$\boxed{\text{Problem 1}}$ Solve the following using KKT conditions

a.)
Minimize $f = 4x_1 - 3x_2 + 2x_1^2 - 3x_1x_2 + 4x_2^2$

$g_1(x): 2x_1 - 1.5x_2 = 5$

KKT Conditions

$\nabla f - \sum \lambda_i \nabla g_i = 0$

$g_i(x) = 0$

$\Rightarrow \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \Rightarrow 4 + 4x_1 - 3x_2 - \lambda(2) = 0$

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0 \Rightarrow -3 - 3x_1 + 8x_2 - \lambda(-1.5) = 0$

$2x_1 - 1.5x_2 - 5 = 0$

$2x_1 - 1.5x_2 - 5 = 0$

In matrix form:

$$\underbrace{\begin{bmatrix} 4 & -3 & -2 \\ -3 & 8 & 1.5 \\ 2 & -1.5 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}}_{b}$$

$\boxed{f^* = 22.5}$

$X = A^{-1}b \Rightarrow X = \begin{bmatrix} 2.5 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} x_1^* = 2.5 \\ x_2^* = 0 \\ \lambda^* = 7 \end{array}}$

This agrees with the graphical optimum

b.) Change $g_1(x) = 2x_1 - 1.5x_2 = 5.1$

$$\Rightarrow \begin{bmatrix} 4 & -3 & -2 \\ -3 & 8 & 1.5 \\ 2 & -1.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 5.1 \end{bmatrix} \Rightarrow \begin{array}{l} x_1^* = 2.55 \\ x_2^* = 0 \\ \lambda^* = 7.1 \end{array}$$

$\Delta f_{actual} = f(2.55, 0) - f(2.5, 0) = 23.205 - 22.5 = \boxed{0.705}$

$\Delta f_{predicted} = \lambda^* \Delta b = 7(0.1) = \boxed{0.7}$

Yes the lagrange multiplier λ accurately predicts the change in objective value.
The difference between the actual and the predicted is only $\boxed{0.005}$

Problem 1

c.) Are the KKT equations for a problem with a quadratic objective and a linear equality constraint always linear?

Yes. If f is quadratic, ∇f will be linear. Since g is also a linear equality constraint, the KKT equations will always be linear ←

What about a problem with a quadratic objective and a linear inequality constraint?

Yes. ∇f will be linear same as before and so will any g (constraint) that is found to be a binding inequality constraint ←

Problem 2 — No problem 2 given?

Problem 3 — Solve the following using the KKT conditions:

Minimize $f(x) = x_1^2 + 2x_2^2 + 3x_3^2$

$g_1(x) = x_1 + 5x_2 = 12$

$g_2(x) = -2x_1 + x_2 - 4x_3 \leq -18$

Step 1 — Put into proper form.

$$\text{Max } f(x) = -x_1^2 - 2x_2^2 - 3x_3^2$$

s.t.

$$g_1(x): \; x_1 + 5x_2 = 12$$

$$g_2(x): \; -2x_1 + x_2 - 4x_3 \leq 18$$

KKT Conditions

$$\nabla f - \sum \lambda_i \nabla g_i = 0 \Rightarrow \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0 \Rightarrow -2x_1 - \lambda_1(1) - \lambda_2(-2) = 0 \quad (1)$$

$$g_i(x) = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0 \Rightarrow -4x_2 - \lambda_1(5) - \lambda_2(1) = 0 \quad (2)$$

$$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial g_1}{\partial x_3} - \lambda_2 \frac{\partial g_2}{\partial x_3} = 0 \Rightarrow -6x_3 - \lambda_1(0) - \lambda_2(-4) = 0. \quad (3)$$

$$g_1(x) = 0 \Rightarrow x_1 + 5x_2 - 12 = 0 \quad (4)$$

$$g_2(x) = 0 \Rightarrow -2x_1 + x_2 - 4x_3 + 18 = 0 \quad (5)$$

Assume g_1, g_2 are both binding and try to solve:

Put into linear matrix form:

$$\underbrace{\begin{bmatrix} -2 & 0 & 0 & -1 & 2 \\ 0 & -4 & 0 & -5 & -1 \\ 0 & 0 & -6 & 0 & 4 \\ 1 & 5 & 0 & 0 & 0 \\ -2 & 1 & -4 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 12 \\ -18 \end{bmatrix}}_{b}$$

Solving $x = A^{-1}b \Rightarrow$

$$\boxed{\begin{aligned} x_1 &= 4.7170 \\ x_2 &= 1.4566 \\ x_3 &= 2.5057 \\ \lambda_1 &= -1.9170 \\ \lambda_2 &= 3.7585 \end{aligned}}$$

$$\boxed{f^* = 45.3283}$$

Since λ_2 corresponding to $g_2(x)$ an inequality constraint is positive, and λ_1 corresponding to $g_1(x)$ can be positive or negative, the KKT conditions are satisfied

$\boxed{\text{Problem 4}}$ Min $f(x) = x_1^2 + x_2$

$g_1(x) = x_1^2 + x_2^2 - 9 = 0$

$g_2(x) = x_1 + x_2^2 - 1 \le 0$

$g_3(x) = x_1 + x_2 - 1 \le 0$

a.) verify that $[-2.3723, -1.8364]$ is a local optimum

$\boxed{\text{Step 1}}$ Put into propper form:

Max $f(x) = -x_1^2 - x_2$

$g_1(x): x_1^2 + x_2^2 - 9 = 0$

$g_2(x): x_1 + x_2^2 - 1 \le 0$

$g_3(x): x_1 + x_2 - 1 \le 0$

$\boxed{\text{Step 2}}$ See which constraints are binding:

$g_1(x)$ is binding since it is an equality constraint

check $g_2(x)$:

$-2.3723 + (-1.8364)^2 - 1 = 0 \Rightarrow \lambda_2 \neq 0$ and $g_2(x)$ is binding

check $g_3(x)$:

$-2.3723 - 1.8364 - 1 = -5.2087 < 0 \Rightarrow \lambda_3 = 0$ and $g_3(x)$ is not binding

$\boxed{\text{Step 3}}$ Write out Lagrange Multiplier Equations

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = -2x_1 - \lambda_1(2x_1) - \lambda_2(1) = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = -1 - \lambda_1(2x_2) - \lambda_2(2x_2) = 0$$

$\boxed{\text{Step 4}}$ Substitute in the point

$$-2(-2.3723) - \lambda_1((2)(-2.3723)) - \lambda_2 = 0 \quad (1)$$

$$-1 - \lambda_1(2(-1.8364)) - \lambda_2(2(-1.8364)) = 0 \quad (2)$$

$$\Rightarrow \lambda_2 = 4.7446\lambda_1 + 4.7446 \quad (3)$$

Sub (3) $\rightarrow$ (2)

$$-1 - \lambda_1(2(-1.8364)) - (4.7446\lambda_1 + 4.7446)(2(-1.8364)) = 0$$

$$\Rightarrow -1 + 3.6728\lambda_1 + 17.4259\lambda_1 + 17.4259 = 0$$

$$\Rightarrow \boxed{\lambda_1 = -0.7785} \Rightarrow \boxed{\lambda_2 = 1.0508}$$

Next Page $\longrightarrow$

Problem 4 Continued

So we have that λ_1 is negative (which is OK since $g_1(x)$ is an equality constraint), and λ_2 is positive (which is required since $g_2(x)$ is an inequality constraint) and so we have a valid set of Lagrange Multipliers which means that we can conclude that the point $[-2.3723, -1.8364]$ is a <u>local optimum</u>

b.) Verify that $[-2.5, -1.6583]$ <u>is not</u> a local optimum

Check if $g_2(x)$ is binding: $-2.5 + (-1.6583)^2 - 1 = -0.75 \Rightarrow g_2(x)$ <u>is not</u> binding and $\lambda_2 = 0$

Check to see if $g_3(x)$ is binding: $-2.5 - 1.6583 - 1 = -5.1583 \Rightarrow g_3(x)$ <u>is not</u> binding and $\lambda_3 = 0$

Set up the KKT equations

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} = -2x_1 - \lambda_1(2x_1) = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} = -1 - \lambda_1(2x_2) = 0$$

Plugging in for x...

$$-2(-2.5) - \lambda_1(2(-2.5)) = 0 \Rightarrow \boxed{\lambda_1 = -1}$$

$$-1 - \lambda_1(2(-1.6583)) = 0 \Rightarrow \boxed{\lambda_1 = 0.3}$$

Since λ_1's are not consistent, this point does not satisfy the KKT equations and <u>cannot</u> be an optimum

c.)

Next Page →

Problem 4 Continued

c.) By inspection, of the contour plot we see that the optimum is somewhere around $[-0.25, -1.1]$ and at this point $g(x) = x_1 + x_2 - 1 \leq 0$ is not a binding constraint. Thus only the constraint $g(x) = x_1 + x_2^2 - 1 \leq 0$ is binding and we must solve for Lambda (λ)

$$\nabla f - \sum \lambda_i \nabla g_i = 0 \Rightarrow \quad \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g}{\partial x_1} = 0 \rightarrow -2x_1 - \lambda(1) = 0$$

$$g_i(x) = 0 \qquad \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g}{\partial x_2} = 0 \rightarrow -1 - \lambda(2x_2) = 0$$

$$g_i(x) = 0 \rightarrow x_1 + x_2^2 = 1$$

In matrix form:

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & -2x_2 \\ 1 & x_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \\ 1 \end{bmatrix}$$

This doesn't help us much since the equations are nonlinear...

Using Matlab Fsolve() to solve the above system of equations

$$\Rightarrow \boxed{x_1^* = -0.2258, \quad x_2^* = -1.1072, \quad \lambda = 0.4516}$$

$$\boxed{f^* = -1.0562}$$

Since λ is positive, we can conclude that this point is a valid local optimum

MATLAB for problem 4 part c

```
clc
clear

x0 = [-0.25, -1.1, 1];

x = fsolve(@myfun, x0)



function F = myfun(x)
F(1) = -2*x(1) - x(3);
F(2) = -1 -x(3)*2*x(2);
F(3) = x(1) + x(2)^2 -1;
end
```

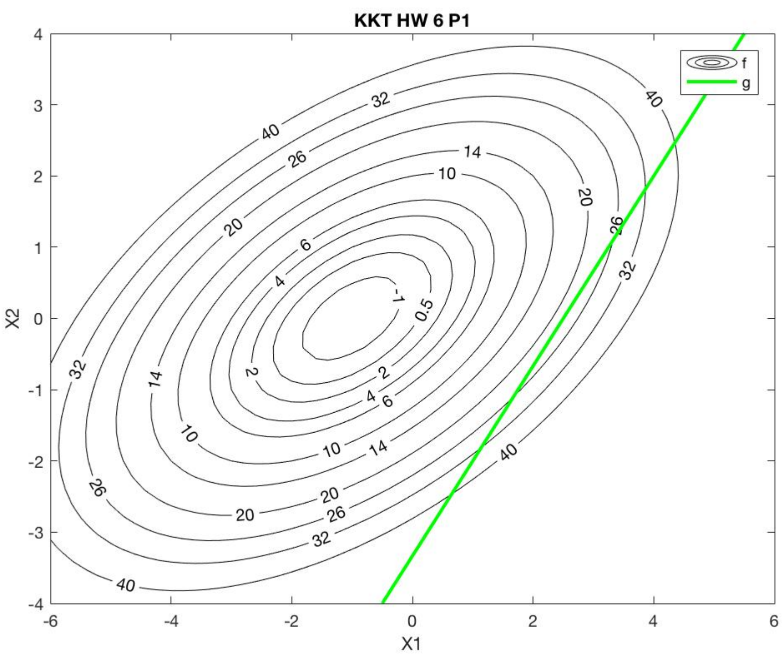
ME 575
Homework #6 Due: March 28 11:50 p.m.
KKT conditions, LaGrange Multipliers

Turn in through Learning Suite. Show your work.

1. (a) (4 points) Solve the following problem using KKT conditions (a contour plot is given):

$$Min\ f = 4x_1 - 3x_2 + 2x_1^2 - 3x_1x_2 + 4x_2^2$$
$$g_1(\mathbf{x}):\quad 2x_1 - 1.5x_2 = 5$$

Does the optimum from the KKT conditions agree with the graphical optimum?



(b) (3) Change the constraint to be,

$$g_1(\mathbf{x}):\quad 2x_1 - 1.5x_2 = 5.1$$

Solve again for the optimum. Does the Lagrange multiplier from (a) accurately predict the change in the objective? Compare the actual change to the predicted change.

(c) (3) Are the KKT equations for a problem with a quadratic objective and a linear equality constraint always linear? Is this true for a problem with a quadratic objective and a linear inequality constraint?

3. (6) Solve the following problem using the KKT conditions:

$$Min\ f(x) = x_1^2 + 2x_2^2 + 3x_3^2$$
$$g_1(x):\ x_1 + 5x_2 = 12$$
$$g_2(x):\ -2x_1 + x_2 - 4x_3 \le -18$$

4. For the problem:

$$Min\ f(x) = x_1^2 + x_2$$

$$g_1(x) = x_1^2 + x_2^2 - 9 = 0$$
$$g_2(x) = x_1 + x_2^2 - 1 \le 0$$
$$g_3(x) = x_1 + x_2 - 1 \le 0$$

A contour plot of this problem looks like:

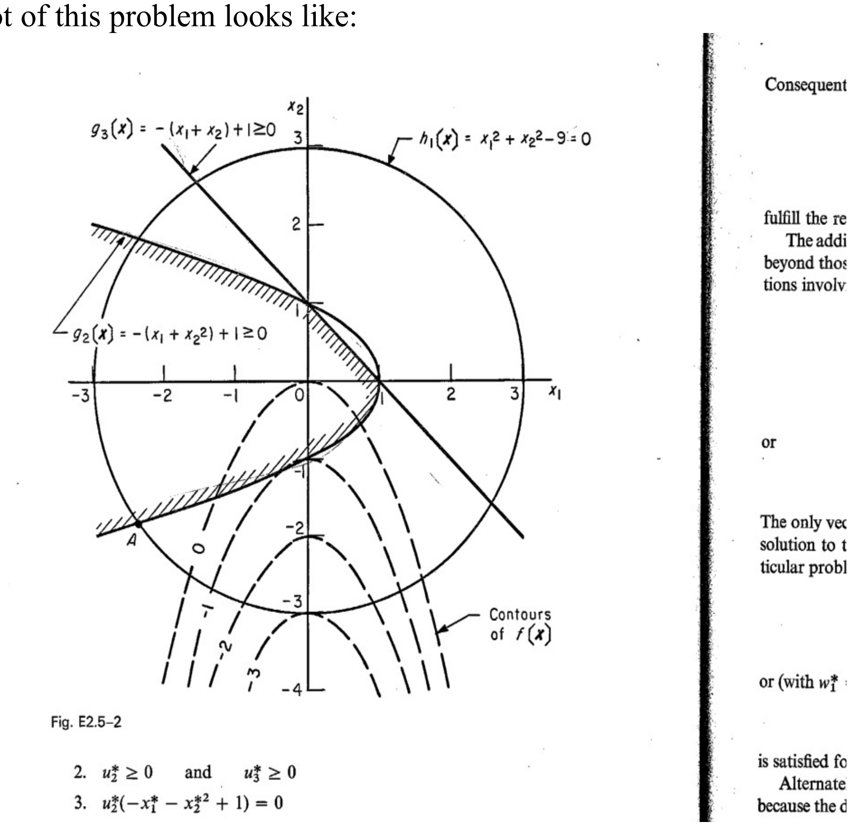


Figure taken from Himmelblau, David, *Applied Nonlinear Programming*. McGraw-Hill.

Using the KKT equations (constraints should be considered satisfied within acceptable round-off):

a. (4) Verify that the point $\lfloor -2.3723, -1.8364 \rfloor$ is a local optimum (point A)

b. (4) Verify that the point $\lfloor -2.5000, -1.6583 \rfloor$ is not a local optimum

c. (6) Drop the equality constraint from the problem. Using the contour plot above to see where the optimum lies (and thereby determine which constraints are binding), solve for the optimum using the KKT conditions. Note the equations will not be linear.