Thm 5.1

- The unique solution to the homogeneous (zero-input) system

$$\dot{x} = A(t)x \quad , \quad x(t_0) = x_0 \in \mathbb{R}^n \quad t \geq 0$$

is given by:

$$x(t) = \Phi(t, t_0)\, x_0 \,, \quad \cancel{x_0 \in \mathbb{R}^n, t \geq 0}$$

where $\Phi(t, t_0)$ is the state transition matrix given by:

$$\Phi(t,t_0) = \underbrace{I + \int_{t_0}^{t} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2) \int_{t_0}^{s_2} A(s_3)\,ds_3\,ds_2\,ds_1 + \ldots}_{\text{Peano-Baker series}}$$

→ Suppose we could solve this infinite integral series (difficult unless we can figure out if it converges) — let's look @ what we would have in terms of properties of the soln - state transition matrix.

Properties of the state transition matrix:

- $\Phi(t, t_0)$ is a way to "transition" the states from the initial values x_0 at time t_0 to $x(t)$ for any future point in time

P#1
$$\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0), \quad \Phi(t_0,t_0) = I, \; t \geq 0$$

To prove this we need Leibnitz rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = f(t,b(t))\frac{db}{dt} - f(t,a(t))\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t,x)}{dt}\,dx$$

(Leibnitz)

5-2 (optional)

$$\frac{d}{dt}\left[\int_{t_0}^{t} A(s_1)\,ds_1\right] = \underbrace{A(t)\frac{dt}{dt}}_{A(t)} + \underbrace{-A(t_0)\frac{dt_0}{dt}}_{0} + \underbrace{\int_{t_0}^{t}\frac{dA(s_1)}{dt}\,ds_1}_{0}$$

0 because A is not a function of t

only part that is a function of t

$$\underbrace{\frac{d}{dt}\int_{t_0}^{t} A(s_1)}_{(A(t)-0+0)}\int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 = A(t)\int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1$$

Proof:

$$\frac{d}{dt}\Phi(t,t_0) = \frac{d}{dt}\left[I + \int_{t_0}^{t} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\int_{t_0}^{s_2} A(s_3)\,ds_3\,ds_2\,ds_1 + \dots\right]$$

$$0 + A(t) + A(t)\int_{t_0}^{t} A(s_2)\,ds_2 + A(t)\int_{t_0}^{t} A(s_2)\int_{t_0}^{s_2} A(s_3)\,ds_2\,ds_3 + \dots$$

$$= A(t)\underbrace{\left[I + \int_{t_0}^{t} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \dots\right]}_{\Phi(t,t_0)}$$

rename integration parameter

$\therefore$

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$$

Also note that —

$$\Phi(t_0,t_0) = I + \int_{t_0}^{t_0} A(s_1)\,ds_1 + \int_{t_0}^{t_0} A(s_1)\int_{t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \dots$$

$$= I + 0 + 0 + \dots$$

$\therefore$

$$\Phi(t_0,t_0) = I$$

— Using this property we can prove Theorem 5.1

let $X(t) = \Phi(t,t_0)X_0$

Then $X(t_0) = \Phi(t_0,t_0)X_0 = X_0$

And

$$\dot{X} = \frac{d}{dt}\left(\Phi(t,t_0)X_0\right) = A(t)\underbrace{\Phi(t,t_0)X_0}_{X(t)}$$

$\therefore$

$$\dot{X}(t) = A(t)X(t)$$

P. #2 Semi-group property

$$\Phi(t,t_1)\Phi(t_1,t_0) = \Phi(t,t_0)$$

$$\cancel{\Phi(t,s) =} \qquad \cancel{\Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau)}$$

– this comes from the fact that our initial time t_0 is completely arbitrary ~~so~~

So we could say

$$x(t) = \Phi(t,t_1)\,x_1 \quad \text{for any } t \geq t_1$$

$$x(t_1) = x_1 = \underbrace{\Phi(t_1,t_0)\,x_0}$$

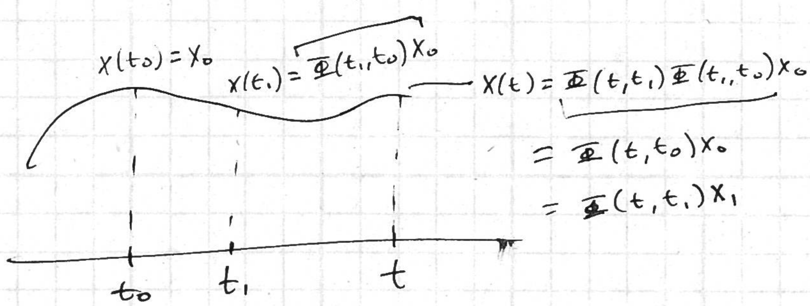
– Plugging-in for x_1 ↓

$$x(t) = \Phi(t,t_1)\big(\Phi(t_1,t_0)\,x_0\big)$$

– This must also equal ↓

$$x(t) = \underbrace{\Phi(t,t_0)}\,x_0 = \underbrace{\Phi(t,t_1)\Phi(t_1,t_0)}\,x_0$$

These must be equal



P #3 For every $t, \tau \geq 0$ $\Phi(t,\tau)$ is nonsingular (i.e. invertible)

w/ $\Phi(t,\tau)^{-1} = \Phi(\tau,t)$

Proof:

$$\Phi(\tau,t)\Phi(t,\tau) = \Phi(\tau,\tau) = I$$

$$\Phi(t,\tau)\Phi(\tau,t) = \Phi(t,t) = I$$

⟶ which means that $\Phi(t,\tau)$ is the inverse of $\Phi(\tau,t)$ and vice-versa (by definition of an inverse).

Thm 5.2 Unique solution of the nonhomogeneous linear system

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x + D(t)u$$

is —

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

(variation of constants formula)

$$y(t) = \underbrace{C(t)\Phi(t,t_0)x_0}_{\text{Homogeneous response}} + \underbrace{\int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau + \boxed{D(t)u(t)}}_{\text{forced response (depends on input } u(t))}$$

$\Phi(t,t_0)$ = state transition matrix

— Feedthrough term
- Instantaneous input has a direct relationship to the output
- In real applications this is very rare. Might happen if have 2 time scales — one really fast and one really slow for purposes of modelling assume fast one is instantaneous.

Proof —

If $t = t_0$

$$x(t) = \underbrace{\Phi(t_0,t_0)}_{I}x_0 + 0 \implies x(t) = x_0$$

otherwise —

$$\dot{x} = \frac{d}{dt}\left(\Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right)$$

$$\frac{d}{dt}\left(\Phi(t,t_0)x_0\right) = A(t)\Phi(t,t_0)x_0$$

$$\frac{d}{dt}\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau = B(t)u(t) + A(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

(using Leibnitz rule)

Adding the two together + rearranging terms

$$\dot{x} = A(t)\underbrace{\left[\Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right]}_{x(t)} + B(t)u(t)$$

Solving w/ Leibniz rule

$$\frac{d}{dt}\int_{t_0}^{t} \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau = \underbrace{\Phi(t,t)}_{I} B(t)\, u(t) \frac{dt}{dt} - (\quad) \frac{dt_0}{dt} + \int_{t_0}^{t} \frac{d}{dt}\left(\Phi(t,\tau)\, B(\tau)\, u(\tau)\right) d\tau$$

$$= B(t)\, u(t) + \int_{t_0}^{t} A(t)\, \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau$$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

$\therefore \quad \dot{x} = A(t)x + B(t)u$

Now look @ $y = C(t)x + D(t)u$

plug in our solution for x

$$y = C(t)\left(\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right) + D(t)u$$

$$= C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} \left(C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right) + D(t)u$$

$\rightarrow$ Shown that these are solutions to the nonhomogeneous LTV system.

Discrete-time Case (Analogous to continuous)

$$x(t+1) = A(t)\,x(t) \qquad x(t_0) = x_0 \in \mathbb{R}^n, \quad t \in \mathbb{N} \text{ (time steps)}$$

~~Solution~~

So —

$$x(t_0+1) = A(t_0)\,x(t_0) = A(t_0)\,x_0$$

$$x(t_0+2) = A(t_0+1)\,x(t_0+1) = A(t_0+1)\,A(t_0)\,x_0$$

$$x(t_0+3) = A(t_0+2)\,x(t_0+2) = A(t_0+2)\,A(t_0+1)\,A(t_0)\,x_0$$

$$\vdots$$

$$x(t) = A(t-1)\,A(t-2)\cdots A(t_0+2)\,A(t_0+1)\,A(t_0)\,x_0$$

$\therefore$ Let

$$\Phi(t,t_0) = \begin{cases} I & t = t_0 \\ A(t-1)A(t-2)\cdots A(t_0+1)A(t_0) & t > t_0 \end{cases}$$

Then

$$x(t) = \Phi(t,t_0)\,x_0 \qquad t \geq t_0$$

Note: $t \geq t_0$ for discrete time. You can't "reverse" time w/ discrete time, but you can w/ continuous.

Properties of the Discrete time state transition matrix:

P #1:

For $t_0 \geq 0$, $\Phi(t, t_0)$ is a unique solution to

$$\Phi(t+1, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I, \quad t \geq t_0$$

→ This falls out from the definition of $\Phi(t, t_0)$

P #2 For every $t \geq s \geq \tau \geq 0$

$$\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau)$$

→ Again, results from the definition of $\Phi(t, t_0)$

Thm 5.3 – The unique solution to the discrete nonhomogeneous equation ~~is~~

$$x(t+1) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

w/ $x(t_0) = x_0 \in \mathbb{R}^n$, $t \in \mathbb{N}$

is —

$$x(t) = \Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)B(\tau)u(\tau)$$

$$y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t, \tau+1)B(\tau)u(\tau) + D(t)u(t)$$

Proof:

→ use definition of $x(t)$ + plug in $x(t+1)$ to see what it yields

$$x(t+1) = \Phi(t+1, t_0)x_0 + \sum_{\tau=t_0}^{t} \Phi(t+1, \tau+1)B(\tau)u(\tau)$$

$$= \Phi(t+1, t)\Phi(t, t_0)x_0 + \underbrace{\Phi(t+1, t+1)}_{I}B(t)u(t) + \sum_{\tau=t_0}^{t-1} \Phi(t+1, \tau+1)B(\tau)u(\tau)$$

$$= \Phi(t+1, t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \underbrace{\Phi(t+1, \tau+1)}_{\hookrightarrow \Phi(t+1, t)\Phi(t, \tau+1)}B(\tau)u(\tau) + B(t)u(t)$$

$$= \Phi(t+1,t)\left[\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1}\Phi(t,\tau+1)B(\tau)u(\tau)\right] + B(t)u(t)$$

$$= \Phi(t+1,t)\,x(t) + B(t)u(t)$$

Put $\Phi(t+1,t) = A(t)$

$$\therefore \quad x(t+1) = A(t)x(t) + B(t)u(t)$$

and $x(t_0) = \Phi(t_0,t_0)x_0 + \sum_{t_0}^{t_0-1}(\;—\;) = x_0$