Problem 1 Solve the following using KKT conditions

a.)
Minimize $f = 4x_1 - 3x_2 + 2x_1^2 - 3x_1x_2 + 4x_2^2$

$g_1(x): 2x_1 - 1.5x_2 = 5$

KKT Conditions

$\nabla f - \sum \lambda_i \nabla g_i = 0$

$g_i(x) = 0$

$\Rightarrow \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \Rightarrow 4 + 4x_1 - 3x_2 - \lambda(2) = 0$

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \Rightarrow -3 - 3x_1 + 8x_2 - \lambda(-1.5) = 0$

$2x_1 - 1.5x_2 - 5 = 0$

$2x_1 - 1.5x_2 - 5 = 0$

In matrix form:

$$\underbrace{\begin{bmatrix} 4 & -3 & -2 \\ -3 & 8 & 1.5 \\ 2 & -1.5 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}}_{b}$$

$\boxed{f^* = 22.5}$

$X = A^{-1}b \Rightarrow X = \begin{bmatrix} 2.5 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_1^* = 2.5 \\ x_2^* = 0 \\ \lambda^* = 7 \end{matrix}}$

This agrees with the graphical optimum

b.) Change $g_1(x) = 2x_1 - 1.5x_2 = 5.1$

$$\Rightarrow \begin{bmatrix} 4 & -3 & -2 \\ -3 & 8 & 1.5 \\ 2 & -1.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 5.1 \end{bmatrix} \Rightarrow \begin{matrix} x_1^* = 2.55 \\ x_2^* = 0 \\ \lambda^* = 7.1 \end{matrix}$$

$\Delta f_{actual} = f(2.55, 0) - f(2.5, 0) = 23.205 - 22.5 = \boxed{0.705}$

$\Delta f_{predicted} = \lambda^* \Delta b = 7(0.1) = \boxed{0.7}$

Yes the lagrange multiplier λ accurately predicts the change in objective value. The difference between the actual and the predicted is only $\boxed{0.005}$

Problem 1

c.) Are the KKT equations for a problem with a quadratic objective and a linear equality constraint always linear?

Yes. If f is quadratic, ∇f will be linear. Since g is also a linear equality constraint, the KKT equations will always be linear

What about a problem with a quadratic objective and a linear inequality constraint?

Yes. ∇f will be linear same as before and so will any g (constraint) that is found to be a binding inequality constraint

Problem 2 — No problem 2 given?

Problem 3 — Solve the following using the KKT conditions:

Minimize $f(x) = x_1^2 + 2x_2^2 + 3x_3^2$

$g_1(x) = x_1 + 5x_2 = 12$

$g_2(x) = -2x_1 + x_2 - 4x_3 \leq -18$

Step 1 — Put into proper form.

$$\text{Max } f(x) = -x_1^2 - 2x_2^2 - 3x_3^2$$

s.t.

$$g_1(x): \; x_1 + 5x_2 = 12$$

$$g_2(x): \; -2x_1 + x_2 - 4x_3 \leq 18$$

KKT Conditions

$$\nabla f - \sum \lambda_i \nabla g_i = 0$$

$$g_i(x) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0 \Rightarrow -2x_1 - \lambda_1(1) - \lambda_2(-2) = 0 \quad (1)$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0 \Rightarrow -4x_2 - \lambda_1(5) - \lambda_2(1) = 0 \quad (2)$$

$$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial g_1}{\partial x_3} - \lambda_2 \frac{\partial g_2}{\partial x_3} = 0 \Rightarrow -6x_3 - \lambda_1(0) - \lambda_2(-4) = 0. \quad (3)$$

$$g_1(x) = 0 \Rightarrow x_1 + 5x_2 - 12 = 0 \quad (4)$$

$$g_2(x) = 0 \Rightarrow -2x_1 + x_2 - 4x_3 + 18 = 0 \quad (5)$$

Assume g_1, g_2 are both binding and try to solve:

Put into linear matrix form:

$$\underbrace{\begin{bmatrix} -2 & 0 & 0 & -1 & 2 \\ 0 & -4 & 0 & -5 & -1 \\ 0 & 0 & -6 & 0 & 4 \\ 1 & 5 & 0 & 0 & 0 \\ -2 & 1 & -4 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 12 \\ -18 \end{bmatrix}}_{b}$$

Solving $x = A^{-1}b \Rightarrow$

$$\boxed{\begin{aligned} x_1 &= 4.7170 \\ x_2 &= 1.4566 \\ x_3 &= 2.5057 \\ \lambda_1 &= -1.9170 \\ \lambda_2 &= 3.7585 \end{aligned}}$$

$$\boxed{f^* = 45.3283}$$

Since λ_2 corresponding to $g_2(x)$ an inequality constraint is positive, and λ_1 corresponding to $g_1(x)$ can be positive or negative, the KKT conditions are satisfied

Problem 4   Min $f(x) = x_1^2 + x_2$

$g_1(x) = x_1^2 + x_2^2 - 9 = 0$

$g_2(x) = x_1 + x_2^2 - 1 \leq 0$

$g_3(x) = x_1 + x_2 - 1 \leq 0$

a.) verify that $[-2.3723, -1.8364]$ is a local optimum

Step 1   Put into propper form:

Max $f(x) = -x_1^2 - x_2$

$g_1(x): x_1^2 + x_2^2 - 9 = 0$

$g_2(x): x_1 + x_2^2 - 1 \leq 0$

$g_3(x): x_1 + x_2 - 1 \leq 0$

Step 2   See which constraints are binding:

$g_1(x)$ is binding since it is an equality constraint

check $g_2(x)$:

$-2.3723 + (-1.8364)^2 - 1 = 0 \Rightarrow \lambda_2 \neq 0$ and $g_2(x)$ is binding

check $g_3(x)$:

$-2.3723 - 1.8364 - 1 = -5.2087 < 0 \Rightarrow \lambda_3 = 0$ and $g_3(x)$ is not binding

Step 3   Write out Lagrange Multiplier Equations

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = -2x_1 - \lambda_1(2x_1) - \lambda_2(1) = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = -1 - \lambda_1(2x_2) - \lambda_2(2x_2) = 0$$

Step 4   Substitute in the point

$$-2(-2.3723) - \lambda_1((2)(-2.3723)) - \lambda_2 = 0 \quad (1)$$

$$-1 - \lambda_1(2(-1.8364)) - \lambda_2(2(-1.8364)) = 0 \quad (2)$$

$$\Rightarrow \lambda_2 = 4.7446\lambda_1 + 4.7446 \quad (3)$$

Sub (3) $\rightarrow$ (2)

$$-1 - \lambda_1(2(-1.8364)) - (4.7446\lambda_1 + 4.7446)(2(-1.8364)) = 0$$

$$\Rightarrow -1 + 3.6728\lambda_1 + 17.4259\lambda_1 + 17.4259 = 0$$

$$\Rightarrow \boxed{\lambda_1 = -0.7785} \Rightarrow \boxed{\lambda_2 = 1.0508}$$

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Problem 4 Continued

So we have that λ_1 is negative (which is OK since $g_1(x)$ is an equality constraint), and λ_2 is positive (which is required since $g_2(x)$ is an inequality constraint) and so we have a valid set of Lagrange Multipliers which means that we can conclude that the point $[-2.3723, -1.8364]$ is a <u>local optimum</u>

b.) Verify that $[-2.5, -1.6583]$ <u>is not a</u> local optimum

Check if $g_2(x)$ is binding: $-2.5 + (-1.6583)^2 - 1 = -0.75 \Rightarrow g_2(x)$ <u>is not</u> binding and $\lambda_2 = 0$

Check to see if $g_3(x)$ is binding: $-2.5 - 1.6583 - 1 = -5.1583 \Rightarrow g_3(x)$ <u>is not</u> binding and $\lambda_3 = 0$

Set up the KKT equations

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} = -2x_1 - \lambda_1(2x_1) = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} = -1 - \lambda_1(2x_2) = 0$$

Plugging in for x...

$$-2(-2.5) - \lambda_1(2(-2.5)) = 0 \Rightarrow \boxed{\lambda_1 = -1}$$

$$-1 - \lambda_1(2(-1.6583)) = 0 \Rightarrow \boxed{\lambda_1 = 0.3}$$

Since λ_1's are not consistent, this point does not satisfy the KKT equations and <u>cannot</u> be an optimum

c.)

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Problem 4 Continued

c.) By inspection, of the contour plot we see that the optimum is somewhere around $[-0.25, -1.1]$ and at this point $g(x) = x_1 + x_2 - 1 \leq 0$ is not a binding constraint. Thus only the constraint $g(x) = x_1 + x_2^2 - 1 \leq 0$ is binding and we must solve for Lambda (λ)

$$\nabla f - \sum \lambda_i \nabla g_i = 0 \Rightarrow \quad \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g}{\partial x_1} = 0 \rightarrow -2x_1 - \lambda(1) = 0$$

$$g_i(x) = 0 \qquad \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g}{\partial x_2} = 0 \rightarrow -1 - \lambda(2x_2) = 0$$

$$g_i(x) = 0 \quad \rightarrow \quad x_1 + x_2^2 = 1$$

In matrix form:

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & -2x_2 \\ 1 & x_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \\ 1 \end{bmatrix}$$

This doesn't help us much since the equations are nonlinear...

Using Matlab Fsolve() to solve the above system of equations

$$\Rightarrow \boxed{x_1^* = -0.2258, \quad x_2^* = -1.1072, \quad \lambda = 0.4516}$$

$$\boxed{f^* = -1.0562}$$

Since λ is positive, we can conclude that this point is a valid local optimum

MATLAB for problem 4 part c

```
clc
clear

x0 = [-0.25, -1.1, 1];

x = fsolve(@myfun, x0)



function F = myfun(x)
F(1) = -2*x(1) - x(3);
F(2) = -1 -x(3)*2*x(2);
F(3) = x(1) + x(2)^2 -1;
end
```