HW#4 12.2, 12.3, 13.1, 14.1, 14.2, 14.3, 15.4

12.3

$$A = \begin{bmatrix} -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & \cdots & -\alpha_{n-1} I_{k\times k} & -\alpha_n I_{k\times k} \\ I_{k\times k} & & & \phi & \\ & \ddots & & & \\ & \phi & & I_{k\times k} & 0_{k\times k} \end{bmatrix} \qquad B = \begin{bmatrix} I_{k\times k} \\ 0_{k\times k} \\ \vdots \\ 0_{k\times k} \end{bmatrix}$$

$$AA = \begin{bmatrix} \alpha_1^2 I_{k\times k} - \alpha_2 I_{k\times k} & \alpha_1\alpha_2 I_{k\times k} - \alpha_3 I_{k\times k} & \cdots & \alpha_1\alpha_{n-1} I_{k\times k} - \alpha_n I_{k\times k} & \alpha_1 \alpha_n I_{k\times k} \\ -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & & & -\alpha_n I_{k\times k} \\ I_{k\times k} & 0_{k\times k} & & \phi & \\ & I_{k\times k} & & & \\ \phi & & \ddots & & \\ & & I_{k\times k} & 0_{k\times k} & 0_{k\times k} \end{bmatrix}$$

$$AAA = \begin{bmatrix} -\alpha_1^3 I_{k\times k} + 2\alpha_1\alpha_2 I_{k\times k} & (\alpha_1\alpha_3 I_{k\times k} - \alpha_4 I_{k\times k} \cdots) & (\alpha_1\alpha_4 I_{k\times k} \cdots) & \cdots \\ \alpha_1^2 I_{k\times k} - \alpha_2 I_{k\times k} & \alpha_1\alpha_2 I_{k\times k} - \alpha_3 I_{k\times k} & \alpha_1\alpha_3 I_{k\times k} - \alpha_4 I_{k\times k} & \\ -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & -\alpha_3 I_{k\times k} & \\ I_{k\times k} & 0_{k\times k} & 0_{k\times k} & \\ & I_{k\times k} & & \\ & & \ddots & \\ & & I_{k\times k} & 0_{k\times k} \end{bmatrix}$$

adds $I_{k\times k}$ each time

$$C = \begin{bmatrix} I_{k\times k} & -\alpha_1 I_{k\times k} & (\alpha_1^2 - \alpha_2) I_{k\times k} & \cdots \\ 0_{k\times k} & I_{k\times k} & -\alpha_1 I_{k\times k} & - \; - \\ \vdots & 0_{k\times k} & I_{k\times k} & - \; - \\ \vdots & \vdots & 0_{k\times k} & - \\ & & \vdots & \ddots \\ 0_{k\times k} & 0_{k\times k} & 0_{k\times k} & I_{k\times k} \end{bmatrix}$$

← upper triangle matrix w/ $I_{k\times k}$ along diagonal = full rank.

13.1 :

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

a) Is it controllable?

$$\mathcal{C} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \operatorname{rank}(\mathcal{C}) = 1 \neq \text{not controllable}$$

$$n = 2$$

b)

$$\operatorname{Im}(\mathcal{C}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\ker(\mathcal{C}^T) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad T^{-1} = \begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{-2} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\bar{A} = T^{-1} A T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \bar{B} = T^{-1} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CT = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \bar{D} = D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

<u>Controllable or Minimal Realization :</u>

$$A_c = -1 \quad B_c = 1 \quad C_c = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad D_c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

14.1

$$P = \begin{bmatrix} Q & \rho S \\ \rho S^T & \rho R \end{bmatrix}$$

- Q, R are both sq. + symmetric + pos-def.

- show that the matrix P is pos-def for a sufficiently small, but positive, ρ.

$$= \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} Q & \rho S \\ \rho S^T & \rho R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^T Q + x_2^T \rho S^T & x_1^T \rho S + x_2^T \rho R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

pos definite

can make ρ arbitrarily small, but don't know that having an arbitrarily large x_2 won't cancel.

need to put in quad form.

$$= x_1^T Q x_1 + x_2^T \rho S^T x_1 + x_1^T \rho S x_2 + x_2^T \rho R x_2 = x_1^T Q x_1 + \rho\left(2x_1^T S^T x_2 + x_2^T R x_2\right)$$

$$x_1^T Q x_1 + \rho\left(x_2^T S' x_1 + x_1^T S x_2\right) + \rho x_2^T R x_2 = \underbrace{x_1^T Q x_1 + 2\rho x_1^T S x_2}_{\text{Complete the sq. for this part}} + \rho x_2^T R x_2$$

$$2\rho x_1^T S x_2$$

~~$x_1^T (Q + 2\rho S$~~

$$(x_1 + w)^T Q (w + x_1) = (x_1^T Q + wQ)(w + x_1) = x_1^T Q w + wQw + x_1^T Q x_1 + wQx_1$$

$$= x_1^T Q x_1 + 2x_1^T Q w + wQw$$

$$w = \rho Q^{-1} S x_2$$

$$\Rightarrow \tfrac{1}{2}(x_1 + 2\rho Q^{-1} S x_2)^T Q (x_1 + 2\rho Q^{-1} S x_2) = \tfrac{1}{2} x_1^T Q x_1 + 2\rho^2 x_2^T S^T Q^{-1} S x_2 + \underline{2\rho x_1^T S x_2}$$

$$\tfrac{1}{2}(x_1 + 2\rho Q^{-1} S x_2)^T Q (\quad) \geq 0 \Rightarrow \tfrac{1}{2} x_1^T Q x_1 + 2\rho^2 x_2^T S^T Q^{-1} S x_2 + 2\rho x_1^T S x_2 \geq 0$$

$$\Rightarrow 2\rho x_1^T S x_2 \geq -\tfrac{1}{2} x_1^T Q x_1 + -2\rho^2 x_2^T S^T Q^{-1} S x_2$$

$\longrightarrow$ plug in for $2\rho x_1^T S x_2$

$x^T P x$

$\cancel{x_1^T Q x_1} \geq x_1^T Q x_1 + \rho x_2^T R x_2 + -\frac{1}{2} x_1^T Q x_1 - 2\rho^2 x_2^T S^T Q^{-1} S x_2$

$x^T P x \geq \frac{1}{2} x_1^T Q x_1 + \rho x_2^T (R - 2\rho S^T Q^{-1} S) x_2$

R is pos definite

if ρ is small enough this entire $(\cdot)$ will be pos-definite.

$(R - 2\rho S^T Q^{-1} S)$

$\longrightarrow$ Right side is then pos-definite for a small enough ρ, and non-zero x.

14.2 - SISO CTI

a)

$$\det(sI - A + Bk) = \det\left(\begin{bmatrix} s+\alpha_1 & s+\alpha_2 & \cdots & s+\alpha_n \\ -1 & 0 & & \phi \\ & \ddots & -1 & \\ \phi & & \ddots & \\ & & -1 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 & \cdots & k_n \\ & & \phi & \\ & & & \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} s+\alpha_1+k_1 & \alpha_2+k_2 & \cdots & \alpha_n+k_n \\ -1 & s & \cdots & 0 \\ & \ddots & \ddots & \phi \\ \phi & & \ddots & s \\ & & -1 & \end{bmatrix}$$

$$= (s+\alpha_1+k_1)s^{n-1} - (\alpha_2+k_2)(-1)(s^{n-2}) + \cdots (-1)^{n+1}(a_n+k_n)(-1)^{n-1}$$

$$= (s+\alpha_1+k_1)s^{n-1} + (\alpha_2+k_2)s^{n-2} + \cdots + (\alpha_{n-1}+k_{n-1})s + \alpha_n + k_n$$

$$= s^n + (\alpha_1+k_1)s^{n-1} + (\alpha_2+k_2)s^{n-2} + \cdots + (\alpha_{n-1}+k_{n-1})s + \alpha_n + k_n$$

b) Desired e-values $= (\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$\det(sI - A + Bk) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) = \prod_{i=1}^{n}(s-\lambda_i)$$

$$= s^n + \gamma_1 s^{n-1} + \gamma_2 s^{n-2} + \cdots + \gamma_n$$

c) let $\gamma_i = (\alpha_i + k_i)$ or $k_i = \gamma_i - \alpha_i$

d) $\det(sI - A + Bk) = (s+1)(s+1)(s+2) = s^3 + 4s^2 + 5s + 2$

$\alpha_1 = -1,\ \alpha_2 = -2,\ \alpha_3 = -3$

$k_1 = 4+1 = 5$

$k_2 = 5+2 = 7$

$k_3 = 2+3 = 5$

$k = [5 \quad 7 \quad 5]$

14.3 : $\det(sI-A) = s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3$

a) $B = T\begin{bmatrix}1\\0\\0\end{bmatrix} = C\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}B & AB & A^2B\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = B$

b) $AT = AC\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix} = A\begin{bmatrix}B & AB & A^2B\end{bmatrix}\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix}$

$= \begin{bmatrix}AB & AB\alpha_1+A^2B & AB\alpha_2+A^2B\alpha_1+A^3B\end{bmatrix}$

$= \begin{bmatrix}AB & (\alpha_1+A)AB & (\alpha_2A+\alpha_1A^2+A^3)B\end{bmatrix}$

$T\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix} = \begin{bmatrix}B & AB & A^2B\end{bmatrix}\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix}\begin{bmatrix}-\alpha_1&-\alpha_2&-\alpha_3\\1&0&0\\0&1&0\end{bmatrix}$

$= \begin{bmatrix}B & AB & A^2B\end{bmatrix}\begin{bmatrix}0&0&-\alpha_3\\1&\alpha_1&0\\0&1&0\end{bmatrix}$

$= \begin{bmatrix}AB & \alpha_1AB+A^2B & -\alpha_3B\end{bmatrix}$

By Cayley-Hamilton theorem:

$s^3+\alpha_1s^2+\alpha_2s+\alpha_3=0 \Rightarrow A^3+\alpha_1A^2+\alpha_2A+\alpha_3=0$

$-\alpha_3 = (\alpha_2A+\alpha_1A^2+A^3)$ plug in

$\Rightarrow \begin{bmatrix}AB & (\alpha_1+A)AB & \cancel{(\alpha_2A+\alpha_1A^2} -\alpha_3B\end{bmatrix} = \begin{bmatrix}AB & \alpha_1AB+A^2B & -\alpha_3B\end{bmatrix}$

$\Rightarrow AT = T\begin{bmatrix}-\alpha_1&-\alpha_2&-\alpha_3\\1&0&0\\0&1&0\end{bmatrix}$

c) $T = C\underbrace{\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix}}_{(*)}$ C = full rank

also full rank

$\det(T) = \det(C)\overbrace{\det(*)}^{=1}$

$= \det(C) \neq 0$

T is nonsingular

d) $C = \begin{bmatrix}1&1&2\\-1&-1&-1\\-1&0&-1\end{bmatrix}$ $T = C\begin{bmatrix}1&\alpha_1&\alpha_2\\0&1&\alpha_1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&-1\\-1&0&2\\-1&1&1\end{bmatrix}$

$\alpha_1=-1,\ \alpha_2=-2,\ \alpha_3=-3$

15.1

a) $\mathcal{C} = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & 0 & -c_3 \\ c_1 & 0 & +c_3 \end{pmatrix}$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} * \underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$(c_1 = 0$ or $c_2 = 0$ or $c_3 = 0) \rightarrow$ i.e. if any of our cols $= 0$, then $\mathcal{C}$ becomes singular

b) $c_1 = c_2 = c_3 = 1$ (among many other options)

c) E-vectors of A: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\longrightarrow$ None of these can be in the kernel of C (Thm. 15.8)

$[c_1 \ c_2 \ c_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 \neq 0$

$[c_1 \ c_2 \ c_3] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow c_2 \neq 0$

$[c_1 \ c_2 \ c_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow c_3 \neq 0$

d) $\longrightarrow$ None of the entries of c should be equal to zero.

* E-vectors will be standard orthonormal vectors $v_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where 1 @ the i^{th} position

If any of the SISO outputs equal to zero then it will be in the kernel of C.