QF 603 – Fixed Income Securities Final Project

Group Member – MQF 2024:

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Part I. Swap Curve Bootstrap

I.A. Bootstrapping OIS Curves

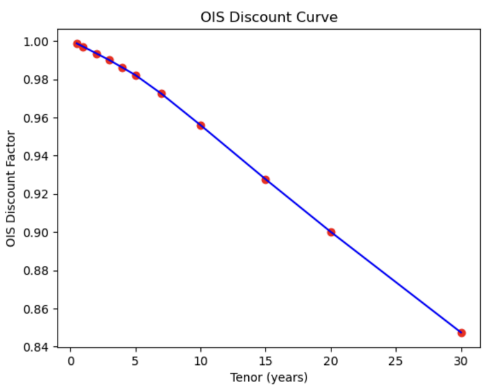
Part I serves as the foundation of this report. It outlines the process of bootstrapping the Overnight Index Swap (OIS) discount curve using the given OIS market data provided. By bootstrapping the discount curve, the goal is to construct the OIS discount factor $D_0(0, T)$ for tenors of 0 to 30 years, in which the collateralized swaps can be valued using the constructed curves with the assumption that overnight interest is paid on posted collateral.

To compute the 6-months OIS discount factor, it is done by simply plugging in the rate of the 6-month OIS rate into the formula: $D_0(0, T) = \frac{1}{1 + 0.5 * r}$, where *r* is the rate. Once the OIS discount factor of 6-months has been computed, using the fixed-float swap parity relationship, it is easier to derive the discount factor for each tenor by solving the rate at which the present value of fixed leg is equal to present value of floating leg.

Assuming, the market is uncollateralized, the fixed leg cash flow is calculated with the market observed OIS rate and multiplied by the sum of all known and interpolated discount factors, whilst floating leg cash flow can be computed in short with the formula of 1 - $D_0(0, T)$; where Brent's method from python function is being used for root-finding to solved for $D_0(0, T)$. However, as the tenor years given are not in fixed increments, we used linear interpolation to estimate the intermediate discount factors. As a result, the OIS discount factors for each given tenor years were obtained as can be seen in *Figure 1*.

Figure 1 - OIS Discount Curves

Tenor	Product	Discount Factor
0.5	OIS	0.998752
1.0	OIS	0.997009
2.0	OIS	0.993531
3.0	OIS	0.990015
4.0	OIS	0.986117
5.0	OIS	0.982184
7.0	OIS	0.972406
10.0	OIS	0.955977
15.0	OIS	0.927611
20.0	OIS	0.900076
30.0	OIS	0.847407



From the graph above, it can be seen that with the linear interpolation, we are able to see reasonably accurate discount factors for missing intermediate tenors. Meanwhile, the downward sloping discount curve is consistent with OIS rate that generally positive and discount factors close to 1 for short tenors also decrease as tenor increases, indicating the less time value of money for a longer payment tenor period in the future. Through the bootstrapping procedure, we are able to successfully construct an OIS discount curve from market data which will serve as a foundation for computing the LIBOR discount factors.

I.B. Bootstrapping LIBOR Curves

Using the Interest Rate Swap (IRS) data, we are able to construct the LIBOR discount factor curve D(0, T), which is used as the reference floating rate for pricing derivatives, assuming that the underlying swap is collateralized with cash and subject to daily OIS discounting.

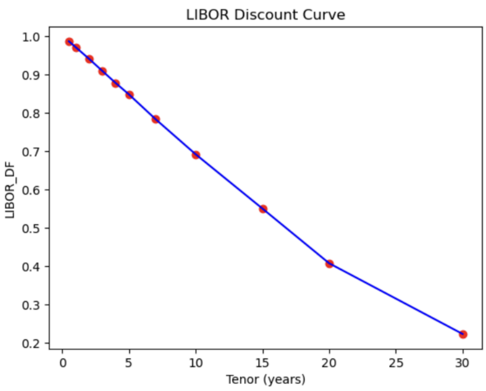
Bootstrapping LIBOR discount factors also follow the principle of fixed-floating swap parity, where the 6-months LIBOR discount factor is being directly computed with the formula: $D(0, 0.5) = \frac{1}{1 + 0.5 * r_{0.5}}$. Once the value is obtained, together with the first two OIS discount factors, it's being used to derived the 1-year LIBOR discount factor with the formula of: $D(0, 1) = \frac{D(0, 0.5)}{1 + 0.5 * f_{6m}}$. Afterwards, for each subsequent tenor, we solve for the unknown LIBOR discount factor $D(0, T_i)$ such that the present value of the fixed leg is equal to the present value of the floating leg, with the given formula as below:

$$PV_{fixed\ leg} = \sum_{j=1}^{i} 0.5 * D^{OIS}(0, T_j) * R_i \qquad PV_{float\ leg} = T_j = D^{OIS}(0, T_j) * \left(\frac{D^{LIBOR}(0, T_{j-1})}{D^{LIBOR}(0, T_j)}\right),$$

We used Brent's root-finding method again to find the unknown discount factor to fill in the intermediate LIBOR discount factors using linear interpolation. Performing all those steps, we are able to capture each LIBOR discount factor accurately which can be seen from *Figure 2*.

Figure 2 - LIBOR Discount Curves

Tenor	Rate	OIS_DF	LIBOR_DF
0.5	0.0250	0.998752	0.987654
1.0	0.0280	0.997009	0.972577
2.0	0.0300	0.993531	0.942179
3.0	0.0315	0.990015	0.910482
4.0	0.0325	0.986117	0.878981
5.0	0.0330	0.982184	0.848989
7.0	0.0350	0.972406	0.784216
10.0	0.0370	0.955977	0.692710
15.0	0.0400	0.927611	0.551081
20.0	0.0450	0.900076	0.408218
30.0	0.0500	0.847407	0.223662



By comparing both LIBOR Discount Curve and OIS Discount Curve for the same tenors (*Figure 3*), it can be seen that the LIBOR discount curve lies below the OIS curve, consistent with a positive spread due to credit and liquidity risk in LIBOR. The shape of the LIBOR curve aligns with financial theory, showing a typical upward-sloping term structure that reflects market expectations of rising interest rates in the future. It also uses OIS discounting, consistent with collateralized valuation practices, supporting accurate pricing under real-world conditions.

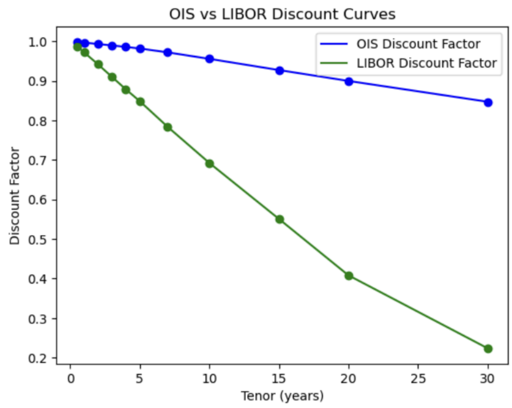


Figure 3 - Comparison of OIS and LIBOR Discount Curves

I.C. Calculating Forward Swap Rates

The combinations of given *A year* expiry and *B year* tenor represent swap contract terms that start at *A year* and will mature at *A + B year*. The corresponding forward swap rate is obtained by equating the fixed rate to the present value of the fixed and floating legs of the swap over the specified future interval, given the formula as below:

$$Forward\ swap\ rate_{i*j} = \frac{floating\ leg_{i->j}}{fixed\ leg_{i->j}},$$

$$floating\ leg_{i->j} = \sum_{t=i+1}^{i+j} D^{OIS}(0,\ t) * \left(\frac{D^{LIBOR}(0, t-1) - D^{LIBOR}(0, t)}{D^{LIBOR}(0, t)}\right)\ fixed\ leg_{i->j} = \sum_{t=i+1}^{i+j} 0.5 * D^{OIS}(0,\ t).$$

The calculated forward swap rate (*Figure 4)* implies that forward rates increase for higher tenors, reflecting upward sloping expectations in the yield curve. Forward rates starting from later expiries (i.e. 10Y) tend to be higher due to market anticipation of long-term rate normalization and as the compensation for swap holders due to the lock agreement on the fixed rate over a longer future period (i.e., demand for higher premium or return for the potential on long duration risk exposure).

Figure 4 - Forward Swap Rates for Different Expiry and Tenor

Tenor	1Y	2Y	3Y	5Y	10Y
Expiry					
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

Part II. Swaption Calibration

II.A. Displaced Diffusion Model

The Black76 Model, as the adjustment on the earlier Black Scholes model, is used for option pricing on futures contracts. In this case, the model uses future or forward rather than spot price as the underlying asset price and assumes to follow log-normally distributed with constant volatility across strike price. However, this inconsistent to the observed market implied volatility that is found to be skewed rather than flat-shaped curve, due to the dynamic of market expectations and risk perceptions on the future underlying asset price as resulted from the uncertainty of the market itself. Therefore, the Displaced Diffusion Model is developed as the adjusted Black76 model for the shifted normal-lognormal market implied volatility.

To construct the implied volatility curve, Displaced Diffusion model uses two parameters, which are implied volatility/sigma (σ) and beta (β) as the adjusted factor for the drift on the underlying assets' price. The optimal parameters could be calibrated so the implied volatility of the model could match with the market observed implied volatility skew. By plotting the initial guess of the beta and implementing the least square estimation process, the optimal parameter of implied volatility and beta for the model could be generated. For the swaption case, we use the observed market implied volatility, forward rate, and the OIS discount curve for the corresponding expiry and tenor terms as the calibration input to the pricing model (*Figure 5*).

The longer swaption expiry terms (e.g., 1Y vs 10Y) for same tenor period (e.g., 1Y), the lower sigma parameter, indicating that volatility of market volatility tend to be smaller compared to short term given market expectations on the mean-reverting forward rate movement in the future. Furthermore, this pattern can also be explained from the liquidity perspective, in which the longer expiry with short tenor option (*10y x 1y* vs *1y x 1y*) is less actively traded hence the price movement itself is also more stable. The same pattern also applies on the for the same expiry term with ranges of tenor. Meanwhile, for the same tenor option with different expiries, the beta is higher for longer expiry terms. This is because of more time space for the forward rate to move / drift, which leads to greater uncertainty, hence the volatility on the model should be scaled for higher beta to capture this higher uncertainty in the market rate.

Figure 5 - Displaced Diffusion Model Calibrated Parameters by Expiry and Tenor

Displaced Diffusion Model Calibrated Sigma

Tenor	1	2	3	5	10
Expiry					
1	0.366383	0.386641	0.373880	0.307546	0.270408
5	0.327257	0.326885	0.319352	0.276239	0.248736
10	0.308141	0.306315	0.302794	0.271029	0.244710

Displaced Diffusion Model Calibrated Beta

Tenor	1	2	3	5	10
Expiry					
1	0.000234	0.002127	0.004175	0.041992	0.169994
5	0.006103	0.024384	0.033407	0.103428	0.097253
10	0.055658	0.080290	0.100007	0.082890	0.080941

II.B. SABR Model

The SABR Model generates implied volatilities that can be used as the parameter input to the Black76 model for swaption pricing. It incorporates stochastic volatility, which better captures the complex volatility patterns and addresses the limitation of constant volatility in traditional Black models. We constructed the SABR model by implementing its implied volatility formula, and set the SABR parameters beta (β) to be fixed at 0.9 to ensure stable calibration, as we assume the interest rate volatility behaves close to 1 under a lognormal model. For the other three settings, we started with an initial guess from setting alpha (α) to 0.1 as a small starting point for volatility, nu (ν) to 0.5 as a moderate guess for how much the volatility might change and rho (ϱ) to -0.5 as the volatility will decrease when rate increase in general, we calibrated the parameters through iterations to fit the market's volatility smile. *Figure 6* indicates the market patterns where longer term swaptions show higher starting volatility and more stable volatility changes.

Figure 6 - SABR Model Calibrated Parameters by Expiry and Tenor

SABR Calibrated Alpha

Tenor	1	2	3	5	10
Expiry					
1	0.139072	0.184649	0.196851	0.178052	0.170838
5	0.166544	0.199505	0.210357	0.190780	0.177208
10	0.177487	0.195232	0.206481	0.201494	0.180771

SABR Calibrated Rho

Tenor	1	2	3	5	10
Expiry					
1	-0.633217	-0.525116	-0.482845	-0.414426	-0.264029
5	-0.585317	-0.546907	-0.549817	-0.509843	-0.438680
10	-0.545617	-0.544287	-0.549047	-0.562481	-0.510189

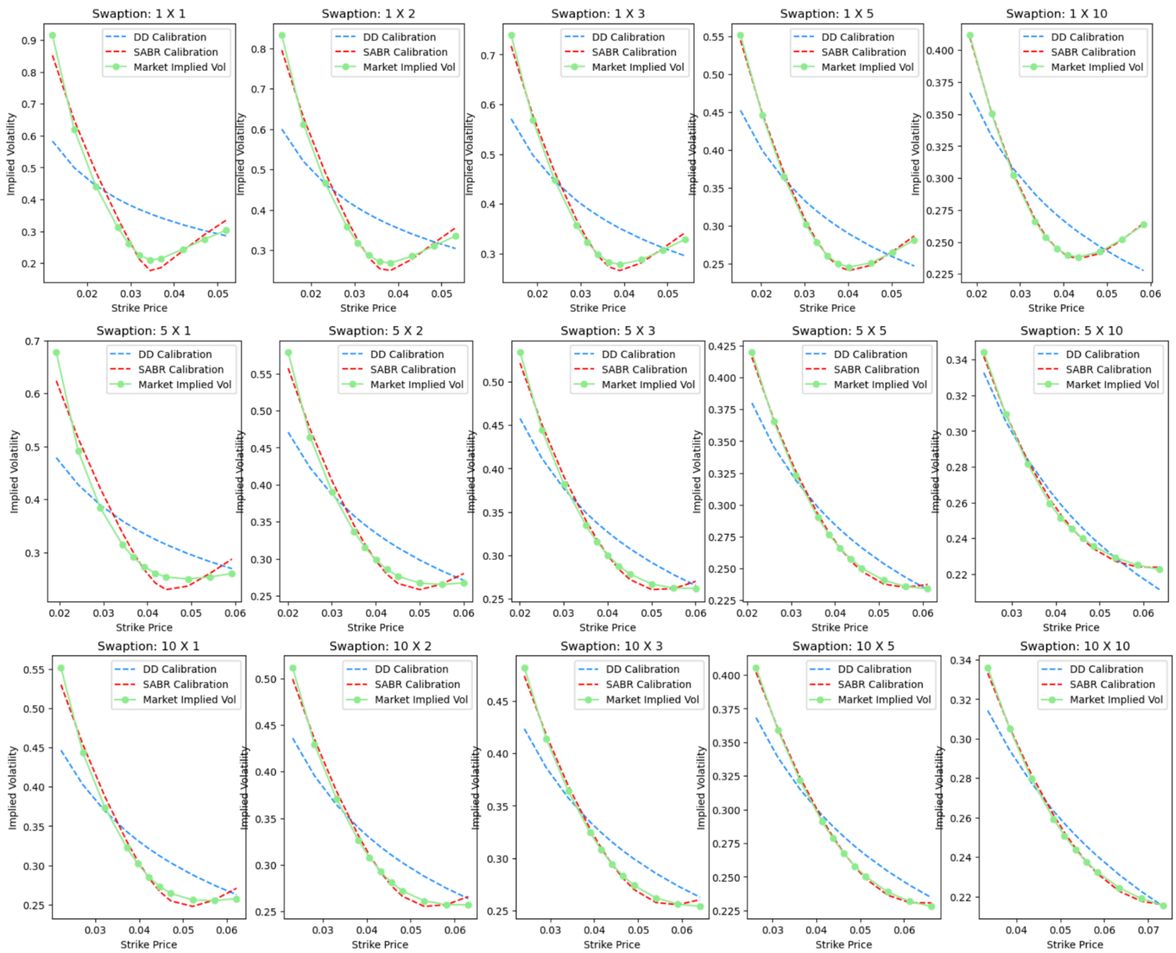
SABR Calibrated Nu

Tenor	1	2	3	5	10
Expiry					
1	2.049453	1.677405	1.438139	1.064877	0.781910
5	1.339876	1.061933	0.936737	0.672606	0.495769
10	1.006947	0.925336	0.867521	0.720021	0.578658

II.C. The Implied Volatility Comparison Across Model

Figure 7 compares the Implied Volatility Smiles of the Displaced Diffusion (DD) Model and SABR Model with market data across different expiries and tenors. The results show that the SABR Model fits the market data closely across all expiry and tenor combinations, capturing the market volatility smile's shape effectively, which is enhanced by interpolating alpha(α), rho(ϱ), nu(ν) parameters for intermediate expires. This procedure ensures the model's applicability across a wider range of swaptions. However, for the Displaced Diffusion Model, which only calibrates the constant sigma (σ) and beta (β), the implied volatility can only fit the market volatility near the at-the-money point and cannot trace the market volatility smile. For the 1Y swaptions, DD's curve is flatter, missing the market volatility, while in 10Y swaptions, it improves slightly but still lags behind the SABR model. In conclusion, the SABR Model's stochastic volatility structure provides a better fit than the DD model for handling long-term swaptions with complex volatility patterns.

Figure 7 - Volatility Smile Comparison for Market, DD, and SABR Swaptions



II.D. Swaption Pricing Case

To derive the payer and receiver swaption pricing, the payoff is similar with the European vanilla call and put option. However, in this case, we use the par swap rate as the fixed rate that makes the present value of the fixed and floating leg from the swaption equal or in other words the value of the underlying interest rate swap is zero at maturity date. In addition, instead of the risk-free account as discount rate, the swaption pricing uses the PVBP as the discount rate or numeraire.

For the long position in the payer swaption, the holder has the right, but not the obligation, to enter the pay fixed and receive floating cash flow in the future, which is the same with the call option on the forward swap rate. The option will be in-the-money (ITM) if the forward swap rate is higher than the par swap rate, since holders will benefit from lower than market payment rate (fixed leg) while at the same time will receive the higher market floating rate. The value of the payer swaption is given below:

$$V_{n,\,N}^{payer}(0) = P_{n+1,N}(0)E^{n+1,N}\left[S_{n,N}(0)\Phi(d_1) - K\Phi(d_2)\right]$$

Meanwhile, for the long position in the receiver swaption, the holder has the right, but not the obligation, to receive fixed and pay floating cash flow in the future at the par swap rate. Similar to the put option payoff, the swaption will be in-the-money (ITM) if the par swap rate is higher than the forward swap rate, given the fixed cash flow that we receive will be higher than the market rate. The value of the receiver swaption is given below:

$$V_{n,\,N}^{rec}(0) = P_{n+1,N}(0)E^{n+1,N}\left[K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)\right]$$

In case of pricing *2y x 10y* payer swaption, we could use the calibrated Displaced Diffusion (e.g., sigma and beta) and SABR model parameters (e.g., alpha, rho, nu) and interpolate it for the intended expiry and tenor terms as the pricing model input without calibrating the swaption itself. For the strike swap rate ranging from 1% to 8%, the payer swaption price between two models are similar (*Figure 8*), with the higher strike swap rate, the lower payer swaption price is. Same arguments with the call option, if the strike swap rate is larger, the amount of payoff that we obtain from the difference between forward swap market rate and the strike swap rate will be smaller, hence less valuable the swaption price is. Therefore, to compensate for the high strike rate, the option should be priced cheaper than option with lower strike, holding all parameters (i.e., forward rate, PVBP, and implied volatility) being equal.

Furthermore, for the second case of pricing *8y x 10y* receiver swaption, we interpolate the calibrated model for each Displaced Diffusion and SABR model parameters as well as input to the Displaced Diffusion and the SABR pricing model. From *Figure 9*, it can be seen that for the same strike swap rate range with the payer swaption, the receiver swaption price will be higher as the strike swap rate increases. This is due to the payoff profile that the receiver swaption holder will receive fixed swap rate cash flow in the future (if the option is being exercised). Hence when the fixed strike swap rate is getting larger than the observed market rate, the amount benefit is also larger for the holder, holding all pricing inputs are equal.

Figure 8 - *2y x 10y* Payer Swaption Price

	Strike	DD Price	SABR Price
0	0.01	0.274237	0.275528
1	0.02	0.186283	0.188691
2	0.03	0.109416	0.109602
3	0.04	0.053037	0.049598
4	0.05	0.020496	0.020332
5	0.06	0.006202	0.010243
6	0.07	0.001464	0.006334
7	0.08	0.000271	0.004451

Figure 9 - *8y x 10y* Receiver Swaption Price

	Strike	DD Price	SABR Price
0	0.01	0.016018	0.018583
1	0.02	0.032873	0.038221
2	0.03	0.060218	0.063442
3	0.04	0.099932	0.099578
4	0.05	0.152375	0.153132
5	0.06	0.216349	0.223481
6	0.07	0.289593	0.303232
7	0.08	0.369494	0.387224

Part III. Convexity Correction

The following formula and assumptions are used to calculate the present value of CMS products. The main formula of the CMS value is given below:

$$\text{CMS PV} = \sum_{i=\text{payment period}}^{N} D(0,T_i) \times \Delta \times \mathbb{E}^T[S_{n,N}(T_i)]$$

Where, $$\mathbb{E}^T[S_{n,N}(T)] = g(F) + \frac{1}{D(0,T)}\left[\int_K^F h''(K)V^{\text{rec}}(K)\,dK + \int_F^\infty h''(K)V^{\text{pay}}(K)\,dK\right]$$

And IRR-settled option price ($V^{Rec}(K)$ *or* $V^{Pay}(K)$) is given by:

$$V_{n,N}(K) = D(0,T) \times \text{IRR}(S_{n,N}(0)) \times \text{Black76}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

Main assumptions used for the model input:

From the dataset, discount factors are provided at discrete standard maturities (e.g., 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, etc.). In order to compute for non-standard tenors (e.g., 1.5Y, 2.5Y), linear interpolation is applied to the zero-coupon discount curve. Forward LIBOR rates are subsequently inferred from these interpolated discount factors via standard bootstrapping techniques. All SABR model parameters (α, β, ρ, ν) used in CMS valuation are also linearly interpolated across both tenor and expiry dimensions. If a given (tenor, expiry) pair lies outside the calibrated SABR grid, nearest-neighbor extrapolation is applied to stabilize the parameter selection.

In calculating implied volatilities using the SABR model, numerical instability ("explosion") is observed at high strike levels due to extrapolation beyond the calibrated range. To mitigate this, strikes above 0.07 are capped at 0.07 within the SABR function. During integration for convexity adjustment, any strike exceeding this threshold is floored to 0.07 to ensure consistent and bounded volatility estimates within the static replication framework.

Based on the above-mentioned assumptions and equations, the PV of the following products are computed:

- PV of a leg receiving CMS *10y* semi-annually over the next 5 years : **0.21**
- PV of a leg receiving CMS *2y* quarterly over the next 10 years : **0.39**

A CMS leg is a collection of CMS rates paid over a period. For example, the PV of a leg receiving CMS *10y* semi-annually over the next 2 years is the sum of the discounted cash flow that pays 10y par swap rate (CMS rate) observed on the semiannual frequency reset date T_i for 2 years.

$$\begin{aligned} PV &= D(0,6m)\cdot 0.5\cdot \mathbb{E}^T\left[S_{6m,10y6m}(6m)\right] \\ &+ D(0,1y)\cdot 0.5\cdot \mathbb{E}^T\left[S_{1y,11y}(1y)\right] \\ &+ D(0,1y6m)\cdot 0.5\cdot \mathbb{E}^T\left[S_{1y6m,11y6m}(1y6m)\right] \\ &+ D(0,2y)\cdot 0.5\cdot \mathbb{E}^T\left[S_{2y,12y}(2y)\right] \end{aligned}$$

A CMS contract paying the wap rate $S_{\{n,N\}}(T)$ at time $T = T_n$ can be expressed as:

$$\frac{V_0}{D(0,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right] \quad \Rightarrow \quad V_0 = D(0,T)\mathbb{E}^T\left[S_{n,N}(T)\right]$$

By the static-replication approach, let $F = S_{\{n,N\}}(0)$ denote the forward swap rate:

$$\begin{aligned} V_0 &= D(0,T)g(F) + h'(F)\left[V^{pay}(F) - V^{rec}(F)\right] \\ &\quad + \int_0^F h''(K)V^{rec}(K)\,dK + \int_F^\infty h''(K)V^{pay}(K)\,dK \\ &= D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)\,dK + \int_F^\infty h''(K)V^{pay}(K)\,dK \end{aligned}$$

Thus,
$$\underbrace{\mathbb{E}^T\left[S_{n,N}(T)\right]}_{\text{CMS Rate}} = g(F) + \frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{rec}(K)\,dK + \int_F^\infty h''(K)V^{pay}(K)\,dK\right]$$

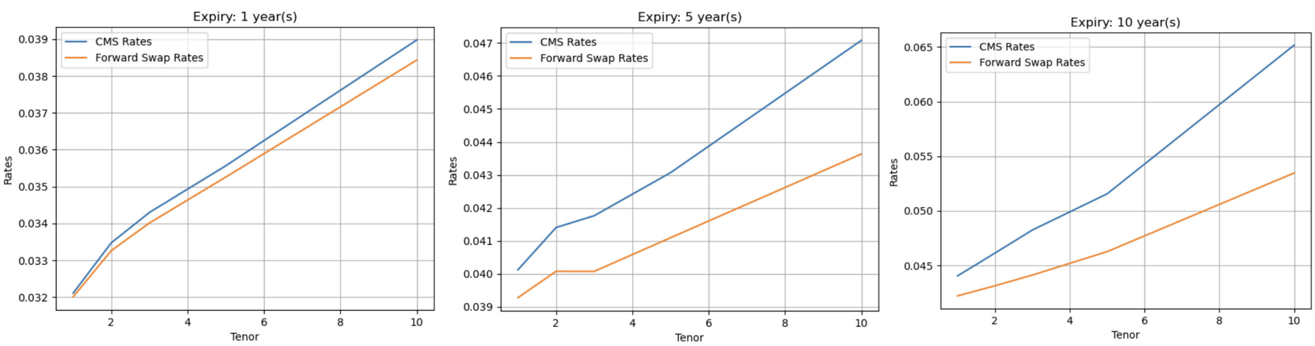
The IRR-settled option pricer ($V^{Rec}(K)$ *or* $V^{Pay}(K)$) is given by:

$$V(K) = D(0,T)\cdot \mathrm{IRR}(S_{n,N}(0))\cdot \mathrm{Black76}(S_{n,N}(0), K, \sigma_{\mathrm{SABR}}, T)$$

Since the payoff function is $g(K) = K$, we have:

$$h''(K) = -\frac{\mathrm{IRR}''(K)\cdot K - 2\cdot \mathrm{IRR}'(K)}{\mathrm{IRR}(K)^2} + \frac{2\cdot \mathrm{IRR}'(K)^2\cdot K}{\mathrm{IRR}(K)^3}$$

Figure 10 - Comparison of Forward Swap Rates and CMS Rates



PART IV. Decompounded Options
IV.A Decompounded Option Valuation

For the first case, we need to value decompounded options with maturity of 5Y and the payoff is $CMY10y^{1/p} - 0.04^{1/q}$. To obtain the present value of the options, we use the status CMS replication formula as below:

$$V_0 = D(0,T)\, E\left[\, CMS10y^{1/p} - 0.04^{1/q}\,\right] \ldots(1)$$

$$V_0 = D(0,T) \int_0^\infty g(K)\, f(K)\, dK \ldots(2)$$

$$V_0 = D(0,T)\, g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] + \int_0^F h''(K)\, V^{rec}(K)\, dk + \int_F^\infty h''(K)\, V^{pay}(K)\, dk$$

Where, $K = F$, $[V^{pay}(F) - V^{rec}(F)] = 0$, therefore we can arrange the equation to be:

$$V_0 = D(0,T)\, g(F) + \int_0^F h''(K)\, V^{rec}(K)\, dk + \int_F^\infty h''(K)\, V^{pay}(K)\, dk \quad \ldots(3)$$

And substituting the equation with below terms to derive the final numerical value:

$$g(K) = F^{1/p} - 0.04^{1/q} \quad ; \quad g'(K) = \frac{1}{P} F^{1/p-1} \quad ; \quad g''(K) = \left(\frac{1}{p^2} - \frac{1}{p}\right) F^{1/p-2}$$

Applying the final formula, we are able to get the final payoff value for the first case of decompounded options of **0.23553.**

IV.B. Decompounded option with new payoff

Similar with previous case, for the second part we need to value decompounded options with different payoff of $(CMY10y^{1/p} - 0.04^{1/q})^+$**.** Since, this payoff is similar to caplet where the $V^{rec}(K) = 0$ **,** the present value formula should be used is below:

$$V_0 = D(0,T)\, g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] + \int_0^F h''(K)\, V^{rec}(K)\, dk + \int_F^\infty h''(K)\, V^{pay}(K)\, dk$$

$$V_0 = h'(F)[V^{pay}(F)] + \int_F^\infty h''(K)\, V^{pay}(K)\, dk \ \ldots (4)$$

Take note that we are dealing with CMS Caplet, so we need to ensure we are working within the correct boundaries for the inequality. Here is a detailed break down:

CMS Payoff Decomposition: $\left(\mathrm{CMS}^{1/p} - K_0\right)^+$, for $K_0 = 0.04^{1/q}$.

Since we are given p=4, q=2, K is 0.2.

We know that CMS payoff now is $\left(\mathrm{CMS}^{1/4} - 0.2\right)^+$, for payoff: $\mathrm{CMS}^{1/4} > 0.2$.

So we have an option at a fixed strike of (**L**) $L = 0.04^{1/2} = 0.2$

for $V^{\text{pay}} = V^{\text{pay}}(L)\, h'(L) + \int_L^\infty h''(K)\, V^{\text{pay}}(K)\, dK.$

Applying the above formula, we obtain the value of the second case decompounded option is **1.05049.**