#1 A has char polynomial $r(\lambda) = (\lambda+1)(\lambda+2)^4$ $(n=5)$

The $\text{rank}(A+2I) = \text{rank}(-2I-A) = 1$

$\Rightarrow$ nullity $(-2I-A) = 4$

$\Rightarrow$ 4 Jordan blocks — so each must be 1×1

$$J = \begin{bmatrix} -1 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & -2 & \\ & & & & -2 \end{bmatrix} \qquad e^{Jt} = \begin{bmatrix} e^{-t} & & & & \\ & e^{-2t} & & & \\ & & e^{-2t} & & \\ & & & e^{-2t} & \\ & & & & e^{-2t} \end{bmatrix}$$

$$\underline{P^{-1} e^{At} P = e^{Jt}}$$

b) $$x = e^{A(t-t_0)} x_0 = P e^{J(t-t_0)} P^{-1} x_0$$

$$x = P \begin{bmatrix} e^{-(t-t_0)} & & & & \\ & e^{-2(t-t_0)} & & & \\ & & e^{-2(t-t_0)} & & \\ & & & e^{-2(t-t_0)} & \\ & & & & e^{-2(t-t_0)} \end{bmatrix} P^{-1} x_0$$

Question # 2

$\dot{z}_1 = 2z_1 + - z_1 z_2$

$\dot{z}_2 = 2z_1^2 - z_2$

a) Eq. Points

$\dot{z}_1 = 0$ when $\boxed{z_1 = 0, \Rightarrow z_2 = 0}$

or $\dot{z}_2 = 0 \Rightarrow 2z_1^2 = z_2 \Rightarrow 2z_1 - z_1(2z_1^2)$

$= 2(z_1 - z_1^3) = 0$

$2z_1(1 - z_1^2) = 0$

3 eq. points : $(0,0)$; $(1,2)$; $(-1,2)$ $\qquad z_1^2 = 1$ $\boxed{z_1 = \pm 1 \\ z_2 = 2}$

b) w/ $z(0) = z^{eq} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ → close to, but not exactly z^{eq}

Solution is : $z(t) = \cancel{\delta z(t) + z^{eq}} \; z^{eq} + \delta z(t)$

$\delta \dot{z}(t) = A \delta z$ w/ $A = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{pmatrix}$

And $\delta z(t) = e^{At} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$

$\Rightarrow z(t) = z^{eq} + e^{At} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$

w/ $A = \begin{bmatrix} 2 - z_1 & -z_1 \\ 4z_1 & -1 \end{bmatrix}_{z^{eq}} \Rightarrow A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ for eq. point $(0,0)$

$A = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix}$ for eq. point $(1,2)$

$A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$ for eq. point $(1,2)$

```
syms t ep1 ep2

ep = [ep1;ep2];

A1=[2 0; 0 -1];
A2=[0 -1; 4 -1];
A3 = [0 1; -4 -1];
zstar1=[0;0]; zstar2=[1;2]; zstar3=[-1;2];

% First eq point
z=zstar1+expm(A1*t)*ep

% not thet if ep1 is not zero, this solution will diverge away from the
% equilibrium

% Second case
z=zstar2+expm(A2*t)*ep


% This is pretty ugly -- using ilaplace to solve makes it much more
% managable, or could simplify this equation:
vpa(simplify(z),3)

syms s
f = inv((s*eye(2)-A2));
z = ilaplace(f)


% Note that this is a damped oscillation which converges back for any
% given ep1 and ep2


% Third case
z=zstar3+expm(A3*t)*ep
% again simplify
vpa(simplify(z),3)

% or use ilaplace
f = inv((s*eye(2)-A2));
z = ilaplace(f)

% This one is also a damped oscillation which will converge
```

```
z =

 ep1*exp(2*t)
  ep2*exp(-t)


z =

 1 + ep1*(exp(- t/2 - (15^(1/2)*t*1i)/2)/2 + exp(- t/2 + (15^(1/2)*t*1i)/2)/2 + (15^(1/2)*exp
(- t/2 - (15^(1/2)*t*1i)/2)*1i)/30 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*1i)/30) - ep2*(
```

```
(15^(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*1i)/15 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*1i
)/15)
 ep1*((15^(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*4i)/15 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)
/2)*4i)/15) + ep2*(exp(- t/2 - (15^(1/2)*t*1i)/2)/2 + exp(- t/2 + (15^(1/2)*t*1i)/2)/2 - (15^
(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*1i)/30 + (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*1i)/30
) + 2


ans =

 ep1*exp(t*(- 0.5 - 1.94i))*(0.5 + 0.129i) + ep1*exp(t*(- 0.5 + 1.94i))*(0.5 - 0.129i) - ep2*
exp(t*(- 0.5 - 1.94i))*0.258i + ep2*exp(t*(- 0.5 + 1.94i))*0.258i + 1.0
     ep1*exp(t*(- 0.5 - 1.94i))*1.03i - ep1*exp(t*(- 0.5 + 1.94i))*1.03i + ep2*(exp(t*(- 0.5
- 1.94i))*(0.5 - 0.129i) + exp(t*(- 0.5 + 1.94i))*(0.5 + 0.129i)) + 2.0


z =

[ exp(-t/2)*(cos((15^(1/2)*t)/2) + (15^(1/2)*sin((15^(1/2)*t)/2))/15),                        -
(2*15^(1/2)*exp(-t/2)*sin((15^(1/2)*t)/2))/15]
[                          (8*15^(1/2)*exp(-t/2)*sin((15^(1/2)*t)/2))/15, exp(-t/2)*(cos((15^(1/
2)*t)/2) - (15^(1/2)*sin((15^(1/2)*t)/2))/15)]


z =

 ep2*((15^(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*1i)/15 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)
/2)*1i)/15) + ep1*(exp(- t/2 - (15^(1/2)*t*1i)/2)/2 + exp(- t/2 + (15^(1/2)*t*1i)/2)/2 + (15^
(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*1i)/30 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*1i)/30
) - 1
 2 + ep2*(exp(- t/2 - (15^(1/2)*t*1i)/2)/2 + exp(- t/2 + (15^(1/2)*t*1i)/2)/2 - (15^(1/2)*exp
(- t/2 - (15^(1/2)*t*1i)/2)*1i)/30 + (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*1i)/30) - ep1*(
(15^(1/2)*exp(- t/2 - (15^(1/2)*t*1i)/2)*4i)/15 - (15^(1/2)*exp(- t/2 + (15^(1/2)*t*1i)/2)*4i
)/15)


ans =

 ep2*exp(t*(- 0.5 - 1.94i))*0.258i - ep2*exp(t*(- 0.5 + 1.94i))*0.258i + ep1*(exp(t*(- 0.5 -
1.94i))*(0.5 + 0.129i) + exp(t*(- 0.5 + 1.94i))*(0.5 - 0.129i)) - 1.0
 2.0 + ep1*exp(t*(- 0.5 + 1.94i))*1.03i + ep2*exp(t*(- 0.5 - 1.94i))*(0.5 - 0.129i) + ep2*exp
(t*(- 0.5 + 1.94i))*(0.5 + 0.129i) - ep1*exp(t*(- 0.5 - 1.94i))*1.03i


z =

[ exp(-t/2)*(cos((15^(1/2)*t)/2) + (15^(1/2)*sin((15^(1/2)*t)/2))/15),                        -
(2*15^(1/2)*exp(-t/2)*sin((15^(1/2)*t)/2))/15]
[                          (8*15^(1/2)*exp(-t/2)*sin((15^(1/2)*t)/2))/15, exp(-t/2)*(cos((15^(1/
2)*t)/2) - (15^(1/2)*sin((15^(1/2)*t)/2))/15)]
```

Question # 3

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} x + 2u$$

a) Using LaPlace

$$\dot{x} = Ax \qquad (sI - A) = \begin{bmatrix} s-1 & 0 \\ -3 & s+2 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} \Rightarrow \qquad (sI-A)^{-1} = \frac{1}{(s-1)(s+2)} \begin{bmatrix} s+2 & 0 \\ 3 & s-1 \end{bmatrix}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{3}{(s-1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}\right\} \Rightarrow \begin{bmatrix} e^{+t} & 0 \\ -e^{-2t} + e^{+t} & e^{-2t} \end{bmatrix}$$ ← check on neg signs.

$$\frac{3}{(s-1)(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)}$$

$$a(s+2) + b(s-1) = 3$$

$$as + bs = 0s \qquad a + b = 0 \qquad a = -b$$

$$2a - b = 3 \qquad -2b - b = 3$$

$$\boxed{b = -1 \\ a = 1}$$

$$\frac{3}{(s-1)(s+2)} = \frac{1}{(s-1)} + \frac{-1}{(s+2)} \Rightarrow \frac{(s+2)-(s-1)}{(s-1)(s+2)} = \frac{3}{(s-1)(s+2)}$$

b) Using Eigenvector/Eigenvalue

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \qquad \det(sI - A) = (s-1)(s+2) = 0$$

$$\lambda = 1, -2$$

e-vector for $\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \qquad -3v_2 + 3v_1 = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-3v_3 = 0$$
$$v_3 = 0$$
$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

e-vector for $\lambda = -2$

$$\begin{bmatrix} -3 & 0 \\ -3 & \cancel{0} \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & \phi \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{At} = V e^{\Lambda t} V^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & 0 \\ e^{t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{-2t} & e^{-2t} \end{bmatrix} \checkmark$$

c) Using Cayley-Hamilton

E-values $\lambda = 1, -2$

$$e^{\lambda_1 t} = \alpha_0 + \alpha_1 \lambda_1 \qquad e^{t} = \alpha_0 + \alpha_1 (1)$$

$$e^{\lambda_2 t} = \alpha_0 + \alpha_1 \lambda_2 \qquad e^{-2t} = \alpha_0 + \alpha_1 (-2)$$

$$\begin{bmatrix} e^{t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \frac{\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}}{-3} \begin{bmatrix} e^{t} \\ e^{-2t} \end{bmatrix}$$

$$\alpha_0 = +\frac{2}{3} e^{t} + \frac{1}{3} e^{-2t}$$

$$\alpha_1 = \frac{1}{3} e^{t} - \frac{1}{3} e^{-2t}$$

$$e^{At} = \alpha_1 A + \alpha_0 I = \left(\frac{1}{3} e^{t} - \frac{1}{3} e^{-2t}\right) \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} + \left(\frac{2}{3} e^{t} + \frac{1}{3} e^{-2t}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \frac{1}{3} e^{t} - \frac{1}{3} e^{-2t} + \frac{2}{3} e^{t} + \frac{1}{3} e^{-2t} & 0 \\ \hline e^{t} - e^{-2t} & -\frac{2}{3} e^{t} + \frac{2}{3} e^{-2t} + \frac{2}{3} e^{t} + \frac{1}{3} e^{-2t} \end{array}\right]$$

$$= \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{-2t} & e^{-2t} \end{bmatrix} \checkmark$$

d) TF $\Rightarrow$

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{3}{(s-1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 2$$

$$= \begin{bmatrix} \frac{2}{s-1} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 2 = \frac{-2}{s-1} + 2 = \frac{-2 + 2(s-1)}{s-1}$$

$$= \frac{2s-4}{(s-1)}$$

e) Internally Stable $\Rightarrow$ No

f) BIBO stable $\Rightarrow$ No

Question # 4 :

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} u$$

$\longrightarrow$ Is the system controllable ?

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B(\tau)^T \Phi(t_0, \tau)^T \, d\tau$$

$$\Phi(t, t_0) = \begin{bmatrix} e^{(t - t_0)} & 0 \\ 0 & e^{2(t-t_0)} \end{bmatrix}$$

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} e^{t_0 - \tau} & 0 \\ 0 & e^{2(t_0-\tau)} \end{bmatrix} \begin{bmatrix} e^{\tau} \\ e^{2\tau} \end{bmatrix} \begin{bmatrix} e^{\tau} & e^{2\tau} \end{bmatrix} \begin{bmatrix} e^{t_0 - \tau} & 0 \\ 0 & e^{2(t_0-\tau)} \end{bmatrix}$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} e^{\tau} e^{t_0 - \tau} \\ e^{2\tau} e^{2(t_0-\tau)} \end{bmatrix} \begin{bmatrix} e^{\tau} e^{(t_0-\tau)} & e^{2\tau} e^{2(t_0-\tau)} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} \cancel{e^{2\tau}} e^{2(t_0-\cancel{\tau})} & \cancel{e^{3\tau}} e^{3(t_0-\cancel{\tau})} \\ \cancel{e^{3\tau}} e^{3(t_0-\cancel{\tau})} & \cancel{e^{4\tau}} e^{4(t_0-\cancel{\tau})} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} e^{2t_0} & e^{3t_0} \\ e^{3t_0} & e^{4t_0} \end{bmatrix} d\tau = \left. \begin{bmatrix} e^{2t_0}\tau & e^{3t_0}\tau \\ e^{3t_0}\tau & e^{4t_0}\tau \end{bmatrix} \right|_{t_0}^{t_1}$$

$$= \begin{bmatrix} e^{2t_0}(t_1 - t_0) & e^{3t_0}(t_1 - t_0) \\ e^{3t_0}(t_1 - t_0) & e^{4t_0}(t_1 - t_0) \end{bmatrix}$$

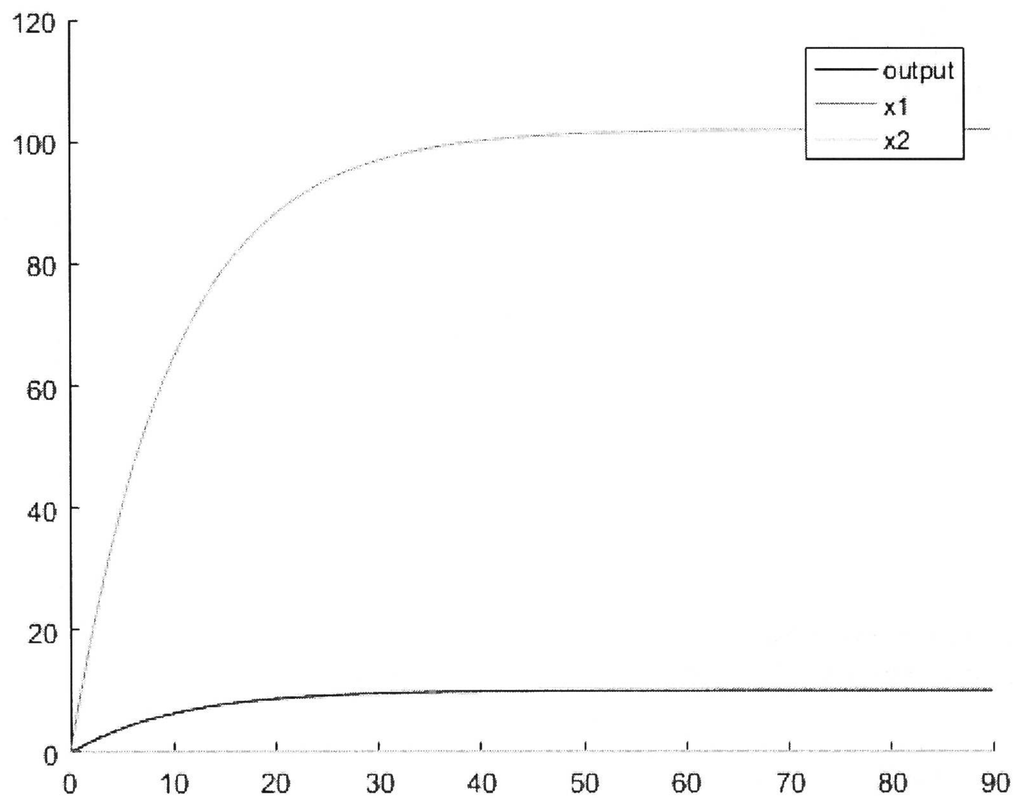
$$\det(W_c(t_0, t_1)) = e^{6t_0}(t_1 - t_0)^2 - e^{6t_0}(t_1 - t_0)^2 = 0$$

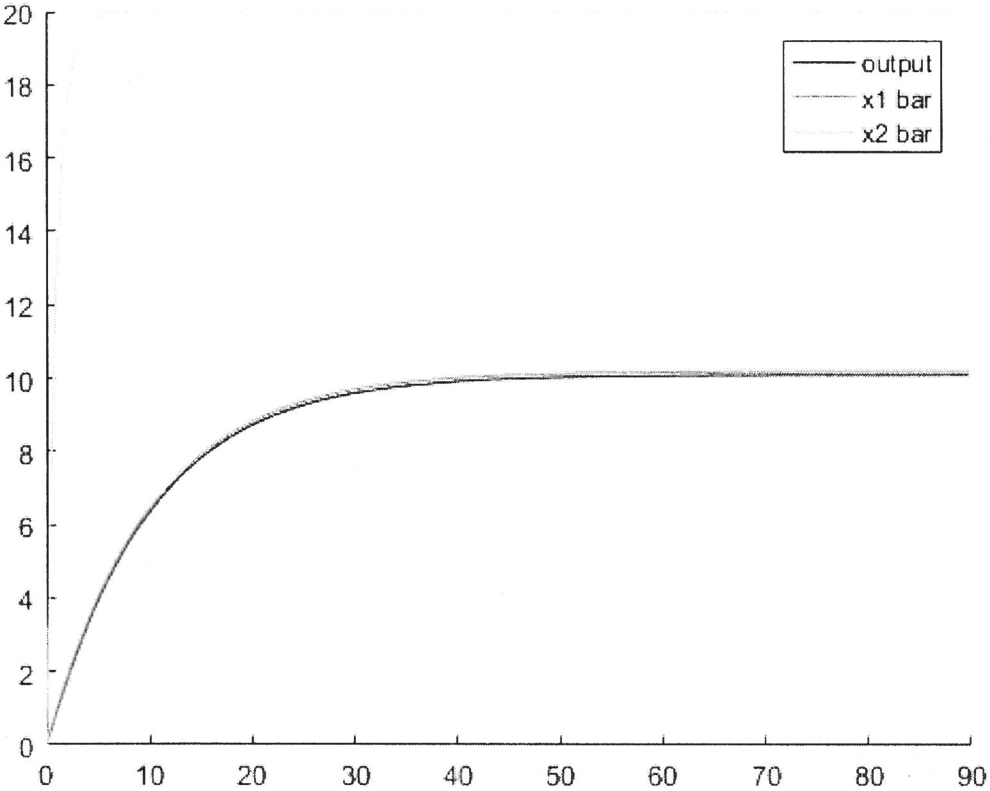
$\Longrightarrow W_c(t_0, t_1)$ is singular $\Longrightarrow$ not controllable

```
% Similarity Transform used to make an op-amp circuit not saturate.
% modified Chen example, p. 99


A = [-0.1 2; 0 -1];
B = [10; .1];
C = [0.1 -1];
D = [0];

sys = ss(A,B,C,D);
[y,t,x]=step(sys);

figure; hold on;
plot(t,y,'b', 'DisplayName','output');
plot(t,x(:,1),'g', 'DisplayName','x1');
plot(t,x(:,2),'y', 'DisplayName','x2');


h = findobj('Color','b');
i = findobj('Color','g');
j = findobj('Color','y');
v = [h(1) i(1) j(1)];
legend(v);




T = [.1 0; 0 200];
T_i = inv(T);

A_bar = T*A*T_i
B_bar = T*B
C_bar = C*T_i
D_bar = D


sys_bar = ss(A_bar,B_bar,C_bar,D_bar);
[y_bar,t_bar,x_bar]=step(sys_bar);

figure; hold on;
plot(t_bar,y_bar,'b', 'DisplayName','output');
plot(t_bar,x_bar(:,1),'g', 'DisplayName','x1 bar');
plot(t_bar,x_bar(:,2),'y', 'DisplayName','x2 bar');

h = findobj('Color','b');
i = findobj('Color','g');
j = findobj('Color','y');
v = [h(1) i(1) j(1)];
legend(v);
```

```
A_bar =

   -0.1000    0.0010
         0   -1.0000
```

```
B_bar =

     1
    20


C_bar =

    1.0000   -0.0050


D_bar =

     0
```





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Question # 6

a) $y(t) = \int_0^1 u(\tau)\,d\tau$ (let $\max_{t\to\infty} u(\tau) < L$ be bounded ($L < \infty$)

$$y(t) \leq \int_0^1 L\,d\tau = L\tau\Big|_0^1 = L$$

$\longrightarrow$ yes, it is BIBO stable

b)

$$\frac{d}{dt}\,\Phi(t,t_0) = A\,\Phi(t,t_0)$$

$$\frac{d}{dt}\,\Phi(t,t_0) = \begin{bmatrix} -e^{-t+t_0} & 0 \\ \frac{1}{4}\left(-4e^{-4t-t_0}\right) & 0 \end{bmatrix} = A\,\Phi(t,t_0)$$

Set $A\,\Phi(t,t_0) = A\,\underbrace{\Phi(t,t)}_{I} = A$

$$= \begin{bmatrix} -e^{0} & 0 \\ -e^{-5t} & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -e^{-5t} & 0 \end{bmatrix} \neq \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix}$$

$\rightarrow$ So, this is not the state transition matrix for this system.

$\rightarrow$ Can also see this by solving the diff eq. of A for $\Phi(t,t_0)$)

c) Not asymptotically stable. The system will converge to a steady state value but not asymptotically so to zero, so only <u>marginally</u> stable.

b) No, it doesn't have to be strictly proper rational. ~~It must be "proper rational";~~ →

~~But~~ D matrix may contain proper rational terms. ~~The~~ If $D = 0$ then it will be strictly proper rational.

- LTI systems can ~~only~~ realize proper rational functions. Does not have to be <u>strictly</u> proper rational.

e)

controllability matrix $C = [B \; AB \; A^2B \; \cdots \; A^{n-1}B]$

$\bar{A} = TAT^{-1}$

$\bar{B} = TB$

$\bar{C} = [TB \;\; TAT^{-1}TB \;\; (TAT^{-1}TAT^{-1})TB \;\; \cdots \;]$

$= [TB \;\; TAB \;\; TA^2B \;\; \cdots \;\; TA^{n-1}B]$

$= T[B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$

$\bar{C} = TC$

$\text{rank}(\bar{C}) = \text{rank}(C)$ if T is nonsingular transformation.