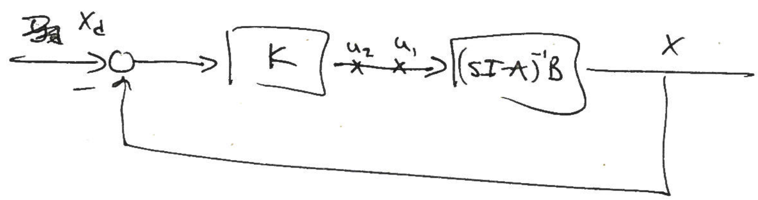
Loop Transfer Recovery:

- Combined LQR/kalman filter loses robustness properties of individual LQR or kalman filter.
- Loop transfer recovery → try to make LQG look like LQR (or vice versa)

LQR:



Loop gain from u_1 to u_2 is:

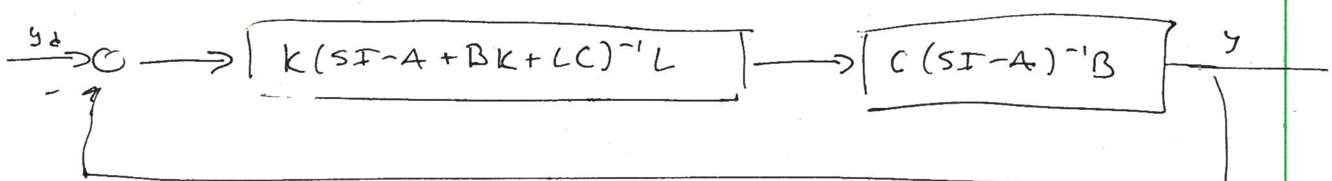
$$H_{LQR} = K(sI-A)^{-1}B$$

(assuming $y_d = 0$ momentarily)

$$\begin{cases} u_2 = +Kx \\ x = (sI-A)^{-1}Bu_1 \end{cases}$$

$$u_2 = -K(sI-A)^{-1}B$$

LQG:



Loop gain from u_1 to u_2 is:

$$H_{LQR} = K\underbrace{(sI-A+BK+LC)^{-1}LC}(sI-A)^{-1}B$$

→ Want to change the variances N & D s.t the matrix

$$(sI-A+BK+LC)^{-1}LC \rightarrow I.$$

→ Sol: Let $D(r) = D_0 + rBB^T$ and let $r \to \infty$

- This will be satisfied as $r \to \infty$ if

  $L = \sqrt{r}\,\tilde{L}$ where $\tilde{L}$ is indepentent of r

- For Kalman filter $L = SC^TN^{-1}$ C & N are indepentent of r.

  Need $S = \sqrt{r}\,\tilde{S}$ w/ $\tilde{S}$ indepentent of r.

- Then as $r \to \infty$ $\frac{S}{r} \to 0$ and $\frac{L}{\sqrt{r}} \not\to 0$

(FARE) $AS + SA^T + D - SC^TN^{-1}CS = 0$

$$\Rightarrow \sqrt{r}A\tilde{S} + \sqrt{r}\tilde{S}A^T + (D_0 + rBB^T) - r\tilde{S}C^TN^{-1}C\tilde{S} = 0$$

(divide by r and let $r \to \infty$, then separate last term)

$$\Rightarrow BB^T = \tilde{S}C^TN^{-1}C\tilde{S}$$

$$BB^T = \left(\frac{L}{\sqrt{r}}N^{1/2}\right)\left(\frac{L}{\sqrt{r}}N^{1/2}\right)^T \qquad w/\ N = N^{1/2}N^{1/2}$$

Let W be some orthogonal matrix $WW^T = I$

Then $BWW^TB^T = \left(\frac{L}{\sqrt{r}}N^{1/2}\right)\left(\frac{L}{\sqrt{r}}N^{1/2}\right)^T$

$$\Rightarrow \frac{L}{\sqrt{r}}N^{1/2} = BW$$

$$\Rightarrow \boxed{L = \sqrt{r}\,BWN^{-1/2}}$$

Lemma: $(I+XY)^{-1}X = X(I+YX)^{-1}$

Proof: pre-multiply by $(I+XY)$ + post-multiply by $(I+YX)$ to get

$$(I+XY)(I+XY)^{-1}X(I+YX) = (I+XY)X(I+YX)^{-1}(I+YX)$$
$$X(I+YX) = (I+XY)X$$
$$X + XYX = X + XYX \quad ✓$$

Theorem #1: As $r \to \infty$

$H_{LQG}(s) \to H_{LQR}(s)$ [Assume same # inputs as outputs]

Proof: $H_{LQR}(s) = \cancel{\ldots}$
$$= K(sI - A + BK + LC)^{-1} LC (sI-A)^{-1} B$$

Let $\Phi(s) = (sI-A)^{-1}$
$$\Phi_c(s) = (sI - A + BK)^{-1}$$
$$F(s) = K(sI - A + BK + LC)^{-1} L$$

Then
$$F(s) = K(sI - A + BK + LC)^{-1} L$$
$$= K(\Phi_c^{-1} + LC)^{-1} L$$
$$= K(\Phi_c^{-1}(I + \Phi_c LC))^{-1} L$$
$$= K(I + \Phi_c LC)^{-1} \Phi_c L$$

Kalman's equality for filtering —

$$[I + H(j\omega)]\, N\, [I + H(j\omega)]^* = N + [C(sI-A)^{-1}]\, D\, [(-sI-A^T)C^T]$$

w/ $H(s) = C(sI-A)^{-1}L$

Letting $D = D_0 + rBB^T$ gives

$$[I + H(j\omega)]\, N\, [I + H(j\omega)]^* = N + [C(j\omega I - A)^{-1}]\,(D_0 + rBB^T)\,[(-j\omega I - A^T)^{-1}C^T]$$

Divides by r & letting $r \to \infty$ gives:

$$H(j\omega)\frac{N}{r} + \frac{N}{r}H^*(j\omega) + H(j\omega)\frac{N}{r}H^*(j\omega) = [C(j\omega - A)^{-1}B][C(j\omega - A)^{-1}B]^*$$

$$\cancel{= \left[C(j\omega I - A)^{-1}\frac{L}{\sqrt{r}}\right] N \left[C(j\omega - A)^{-1}\frac{L}{\sqrt{r}}\right]^*}$$

(Remember $BB^T = \left(\frac{L}{\sqrt{r}}N^{1/2}\right)\left(\frac{L}{\sqrt{r}}N^{1/2}\right)^T$

Apply Lemma: $(I + XY)^{-1}X = X(I + YX)^{-1}$

on $F(s) = K\Phi_c L(I + C\Phi_c L)^{-1}$ *

w/ $X = \Phi_c L$ & $Y = K$

And $L \to \sqrt{r}\, BWN^{-1/2}$

$\rightarrow$

$$\Rightarrow F(s) = \cancel{\sqrt{r}K}\ \sqrt{r}\,K\Phi_c BWN^{-1/2}\left(I + \sqrt{r}\,C\Phi_c BWN^{-1/2}\right)^{-1}$$

$$= K\Phi_c BWN^{-1/2}\left(\frac{1}{\sqrt{r}}I + C\Phi_c BWN^{-1/2}\right)^{-1}$$

$$\cancel{= K\Phi_c BWN^{-1/2}\left(\frac{1}{\sqrt{r}}\right.}$$

$$= K\Phi_c BWN^{-1/2}N^{1/2}W^{-1}(C\Phi_c B)^{-1}$$

$$= K\Phi_c B(C\Phi_c B)^{-1}$$

$$\Phi_c B = (sI - A + BK)^{-1}B$$

$$= (\Phi^{-1} + BK)^{-1}B = \left[\Phi^{-1}(I + \Phi BK)\right]^{-1}B$$

$$= (I + \Phi BK)^{-1}\Phi B$$

Use lemma again w/ $X = \Phi B$ & $Y = K$

$$\Phi_c B = \Phi B(I + K\Phi B)^{-1}$$

In the limit —

$$F(s) = K\Phi B(I + K\Phi B)^{-1}\left[C\Phi B(I + K\Phi B)^{-1}\right]^{-1}$$

$$= K\Phi B(I + K\Phi B)^{-1}(I + K\Phi B)(C\Phi B)^{-1}$$

$$= K\Phi B\left[C\Phi B\right]^{-1}$$

$\therefore$ As $r \to \infty$

plug in

$$H_{LQG}(s) = F(s)C\Phi B = K\Phi B\left[C\Phi B\right]^{-1}C\Phi B$$

$$= K\Phi B = K(sI - A)^{-1}B$$

$$= H_{LQR}(s)$$

* will get same result if instead of letting the process covariance be:

$$D = D_0 + r B B^T$$

we let the state weighting matrix be:

$$Q = Q_0 + r C^T C$$

while keeping the KF gain fixed.