Politecnico di Milano

FINANCIAL ENGINEERING

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Assignment 7

Group 11

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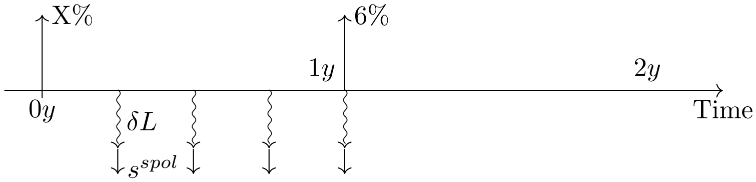
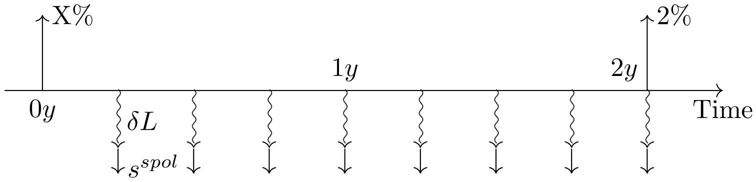
1 Case Study: Certificate Pricing

In this case study we are given two different term-sheets, the first being a bank's issue and the second an hedging swap.

The situation is the following: bank XX issues this particular product that has an exotic coupon, whose rules we will describe below. However, said bank does not have the ability to hedge against this exotic risk in-house.
Therefore, it stipulates a structured swap contract with an investment bank, paying quarterly the Euribor 3m plus a spol and receiving from the IB the aforementioned exotic coupon and an upfront X% at start date.
This means that, for our case study, we will just focus on the swap, forgetting the bank's issue.



if Stoxx50 < K at 1y



if Stoxx50 ≥ K at 1y

The coupons of the bank's issue are as follows:

$$\begin{cases} 6\% & \text{if Stoxx50} < \text{K at 1y (Coupon Reset Date)} \\ 2\% & \text{otherwise, at 2y (Final Coupon Clause)} \end{cases}$$

where Stoxx50 is a stock with given $S_0 = 2973.87$, the strike K = 3200 and Coupon Reset Date means 2 business days prior to the respective Coupon Payment Date, which is at exactly 1 year.
In the contract there's also an Early Redemption clause, that means that if the contract's maturity is 2 years, the swap is over if we receive the 6% coupon at 1y.

1.1 Value the Upfront with NIG model

Calibrate the model parameters

We first model the Stoxx50 dynamics following a NIG model, whose dynamics for the forward are the following; taking $F(t,t) = f_0\, e^{f_t}$, we have:

$$f_t = \mu - \sigma^2 \Delta t \left(\frac{1}{2} + \eta\right) G + \sigma\sqrt{\Delta t} \cdot g \cdot \sqrt{G} \tag{1}$$

where G is an Inverse Gaussian random variable with mean = 1 and variance = $\Delta t/\kappa$ and g is a standard Gaussian random variable. Moreover, we have:

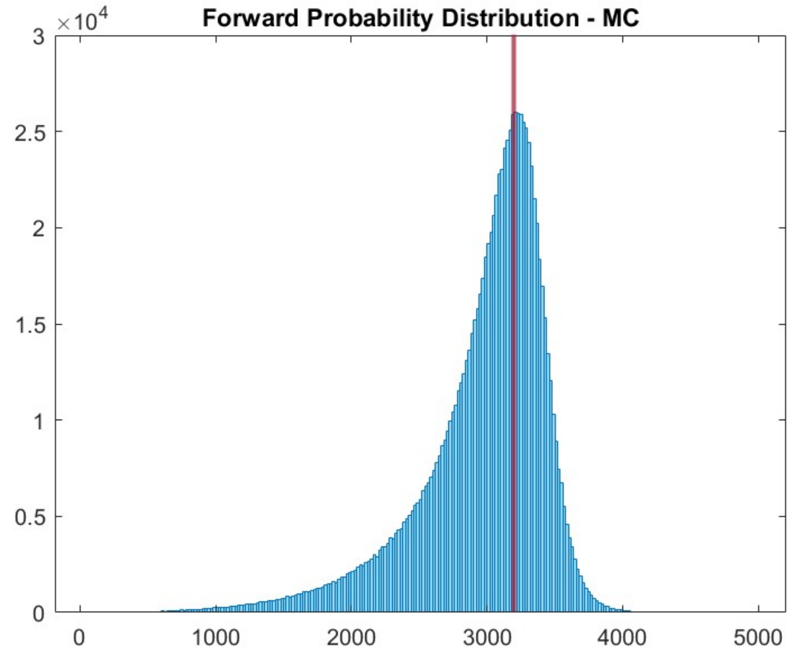
$$\mu = -\log \hat{\mathscr{L}}[\eta] = -\frac{\Delta t}{\kappa}\left[1 - \sqrt{1 + 2\kappa\eta\sigma^2}\right]$$

To calibrate the parameters of the NIG model, η, κ, σ, we perform the same procedure as in the Assignment 5, minimizing the distance between market Black prices and NIG model prices.

Probability of early redemption

To compute the probability of having $S_T < K$ at 1y we then follow two different approaches.

The first one simply performs MC simulation of the underlying's evolution in time, from 0 to the (first) reset date, using the dynamics specified in eq.1 and then checking the condition.



For the second one, instead, we observe that the probability $P(S_T < K)$ can be seen as 1 minus the price of a digital option in 0. Indeed, the payoff of the digital is:

$$\begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

So, if we compute the digital price, that is we take the expected value of the payoff above, we get $P(S_T \geq K)$, which is 1 minus the wanted probability.
From the given Lewis paper, we obtain the probability with the following formula:

$$P(S_t < K) = 1 - \left(\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{iuk}\ \Phi_T(u)}{iu}\right] du\right)$$

We report in the following table the numerical results obtained for the probability of the event $S_t < K$:

$P(S_t < K)$ - **NIG model**			
	MC	**MC Confidence Interval**	**Closed formula**
Upfront	63.9159%	[63.8218%, 64.0100%]	63.9591%

Value the upfront

We can now move on to the computation of the two legs of the hedging swap.
Now that we know the probability of the event $S_t < K$, we can consider the two possible scenarios separately, and then compute the expected value.

Let us first compute the NPV of the floating leg (payed by the Bank XX): we will use the telescopic sum trick to handle the Euribor payments and then discount the spol using Spot discount factors obtained via bootstrap.

$$NPV_{XX}(scenario\ 1) = 1 - B(t_0,t_4) + \sum_{i=1}^{4} \delta(t_0,t_i) s^{spol} B(t_0,t_i) \quad \text{if Stoxx50} < \text{K at 1y}$$

$$NPV_{XX}(scenario\ 2) = 1 - B(t_0,t_8) + \sum_{i=1}^{8} \delta(t_0,t_i) s^{spol} B(t_0,t_i) \quad \text{if Stoxx50} \geq \text{K at 1y}$$

Therefore the expected NPV is:

$$NPV_{XX} = P(S_t < K) NPV_{XX}(scenario\ 1) \ + \ (1 - P(S_t < K)) NPV_{XX}(scenario\ 2)$$

We remark that the $\delta(t_0,t_i)$ follow an ACT/360 convection.

With the same reasoning we compute the NPV for the fixed leg (payed by IB):

$$NPV_{IB} = X\% \left(P(S_t < K) \cdot 6\% \ + \ (1 - P(S_t < K) \cdot 2\% \right)$$

At last, asking that $NPV_{XX} = NPV_{IB}$, we can deduce the value of the upfront X%:

Upfront X% - 2 years - NIG model		
	Upfront X%	**Confidence Interval**
Upfront	2.4886%	[2.4805%, 2.4967%]

By multiplying for the principal amount, we can get the value of the first payment made by the IB, according to the NIG model: €2.5 MLN.

1.2 Value the Upfront with a different model

We can also price the swap selecting a different model for the underlying Stoxx50, the VG model. It has the same forward dynamics as in eq.1, except for the Laplace exponent parameter which is now:

$$\mu = -\log \hat{\mathscr{L}}[\eta] = \frac{\Delta t}{\kappa} log(1 + \kappa\eta\sigma^2)$$

Firstly, we compute the probability $P(S_t < K)$ at 1 year as before, obtaining:

$P(S_t < K)$ **- VG model**			
	MC	**MC Confidence Interval**	**Closed formula**
Upfront	63.9414%	[63.8473%, 64.0355%]	63.9276%

Having obtained this, we can now perform the same steps as above, obtaining an upfront of:

Upfront X% - 2 years - VG model		
	Upfront X%	**Confidence Interval**
Upfront	2.4864%	[2.4783%, 2.4945%]

By multiplying for the principal amount, we can get the value of the first payment made by the IB, according to the VG model: €2.5 MLN.

We note that the difference between the upfronts computed through the VG model and the NIG model is a little over €2K, with the NIG model giving the higher result.
In the end we can consider these two approaches as equivalent, as they give such similar results.

1.3 Value the Upfront in the case of 3-year expiry

We now extend the length of the issue, adding 1 year to its maturity and having these coupons:

$$\begin{cases} 6\% & \text{if Stoxx50} < \text{K at 1y (Coupon Reset Date)} \\ 6\% & \text{if Stoxx50} < \text{K at 2y (Coupon Reset Date)} \\ 2\% & \text{otherwise, at 3y} \end{cases}$$

The Early Redemption clause can now be exercised only at the second year.

We therefore need to compute the probabilities in the new four possible scenarios: $P(S_1 \geq K, S_2 \geq K)$, $P(S_1 \geq K, S_2 < K)$, $P(S_1 < K, S_2 \geq K)$, $P(S_1 < K, S_2 < K)$.
We find them through MC simulation of the underlying, using the NIG underlying's evolution over two consecutive years:

Probabilities in 4 scenarios - NIG model				
	$P(S_1 \geq K, S_2 \geq K)$	$P(S_1 \geq K, S_2 < K)$	$P(S_1 < K, S_2 \geq K)$	$P(S_1 < K, S_2 < K)$
Upfront	25.7129%	10.3712%	16.4445%	47.4714%

We also create a new function that computes the NPV in the case of a swap contract with maturity up to 3 years, to account for the four different scenarios. The swap goes until maturity only in the case ($S_1 \geq K, S_2 \geq K$), giving a coupon of 2% at 3 years, while in the case ($S_1 < K, S_2 < K$) two coupon at 6% are paid. Finally, in the two remaining cases a single 6% coupon is paid, alternatively at 1 year or at 2 year.

We then find an upfront X% equal to:

Upfront X% - 3 years - NIG model
3.6892%

By multiplying for the principal amount, we can get the value of the first payment made by the IB, according to the NIG model in the case of 3 year expiry: €3.7 MLN.

1.4 Black Formula

Finally, we implement the second approach using Black formula to price the calls.
So we consider two strike values, K and K+ϵ, their respective market volatilities (interpolated from the market volatility surface), and time to maturity equal to the time to the reset date. The prices obtained with the Black formula are then subtracted and divided by ϵ to get the price of the digital option, i.e. the probability of $S_t \geq K$.
The found probability for the event $S_t < K$ is 67.3616%.

Therefore, with the same procedure as before, we get the upfront percentage as: 2.1923%.
By multiplying for the principal amount, we can get the value of the first payment made by the IB, according to the Black formula: €2.2 MLN.

We comment that this result is quite different from the one obtained with the NIG/VG models, and this is because the Black formula does not keep into account the digital risk, which is fundamental in a case study such as this one.

2 Exercise: Bermudan Swaption Pricing via Hull-White

In this exercise we consider a 10y Bermudan yearly Payer Swaption strike K 5% non-call 2 (i.e. the swaption can be exercised every year starting from the 2y, the underlying swap ends on the 19th of

February 2018).
Our aim is to price this option using a single curve framework and a 1-factor Hull-White model, with mean reversion speed $\alpha = 11\%$ and volatility $\sigma = 0.8\%$ for its OU process.

2.1 Price via a trinomial tree

At first we chose, as a variable that controls the precision of the grid, the number of steps for our 10 year interval to be divided in.

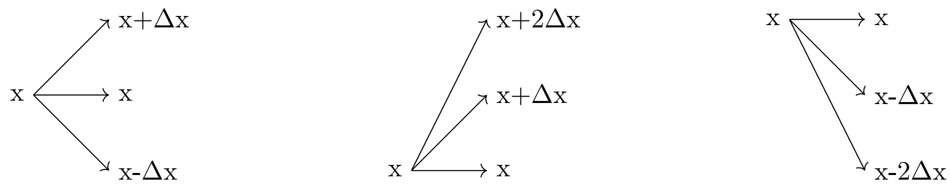
The first step is to set up the needed parameters:

$$\hat{\mu} = 1 - e^{-\alpha dt} \qquad \hat{\sigma} = \sigma\sqrt{\frac{1-e^{-2\alpha dt}}{2\alpha}} \qquad \hat{\sigma}^* = \frac{\sigma}{\alpha}\sqrt{dt - 2\frac{1-e^{-2\alpha dt}}{\alpha} + \frac{1-e^{-2\alpha dt}}{2\alpha}}$$

$$l_{max} = \left\lceil \frac{(1-\sqrt{2/3})}{\hat{\mu}} \right\rceil \qquad l_{min} = -l_{max} \qquad \Delta x = \sqrt{3}\ \hat{\sigma}$$

We also define the probabilities of moving upward, downwards and of staying level in the three possible scenarios: this tree is indeed limited vertically, and once the variable x reaches the maximum (or minimum) height it will only stay level, or go down by one or two steps.

scenario: **case A** **case B** **case C**



$$p_{up} = \frac{1}{2}\left(\frac{1}{3} - l\hat{\mu} + (l\hat{\mu})^2\right) \qquad p_{up} = \frac{1}{2}\left(\frac{1}{3} + l\hat{\mu} + (l\hat{\mu})^2\right) \qquad p_{up} = \frac{1}{2}\left(\frac{7}{3} - 3l\hat{\mu} + (l\hat{\mu})^2\right)$$

$$p_m = \frac{2}{3} - (l\hat{\mu})^2 \qquad p_m = -\frac{1}{3} - 2l\hat{\mu} - (l\hat{\mu})^2 \qquad p_m = -\frac{1}{3} + 2l\hat{\mu} - (l\hat{\mu})^2$$

$$p_{down} = \frac{1}{2}\left(\frac{1}{3} + l\hat{\mu} + (l\hat{\mu})^2\right) \qquad p_{down} = \frac{1}{2}\left(\frac{7}{3} + 3l\hat{\mu} + (l\hat{\mu})^2\right) \qquad p_{down} = \frac{1}{2}\left(\frac{1}{3} - l\hat{\mu} + (l\hat{\mu})^2\right)$$

Now that we have everything we need, we can simulate the dynamics of the Ornstein-Uhlenbeck variable X, which represents our interest rate.
From the values of x_i (at time t_i, one for every node) we want to obtain the forward ZC bonds $B(t_i, t_i, t_{i+\tau})$, as follows:

$$B(t_i, t_i, t_{i+\tau}) = B(t_0, t_i, t_{i+\tau})\cdot\exp\left\{-x_i\frac{\sigma_{HJM}(0,\tau)}{\sigma} - \frac{1}{2}\int_{t_0}^{t_i}[\sigma_{HJM}(u, t_i+\tau)^2 - \sigma_{HJM}(u, t_i)^2]\,du\right\} \quad (2)$$

$$\sigma_{HJM}(u,t) = \sigma\ \frac{1-e^{-\alpha(t-u)}}{\alpha}$$

where the $B(t_0, t_i, t_{i+\tau})$ do not depend on the realization of x, and are instead obtained as forward discount factors from the bootstrap made on the spot discount factors.

When posing in 2 t_i=2,3,4,5,6,7,8,9 and τ equal to all the integer values bigger than t_i up to 10, we get discounts needed in the computation of the swap rate.
Repeating this procedure for every possible value of x in the tree, and for $\tau = dt$, we get another tree with the values of these discount factors, to be used in the following equation.

Now we can compute a tree of stochastic discount factors $D(t_i, t_i + dt)$, needed to discount the future payoffs of our option:

$$D(t_i, t_i + dt) = B(t_i, t_i + dt) \cdot \exp\left(-\frac{1}{2}(\hat{\sigma}^*)^2 - \frac{\hat{\sigma}^*}{\hat{\sigma}}[\Delta x_{i+1} + \hat{\mu} x_i]\right) \tag{3}$$

Here we remind that $B(t_i, t_i + dt)$ is exactly the $B(t_i, t_i, t_{i+dt})$ computed above.

At last we can finally start using the tree to move backwards in time and compare the swap rate available in the market every year with the strike value, and then decide whether to exercise the option or not, were we to find ourselves in that particular state of the world.

We start with computing the payoff of the option at the 9^{th} year: this is because at year 9 I have the right to enter in the swap which has current time to maturity equal to 1, and this is the last chance to enter in this swap because it will expire next year.

The payoff of a swaption with exercising time t_α and expiry t_ω and strike value K is defined as:

$$\text{Payoff}(t_\alpha) = BPV_{\alpha,\omega}(t_\alpha)[S_{\alpha,\omega}(t_\alpha) - K]^+$$

$$S_{\alpha,\omega}(t_\alpha) = \frac{1 - B(t_\alpha, t_\omega)}{BPV_{\alpha,\omega}(t_\alpha)} \qquad BPV_{\alpha,\omega}(t_\alpha) = \sum_{i>\alpha}^{\omega} \delta(t_{i-1}, t_i) B(t_\alpha, t_i)$$

where the $\delta(t_{i-1}, t_i)$ follow a 30/360 convention, the i takes yearly values, the $B(t_\alpha, t_i)$ are discounts obtained in 2.
We note that because $B(t_\alpha, t_i)$ depends on x_α, here we get a vector of payoffs, one for each possible state of the world at the year t_α.

Once we have computed the payoff vector at the 9^{th} year, we enter into a loop that implements the following instructions.
We move backwards of a time step dt, and for the whole vector x_i we compute the discounted value of the expected payoff (relative to time t+dt), using the probabilities defined earlier, keeping in mind the three scenarios explained above. We note that we use the stochastic discount factor introduced in 3

$$\text{Expected Discounted Payoff} = D(t_i, t_i + dt)\mathbb{E}_i[\text{Payoff}(t_i)]$$

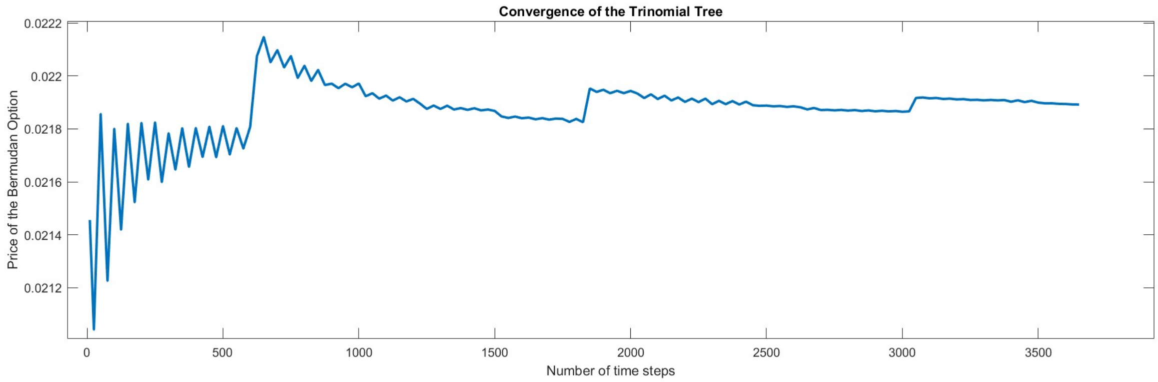
When we go back to a time instant t_i that corresponds to 2...8 years, we not only discount the expected payoff, but we also compute the swap rate that would be available in the market at t_i and the corresponding payoff of the swaption, just as we did for the 9^{th} year.

Then we compare the discounted expected payoff with the so called 'Intrinsic Value' just computed, and take the maximum of the two as the new starting point for the next loop.
This comparison between the continuation value and the intrinsic value is what characterizes the Bermudan option, and is done only at yearly time instants, starting from the second, just as the text described.

At the very end we finally get a numerical value that represents the current price of this particular Bermudan swaption.

Changing the value of N we get the following results:
The number of time steps is spread across 10 years: so 10 time steps equal to yearly steps, 120 equal to monthly steps, and so forth.
As expected for low values, like under 1000, the output price of the Bermudan is not reliable at all, and it varies greatly.
Instead as the number of time steps increases we can say that there is convergence.

Thanks to this analysis, we can say that a valid and reliable choice of N should fall after 3000, and that the price of our bermudan option is close to €0.02189.



2.2 Check on the implementation of the tree and range with Jamshidian approach

To verify the tree's implementation, we calculate both an upper and a lower bound to confirm the bermudan price's range.
We define the upper bound as the aggregate of prices for the swaptions exercisable at point 'a'. This value is expected to exceed the bermudan option's price because we would only enter in one swap, versus to buying them all.
Conversely, for the lower bound, we identify it as the maximun among all swaption prices, because in the bermudan case we will enter in the most favorable swap and we will also pay a premium attributed to the right we have to choose the swap we wish to enter into .
To price these options, we employ the Jamshidian method. Recognizing that a swaption's price aligns with that of a coupon bond put with a strike of 1, we begin our analysis with the related coupon bond call formula. Leveraging a Put-Call parity relationship, we then deduce the put price.

$$C(t_0) = E_0[D(t_0, t_\alpha)(P(t_\alpha; \underline{c}, \underline{t}) - 1)^+] \tag{4}$$

In particular, we have the following definition:

$$P(t; \underline{c}, \underline{t}) = \sum_{i=\alpha+1}^{\omega} c_i B(t; t_\alpha, t_i)$$

$$\begin{cases} t_\alpha = \text{starting date, in our case is from } 2^{nd} \text{ to } 9^{th} \text{ year} \\ t_i = \text{payment dates} \\ t_\omega = \text{ end date, in our case is } 10^{th} \text{ year} \\ c = \text{coupon, in our case is } 5\% \\ c_i = c\delta_i \text{ for } i < \omega \\ c_\omega = 1 + c\delta_\omega \end{cases}$$

Now we focus on the second factor of the product in the summation. To implement $B(t; t_\alpha, t_i)$ we use the forward ZC bond via Hull-White model:

$$B(t; t_\alpha, t_i) = B(t_0; t_\alpha, t_i) \exp\{-x_\alpha \frac{\sigma(0, t_i - t_\alpha)}{\sigma} - \frac{1}{2}\int_{t_0}^{t_\alpha} [\sigma(u, t_i)^2 - \sigma(u, t_\alpha)^2]\, du\}$$

$$\sigma(s, t) := \frac{\sigma}{a}(1 - e^{-a(t-s)})$$

In our case, $t = t_\alpha$ and we notice that this forward discount depends on x_α . So we add the following notation: $B(t_\alpha; t_\alpha, t_i) = B_{\alpha i}(x_\alpha)$. We manipulate 4 mathematically to obtain the price as a function of a ZCB call:

$$\begin{aligned} C(t_0) &= E_0[D(t_0,t_\alpha)[\sum_{i=\alpha+1}^{\omega}(c_i B_{\alpha i}(x_\alpha)) - 1]\mathbf{I}_{x_\alpha < x^*}] = \\ &= E_0[D(t_0,t_\alpha)[\sum_{i=\alpha+1}^{\omega} c_i(B_{\alpha i}(x_\alpha) - B_{\alpha i}(x^*))\mathbf{I}_{x_\alpha < x^*}]] = \\ &= \sum_{i=\alpha+1}^{\omega} c_i E_0[D(t_0,t_\alpha)[B_{\alpha i}(x_\alpha) - B_{\alpha i}(x^*)]^+] \end{aligned}$$

x^* is the value of the dynamic such that $P(t_\alpha = K$, in our case K = 1. To compute this we use the Matlab command *'fzero'* in our constructed function textit'swaptionPriceJamshidian'. The expected value of last equation above is clearly a Call option on a ZCB. We are in a gaussian HJM framework so we can use the following closed formula to evaluate it :

$$Call_{ZC}(t_0) = E_0[D(t_0,t_\alpha)[B_{\alpha i}(x_\alpha) - B_{\alpha i}(x^*)]^+] = B(t_0,t_\alpha)[B(t_0;t_\alpha,t_i)N(d_1) - B_{\alpha i}(x^*)N(d_2)]$$

$$d_{1,2} = \frac{log(\frac{B(t_0;t_\alpha,t_i)}{B_{\alpha i}(x^*)})}{V\sqrt{t_\alpha - t_0}} \pm \frac{1}{2}V\sqrt{t_\alpha - t_0} \qquad V := \sqrt{\frac{1}{t_\alpha - t_0}\int_{t_0}^{t_\alpha}(\sigma(u,t_i) - \sigma(u,t_\alpha))^2\,du}$$

After the evaluation of the $Call_{ZC}$ and consequently of the $C(t_0)$, we can apply the Put-Call parity written in the Jamshidian paper to find the price of the Coupon Bond Put equal to our swaptions:

$$\textbf{Put-Call parity:} \qquad SP_{\alpha\omega}(t_0) = P(t_0) = C(t_0) + B(t_0,t_\alpha) - \sum_{i=\alpha+1}^{\omega} c_i B(t_0,t_i)$$

Swaption prices with different t_α and $t_\omega = 10y$								
t_α	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**
price	0.008196	0.012043	0.014244	0.014814	0.014018	0.011931	0.008761	0.004728

The interval is [0.014814; 0.088734] where the lower bound is the maximum between all swaption prices and the upper is the sum of the prices above. So we verify that bermudan price found (0.02189) is in this range.

Moreover, we observe that the price of a 9y-10y swaption calculated using the Jamshidian formula is near to the European option evaluated with the tree:

Prices comparison	
Swaption 9y10y - Jamshidian	**European option - tree**
0.004728	0.004761