\ x \end{bmatrix}$

$$\dot{X}_a = \begin{bmatrix} \dot{\xi} \\ \dot{x} \end{bmatrix} = \overbrace{\begin{bmatrix} A_b & 0 \\ BC_b & A \end{bmatrix}}^{A_a} \begin{bmatrix} \xi \\ x \end{bmatrix} + \overbrace{\begin{bmatrix} B_b \\ \phi \end{bmatrix}}^{B_a} v$$

$$y = \underbrace{\begin{bmatrix} 0 & C \end{bmatrix}}_{C_a} \begin{bmatrix} \xi \\ x \end{bmatrix}$$

Design K_a + L_a using the "Augmented" plant dynamics. (A_a, B_a, C_a)

Compensator dynamics (G_1):

$$\dot{z} = (A_a - B_a K_a - L_a C_a) z - L_a \tilde{y}$$

$$v = -K_a z$$

→ Now absorb these dynamics back into the compensator:

$$\dot{\xi} = A_b \xi + B_b v$$

$$u = C_b \xi$$

— temp. grouped w/ plant while designed K_a + L_a, but really need to pull into the compensator.

<u>Compensator Dynamics:</u>

$$\frac{d}{dt}\begin{bmatrix} z \\ \xi \end{bmatrix} = \begin{pmatrix} A_a - B_a K_a - L_a C_a & 0 \\ -B_b K_a & A_b \end{pmatrix}\begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} -L_a \\ 0 \end{bmatrix}\tilde{y}$$

$$u = \begin{bmatrix} 0 & C_b \end{bmatrix}\begin{bmatrix} z \\ \xi \end{bmatrix}$$

→ For tracking a step:

$$b(s) = s$$

→ The ~~all~~ forced dynamics

$\dot{\xi} = v, \quad u = \xi \quad \Rightarrow$ eq. to $\dot{u} = v$

$$\left.\begin{aligned} \dot{\xi} &= A_b \xi + B_b v \\ u &= C_b \xi \end{aligned}\right\} \qquad \begin{aligned} A_b &= 0 \\ B_b &= I \\ C_b &= I \end{aligned}$$

The augmented plant becomes:

$$A_a = \begin{bmatrix} A_b & 0 \\ B C_b & A \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$$

$$B_a = \begin{bmatrix} B_b \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$C_a = \begin{bmatrix} 0 & C \end{bmatrix} = \begin{bmatrix} 0 & C \end{bmatrix}$$

→ Augments the loop dynamics w/ an integrator on each channel.

→ works for steps, ramps, sinusoids, etc.

→ Generally can't predict complete spectrum + can't augment system w/ "everything".

→ once have design – how good is it for other $y_d(t)$?
(Fourier → break $y_d(t)$ into freq. components)

– Could break up into a freq. response, + look @ how good it may be @ tracking high freq. vs. low freq.

If $y_d(t) = y_o e^{j\omega t}$

$$e_{ss}(t) = \left(S(j\omega)\, y_o\right) e^{j\omega t} = e_o e^{j\omega t}$$

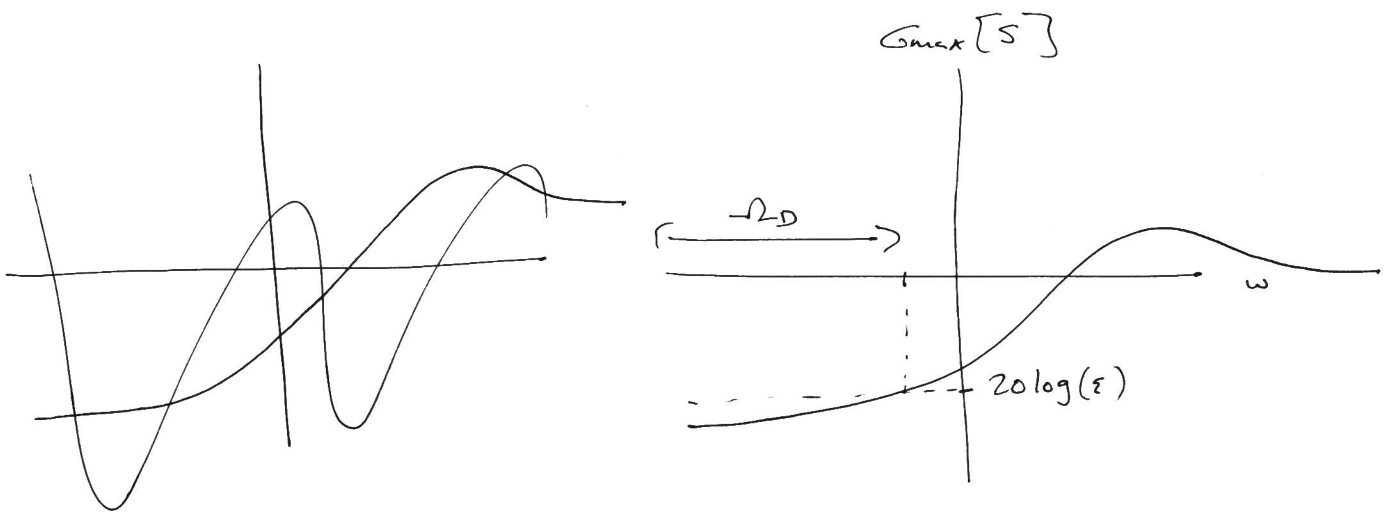
Want:

$$\| e_o \| \le \varepsilon \ll 1 \rightarrow \text{for wide range of freq.}$$

$$\| e_o \|_2 \le \underbrace{\sigma_{max}\left[S(j\omega)\right]}_{} \| y_o \|_2$$

↳ make this small over a range of freq.

→ recall $\|A\|_2 = \sigma_{max}[A]$ → largest singular value.



Typical requirement:
- keep $\sigma_{max}\left[S(j\omega)\right] \le \varepsilon \ll 1$ for all $\omega \in \Omega_d$
- Design compensator to "push down" $\sigma_{max}\left[S(j\omega)\right]$ for $\omega \in \Omega_d$
- there is a trade-off here – if we push down in one freq. range will pop up in another region.

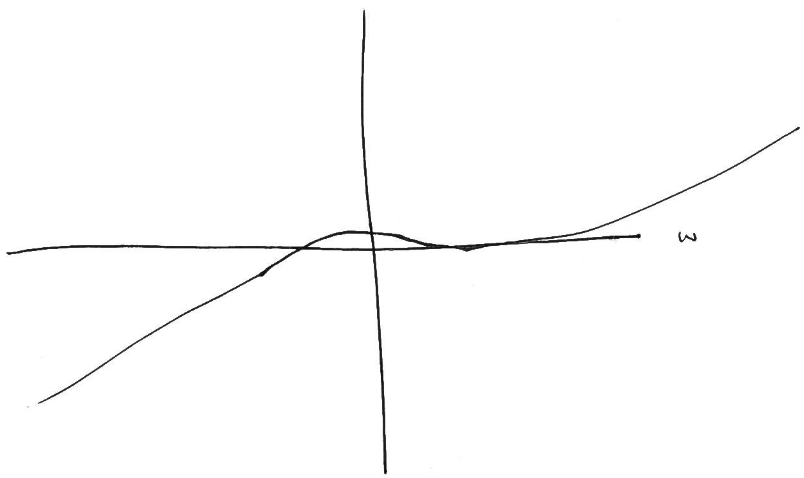
– Similar consideration apply to determining the actual amount of control to track a specific $y_d(t)$.

$$u(s) = G_c(s)\,\overset{\tilde{\psi}(s)}{\cancel{e(s)}} = \underbrace{G_c(s)\,S(s)}\,y_d(s)$$

↳ look @ singular values of this product.

→ Gives an idea of how much control is required to track a specific $y_d(t)$.

$\sigma_{max}\left[G_c(j\omega)\,S(j\omega)\right]$



– would like to push this down to achieve

– Another loop shaping design problem.

– Less control usually equals worse tracking (so pushing down both plots can be competing).

→ Separation principle: As observer converges to zero can have problems where need to use a lot of control to track the observer transients. (For regulator problem can let transients die out before turning on control.)

→ Discontinuous change in $y_d(t)$ will excite new observer transients $\overset{\text{could equal}}{=}$ large control transients.