9.1

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u, \quad y = [1 \ 1 \ 0] x + u$$

a) TF: $\hat{G}(s) = C(sI - A)^{-1} B + D$

$$= [1 \ 1 \ 0] \begin{bmatrix} \frac{1}{s+2} & 0 & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1$$

$$= \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s-1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1 = \frac{1}{s+2} + 1 = \frac{s+3}{s+2}$$

b) A is unstable, has a pos e-value

c) The system is BIBO stable - All poles of TF have neg. real part.

10.1

a) Prove by contradiction -

- Assume the null space of $\begin{bmatrix} A - \lambda I \\ c \end{bmatrix}$ has a non zero e-vector
- Prove that this makes the Θ matrix singular.

$$\begin{bmatrix} A - \lambda I \\ c \end{bmatrix} v = \phi \quad \text{for } v \neq 0 \implies \begin{matrix} Av = \lambda v \\ cv = 0 \end{matrix}$$

- Then if $\Theta v = 0$ is singular (this is a linear combination of the column vectors - so only equals zero if Θ is singular)

$$\mathcal{O}v = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = \begin{bmatrix} Cv \\ CAv \\ \vdots \\ CA^{n-1}v \end{bmatrix} = \begin{bmatrix} 0 \\ C\lambda v \\ \vdots \\ CA^{n-2}\lambda v \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda Cv \\ \vdots \\ \underbrace{\lambda CA^{n-2}v}_{\lambda^{n-1}Cv} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore$ $\mathcal{O}$ is singular $\Rightarrow$ if $\mathcal{O}$ is nonsingular then v must be a zero vector.

b) show that if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ e-vector of H w/ $j\omega$ e-value.

i.e. $Hx = \lambda x = j\omega x$

$$[x_2^* \;\; x_1^*] Hx + (Hx)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

$$[x_2^* \;\; x_1^*](j\omega x) + (j\omega x)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= j\omega\left(x_2^* x_1 + x_1^* x_2\right) + -j\omega [x_1^* \;\; x_2^*] \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$= j\omega\left(\cancel{x_2^* x_1} + \cancel{x_1^* x_2}\right) - j\omega\left(\cancel{x_1^* x_2} + \cancel{x_2^* x_1}\right) = 0$$

c) If $Hx = j\omega x$ then $b^T x_2 = 0$ + $Cx_1 = 0$

$$[x_2^* \quad x_1^*] \begin{bmatrix} A & -bb^T \\ -c^Tc & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1^* \quad x_2^*] \begin{bmatrix} A & -bb^T \\ -c^Tc & -A^T \end{bmatrix}^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

$$[x_2^* A - x_1^* c^T c \quad -x_2^* bb^T - x_1^* A^T] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [x_1^* \quad x_2^*] \begin{bmatrix} A^T & -c^T c^* \\ -b^T b^* & -A^* \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

$$x_2^* A x_1 - x_1^* c^T c x_1 - x_2^* bb^T x_2 - x_1^* A^T x_2 + [x_1^* A^T - x_2^* b^T b^* \quad -x_1^* c^T c^* - x_2^* A^*] \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$

$$x_2^* \cancel{A} x_1 - x_2^* bb^T x_2 - x_1^* c^T c x_1 - x_1^* \cancel{A^T} x_2 + x_1^* \cancel{A^T} x_2 - x_2^* b^T b x_2 - x_1^* c^T c x_1 - x_2^* \cancel{A} x_1 = 0$$

$$-2x_2^* bb^T x_2 - 2x_1^* c^T c x_1 = 0$$

$$-2(b^T x_2)^* (b^T x_2) - 2(c x_1)^* (c x_1) = 0$$

$$-2\|b^T x_2\|^2 - 2\|c x_1\|^2 = 0$$

$\rightarrow$ norms must be positive so

$b^T x_2 = 0$ + $c x_1 = 0$

b)

$$H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = j\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{bmatrix} A & -bb^T \\ -c^Tc & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = j\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A x_1 - \cancel{bb^T} x_2 = j\omega x_1 \qquad A x_1 = j\omega x_1$$

$$-\cancel{c^T}c x_1 - A^T x_2 = j\omega x_2 \qquad -A^T x_2 = j\omega x_2$$

$$\begin{pmatrix} A - \lambda I \\ c \end{pmatrix} x_1 = \left.\begin{matrix} A x_1 - \overset{j\omega}{\overset{\downarrow}{\lambda}} x_1 = 0 \\ c x_1 = 0 \end{matrix}\right\}$$

Have an e-vector $x_1 \neq 0$ in the null space of $\begin{pmatrix} A - \lambda I \\ c \end{pmatrix}$

$$\begin{pmatrix} A^T - \lambda I \\ b^T \end{pmatrix} x_2 = \left.\begin{matrix} A^T x_2 - \overset{-j\omega}{\overset{\swarrow}{\lambda}} x_2 = 0 \\ b^T x_2 = 0 \end{matrix}\right\}$$

Have an e-vector $x_2 \neq 0$ in the null space of $\begin{pmatrix} A^T - \lambda I \\ b^T \end{pmatrix}$

→ If there are e-values of H over the Im axis then there are e-vectors (non-zero) in the null space of $\begin{bmatrix} A - \lambda I \\ c \end{bmatrix}$ & $\begin{bmatrix} A^T - \lambda I \\ b^T \end{bmatrix}$.

→ Therefore if there are no e-vectors (non-zero) in $\begin{pmatrix} A - \lambda I \\ c \end{pmatrix}$ + $\begin{bmatrix} A^T - \lambda I \\ b^T \end{bmatrix}$ then H has no e-values over the Im axis

Problem # 3

1.

$$x_1 = h \qquad x_2 = v \qquad u = \frac{f}{m} - g$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. Controllibility Gramian : (LTI system)

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0 - \tau)} B B^T e^{A^T(t_0 - \tau)}$$

$$e^{A(t-t_0)} = \mathcal{L}^{-1}\{(sI - A)\}$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} \qquad (sI-A)^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$e^{A(t-t_0)} = \begin{bmatrix} 1 & (t-t_0) \\ 0 & 1 \end{bmatrix} \qquad e^{-A(t-t_0)} = (e^{At})^{-1} = \begin{bmatrix} 1 & -(t-t_0) \\ 0 & 1 \end{bmatrix}$$

$$W_c(t_0, t_1) = \int_0^{t_1 - t_0} \begin{bmatrix} 1 & -\sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma \\ 0 & 1 \end{bmatrix}^T d\sigma$$

$$= \int_0^{t_1 - t_0} \begin{bmatrix} -\sigma \\ 1 \end{bmatrix} \begin{bmatrix} -\sigma & 1 \end{bmatrix} d\sigma = \int_0^{t_1 - t_0} \begin{bmatrix} \sigma^2 & -\sigma \\ -\sigma & 1 \end{bmatrix} d\sigma$$

$$= \begin{bmatrix} \frac{\sigma^3}{3} & -\frac{\sigma^2}{2} \\ -\frac{\sigma^2}{2} & \sigma \end{bmatrix} \Bigg|_0^{t_1 - t_0} = \begin{bmatrix} \frac{(t_1 - t_0)^3}{3} & -\frac{(t_1 - t_0)^2}{2} \\ -\frac{(t_1 - t_0)^2}{2} & t_1 - t_0 \end{bmatrix}$$

Reachability Gramian:

$$W_R(t_0,t_1) = \int_0^{t_1-t_0} e^{A\sigma} B B^T e^{A^T\sigma}\, d\sigma$$

$$= \int_0^{t_1-t_0} \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix} d\sigma$$

$$= \int_0^{t_1-t_0} \begin{bmatrix} \sigma \\ 1 \end{bmatrix} \begin{bmatrix} \sigma & 1 \end{bmatrix} d\sigma$$

$$= \int_0^{t_1-t_0} \begin{bmatrix} \sigma^2 & \sigma \\ \sigma & 1 \end{bmatrix} d\sigma = \begin{bmatrix} \frac{\sigma^3}{3} & \frac{\sigma^2}{2} \\ \frac{\sigma^2}{2} & \sigma \end{bmatrix} \Bigg|_0^{t_1-t_0}$$

$$= \begin{bmatrix} \frac{(t_1-t_0)^3}{3} & \frac{(t_1-t_0)^2}{2} \\ \frac{(t_1-t_0)^2}{2} & (t_1-t_0) \end{bmatrix}$$

3. $\text{Im}\left[W_c(t_0,t_1)\right] = \text{Im}\begin{pmatrix} \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ -\frac{\Delta t^2}{2} & \Delta t \end{pmatrix} = \begin{bmatrix} \frac{\Delta t^3}{3} \\ -\frac{\Delta t^2}{2} \end{bmatrix}, \begin{pmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{pmatrix}$

let $\Delta t = t_1 - t_0$

$\det(W_c(t_0,t_1)) = \frac{\Delta t^4}{3} + \frac{\Delta t^4}{4} \neq 0$ So two cols are linearly independent & form a set of basis vectors

$\text{rank}(W_c(t_0,t_1)) = 2$ $\quad \ker(W_c(t_0,t_1)) = \phi$

$\text{nullity}(W_c(t_0,t_1)) = 0$

$\text{Im}\left[W_R(t_0,t_1)\right] = \text{Im}\begin{pmatrix} \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^2}{2} & \Delta t \end{pmatrix} = \begin{pmatrix} \frac{\Delta t^3}{3} \\ \frac{\Delta t^2}{2} \end{pmatrix}, \begin{pmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{pmatrix}$

$\det(W_R(t_0,t_1)) = \frac{\Delta t^4}{3} - \frac{\Delta t^4}{4} \neq 0$ - So cols are linearly indp. & both are in the Im of W_R

$\ker\left[W_R(t_0,t_1)\right] = \phi$ because $\ker\left[W_R(t_0,t_1)\right] = \begin{pmatrix} \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^2}{2} & \Delta t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\rightarrow$ this is only true if $x_1 = x_2 = 0$, thus only have trivial $x = 0$ vector in nullspace or ker of $W_R(t_0,t_1)$

$\text{rank}(W_R(t_0,t_1)) = 2$

$\text{nullity}(W_R(t_0,t_1)) = 0$

4) Controllability matrix —

$\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \text{rank}(\mathcal{C}) = 2$ $\quad \text{Im}(\mathcal{C}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{nullity}(\mathcal{C}) = 0$

5) $\mathcal{C}[t_0,t_1] \in \mathbb{R}^2$ (controllable subspace)

$\mathcal{R}[t_0,t_1] \in \mathbb{R}^2$ (Reachable Subspace)

System is fully controllable & reachable.