Linear Quadratic Regulator (LQR):

Given an LTI system the Linear Quatratic Regulator ~~Problem~~ (LQR) problem is to find u(t) to minimize —

$$J_{LQR} := \int_0^\infty \left(y^T(t)\, Q\, y(t) + u^T(t)\, R\, u(t)\right) dt$$

Where $Q = Q^T \geq 0$ + $R = R^T \geq 0$

For SISO case: $J_{LQR} = \int_0^\infty \left(\underbrace{q\,\|y(t)\|^2}_{\text{energy in the output}} + \underbrace{r\,\|u(t)\|^2}_{\text{Energy in the input (control signal)}}\right) dt$

→ We are trying to minimize both energies.

→ Q + R are control knobs. Trade-off to decrease output energy, have to increase the control signal + vice-versa.

1) If Q = large relative to R ~~will~~ then the best way to minimize J_{LQR} is to get a very small output (y) — even if that means getting a large input (u).

2) If Q = small (relative to R) then to minimize cost function want a very small control (even @ cost of the output energy).

Feedback Invariants:

- Given an initial condition $X(0)$ and an input signal $u(t)$ which is a function of time, the solution to the LTI system is a function of time.
- A function of functions is called a functional.

(function inside a function)

For example $H(\overbrace{X(\cdot)}) = \int_0^\infty X^T(t)\, Q\, X(t)\, dt$

is a functional. (A functional maps functions to scalar values).

For the LTI system we say the a functional $H(x(\cdot); u(\cdot))$ is a feedback invariant ~~for the CLTI~~ if, when computed along the trajectory of the system, its value depends only on $X(0)$ and not on $u(t)$, i.e it is invariant to $u(t)$.

Prop 10.1: For every symmetric matrix $P = P^T$,

$$H(x(\cdot); u(\cdot)) = -\int_0^\infty \Big[(A\,x(t) + Bu(t))^T P\, x(u) + X^T(t)\, P\, (\cancel{Ax})\, A\,x(t) + Bu(t))\Big] dt$$

is a feedback invariant if $\lim_{t \to \infty} X(t) = 0$

Proof:

$$H(x(\cdot);u(\cdot)) = -\int_0^\infty [\dot{x}^T P x + x^T P \dot{x}]\,dt$$

$$= -\int_0^\infty \frac{d(x^T(t)Px(t))}{dt}\,dt$$

$$= x(0)^T P x(0) - \underbrace{\lim_{t\to\infty} x(t)^T P x(t)}_{\text{requires this to go to zero.}}$$

$$= x(0)^T P x(0) \quad \text{(only dependent on } x(0)\text{, invariant to } u(t)\text{)}$$

Feedback Invariants in Optimal Control:

Suppose — (we can express the optimal control as)

$$J = H(x(\cdot);u(\cdot)) + \int_0^\infty \Lambda(x(t),u(t))\,dt$$

where H is a feedback invariant + the function $\Lambda(x,u)$ has the property:

$$\underbrace{\min_{u\in\mathbb{R}^k}}_{\to \text{Not time-varying}} \Lambda(x,u) = 0$$

Then since $H(x(\cdot);u(\cdot))$ is independent of $u(\cdot)$,

$$\min_{u(\cdot)} J = H(x(\cdot),u(\cdot)) + \min_{u(\cdot)} \int_0^\infty \Lambda(x(t),u(t))\,dt$$

Then choose $u(t)$ as

$$u(t) = \arg\min_{u\in\mathbb{R}^k} \Lambda(x,u)$$

and

$$\min_{u(\cdot)} J = \underbrace{H(x(\cdot);u(\cdot))}_{\text{- unaffected by } u(\cdot)\text{, so can't get } J \text{ any smaller than this.}}$$

LQR Derivation:

$$y = CX$$

$$J_{LQR} = \int_0^\infty (x^T C^T Q C x + u^T R u)\,dt$$

$$= \int_0^\infty (x^T C^T Q C x + u^T R u)\,dt + H(x(\cdot);u(\cdot)) - H(x(\cdot);u(\cdot))$$

$$= H(x(\cdot);u(\cdot)) + \int_0^\infty (x^T C^T Q C x + u^T R u)\,dt$$

$$+ \underbrace{\int_0^\infty \Big(\underbrace{(Ax+Bu)}_{\dot{x}}{}^T P x + x^T P (Ax+Bu)\Big)\,dt}_{-H(x(\cdot);u(\cdot))}$$

$$= H(x(\cdot);u(\cdot))$$
$$+ \int_0^\infty \Big[x^T C^T Q C x + u^T R u + x^T A^T P x + u^T B^T P x + x^T P A x + x^T P B u\Big]\,dt$$

$$= H(x(\cdot);u(\cdot))$$
$$+ \int_0^\infty \Big[x^T (A^T P + PA + C^T Q C) x + \underbrace{u^T R u + 2u^T B^T P x}_{\text{complete the sq. in } u}\Big]\,dt$$

Example of computing the sq: ~~(Scaler)~~

Scaler:

$$az^2 + bz$$

Want to write as: $a(z+c)^2 = a(z^2 + 2cz + c^2)$

$$= az^2 + 2acz + ac^2$$

$\therefore$ pick c s.t $2ac = b \Rightarrow c = \frac{b}{2a}$

$$\left[az^2 + bz = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} \right]$$

matrix:

$$u^T R u + 2 u^T B^T P x$$

we would like to write in terms of

$$(u+w)^T R (u+w) = u^T R u + u^T R w + w^T R u + w^T R w$$
$$= u^T R u + 2 u^T R w + w^T R w$$

$\therefore$ Pick w s.t. $Rw = B^T P x \Rightarrow w = R^{-1} B^T P x$

$$\Rightarrow u^T R u + 2 u^T B^T P x = (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) - x^T P B R^{-1} B^T P x$$

(P, R = symmetric)

$$\therefore J_{LQR} = H(x(\cdot); u(\cdot)) + \int_0^\infty \Big[x^T (A^T P + PA + C^T Q C) x + (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) - x^T P B R^{-1} B^T P x \Big] dt$$

$$= H(x(\cdot); u(\cdot)) + \int_0^\infty \Big[x^T (A^T P + PA + C^T Q C - P B R^{-1} B^T P) x + (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) \Big] dt$$

Select $\Big(u(t) = -R^{-1} B^T P x(t) \Big]$ = optimal control

where $P = P^T$ satisfies —

$$\underbrace{A^T P + PA + C^T Q C - P B R^{-1} B^T P}_{\text{Algebraic Riccati Equation}} = 0$$

To give:

$$J_{LQR} = H(x(\cdot), u(\cdot)) = x^T(0) P (x(0) \leftarrow \text{optimal cost.}$$

Closed loop system:

$$\dot{x} = Ax + Bu = Ax - BR^{-1}B^TPx$$
$$= (A - BR^{-1}B^TP)x$$

Open Questions:

1. Under what conditions can we find a P to solve the (ARE)?

   (Intuitively, this should be true as long as there is an input u(t) that can take y(t) to zero w/ finite energy)

2. Is the solution unique?

3. When does the closed loop system $(A - BR^{-1}B^TP)$ provide e-values in the LHP?