6.2

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad e^{At} = \mathcal{L}^{-1}\{\underbrace{(SI-A)^{-1}}_{\bar{A}}\} \qquad \bar{A} = \begin{pmatrix} s-1 & -1 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & s-1 \end{pmatrix}$$

$$\bar{A}^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\underbrace{\mathcal{L}^{-1}\{\bar{A}^{-1}\} = e^{At} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}}$$

$$A^t = Z^{-1}\left[z(zI-A)^{-1}\right] \Rightarrow Z^{-1}\left\{\begin{bmatrix} \frac{z}{z-1} & \frac{z}{(z-1)^2} & 0 \\ 0 & \frac{z}{z-1} & 0 \\ 0 & 0 & \frac{z}{z-1} \end{bmatrix}\right\}$$

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u[t](1)^t = 1 \text{ for } t > 0$$

$$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = t$$

$$A^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{A}^{-1} = \begin{pmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & \frac{1}{s(s-1)^2} \\ 0 & \frac{1}{s} & \frac{1}{s(s-1)} \\ 0 & 0 & \frac{1}{s-1} \end{pmatrix}$$

$$e^{A_2 t} = \mathcal{L}^{-1}\{\bar{A}^{-1}\} = \begin{bmatrix} e^t & -1+e^t & 1+(t-1)e^t \\ 0 & 1 & -1+e^t \\ 0 & 0 & e^t \end{bmatrix}$$

found using Partial frac. expansion

Partial Frac Expansion:

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)} = 1 \qquad \begin{matrix} A+B=0 & A=-1 \\ -A=1 & B=1 \end{matrix}$$

$$\frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$A(s^2-2s+1) + Bs(s-1) + Cs = 1$$

$$As^2 - 2As + A + Bs^2 - Bs + Cs = 1$$

$$-2s + s + cs$$

$$A = 1, \quad B = -1, \quad C = 1$$

$$\Rightarrow \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}\right\} = 1 - e^t + te^t = 1 + (t-1)e^t$$

$$A_2^t = \mathcal{Z}^{-1}\left[z(zI-A_2)^{-1}\right] = \begin{bmatrix} \frac{z}{z-1} & \frac{z}{z(z-1)} & \frac{z}{z(z-1)^2} \\ 0 & \cancel{z}/\cancel{z} & \frac{1}{z-1} \\ 0 & 0 & \frac{z}{z-1} \end{bmatrix}$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1, \quad t > 0$$

$$\mathcal{Z}^{-1}\left\{\frac{1}{(z-1)^2}\right\} = t-1, \quad t > 0$$

$$\mathcal{Z}^{-1}\left\{\frac{1}{z-1}\right\} = 1, \quad t > 0$$

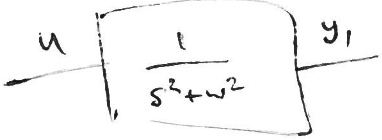
$$A_2^t = \begin{pmatrix} 1 & 1 & (t-1) \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad e^{A_3 t} = \mathcal{L}^{-1}\{\underbrace{(sI-A)}_{\bar{A}_3}{}^{-1}\} = \mathcal{L}^{-1}\{\bar{A}_3^{-1}\}$$

$$\bar{A}_3^{-1} = \begin{bmatrix} \frac{1}{s-2} & 0 & 0 & 0 \\ \frac{2}{(s-2)^2} & \frac{1}{s-2} & 0 & 0 \\ 0 & 0 & \frac{1}{s-3} & \frac{3}{(s-3)^2} \\ 0 & 0 & 0 & \frac{1}{s-3} \end{bmatrix} \qquad \mathcal{L}^{-1}\{\bar{A}^{-1}\} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 2te^{2t} & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & 3te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

$$A_3^t = Z^{-1}\left\{\begin{bmatrix} \frac{z}{z-2} & 0 & 0 & 0 \\ \frac{2z}{(z-2)^2} & \frac{z}{z-2} & 0 & 0 \\ 0 & 0 & \frac{z}{z-3} & \frac{3z}{(z-3)^2} \\ 0 & 0 & 0 & \frac{z}{z-3} \end{bmatrix}\right\} = \begin{bmatrix} 2^t & 0 & 0 & 0 \\ t2^t & 2^t & 0 & 0 \\ 0 & 0 & 3^t & t3^t \\ 0 & 0 & 0 & 3^t \end{bmatrix}$$

# 7.1 Put System in Jordan Normal Form




$$\hat{g}(s) = \frac{1}{s^2+w^2}$$

$\beta_1 = 0,\ \beta_2 = 1$
$\alpha_1 = 0,\ \alpha_2 = w^2$

$$A = \begin{bmatrix} 0 & -w^2 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\dot{x}_1 = \begin{bmatrix} 0 & -w^2 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} x_1$$

---


$$\hat{g}(s) = \frac{1}{s}$$

$\beta_1 = 1$
$\alpha_1 = 0$

$A = \begin{bmatrix} 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \end{bmatrix}$
$C = \begin{bmatrix} 1 \end{bmatrix}$

$$\dot{x}_2 = \begin{bmatrix} 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \end{bmatrix} y_1, \quad y_2 = \begin{bmatrix} 1 \end{bmatrix} x_2$$

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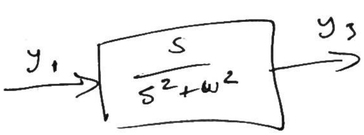


$$\hat{g}(s) = \frac{s}{s^2+w^2}$$

$\beta_1 = 1,\ \beta_2 = 0$
$\alpha_1 = 0,\ \alpha_2 = w^2$

$$A = \begin{bmatrix} 0 & -w^2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x}_3 = \begin{bmatrix} 0 & -w^2 \\ 1 & 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y_1, \quad y_3 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_3$$

Combine systems into a single representation ——

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} x_1$$

$$\dot{x}_3 = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} x_1 = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_1$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\omega^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\omega^2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Put in Jordan Form: Find e-values + e-vektors

$$\det(sI-A) = \det\begin{pmatrix} s & w^2 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & -1 & 0 & s & w^2 \\ 0 & 0 & 0 & -1 & s \end{pmatrix} =$$

$$s^5 + 2s^3w^2 + sw^2 = 0$$

$$s(s^4 + 2s^2w^2 + w^4) = 0$$

$$s(s^2+w^2)^2 = 0$$

$$s = 0,\ \underbrace{wi}_{\text{mult } 2},\ \underbrace{-wi}_{\text{mult } 2}$$

E-value for $\lambda_1 = 0$

$(\lambda_1 I - A)v = 0$

$$\begin{bmatrix} 0 & w^2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & w^2 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \\ v_1 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \phi \\ \\ \end{bmatrix}$$

$w^2 v_{12} = 0$

$-v_{11} = 0$

$-v_{12} = 0$

$-v_{12} + w^2 v_{14} = 0$

$-v_{14} = 0$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

w/ $\lambda_2 = wi$

$$\begin{bmatrix} wi & w^2 & 0 & 0 & 0 \\ -1 & wi & 0 & 0 & 0 \\ 0 & -1 & wi & 0 & 0 \\ 0 & -1 & 0 & wi & w^2 \\ 0 & 0 & 0 & -1 & wi \end{bmatrix} \begin{bmatrix} \\ v_2 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \phi \\ \\ \end{bmatrix}$$

$wi v_{21} + w^2 v_{22} = 0$

$-v_{21} + wi\, v_{22} = 0$

$-v_{22} + wi\, v_{23} = 0$

$-v_{22} + wi v_{24} + w^2 v_{25} = 0$

$-v_{24} + wi v_{25} = 0$

$v_{25} = 1$

$v_{24} = wi$

$v_{22} = 0$

$v_{21} = 0$

$v_{23} = 0$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ wi \end{bmatrix}$$

w/ $\lambda_3 = -wi$

$$\begin{pmatrix} -wi & w^2 & 0 & 0 & 0 \\ -1 & -wi & 0 & 0 & 0 \\ 0 & -1 & -wi & 0 & 0 \\ 0 & -1 & 0 & -wi & w^2 \\ 0 & 0 & 0 & -1 & -wi \end{pmatrix} \begin{pmatrix} \\ V_3 \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \phi \\ \\ \end{pmatrix}$$

$-wiV_{31} + w^2 V_{32} = 0$

$-V_{31} - wiV_{32} = 0$

$-V_{32} - wiV_{33} = 0$

$-V_{32} + -wiV_{34} + w^2 V_{35} = 0$

$-V_{34} \mp wiV_{35} = 0$

$V_{35} = 1$

$V_{34} = wi$

$V_{32} = V_{33} = V_{31} = 0$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -wi \\ 1 \end{pmatrix}$$

→ Have 3 e-vectors – need to have 2 generalized e-vectors, from 2 multiple e-values.

→ Try w/ $\lambda_2 = wi$

$AV_4 = \lambda_2 V_4 + V_2$

$(\lambda_2 I - A)V_4 = V_2$

$wiV_{41} + w^2 V_{42} = 0$

$-V_{41} + wiV_{42} = 0$

$-V_{42} + wiV_{43} = 0$

$-V_{44} + wiV_{45} = \cancel{0}\ wi$

$-V_{42} + wiV_{44} + w^2 V_{45} = 1$

$$V_4 = \begin{bmatrix} iw(w^2-1) \\ w^2-1 \\ \dfrac{-i(w^2-1)}{w} \\ 0 \\ 1 \end{bmatrix}$$

$V_{45} = 1,\ V_{44} = 0,\ w^2 - 1 = V_{42}$

$V_{41} = wi(w^2-1)$

$wiV_{41} + w^2(w^2-1) = 0$

$V_{41} = iw(w^2-1)$

$wiV_{43} = (w^2-1)$

$V_{43} = -\dfrac{i(w^2-1)}{w}$

w/ $\lambda_3 = -\omega i$, find $V_5$

$$-\omega i V_{51} + \omega^2 V_{52} = 0$$

$$-V_{51} - \omega i V_{52} = 0$$

$$-V_{52} - \omega i V_{53} = 0$$

$$-V_{52} - \omega i V_{54} + \omega^2 V_{55} = -\omega i$$

$$-V_{54} - \omega i V_{55} = 1$$

$$V_{55} = -1$$

$$V_{54} = \omega i - 1$$

$$\cancel{V_{54}}$$

$$-V_{52} - \omega i(\omega i - 1) - \omega^2 = -\omega i$$

$$V_{52} = -\omega i(\omega i - 1) - \omega^2 + \omega i$$

$$= \cancel{\omega^2} + \omega i + \omega i - \cancel{\omega^2}$$

$$= 2\omega i$$

$$V_{53} = \frac{2\omega i}{\omega i} = 2$$

$$-V_{51} - \omega i(2\omega i) = 0$$

$$V_{51} = 2\omega^2$$

$$V_5 = \begin{bmatrix} 2\omega^2 \\ 2\omega i \\ 2 \\ \omega i - 1 \\ -1 \end{bmatrix}$$

- Shows that the Jordan blocks are $(2\times 2)$:

$$J_1 = [0]$$

$$J_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} \lambda_3 & 1 \\ 0 & \lambda_3 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \omega i & 1 & 0 & 0 \\ 0 & 0 & \omega i & 0 & 0 \\ 0 & 0 & 0 & -\omega i & 1 \\ 0 & 0 & 0 & 0 & -\omega i \end{bmatrix}$$

8.2

$\|A\|_1 \rightarrow$ choose vector to preserve the max column sum of A.

$\|A\|_\infty \rightarrow$ Choose vector to preserve abs value of the max row.

$\|A\|_2 \rightarrow$ Since this norm is determined by an e-value relationship, the $x$ which achieves equality will be determined by the corresponding e-vector.

i.e. $(\lambda_{max} I - A^T A)\, x = 0$

$\hookrightarrow$ find $x$

8.3 Prove if e-values of $A$ have strictly neg real parts then $\|e^{At}\| \le ce^{-\lambda t}, \forall t \in \mathbb{R}$

$$x(t) = P^{-1} e^{Jt} P x_0$$

$$\|x(t)\| \le \|P^{-1}\| \|e^{Jt}\| \|P\| \|x_0\|$$

$$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{n_i-1}/(n_i-1)! \\ 0 & 1 & \cdots & \cdots & t^{n_i-2}/(n_i-2)! \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix}$$

w/ $$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & \\ & e^{J_2 t} & \phi \\ \phi & & \ddots \\ & & & e^{J_\ell t} \end{pmatrix}$$

→ Pick the $\infty$-norm — will be largest for the Jordan block w/ the highest # of generalized e-vectors. Arbitrarily call this the $i^{th}$ e-value/Jordan block.

neg because $\lambda_i < 0$

$$\|e^{J_i t}\| = e^{\lambda_i t} \sum_{k=0}^{n} \frac{t^k}{k!} = \underbrace{\sum_{m=0}^{\infty} \frac{(\lambda_i t)^m}{m!}} \sum_{k=0}^{n} \frac{t^k}{k!} \le \beta e^{-\alpha t}$$

→ this part dominates because it is an infinite sum + will have higher order terms.

→ exponential will dominate over polynomial.

$$\|x(t)\| \le \underbrace{\|P^{-1}\| \|P\| \beta}_{c} e^{-\alpha t} \|x_0\| \le ce^{-\alpha t} \|x_0\|$$

8.4

a) compute the state transition matrix $\Phi(t,t_0)$

$$\Phi(t,0)\,\Phi(0,t_0) = \Phi(t,t_0) \quad w/\ \Phi(0,t_0) = \Phi^{-1}(t_0,0)$$

$$\Phi(t_0,0) = \begin{bmatrix} e^{t_0}\cos 2t_0 & e^{-2t_0}\sin 2t_0 \\ -e^{t_0}\sin 2t_0 & e^{-2t_0}\cos 2t_0 \end{bmatrix}$$

$$\det(\Phi(t_0,0)) = e^{-t_0}\cos^2 2t_0 + e^{-t_0}\sin^2 2t_0 = e^{-t_0}$$

$$\Phi^{-1}(t_0,0) = e^{t_0}\begin{bmatrix} e^{-2t_0}\cos 2t_0 & -e^{-2t_0}\sin 2t_0 \\ e^{t_0}\sin 2t_0 & e^{t_0}\cos 2t_0 \end{bmatrix} = \begin{bmatrix} e^{-t_0}\cos 2t_0 & -e^{-t_0}\sin 2t_0 \\ e^{2t_0}\sin 2t_0 & e^{2t_0}\cos 2t_0 \end{bmatrix}$$

$$= \Phi(0,t_0)$$

$$\Phi(t,0)\,\Phi(0,t_0) = \begin{bmatrix} e^{t}\cos 2t & e^{-2t}\sin 2t \\ -e^{t}\sin 2t & e^{-2t}\cos 2t \end{bmatrix}\begin{bmatrix} e^{-t_0}\cos 2t_0 & -e^{-t_0}\sin 2t_0 \\ e^{2t_0}\sin 2t_0 & e^{2t_0}\cos 2t_0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t-t_0}\cos 2t\cos 2t_0 + e^{-2(t-t_0)}\sin 2t\sin 2t_0 & -e^{t-t_0}\cos 2t\sin 2t_0 + e^{-2(t-t_0)}\sin 2t\cos 2t_0 \\ -e^{t-t_0}\sin 2t\cos 2t_0 + e^{-2(t-t_0)}\cos 2t\sin 2t_0 & e^{t-t_0}\sin 2t\sin 2t_0 + e^{-2(t-t_0)}\cos 2t \times \cos 2t_0 \end{bmatrix}$$

b) compute $A(t) \Rightarrow \frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$ 

$$\frac{d}{dt}\Phi(t,t_0) = \cancel{e^{t-t_0}\cos 2t\cos 2t_0} \begin{bmatrix} \frac{d}{dt}\Phi(t,t_0)_{11} & \frac{d}{dt}\Phi(t,t_0)_{12} \\ \frac{d}{dt}\Phi(t,t_0)_{21} & \frac{d}{dt}\Phi(t,t_0)_{22} \end{bmatrix}$$

$$\frac{d}{dt}\Phi(t,t_0)_{11} = e^{t-t_0}\cos 2t\cos 2t_0 \mp e^{t-t_0}\,2\sin 2t\cos 2t_0 - 2e^{-2(t-t_0)}\sin 2t\sin 2t_0$$
$$+2e^{-2(t-t_0)}\cos 2t\sin 2t_0$$

$$= e^{t-t_0}\left(\cos 2t\cos 2t_0 - 2\sin 2t\cos 2t_0\right) + 2e^{-2(t-t_0)}\left(\cos 2t\sin 2t_0 - \sin 2t\sin 2t_0\right)$$

$$\frac{d}{dt}\Phi(t,t_0)_{12} = -e^{t-t_0}\cos 2t\sin 2t_0 + 2e^{t-t_0}\sin 2t\sin 2t_0$$
$$-2e^{-2(t-t_0)}\sin 2t\cos 2t_0 + 2e^{-2(t-t_0)}\cos 2t\cos 2t_0$$

$$= e^{t-t_0}\sin 2t_0\left(2\sin 2t - \cos 2t\right) + 2e^{-2(t-t_0)}\cos 2t_0\left(\cos 2t - \sin 2t\right)$$

$$\frac{d}{dt}\Phi(t,t_0)_{21} = -e^{t-t_0}\sin 2t\cos 2t_0 - 2e^{t-t_0}\cos 2t\cos 2t_0$$
$$-2e^{-2(t-t_0)}\cos 2t\sin 2t_0 + e^{-2(t-t_0)}(-2)\overset{\sin}{\cancel{\cos}}\,2t\sin 2t_0$$

$$= -e^{t-t_0}\cos 2t_0\left(\sin 2t + 2\cos 2t\right)$$
$$-2e^{-2(t-t_0)}\sin 2t_0\left(\cos 2t + \sin 2t\right)$$

$$\frac{d}{dt}\Phi(t,t_0)_{22} = e^{t-t_0}\sin 2t\sin 2t_0 + e^{t-t_0}\,2\cos 2t\sin 2t_0$$
$$-2e^{-2(t-t_0)}\cos 2t\cos 2t_0 + e^{-2(t-t_0)}(-2)\overset{\sin}{\cancel{\cos}}\,2t\cos 2t_0$$

$$= e^{t-t_0}\sin 2t_0\left(\sin 2t + 2\cos 2t\right) - 2e^{-2(t-t_0)}\cos 2t_0\left(\cos 2t + \sin 2t\right)$$

(2b)

Plus in - ~~$\Phi(t,t)$~~ $t$ for $t_0$ because $\Phi(t,t) = I$

~~And~~ will leave ~~A~~ —

$$A_{11} = \cos^2 2t - 2\sin 2t\cos 2t + 2\cos 2t\sin 2t - 2\sin^2 2t$$

$$= \cos^2 2t - 2\sin^2 2t = -2 + 3\cos^2 2t = -\frac{1}{2} + \frac{3}{2}\cos 4t$$

$$A_{12} = \sin 2t(2\sin 2t - \cos 2t) + 2\cos 2t(\cos 2t - \sin 2t)$$

$$= 2\sin^2 2t - \sin 2t\cos 2t + 2\cos^2 2t - 2\sin 2t\cos t$$

$$= 2 - 3\sin 2t\cos 2t = 2 - \frac{3}{2}\sin 4t$$

$$A_{21} = -\cos 2t(\sin 2t + 2\cos 2t) - 2\sin 2t(\cos 2t + \sin 2t)$$

$$= -\cos 2t\sin 2t - 2\cos^2 2t - 2\sin 2t\cos 2t - 2\sin^2 2t$$

$$= -2 - 3\sin 2t\cos 2t = -2 - \frac{3}{2}\sin 4t$$

$$A_{22} = \sin 2t(\sin 2t + 2\cos 2t) - 2\cos 2t(\cos 2t + \sin 2t)$$

$$= \sin^2 2t + \cancel{2\sin 2t\cos 2t} - 2\cos^2 2t - \cancel{2\sin 2t\cos 2t}$$

$$= \sin^2 2t - 2\cos^2 2t = 1 - 3\cos^2 2t = -\frac{1}{2} - \frac{3}{2}\cos 4t$$

$$A(t) = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}\cos 4t & 2 - \frac{3}{2}\sin 4t \\ -2 - \frac{3}{2}\sin 4t & -\frac{1}{2} - \frac{3}{2}\cos 4t \end{bmatrix}$$

d) ~~Stable?~~

$\rightarrow$ ~~Not stable, $\Phi(t,t_0)$ is unbounded~~

c) Find char pol. of $A \Rightarrow \det(sI - A) = 0$

$$\left(s + \tfrac{1}{2} \mp \tfrac{3}{2}\cos 4t\right)\left(s + \tfrac{1}{2} + \tfrac{3}{2}\cos 4t\right) - \left(+2 + \tfrac{3}{2}\sin 4t\right)\left(-2 + \tfrac{3}{2}\sin 4t\right)$$

$$= s^2 + s\left(\tfrac{1}{2} - \tfrac{3}{2}\cos 4t\right) + s\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 4t\right) + \left(\tfrac{1}{2} - \tfrac{3}{2}\cos 4t\right)\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 4t\right)$$

$$- \left(-4 - \cancel{3\sin 4t} + \cancel{3\sin 4t} + \tfrac{9}{4}\sin^2 4t\right)$$

$$= s^2 + s + \tfrac{1}{4} - \cancel{\tfrac{3}{4}\cos 4t} + \cancel{\tfrac{3}{4}\cos 4t} - \tfrac{9}{4}\cos^2 4t + 4 - \tfrac{9}{4}\sin^2 4t$$

$$= s^2 + s + \tfrac{1}{4} + \tfrac{16}{4} - \tfrac{9}{4} = s^2 + s + 2$$

w/ roots (e-values) $s^2 + s + 2 = 0$

$$\frac{-1 \pm (1-8)^{1/2}}{2} = \boxed{-\frac{1}{2} \pm \frac{i\sqrt{7}}{2}}$$

d) Stable?

$\rightarrow$ Not stable, $\Phi(t, t_0)$ is unbounded.

8.6

Two ways to show this is true.

① Use modified Lyap eq to imply that $eig(A) < -\mu$

$$A^T P + PA + 2\mu P = -Q$$

$$= (A + \mu I)^T P + P(A + \mu I) = -Q$$

→ so Lyap eq. holds for $(A + \mu I)$ and

$(A + \mu I)$ = new stability matrix ⇒ $eig(A) < -\mu$

② Show that if $eig(A + \mu I) < 0$ then the Lyap equation becomes $A^T P + PA + 2\mu P = -Q$.

$$Ax = \lambda x \iff (A + \mu I)x = (\mu + \lambda)x$$

→ All e-values of $A + \mu I$ can be obtained from those of $A$ by adding $\mu$. So all e-values of $A$ have real part $< -\mu$.

Let $\bar{A} = (A + \mu I)$ & plug new A matrix into Lyap eq.

$$(A + \mu I)^T P + P(A + \mu I) = -Q$$

$$= A^T P + \mu I P + PA + P\mu I = -Q$$

$$= A^T P + PA + 2\mu P = -Q$$

8.7

a) $\dot{x}_1 = -x_1 + x_1(x_1^2 + x_2^2)$ eq. point $x_1 = x_2 = 0$

$\dot{x}_2 = -x_2 + x_2(x_1^2 + x_2^2)$

$$A = \begin{bmatrix} -1 + 3x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & -1 + 3x_2^2 + x_1^2 \end{bmatrix}_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

e-values $= \det(sI - A) = 0$

$\Rightarrow \det\begin{pmatrix} s+1 & 0 \\ 0 & s+1 \end{pmatrix} = 0$ $(s+1)^2 = 0$

e-values $= -1$ w/ multiplicity $= 2$

$\rightarrow$ system is stable if start close enough

b) $\ddot{w} + g(w)\dot{w} + w = 0$ w/ eq. point $w = \dot{w} = 0$

let $x_1 = w \Rightarrow \dot{x}_1 = \dot{w} = x_2$ $f_1 = \dot{x}_1 = x_2$

$x_2 = \dot{w} \Rightarrow \dot{x}_2 = -g(w)\dot{w} - w$ $f_2 = \dot{x}_2 = -g(w)\dot{w} - w$

$= -g(x_1)x_2 - x_1$ $= -g(x_1)x_2 - x_1$

$\rightarrow$ for what values of $g(0)$ can we guarantee convergence?

$$A = \begin{bmatrix} 0 & 1 \\ -1 \mp \frac{dg}{dx_1}x_2 & -g(x_1) \end{bmatrix}_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix}$$

Stability $\rightarrow$ check e-values $\det(sI - A) = 0$

$s(s + g(0)) + 1 = 0 \Rightarrow s^2 + s\,g(0) + 1 = 0$

$s = \dfrac{-g(0) \pm (g^2(0) - 4)^{1/2}}{2} \Rightarrow$ if $g(0) > 0$ then this system will be stable.