Common Norms (vectors):

2-norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \quad \text{(Euclidean norm)}$$

1-norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Infinity-norm

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

- Can sort of see this, because as p gets larger the biggest element is going to dominate.

P-norm

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p \right]^{1/p} \quad \text{where} \quad \|x\|_\infty = \lim_{p \to \infty} \|x\|_p$$

absolute value signs

~~Matrix Induced Norms~~:

$$\cancel{\|A\|_i = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|}$$

Equivalent Matrix Norms:

2-norm

$$\cancel{\|x\|_2} \quad \|A\|_2 = \left(\lambda_{max} \underbrace{(A^T A)}_{} \right)^{1/2} = \sigma_{max}(A)$$

largest singular value of A

$\lambda_{max}(\cdot)$ = maximum e-value

ensures the matrix is pos. definite.

1-norm

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

j = col
i = row
- max absolute value of the col sum

(compute the 1 norm w/ each column & take the max)

∞-norm

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$ - max absolute row sum (compute 1-norm over rows)

Frobenius-norm

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} = \left(\sum_{i=1}^{n} \sigma_i [A]^2 \right)^{1/2}$$

Example:

$$A = \begin{array}{cc} \begin{bmatrix} -3 & 5 & 7 \\ 2 & 6 & 4 \\ 1 & 2 & 2 \end{bmatrix} & \begin{array}{c} 15 \\ 12 \\ 5 \end{array} \\ \begin{array}{ccc} 6 & 13 & 13 \end{array} & \end{array}$$

$$\|A\|_1 = \max(|-3|+2+1,\ 5+6+2,\ 7+4+2) = (6, 13, 13) = 13$$

$$\|A\|_\infty = \max(15, 12, 5) = 15$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)} = 11.43$$

$$\|A\|_F = \left(\sum_i^m \sum_j^n a_{ij}^2 \right)^{1/2} = 12.17$$

- All matrix norms are equivalent. They can be upper and lower bounded by any other type of norm times a constant.

$$r\,\|A\|_\alpha \le \|A\|_\beta \le s\,\|A\|_\alpha \qquad r,s>0$$

(tells us that it doesn't matter which norm we use — if easier to calculate one norm then can use that and rest holds)

- Matrix norms are also submultiplicative:

$$\|AB\|_\alpha \le \|A\|_\alpha \|B\|_\alpha \qquad \alpha \in \{1,2,\infty,F\}$$

- Induced Norms: –

$$\Rightarrow \quad \|Ax\|_\alpha \le \|A\|_\alpha \|x\|_\alpha \quad \forall x$$

$$\|A\|_\alpha \ge \max_{x\neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha} = \max_{\|x\|=1} \|Ax\|_\alpha$$

$y = Ax \longrightarrow$ ~~we~~ want to know – what is the size of y in terms of the size of x

i.e. $\|y\| \overset{?}{\Longleftrightarrow} \|x\|$

(what is the relationship)

~~wrt y~~

$\Longrightarrow$ use submultiplicative property

$$\|y\|_\alpha \le \underbrace{\|A\|_\alpha}_{} \|x\|_\alpha$$

$\longrightarrow$ Induced norm is the maximum that x can be stretched – so resulting y must be less than or equal to that.

~~Matrix induced norm~~

- Matrix induced norm – How much does ~~a~~ the matrix A shrink or stretch ~~a vector~~ (or amplify) the x vector to generate the vector y.

Lyapunov Stability:

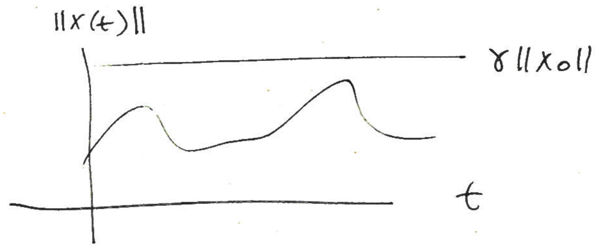
LTV System: $\dot{x} = A(t)x + B(t)u \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,$

$y = C(t)x + D(t)u \qquad y \in \mathbb{R}^m$

Definitions:

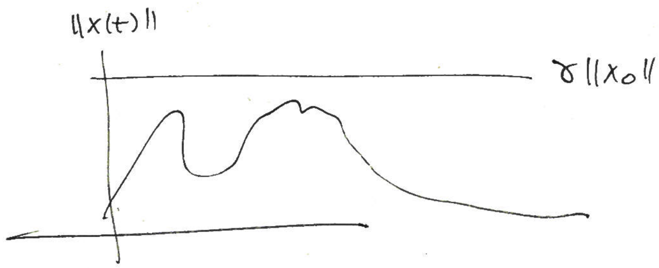
① **Marginally Stable** — The LTV system is (marginally) stable in the sense of Lyapunov (ISL stability) (internally stable) if for every $x(t_0) = x_0$, $x(t) = \Phi(t, t_0)\, x_0$, $\forall t \geq 0$ is uniformly bounted.

i.e. $\|\Phi(t, t_0)\| \leq \gamma$ for some $\gamma < \infty$



- size of solution can be no larger than size of the initial condition multiplied by a fixed constant.
- i.e. $\|x(t)\| \leq \|\Phi(t, t_0)\|\, \|x_0\| \leq \gamma \|x_0\|$

② The LTV systym is **asymptotically stable** if it is marginally stable and $x(t) \to 0$ as $t \to \infty$



③ The LTV System is **exponentially stable** if $\exists\ c, \lambda > 0$

s.t $\|x(t)\| \leq c\, e^{\lambda(t - t_0)} \|x(t_0)\| \quad \forall t \geq t_0$

~~$(sI - A)v = 0$~~ ~~$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$~~

$$e^{J_i t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & + \cdots \\ 0 & 1 & t & \frac{t^2}{2!} & + \cdots \\ 0 & 0 & 1 & t & \cdots \\ & & & \ddots & \end{pmatrix}$$

① When e-values of A have strictly neg real parts then $e^{J_i t} \to 0$ as $t \to \infty$ and $e^{At} \to 0$ as $t \to \infty$

② When all e-values have negative or zero real parts + the blocks corresponding to e-values w/ zero real parts are 1×1 then $e^{J_i t}$ remains bounded as $t \to \infty$ + e^{At} is bounded as $t \to \infty$

w/ zero part $e^{\lambda_i t} = 1$ + $e^{J_i t} = \begin{pmatrix} 1 & t & t^2/2! & + \cdots \\ 0 & 1 & t & + \cdots \\ \vdots & & & \end{pmatrix}$

If have a 1×1 block

$$e^{J_i t} = [1] \Rightarrow \text{bounded}$$

If we have a 2×2 block

$$e^{J_i t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \Rightarrow \text{not bounded}$$

Bigger, problem gets worse.

④ The LTV system is unstable if it is not marginally stable in the sense of Lyapunov.

Note: The only only part of the state space representation that matters is the A matrix (~~Φ~~ through the state transition matrix $\Phi(t,t_0)$).

$\longrightarrow$ This is why called Internal Stability. Only Homogeneous solution comes into play.

E-value conditions for Lyapunov stability: (Thm 8.1)

LTI system: $\dot{x} = Ax, \quad x(0) = x_0$

Thm: The LTI system is

① Marginally stable iff all eigenvalues of A have negative or zero real parts + all the Jordan blocks corresponding to e-values w/ zero real parts are (1×1).

② Asymptotically stable iff all e-values have strictly negative real parts: $Re\{\lambda_i\} < 0 \ \forall \lambda_i$

③ ~~Eigen~~ Exponentially stable iff all e-values have strictly negative real parts: $Re\{\lambda_i\} < 0, \forall \lambda_i$

④ Unstable iff at least one e-value has a positive real part or zero w/ Jordan block larger than 1×1

Note: If all e-values of A have strictly negative real parts, all entries of e^{At} converge to zero exp. fast.

$\|e^{At}\| \to 0$ exp fast

$\|e^{At}\| \le c e^{-\lambda t}, \quad \forall t \in \mathbb{R}$

- λ = abuse of notation — not equal to an e-value here.
- Not a formal proof — will be proving this in Homework problem (8.3)

$$\|x(t)\| = \|e^{A(t-t_0)} x_0\| \le \|e^{A(t-t_0)}\| \, \|x_0\| \le c e^{-\lambda(t-t_0)} \|x_0\|$$

→ For <u>LTI</u> systems exponential stability and asymptotic stability are equivalent.

Note: These definitions w/ e-values only work for <u>LTI</u> systems. Even if the e-values do not depend on time.

- Discrete systems have equivalent conditions, but for LTI, discrete systems the e-values only must ~~be~~ $|\lambda_i| \le 1$ (magnitude less than 1) rather then $\lambda_i \le 0$ as w/ continuous.

Positive Definite Matrices:

A symetric $n \times n$ matrix Q is positive definite if

$$x^T Q x > 0 \qquad \forall x \in \mathbb{R}^n, \ x \neq \{0\}$$

negative definite if $\quad x^T Q x < 0 \qquad \forall x \in \mathbb{R}^n, \ x \neq \{0\}$

positive semi-definite if $\quad x^T Q x \geq 0 \qquad \forall x \in \mathbb{R}^n$

negative semi-definite if $\quad x^T Q x \leq 0 \qquad \forall x \in \mathbb{R}^n$

The following are equivalent statements:

1. Q is positive-definite
2. All e-values are strictly positive
3. The determinants of all upper left submatrices of Q are positive.
4. $\exists$ nonsingular H (sq. root of Q) s.t.

$$Q = H^T H$$

Fact: If Q is pos-def

$$0 < \underbrace{\lambda_{min}[Q]}_{\text{smallest e-value}} \|x\|^2 \leq x^T Q x \leq \underbrace{\lambda_{max}[Q]}_{\text{largest e-value}} \|x\|^2 \qquad \forall x$$

Lyapunov Stability theorem (Homogeneous (TI)):

Equivalent Statements: (Theorem 8.2 in book)

1. The system is asymptotically stable
2. The system is exponentially stable
3. All e-values of A have strictly negative real parts (A is Hurwitz)
4. For every symetric pos-definite matrix Q, there exists a Solution P to the following Lyapunov eq:

$$A^T P + PA = -Q \quad \text{w/ } P = \text{symetric + pos-def.}$$

5. There exists a symetric pos-def matrix for which the Lyapunov inequality holds:

$$A^T P + PA < 0$$

Proof: Thm. 8.1 shows $(1) \Leftrightarrow (2) \Leftrightarrow (3)$

We will show $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$

$(2) \Rightarrow (4)$

– Assume that $\exists\, c, \lambda > 0$ s.t. $\|x(t)\| \le c e^{-\lambda t} \|x_0\|$

or $\|e^{At}\| \le c e^{-\lambda t}$

Define: $$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$$ w/ Q = pos-def + symmetric

– we will show that P is the unique, symmetric, pos-def solution of statement #4 (thm. 8.2)

Is it a solution?

$$A^T P + PA = \int_0^\infty \left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A\right) dt$$

remember A commutes w/ its exponential.

But – $$\frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A$$

$$A^T P + PA = \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt = \left[e^{A^T t} Q e^{At}\right]_0^\infty$$

$$= \underbrace{\lim_{t\to\infty}\left(e^{A^Tt} Q e^{At}\right)}_{\text{goes to zero because of asymptotic stability.}} - e^{A^T(0)} Q \underbrace{e^{A(0)}}_{\text{Identity matrix}}$$

$$= -Q$$

$\therefore$ P is a solution to $A^TP + PA = -Q$

Is P symmetric?

$$P^T = \int_0^\infty \left(e^{A^Tt} Q e^{At}\right)^T dt = \int_0^\infty \left(e^{At}\right)^T Q^T \left(e^{A^Tt}\right)^T dt$$

Q is symmetric & $\left(e^{At}\right)^T = e^{A^Tt}$

$$= \int_0^\infty e^{A^Tt} Q e^{At} \, dt = P$$

$\therefore$ P is symmetric

Is P pos-definite?

$$z^TPz = \int_0^\infty z^T e^{A^Tt} Q e^{At} z \, dt$$

let $w(t) = e^{At} z$

then $$z^TPz = \int_0^\infty w^T(t) \, Q \, w(t) \, dt$$

Since Q is pos-definite, $w^T(t) Q w(t) > 0$

$\therefore$ $z^TPz > 0$ & P is pos-definite

Is P a <u>unique</u> solution?

Suppose $\bar{P}$ is another solution, then

$A^TP + PA = -Q$ and $A^T\bar{P} + \bar{P}A = -Q$

$\Rightarrow A^T(P-\bar{P}) + (P-\bar{P})A = 0$ (subtract the two)

$\Rightarrow e^{A^Tt}A^T(P-\bar{P})e^{At} + e^{A^Tt}(P-\bar{P})Ae^{At} = 0$ (multiply R.H.S. by e^{At} & L.H.S by e^{A^Tt})

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{e^{A^Tt}(P-\bar{P})e^{At}}_{\text{must be constant}}\Big) = 0$$

And since $e^{At} \to 0$ as $t \to \infty$ it must be that

$$e^{A^T t}(P-\bar{P})e^{At} = 0, \quad \text{but since } e^{At} \neq 0 \text{ for finite } t$$

Then $P - \bar{P} = 0$ and $\underline{P = \bar{P}}$

$\therefore$ P is a unique solution.

(4) $\Rightarrow$ (5)

Let $Q = I$ in $P = \int_0^\infty e^{A^T t} Q e^{At} dt = \int_0^\infty e^{A^T t} e^{At}$

Then solution for $A^T P + PA = -I < 0$

or Q = pos-defite so $-Q < 0$

and $A^T P + PA = -Q < 0$

To show (5) $\Rightarrow$ (2) we need a lemma

Lemma (8.1): Comparison lemma

Let $v(t)$ be a differentiable scalar signal s.t.

$$\dot{v}(t) \leq \mu v(t) \quad \forall t \geq t_0$$

Then
$$v(t) \leq e^{\mu(t-t_0)} v(t_0)$$

Proof (of Lemma 8.1):

Define a new signal $u(t) = e^{-\mu(t-t_0)} v(t), \quad \forall t \geq t_0$

Take the derivative: $\dot{u}(t) = -\mu e^{-\mu(t-t_0)} + e^{-\mu(t-t_0)} \underbrace{\dot{v}(t)}_{\dot{v}(t) \leq \mu v(t)}$

$\Rightarrow$ $\dot{u}(t) \leq -\mu e^{-\mu(t-t_0)} v(t) + \mu e^{-\mu(t-t_0)} v(t) = 0$ (terms cancel)

$\Rightarrow$ $\dot{u}(t) \leq 0$ $\Rightarrow$ $u(t)$ is non-increasing. Signal must stay same or ~~get~~ decrease.

$u(t) = e^{-\mu(t-t_0)} v(t) \leq \underbrace{u(t_0)}_{\text{can't get bigger than our starting point (always decreasing)}} = v(t_0) \longrightarrow$ by definition, plug in t_0 for t.

$$\Rightarrow \quad v(t) \le e^{\mu(t-t_0)} v(t_0)$$

because $\left(e^{-\mu(t-t_0)} v(t) \le v(t_0) \right)$

move to other side

$(5) \Rightarrow (2)$

Assume $P = P^T > 0$ satisfies $A^T P + PA < 0$

Let $Q := -(A^T P + PA) > 0$

Let $v(t) = x^T(t) P x(t) \ge 0, \quad \forall t \ge 0$

Then $\dot{v}(t) = \dot{x}^T P x + x^T P \dot{x} = \cancel{x^T (A^T P + PA)} x$

(because) $\dot{x} = Ax$

$$= x^T A^T P x + x^T P A x = x^T \underbrace{(A^T P + PA)}_{-Q} x$$

$$= -x^T Q x \le 0 \quad \forall t \ge 0$$

$\Rightarrow$ $v(t)$ is a non-increasing signal

$\Rightarrow$ $v(t) = x^T(t) P x(t) \le v(0) = x^T(0) P x(0)$

Also, since

$$\lambda_{min}(Q) \|x(t)\|^2 \le x^T(t) Q x(t) \le \lambda_{max}(Q) \|x(t)\|^2$$

we have

$$\dot{v} = -x^T Q x \le -\lambda_{min}(Q) \|x(t)\|^2$$

Also since

$$\lambda_{min}(P) \|x(t)\|^2 \le \underbrace{v(t) = x^T P x}_{\text{def of } v(t)} \le \lambda_{max}(P) \|x(t)\|^2$$

$$\Rightarrow \|x(t)\|^2 \le \frac{x^T P x}{\lambda_{min}(P)} = \frac{v(t)}{\lambda_{min}(P)} \le \underbrace{\frac{v(0)}{\lambda_{min}(P)}}_{} \quad \forall t \ge 0$$

v is non-increasing

→ this shows the system is stable, will ~~not~~ always be less than an initial constant

→ To show exp. stability

$$\dot{V} \le -\lambda_{min}(Q)\,\|x(t)\|^2 \le -\frac{\lambda_{min}(Q)}{\lambda_{min}(P)} V(t)$$

plug in for $\|x\|^2$

$\therefore$ By the comparison lemma

$$V(t) \le e^{-\lambda(t-t_0)} V(t_0) \quad \text{where} \quad \lambda \triangleq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$$

Because –

$$\|x(t)\|^2 \le \frac{V(t)}{\lambda_{min}(P)} \le \frac{1}{\lambda_{min}(P)} e^{-\lambda(t-t_0)} V(t_0)$$

$$V(t_0) = x^T(t_0) P x(t_0) \le \lambda_{max}(P)\,\|x(t_0)\|^2$$

$$\Rightarrow \|x(t)\|^2 \le \frac{\lambda_{max}(P)}{\lambda_{min}(P)} e^{-\lambda(t-t_0)} \|x(t_0)\|^2$$

→ we have shown exp. stability (2) assuming (5)

Stability of locally linearized system:

- Given a nonlinear system $\dot{x} = f(t)$ w/ equilibrium s.t. $f(x^{eq}) = 0$.
- Let $\delta x = x - x^{eq}$ and $A = \left.\frac{\partial f}{\partial x}\right|_{x^{eq}}$

Then the linearized system is:

$$\dot{\delta x} = A\,\delta x \qquad (*)$$

Theorem 8.5: Assume $f(x)$ is twice differentiable. If linearized system $(*)$ is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around x^{eq}, and constants $c, \lambda > 0$ s.t.

$x(t_0) \in B$ implies that

$$\| x(t) - x^{eq} \| \leq c e^{-\lambda(t-t_0)} \| x(t_0) - x^{eq} \|$$

Proof: From taylor's theorem, f ~~twice~~ twice differentiable implies $f(x) = f(x^{eq}) + A\delta x + O(\|\delta x\|)^2$

$\exists\, c, \bar{B}$

~~let~~ $r(x) := f(x) - f(x^{eq}) - A\delta x \leq c \|\delta x\|^2 \quad \forall x \in \bar{B}$

- Since the linearized system is exp. stable, there exists a pos-def matrix P, for which

$$A^T P + PA = -I$$

Let $v(t) = \delta x^T P\, \delta x$, then

$$\begin{aligned}
\dot{v} &= \dot{\delta x}^T P \delta x + \delta x^T P \dot{\delta x} \\
&= (\dot{x} - \dot{x}_{eq})^T P (x - x_{eq}) + (x - x_{eq})^T P (\dot{x} - \dot{x}_{eq}) \\
&= \cancel{f^T P \delta x} \; f^T P \delta x + \delta x^T P f \\
&= \cancel{(A + \delta x + r)} \; (A\delta x + r(x))^T P \delta x + \delta x^T P (A\delta x + r(x)) \\
&= \delta x^T (A^T P + PA) \delta x + r^T P \delta x + \delta x^T P r
\end{aligned}$$

$$= \delta x^T \underbrace{(A^T P + P A)}_{-I} \delta x + \underbrace{r^T P \delta x + \delta x^T P r}_{(\delta x^T P r)^T + (\delta x^T P r)}$$

because P is symmetric

$(\delta x^T P r)$ = single value 1x1 matrix so it is automatically symmetric and

$$(\delta x^T P r)^T = (\delta x^T P r)$$

$$\dot{V} \leq -\|\delta x\|^2 + 2\,\delta x^T P r$$

$$\leq -\|\delta x\|^2 + 2\,\|\delta x\|\,\|P\|\,\|r\|$$

If δx is small enough to ensure that

$$-\|\delta x\| + 2\,\|P\|\,\|\delta x\|\,\|r(x)\| \leq -\tfrac{1}{2}\|\delta x\|^2 \qquad (\cancel{**})$$

Then $\dot{V} \leq -\tfrac{1}{2}\|\delta x\|^2$

but $\lambda_{max}(P)\|\delta x\|^2 \leq V(t) = \delta x^T P \delta x \leq \lambda_{max}(P)\|\delta x\|^2$

$$\Rightarrow \quad -\tfrac{1}{2}\|\delta x\|^2 \leq -\frac{1}{2\lambda_{max}(P)} V$$

$$\Rightarrow \quad \dot{V} \leq \frac{-1}{2\lambda_{max}(P)} V$$

Then by the comparison lemma $V(t) \to 0$ exp. fast

$\Rightarrow \delta x(t) \to 0$ exp. fast

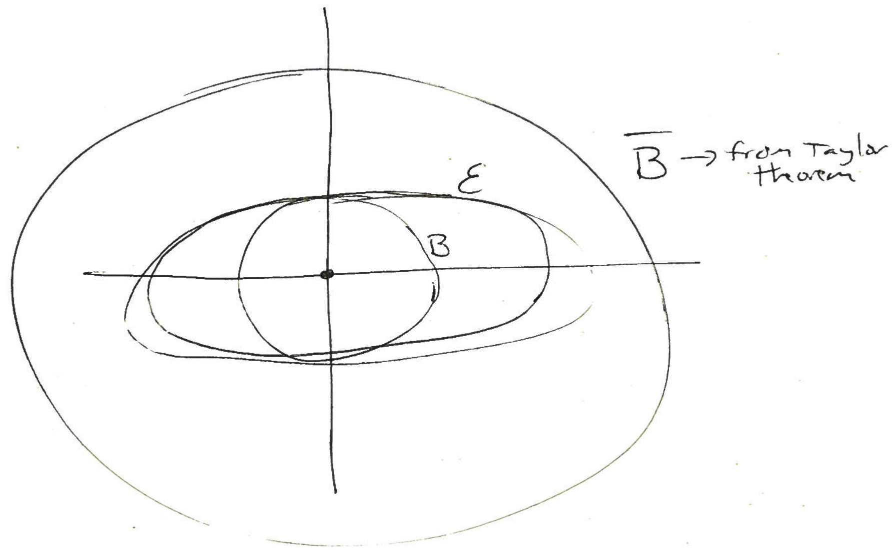
So we need to ensure our assumption $(\cancel{**})$ holds.

$$-\|\delta x\| + 2\,\|P\|\,\|\delta x\|\,\|r(x)\| \leq -\tfrac{1}{2}\|\delta x\|^2$$

Define

$$\mathcal{E} \triangleq \{ x \in \mathbb{R}^n : (\delta x)^T P \delta x \le \varepsilon \}$$

and pick ε small enough so that $\mathcal{E} \subset \overline{B}$



Then for all $\delta x \in \mathcal{E}$ we have $\|r(x)\| \le c \|\delta x\|^2$

So —

$$-\|\delta x\|^2 + 2\|P\|\|\delta x\| \underbrace{\|r(x)\|}_{\le c\|\delta x\|^2} \le (-1 + 2\|P\|\|\delta x\| c)\|\delta x\|^2$$

$$= -(1 - 2c\|P\|\|\delta x\|)\|\delta x\|^2$$

– Pick $B \subset \mathcal{E}$ so that

$$1 - 2c\|P\|\|\delta x\| \ge 1/2$$

i.e. $\|\delta x\| \le \dfrac{1}{4c\|P\|}$ i.e B has a radius $\le \dfrac{1}{4c\|P\|}$

$\therefore \quad \delta x \in B \implies -\|\delta x\|^2 + 2\|P\|\|\delta x\|\|r(x)\| \le -\frac{1}{2}\|\delta x\|$

⟶ Any ~~point~~ solution $x(t)$ that starts w/in radius $\frac{1}{4c\|P\|}$ of the eq. point will converge exp. fast.

(c is a bound of how big the remainder term $r(x)$ can get – $\|r(x)\| \le c\|\delta x\|^2$)