Midterm Exam: Nonlinear Systems
Winter 2018

For this exam you may use your notes, text book, any material posted on our class learning suite website, Wikipedia, and Matlab or the equivalent (Python, Julia, etc.). The materials you use must be your own, i.e., you cannot borrow another classmates notes or simulation code. You may not use the internet.

To get full credit you must *show all your work* and *clearly mark your answers*. If you can complete a problem without doing any derivations, then *explain why this is true*. The exam is due at the beginning of class on Monday 3/12/18.

There are seven questions to this exam which are spread out over 3 pages (including this cover sheet).

1. Consider the following nonlinear oscillator, where $x \in R$ and $\mu \in R$:

$$\ddot{x} + (x^2 + \dot{x}^2 - \mu)\dot{x} + x = 0.$$

   a) Rewrite the system in state-space form, find its equilibrium point(s), and linearize about each one.
   b) Compare the solutions and phase portraits of the nonlinear system and its linearization(s) for $\mu < 0, \mu = 0, and\ \mu > 0$.

2. For the function $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$ with $a, b \in R$, find whether $f(x)$ is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

3. **True or False:** Let $x = [x_1, x_2]^T \in R^2$. If false, then explain your reasoning.
   a. $V(x) = x^T P x$, where $P = P^T$, is positive definite if and only if P is positive definite.
   b. $V(x) = \frac{1}{2}(x_1 + x_2)^2$ is radially unbounded.
   c. If $\dot{V}(\boldsymbol{x}) = -x_1^2$ along solutions of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, then $\dot{V}$ is negative definite.
   d. If a system $\dot{\boldsymbol{x}} = f(x)$ is **not** continuously differentiable, then we can't guarantee that there will be a unique solution, $x(t)$.

4. For the second-order system

$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = u,$$

Find a feedback control law $u = u(x_1, x_2)$ so that the closed-loop system will have the origin as a globally asymptotically stable equilibrium point.

5. Consider a rigid spacecraft that evolves according to the Euler equations,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + M_1$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 + M_2$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T \in R^3$ is the angular velocity with respect to an inertial frame from a body-fixed frame. And $I_1 > I_2 > I_3 > 0$ are the principle moments of inertial and $\boldsymbol{M} = [M_1, M_2, M_3]^T$ is the external torque applied by means of a rocket thruster.

a) Suppose that the torques apply the feedback control $M_i = -k_i\omega_i$, where k_1, k_2, k_3 are all positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

b) Now suppose that the spacecraft is subjected to a constant external disturbance $\boldsymbol{d} = [d_1, d_2, d_3]^T$, modifying the dynamics to:

$$
\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + M_1 + d_1 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + M_2 + d_2 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + M_3 + d_3,
\end{aligned}
$$

And implement the dynamic state feedback control law given by:

$$
\begin{aligned}
\boldsymbol{M} &= -K\boldsymbol{\omega} - \boldsymbol{z} \\
\dot{\boldsymbol{z}} &= \boldsymbol{\omega}.
\end{aligned}
$$

Where $K > 0$. Prove that this controller yields a stable closed-loop system and ensures asymptotic convergence of $\boldsymbol{\omega}(t)$ to zero for any $\boldsymbol{d}$ and any initial conditions $\boldsymbol{\omega}(0)$.

6. Which, if any, of the state variables for the following system are guaranteed to converge to zero:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + x_1x_3 \\
\dot{x}_2 &= -x_1 - x_2 + x_2x_3 \\
\dot{x}_3 &= -x_1^2 - x_2^2
\end{aligned}
$$

7. Given the following system:

$$
\begin{aligned}
\dot{x}_1 &= -x_2^3 \\
\dot{x}_2 &= x_1 - x_2
\end{aligned}
$$

What is the strongest stability condition you can say about it? Is it stable, asymptotically stable, exponentially stable, globally asymptotically stable, globally exponentially stable?