QF620 - Stochastic Modelling in Finance
Final Project Report– Group 5

Bianca Lucia Santoso (01516431), Chang Wen-Yu (01505913), Dragos Horge (01507705), Maureen Calista Chandra (01498515), Merlyvia Mualim (01458745), Nethania Abigail (01520514)

1 ANALYTICAL OPTION FORMULAE

1.1 Black-Scholes Model

Under the Black-Scholes model, the stock price process follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Now, solve the SDE by first applying Itô's formula to the function $X_t = f(S_t) = \log(S_t)$:

$$dX_t = (r - \frac{\sigma^2}{2})dt + \sigma dW_t$$

Integrating both sides and substituting for X_t, we arrive at

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}, \quad W_T \sim N(0,T)$$

The inequality to satisfy when $S_T = K$,

$$x^* = \frac{\log\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

A financial contract pays V_T on the expiry date T, to value the option price we have,

$$V_0 = e^{-rT}\,\mathbb{E}[V_T]$$

We have d_1 and d_2,

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Table 1 – Option pricing formulae for Black-Scholes model

Option Types	Payoff (V_T)	Pricing Formulae
Vanilla Call	$(S_T - K)^+$	$V_C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$
Vanilla Put	$(K - S_T)^+$	$V_P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$
Digital Cash-or-Nothing Call	$1_{S_T>K}$	$V_C = e^{-rT}\Phi(d_2)$
Digital Cash-or-Nothing Put	$1_{S_T<K}$	$V_P = e^{-rT}\Phi(-d_2)$
Digital Asset-or-Nothing Call	$S_T 1_{S_T>K}$	$V_C = S_0\Phi(d_1)$
Digital Asset-or-Nothing Put	$S_T 1_{S_T<K}$	$V_P = S_0\Phi(-d_1)$

1.2 Bachelier Model

Under Bachelier model, the stock price process is defined as:

$$dS_t = \sigma dW_t$$

Integrating both sides, we arrive at

$$S_T = S_0 + \sigma W_T, \quad W_T \sim N(0,T)$$

The inequality to satisfy when $S_T = K$,

$$x^* = \frac{K - S_0}{S_0 \sigma \sqrt{T}}$$

A financial contract pays V_T on the expiry date T, to value the option price we have,

$$V_0 = e^{-rT}\, \mathbb{E}[V_T]$$

We have d_1 ,

$$d_1 = \frac{S_0 - K}{S_0 \sigma \sqrt{T}}$$

Table 2 – Option pricing formulae for Bachelier model

Option Types	Payoff (V_T)	Pricing Formulae
Vanilla Call	$(S_T - K)^+$	$V_C = e^{-rT}\,[(S_0 - K)\Phi(d_1) + \sigma\sqrt{T}\phi(d_1)]$
Vanilla Put	$(K - S_T)^+$	$V_P = e^{-rT}\,[(K - S_0)\Phi(-d_1) + \sigma\sqrt{T}\phi(d_1)]$
Digital Cash-or-Nothing Call	$1_{S_T > K}$	$V_C = e^{-rT}\Phi(d_1)$
Digital Cash-or-Nothing Put	$1_{S_T < K}$	$V_P = e^{-rT}\Phi(-d_1)$
Digital Asset-or-Nothing Call	$S_T 1_{S_T > K}$	$V_C = e^{-rT}\,[S_0\Phi(d_1) + \sigma\sqrt{T}\phi(d_1)]$
Digital Asset-or-Nothing Put	$S_T 1_{S_T < K}$	$V_P = e^{-rT}\,[S_0\Phi(-d_1) - \sigma\sqrt{T}\phi(d_1)]$

1.3 Black Model

Applying Itô's formula with function $f(t, S_t) = S_t e^{r(T-t)}$ to Black-Scholes price process gives the stochastic differential equation for the forward price,

$$dF_t = \sigma F_t dW_t$$

Now, solve the SDE by first applying Itô's formula to the function $X_t = f(F_t) = \log(F_t)$:

$$dX_t = -\frac{\sigma^2}{2} dt + \sigma dW_t$$

Integrating both sides and substituting for X_t, we arrive at

$$F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}, \quad W_T \sim N(0, T)$$

The inequality to satisfy when $F_T = K$,

$$x^* = \frac{\log\left(\frac{K}{F_0}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

A financial contract pays V_T on the expiry date T, to value the option price we have,

$$V_0 = e^{-rT}\, \mathbb{E}[V_T]$$

We have d_1 and d_2,

$$d_1 = \frac{\log\left(\frac{F_0}{K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Table 3 – Option pricing formulae for Black model

Option Types	Payoff (V_T)	Pricing Formulae

Vanilla Call	$(F_T - K)^+$	$V_C = e^{-rT}[F_0\Phi(d_1) - K\Phi(d_2)]$
Vanilla Put	$(K - F_T)^+$	$V_P = e^{-rT}[K\Phi(-d_2) - F_0\Phi(-d_1)]$
Digital Cash-or-Nothing Call	$1_{F_T>K}$	$V_C = e^{-rT}\Phi(d_2)$
Digital Cash-or-Nothing Put	$1_{F_T<K}$	$V_P = e^{-rT}\Phi(-d_2)$
Digital Asset-or-Nothing Call	$F_T 1_{F_T>K}$	$V_C = e^{-rT}F_0\Phi(d_1)$
Digital Asset-or-Nothing Put	$F_T 1_{F_T<K}$	$V_P = e^{-rT}F_0\Phi(-d_1)$

1.4 Displaced-Diffusion Model

The SDE for displaced-diffusion process,

$$dF_t = \sigma[\beta F_t + (1-\beta)\,F_0]dW_t$$

Now, solve the SDE by first applying Itô's formula to the function $X_t = f(F_t) = \log\,[\beta F_t + (1-\beta)\,F_0]$,

$$dX_t = -\frac{\beta^2\sigma^2}{2}dt + \beta\boldsymbol{\sigma d}W_t$$

Integrating both sides and substituting for X_t, we arrive at

$$F_T = \frac{F_0}{\beta}e^{-\frac{\beta^2\sigma^2 T}{2}+\beta\sigma W_T} - \frac{1-\beta}{\beta}F_0, \quad W_T \sim N(0,T)$$

The inequality to satisfy when $F_T = K$,

$$x^* = \frac{\log\left(\dfrac{K + \frac{1-\beta}{\beta}F_0}{\frac{F_0}{\beta}}\right) + \dfrac{\beta^2\sigma^2 T}{2}}{\beta\sigma\sqrt{T}}$$

A financial contract pays V_T on the expiry date T, to value the option price we have,

$$V_0 = e^{-rT}\,\mathbb{E}[V_T]$$

We have d_1 and d_2,

$$d_1 = \frac{\log\left(\dfrac{\frac{F_0}{\beta}}{K + \frac{1-\beta}{\beta}F_0}\right) + \frac{\beta^2\sigma^2 T}{2}}{\beta\sigma\sqrt{T}} \qquad d_2 = d_1 - \beta\sigma\sqrt{T}$$

Table 4 – Option pricing formulae for Displaced-Diffusion model

Option Types	**Payoff (V_T)**	**Pricing Formulae**
Vanilla Call	$(F_T - K)^+$	$V_C = e^{-rT}\left[\frac{F_0}{\beta}\Phi(d_1) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$
Vanilla Put	$(K - F_T)^+$	$V_P = e^{-rT}\left[\left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(-d_2) - \frac{F_0}{\beta}\Phi(-d_1)\right]$
Digital Cash-or-Nothing Call	$1_{F_T>K}$	$V_C = e^{-rT}\Phi(d_2)$
Digital Cash-or-Nothing Put	$1_{F_T<K}$	$V_P = e^{-rT}\Phi(-d_2)$
Digital Asset-or-Nothing Call	$F_T 1_{F_T>K}$	$V_C = e^{-rT}\frac{F_0}{\beta}\Phi(d_1)$
Digital Asset-or-Nothing Put	$F_T 1_{F_T<K}$	$V_P = e^{-rT}\frac{F_0}{\beta}\Phi(-d_1)$

2 MODEL CALIBRATION

2.1 Market Model

Based on the given case of S&P 500 (SPX) observed index value of 3,662.45 and SPDR S&P 500 Exchange Traded Fund (SPY) stock price of 366.02 on 1 Dec 2020, the implied volatility for both option can be generated using the Black Scholes (BS) Model. The implied volatility parameter is calculated for three different maturities (e.g., 17, 45, and 80 days) that fit the ATM call and put option. As presented in Figure XX, the implied volatility is higher for lower strikes given the market is more liquid for the ITM call and OTM put option, which the price of both options are more expensive. This could be explained due to the market perception on the potential economic movement, whereas they demand more for OTM put options during the bearish condition as a hedging tool and ITM call options during the bullish condition.

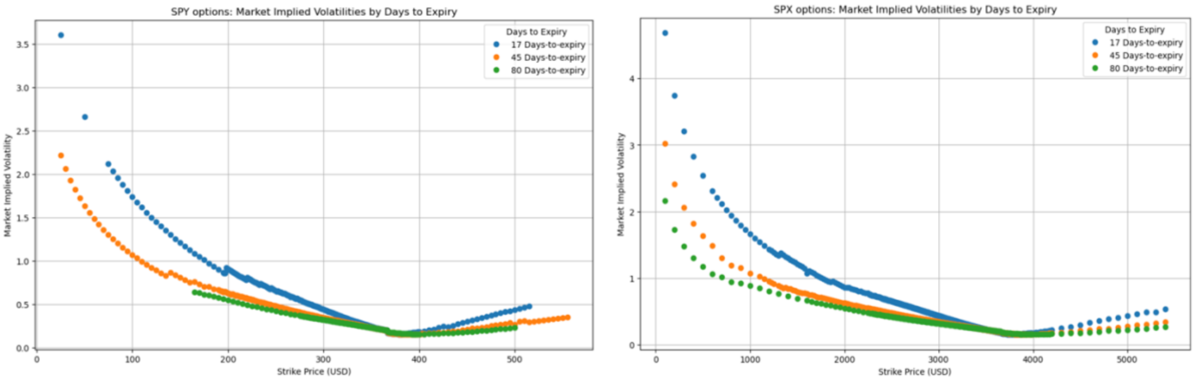

Figure 1 – SPX and SPY Market Implied Volatilities By Maturities

2.2 Displaced Diffusion Model

In response to the market behaviour post 1987 Black Monday Crash, Displaced-Diffusion (DD) model is developed from the base of BS model by introducing the displacement parameter (β) that adjust for the drift component on the underlying assets' price on the BS model. This will adjust the mean return of the asset given the price dynamics in the financial market that often reflect bias such as skewness or kurtosis in the distribution.

There are two important parameters in the DD model that influence the underlying asset price which are the implied volatility (σ) and beta (β). These optimal parameters could be generated using the calibration process so the implied volatility of the DD model could match with the implied volatility observed in the market. By plotting the initial guess of the beta parameter into the calibration model, the optimal parameter of implied volatility and beta for DD model could be generated.

Table 5 – Displaced Diffusion Parameter for SPX (left) and SPY (right)

Days to Expiry	Initial Guess	Sigma	Beta
17	0.7	0.200906	4.329174e-07
	0.8	0.200906	6.846765e-07
	0.9	0.200906	1.288485e-06
45	0.7	0.197218	7.920700e-12
	0.8	0.197218	6.570237e-10
	0.9	0.197218	1.302556e-06
80	0.7	0.200240	2.006767e-06
	0.8	0.200240	1.865006e-07
	0.9	0.200240	1.698395e-08

Days to Expiry	Initial Guess	Sigma	Beta
17	0.7	0.174485	4.014463e-08
	0.8	0.174485	9.384401e-08
	0.9	0.174485	1.222742e-07
45	0.7	0.184910	1.364602e-11
	0.8	0.184910	9.186377e-07
	0.9	0.184910	3.726754e-07
80	0.7	0.193747	3.729998e-09
	0.8	0.193747	4.311220e-08
	0.9	0.193747	2.275977e-07

To illustrate the volatility smile plot comparison between market implied volatility and DD model implied volatility across the maturity days, we used 0.9 as our initial guess for beta parameter values. As can be seen from Figure 1, in both options cases, the volatility plot generated by DD model is relatively flat compared to the market volatility plot within all three maturity days. This implies that DD model still could not capture well the asymmetric distribution of the market return since it assumes that the volatility is constant and independent of strikes and maturities. Therefore, DD model unables to fit the market implied volatility from the observed data that usually involves more complex or sudden movements.

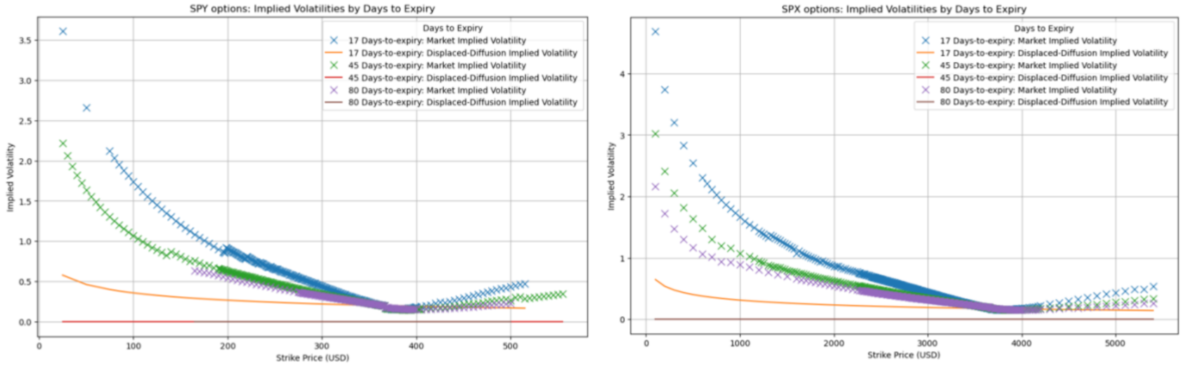

Figure 1 – Displaced-Diffusion Model of Volatility Smile Plot for SPX and SPY

In addition, to see any differences in the volatility smile plot for any initial guess beta parameter values, we tried to plot for each different maturities period by using values of 0.7 and 0.8 (Appendix B1). Based on the results, there are no changes in the implied volatility DD model for different beta values. This confirms that given changes in the adjusted drift in the stock price model, DD always assumes the implied volatility to be constant and unable to capture all of the implied volatility skewness such as occurred in the observed market. This further leads to overfitting especially for shorter days of expiry.

2.3 SABR Model

Similar to DD model, the Stochastic Alpha Beta Rho (SABR) model is an extension of the BS framework. The SABR model is designed to account for observed market phenomena such as volatility smiles and skews that are not captured by the BS model. Stochastic volatility dynamics is introduced in this model that serves its purpose of enhancing its flexibility in modeling the implied volatility surface.

The SABR model incorporates three key parameters that influence the underlying asset price, which includes alpha (α), rho (ρ), and nu (ν). Alpha governs the level of volatility, controlling the overall height of the curve, while rho dictates the correlation between asset price and its volatility that influence the curve's skew. Nu represents the volatility of volatility that determines the degree of the smile exhibited by the volatility curve. These parameters are essential in capturing the market's implied volatility patterns and could be generated using the calibration process to determine the optimal values such that the SABR model's implied volatility matches the observed market data.

In practice, an initial guess for these parameters is provided, and an iterative calibration algorithm is applied. This optimization process ensures that the SABR model aligns with market-implied volatility across different strike prices and maturities. By plotting the implied volatility surface generated during the calibration against the observed market surface, the optimal SABR parameters can be identified.

Table 6 – SABR Parameter for SPX (up) and SPY (down)

Days to expiry	F	alpha	beta	rho	nu
17	366.041395	0.665402	0.7	-0.411900	5.249981
45	366.112568	0.908133	0.7	-0.488779	2.728516
80	366.197038	1.120924	0.7	-0.632939	1.742225

Days to expiry	F	alpha	beta	rho	nu
17	3662.664085	1.212290	0.7	-0.300900	5.459761
45	3663.376249	1.816504	0.7	-0.404302	2.790158
80	3664.221473	2.140133	0.7	-0.574934	1.841747

The SABR model's flexibility and ability to capture complex market dynamics make it a preferred choice for pricing and risk management in derivative markets, especially for options with non-standard features or markets exhibiting pronounced skew and smile effects.

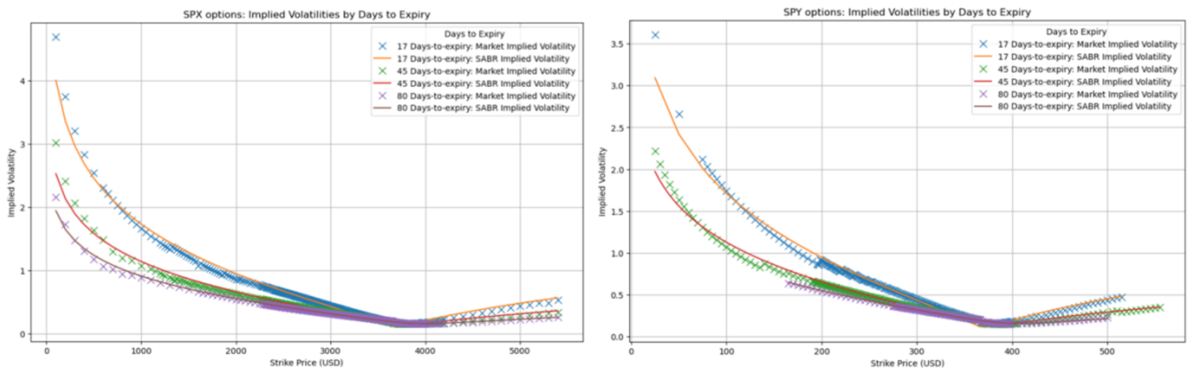

Figure 2 – SABR Model of Volatility Smile Plot for SPX and SPY

The plots above illustrate that, for each maturity day(s), the SABR model's implied volatility curves closely align with the shape and asymmetry of the market-implied volatility curves, demonstrating its robustness and accuracy.

When comparing with DD implied volatility plot (Figure XX and XX), it is evident that the SABR model provides a significantly better fit to the market-implied volatilities compared to the DD model, given the DD beta value that always held constant (capped at zero) which does not produce sufficient skewness to the plot. This improved performance arises from the SABR model's three key parameters, which work together to capture complex features of the volatility surface, including skewness and smile effect. This highlights the model's ability to adapt to variations in volatility driven by market dynamics.

Consequently, the SABR model's capacity in accurately representing complex volatility structures makes it a preferred choice for pricing and hedging in markets characterized with strong skew and smile effects. This capability is particularly beneficial in scenarios where traditional models (i.e. DD and Black-Scholes) fail to capture the complexities of market-implied volatilities.

3 STATIC REPLICATION

Suppose on 1-Dec-2020, we need to evaluate an exotic European derivative expiring on 15-Jan-2021 which pays:

3.1 Payoff function

$$\boldsymbol{S_T^{1/3} + 1.5\,log\,(S_T) + 10.0}$$

$$h(S_T) = K^{1/3} + 1.5\,log\,S_T + 10, \quad h'(S_T) = \frac{1}{3}{S_T}^{-2/3} + \frac{1.5}{S_T}, \quad h''(S_T) = -\frac{2}{9}{S_T}^{-5/3} - \frac{1.5}{{S_T}^2}$$

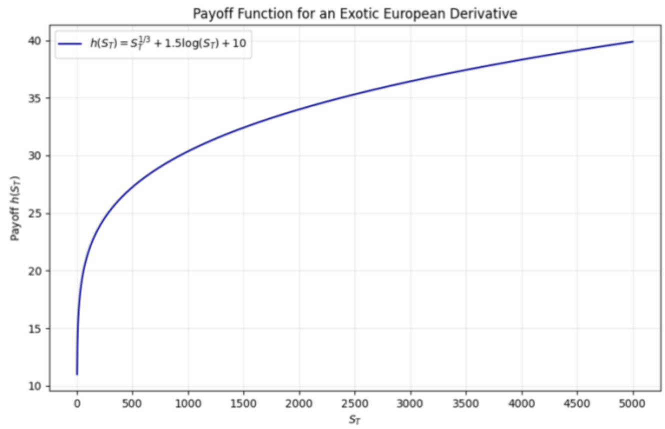

Figure 3 – Payoff function for an exotic European derivative

3.1.1 Black-Scholes Model

The Black-Scholes model assumes that the price of the underlying asset S_T follows:

$$S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma W_T}, \quad W_T \sim N(0, T)$$

The pricing function of the derivative with the given exotic payoff is calculated by discounting the expected payoff under the risk-neutral measure:

$$V_0 = e^{-rT}\, \mathbb{E}[S_T^{1/3} + 1.5\, log\, (S_T) + 10.0\,]$$

$$V_0 = e^{-rT}\, [S_0^{\frac{1}{3}}\, (e^{(r - \frac{\sigma^2}{3})\frac{T}{3}}) + 1.5(log(S_0 + (r - \frac{\sigma^2}{2})T\,) + 10\,]$$

3.1.2 Bachelier Model

The Bachelier model assumes the underlying asset price S_T follows:

$$S_T = S_0 + S_0\, \sigma\, W_T$$

The pricing function based on Bachelier model is given by:

$$V_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\frac{-1}{\sigma\sqrt{T}}}^{\infty} \left[(\, S_0 + S_0\, \sigma \sqrt{T}\, x)^{1/3} + 1.5\, log\, (\, S_0 + S_0\, \sigma \sqrt{T}\, x) + 10]\, e^{-\frac{x^2}{2}} \right] dx$$

Assuming at-the-money (ATM) options, the σ used in the pricing function can be calculated using:

$$V_c = S_0 e^{-rT} \sigma \sqrt{\frac{T}{2\pi}}$$

3.1.3 Static Replication using Calibrated SABR model

The SABR model is designed to capture stochastic volatility and uses the following key equation for volatility:

$$\sigma_{SABR} = f\ (\alpha, \rho, \nu)$$

The pricing function based on static-replication is given by:

$$V_0 = e^{-rT}\ h(F) + \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\ dK$$

3.2 Model-Free Integrated Variance

The model-free integrated variance equation provides a non-parametric measure of the total variance over the option's lifetime. The "model-free" integrated variance equation under static replication is given by:

$$\sigma_{MF}^2 T = \mathbb{E}[\int_0^T \sigma_t^2\,dt] = 2e^{rT}\ (\int_0^F \frac{P(K)}{K^2}dK + \int_F^\infty \frac{C(K)}{K^2}dK)$$

Using the above formula to derive the pricing for 3 different models, Black-Scholes, Bachelier and Static Replication of European payoff (using calibrated SABR model). For each model, P(K) and C(K) are calculated using their respective pricing formulas, with parameters obtained from previous analysis.

3.3 Results

The table below summarizes contract prices derived under the three models. For SPX and SPY, the SABR-based static replication consistently yields higher prices due to its stochastic volatility component. The Black-Scholes and Bachelier models produce comparable prices, with slight differences arising from their treatment of volatility. This comparison underscores the impact of model assumptions on pricing exotic derivatives.

***Table** 7* – Derivative Contract Pricing

Model			**Black Scholes**		**Bachelier**		**Static Replication SABR (price)**	
Stock			**SPX**	**SPY**	**SPX**	**SPY**	**SPX**	**SPY**
1	Payoff Function	Vo	37.70490	25.99425	37.70310	25.99327	37.71051	25.99715
		σ	0.18491	0.19722	0.18551	0.19665		
2	Model-Free	Price	0.00422	0.00480	0.00424	0.00480	0.00635	0.00602
		σ_{MF}	0.18491	0.19722	0.18551	0.19665		

4 DYNAMIC HEDGING

The dynamic hedging strategy for the call option is set as follows:

$$Ct = \phi t St - \psi t\ Bt$$

$$\phi t = \Delta t = \frac{\partial C}{\partial S} = \Phi\left(\frac{log\frac{St}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \qquad \psi t Bt = -Ke^{-r(T-t)}\Phi\left(\frac{log\frac{St}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

In the above formula, C_t represents the call option value at time t, where delta $\varphi_t = \partial C / \partial S$, shows the call option sensitivity to stock price changes. Delta movement indicates the adjustment needed in ϕ_t stock and ψ_t bond positions to maintain the hedge. The replicated portfolio can offset its exposure to stock price changes by rebalancing these positions, showing that dynamic hedging effectively protects the call option.

We consider the option with a maturity of T=1/12 (1 month) over 21 trading days per month and rebalancing times of N=21 and N=84. The stock price follows a stochastic process and was generated under 50,000 simulated paths to represent different price movements. At each rebalancing point, the holdings are adjusted based on the updated Delta. N=21 represents 21 hedging times over the call option's life, while N=84 indicates a higher rebalancing frequency.

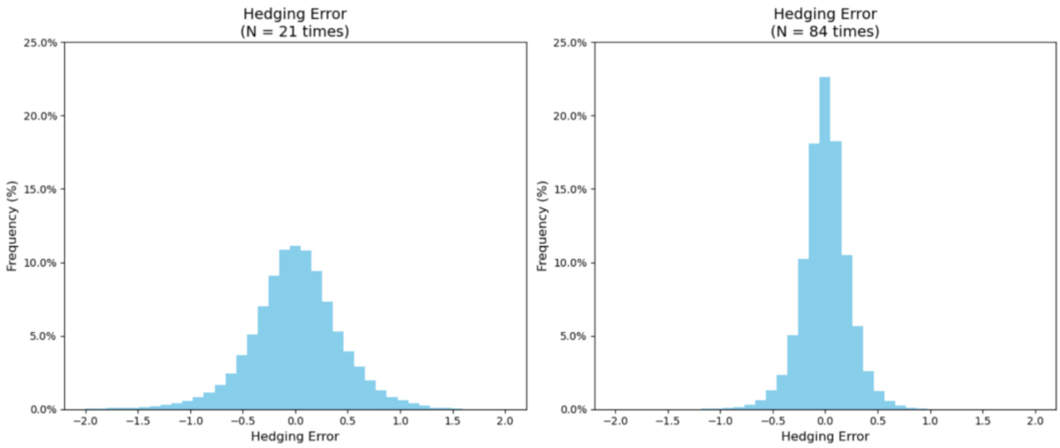

Figure 3 – Histogram of Hedging Error (N=21, 84)

Figure 3 shows the hedging error histogram for two rebalancing times. Based on the result, both hedging errors are mostly concentrated to 0, indicating the dynamic hedging strategy is effective in most cases. In addition, the hedging error distribution is wider for N=21, compared to higher rebalancing times N=84, this shows that lower rebalancing frequency results in larger hedging errors.

Table 6 – Statistical Analysis of Hedging Error (N=21, 84)

	Mean of Hedging Error	Std of Hedging Error	Min Hedging Error	Max Hedging Error
Hedging Times				
21	-0.002962	0.426702	-3.284642	1.688436
84	0.000690	0.217406	-1.650525	1.044974

Table 6 presents the statistical analysis of the Hedging Error under 21 and 84 hedge times. As seen from the chart, the standard deviation of hedging error for N=84 is twice lower than that of N=21. This suggests that higher hedging frequency reduces the error volatility, and hence produces more stable hedging performance.