

Quantization of a Free Scalar Field

Consider the free scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

that we considered before. ϕ satisfies the Klein-Gordon equation

$$(\square + m^2) \phi = 0 \quad \leftarrow \text{This has solution } e^{\pm i k \cdot x} \quad k \cdot x = k_x^n \\ k^n = (E(\vec{k}), \vec{k})$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \quad = (E(\vec{k}), -\vec{k}) \\ = (\omega_{k_e}, -\vec{k})$$

The general solution of the Klein-Gordon equation is

$$\phi(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [a(\vec{k}) e^{-ik \cdot x} + a^*(\vec{k}) e^{ik \cdot x}]$$

Then

$$\pi(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [-iE(\vec{k})] [a(\vec{k}) e^{-ik \cdot x} - a^*(\vec{k}) e^{ik \cdot x}]$$

To quantize the field, we follow the usual canonical procedure:

$$\{ \phi, \pi \} \rightarrow ; [\hat{\phi}, \hat{\pi}]$$

For scalar field theory

$$\bar{\Pi}(x) = \frac{\partial^2}{\partial \phi \partial \phi}(S)$$

and

$$\mathcal{H} = \bar{\Pi}(x) \frac{\partial}{\partial} \phi(x) - \mathcal{L}$$

Consider

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

then

$$\bar{\Pi}(x) = \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi}$$

$$= \frac{1}{2} \frac{\partial}{\partial} (\partial_\mu \phi) (\partial_\mu \phi)$$

$$= \frac{1}{2} \frac{\partial}{\partial} (\partial_\mu \phi) \underbrace{(\partial_\mu \phi \gamma^\mu \partial_\mu \phi)}_{\gamma^\mu \partial_\mu \phi = \partial_\mu \phi}$$

$$= \partial_\mu \phi$$

Then

$$\mathcal{H} = \underbrace{\dot{\phi}^2}_{\bar{\Pi}^2} - \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{1}{2} m^2 \phi^2$$

(16)

$$= \frac{1}{2} \bar{u}^2 - \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} \bar{u}^2 - \frac{1}{2} \gamma^{ij} \nabla_i \phi \nabla_j \phi + \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} \bar{u}^2 + \frac{1}{2} |\vec{\nabla} \phi|^2 + \frac{1}{2} m^2 \phi^2$$

Substituting the assumed solution for $\phi(x)$ into the KG equation:

$$(\square + m^2) \phi = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [a(\vec{k})(\square + m^2) e^{-ik_x x} + a^*(\vec{k})(\square + m^2) e^{ik_x x}]$$

But

$$\begin{aligned} (\square + m^2) e^{-ik_x x} &= \\ &= \left(\frac{\vec{k}^2}{\Delta t^2} - \frac{\vec{k}_x^2}{\lambda_x^2} - \frac{\vec{k}_y^2}{\lambda_y^2} - \frac{\vec{k}_z^2}{\lambda_z^2} + m^2 \right) e^{-iE(\vec{k})t} e^{ik_x x} e^{ik_y y} e^{ik_z z} \\ &= \left(\frac{\vec{k}^2}{\Delta t^2} + |\vec{k}|^2 + m^2 \right) e^{-iE(\vec{k})t} e^{i\vec{k} \cdot \vec{x}} \end{aligned}$$

and

$$\begin{aligned} (\square + m^2) a(\vec{k}) e^{-ik_x x} &= \\ &= \left(\frac{\vec{k}^2}{\Delta t^2} + |\vec{k}|^2 + m^2 \right) a(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} \\ &\equiv \left(\frac{\vec{k}^2}{\Delta t^2} + \omega_k^2 \right) a(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} \quad \omega_k \equiv [\lambda_k^2 + m^2]^{1/2} \\ &= E(\vec{k}) \end{aligned}$$

Then, the KG equation becomes

$$\left(\frac{\vec{k}^2}{\Delta t^2} + \omega_k^2 \right) a(\vec{k}, t) = 0$$

whose solution is

$$a(\vec{k}, t) = a(\vec{k}) e^{-i\omega_n t}$$

Following the same procedure for $a^*(\vec{k}, t)$, we find

$$a^*(\vec{k}, t) = a^*(\vec{k}) e^{i\omega_n t}$$

Let we have been able to determine the exact time dependence for $a(\vec{k}, t)$ and $a^*(\vec{k}, t)$ in the free field case.

$$a(\vec{k}) \rightarrow \hat{a}(\vec{k})$$

$$a^*(\vec{k}) \rightarrow \hat{a}^+(\vec{k})$$

In particular, we impose

$$[\hat{q}(\vec{x}, t), \hat{\pi}(\vec{x}', t)] = i \delta^{(2)}(\vec{x} - \vec{x}')$$

What does this imply about $[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')]$?

To answer this question, compute

$$\hat{q}(\vec{x}, t) \hat{\pi}(\vec{x}', t)$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [\hat{a}(\vec{k}) e^{-iE(\vec{k})t} e^{i\vec{k} \cdot \vec{x}} + \hat{a}^+(\vec{k}) e^{iE(\vec{k})t} e^{-i\vec{k} \cdot \vec{x}}]$$

$$* (-i) \int \frac{d^3 \vec{k}'}{(2\pi)^3} \frac{E(\vec{k}'')}{\sqrt{2E(\vec{k}'')}} [\hat{a}(\vec{k}') e^{-iE(\vec{k}')t} e^{i\vec{k}' \cdot \vec{x}'} - \hat{a}^+(\vec{k}'') e^{iE(\vec{k}'')t} e^{-i\vec{k}'' \cdot \vec{x}'}]$$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{d^3 \vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{E(\vec{k}'')}{\sqrt{2E(\vec{k}'')}}$$

$$\times [\hat{a}(\vec{k}) \hat{a}^+(\vec{k}')] e^{-i[E(\vec{k})+E(\vec{k}')]t} e^{i\vec{k} \cdot \vec{x}} e^{i\vec{k}' \cdot \vec{x}'}$$

$$+ \hat{a}(\vec{k}) \hat{a}^+(\vec{k}'') e^{-i[E(\vec{k})-E(\vec{k}'')]t} e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k}'' \cdot \vec{x}'}$$

$$+ \hat{a}^+(\vec{k}) \hat{a}^+(\vec{k}') e^{i[E(\vec{k})-E(\vec{k}')]t} e^{-i\vec{k} \cdot \vec{x}} e^{i\vec{k}' \cdot \vec{x}'}$$

(3)

$$-\hat{a}^+(\vec{k}) \hat{a}^+(\vec{k}') e^{i[E(\vec{k})+E(\vec{k}')]t} e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'}]$$

Now compare

$$\hat{\pi}(\vec{x}', t) \hat{\phi}(\vec{x}, t)$$

$$= (-i) \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{d^3 \vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{1}{\sqrt{2E(\vec{k}')}} E(\vec{k}')$$

$$\times \left[\hat{a}(\vec{k}') \hat{a}^+(\vec{k}) e^{-i[E(\vec{k}') + E(\vec{k})]t} e^{i\vec{k}'\cdot\vec{x}'} e^{-i\vec{k}\cdot\vec{x}} \right.$$

$$+ \hat{a}(\vec{k}') \hat{a}^+(\vec{k}) e^{-i[E(\vec{k}') - E(\vec{k})]t} e^{i\vec{k}'\cdot\vec{x}'} e^{-i\vec{k}\cdot\vec{x}}$$

$$- \hat{a}^+(\vec{k}') \hat{a}^+(\vec{k}) e^{-i[E(\vec{k}') - E(\vec{k})]t} e^{-i\vec{k}'\cdot\vec{x}'} e^{i\vec{k}\cdot\vec{x}}$$

$$- \hat{a}^+(\vec{k}') \hat{a}^+(\vec{k}) e^{-i[E(\vec{k}') + E(\vec{k})]t} e^{-i\vec{k}'\cdot\vec{x}'} e^{-i\vec{k}\cdot\vec{x}} \right]$$

Then

$$\hat{\phi}(\vec{x}, t) \hat{\pi}(\vec{x}', t) - \hat{\pi}(\vec{x}', t) \hat{\phi}(\vec{x}, t)$$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{d^3 \vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{1}{\sqrt{2E(\vec{k}')}} E(\vec{k}')$$

$$\times \left[- \hat{a}(\vec{k}) \hat{a}^+(\vec{k}') e^{-i[E(\vec{k}) - E(\vec{k}')]t} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \right. \quad (1)$$

$$+ \hat{a}^+(\vec{k}) \hat{a}^+(\vec{k}') e^{i[E(\vec{k}) - E(\vec{k}')]t} e^{-i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \quad (2)$$

(4)

$$- \hat{a}(\vec{k}') \hat{a}^+(\vec{k}) e^{-i[E(\vec{k}') - E(\vec{k})]t} e^{i\vec{k}' \cdot \vec{x}'} e^{-i\vec{k} \cdot \vec{x}}$$

(3)

$$+ \hat{a}^+(\vec{k}') \hat{a}(\vec{k}) e^{i[E(\vec{k}') - E(\vec{k})]t} e^{-i\vec{k}' \cdot \vec{x}'} e^{i\vec{k} \cdot \vec{x}}]$$

(2)

if

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = 0$$

$$[\hat{a}^+(\vec{k}), \hat{a}^+(\vec{k}')] = 0$$

Further, if

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

then

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{x}', t)]$$

$$= i \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{d^3 \vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k}')}} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times [(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')] e^{-i[E(\vec{k}) - E(\vec{k}')]t} e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k}' \cdot \vec{x}'} \quad ① + ④$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k}) e^{i[E(\vec{k}) - E(\vec{k}')]t} e^{-i\vec{k} \cdot \vec{x}} e^{i\vec{k}' \cdot \vec{x}'}] \quad ③ + ②$$

$$= i \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} [e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} + e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}] = i \delta^{(3)}(\vec{x} - \vec{x}')$$

Therefore, if we impose

$$[\hat{f}(\vec{x}, t), \hat{\pi}(\vec{x}, t)] = i \delta^{(3)}(\vec{x} - \vec{x}')$$

then

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$

which is the commutation relation for the creation and annihilation operators of the harmonic oscillator!

HOMEWORK 1 :

1. Define the Hamiltonian for the real scalar field $\hat{\phi}$. Interpret the result. What is the number operator in this case?
2. Given the Hamiltonian, show that $\hat{\phi}$ satisfies the Heisenberg equations of motion.
3. Show that $\hat{a}(\vec{k})$ and $\hat{a}^+(\vec{k})$ are creation and annihilation operators for scalar particles in mode \vec{k} of the quantized scalar field.

The above homework problems will further support the interpretation of $\hat{a}(\vec{k})$ and $\hat{a}^+(\vec{k})$ as annihilation and creation operators respectively of particles (in this case scalar, spin 0 particles) of momentum \vec{k} - i.e., of quanta in mode \vec{k} of the quantized field $\hat{\phi}$.

The action of $\hat{a}(\vec{k})$ and $\hat{a}^+(\vec{k})$ on the vacuum are

$$\overline{[2E(\vec{k}) \hat{a}(\vec{k})]} |0\rangle = 0 \quad (\text{this is the same as } \hat{a}(\vec{k})|0\rangle = 0)$$

(6)

$$\sqrt{2E(\vec{k})} \hat{a}^+(\vec{k}) |0\rangle = |\vec{k}\rangle$$

The factor $\sqrt{2E(\vec{k})}$ is a convenient relativistic normalization.

To develop a better understanding of the quantized scalar field operator, let's compute

$$\langle \vec{x} | \hat{\phi}(x) | 0 \rangle$$

$$= \langle 0 | \sqrt{2E(\vec{x})} \hat{a}(\vec{x}) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [\hat{a}(\vec{k}) e^{-ik \cdot x} + \hat{a}^+(\vec{k}) e^{ik \cdot x}] | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[\frac{2E(\vec{x})}{2E(\vec{k})} \right]^{1/2} e^{-ik \cdot x} \langle 0 | \hat{a}(\vec{x}) \hat{a}^+(\vec{k}) | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[\frac{2E(\vec{x})}{2E(\vec{k})} \right]^{1/2} e^{ik \cdot x} \langle 0 | [\hat{a}^+(\vec{k}) \hat{a}(\vec{x}) + 2\pi \delta^{(3)}(\vec{k} - \vec{x})] | 0 \rangle$$

$$= \int d^3k \left[\frac{2E(\vec{x})}{2E(\vec{k})} \right]^{1/2} e^{ik \cdot x} \underbrace{\delta^{(3)}(\vec{k} - \vec{x})}_{=1} \langle 0 | 0 \rangle$$

$$= e^{-i\vec{x} \cdot \vec{x}}$$

$$= e^{iE(\vec{x})t} e^{-i\vec{x} \cdot \vec{x}}$$

$$= e^{iE(\vec{x})t} \langle \vec{x} | \vec{x} \rangle$$

Thus

(7)

$$\hat{\phi}(x)|0\rangle = e^{iE(\vec{p})t}|\vec{x}\rangle$$

- i.e., the operator creates a free spin-0 particle of mass m and energy $E(\vec{p})$ (momentum \vec{p}) at \vec{x} (i.e., in the position eigenstate $|\vec{x}\rangle$).

Since $\hat{\phi}(x)$ is Hermitian

$$\hat{\phi}^+(x) = \hat{\phi}(x)$$

We have

$$\langle x| = (\langle x|)^+ = (\hat{\phi}(x)|0\rangle)^+ = \langle 0|\hat{\phi}^+(x) = \langle 0|\hat{\phi}(x)$$

Then the wave function of a particle in state $|4\rangle$ is

$$\psi(x) = \langle x|4\rangle = \langle 0|\hat{\phi}(x)|4\rangle$$

Let's consider the time evolution of $\psi(x)$

$$\frac{d}{dt} \psi(x) = \frac{d}{dt} \langle 0|\hat{\phi}(x)|4\rangle$$

$$= \langle 0|\frac{d}{dt}\hat{\phi}(x)|4\rangle$$

$$= \langle 0| \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} [-iE(\vec{k})] [\hat{a}(\vec{k})e^{-ik\cdot x} - \hat{a}^+(\vec{k})e^{ik\cdot x}] |4\rangle$$

$$= -i \langle 0 | \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E(k)}} [\vec{k}^2 + m^2]^{1/2} [\hat{a}(\vec{k}) e^{-ik \cdot x} - \hat{a}^\dagger(\vec{k}) e^{ik \cdot x}] | \psi \rangle$$

8a

for our
alternative
definition

$$= -i \langle 0 | [-\vec{\nabla}^2 + m^2]^{1/2} \hat{\psi}(x) | \psi \rangle$$

$$= -i [-\vec{\nabla}^2 + m^2]^{1/2} \langle 0 | \hat{\psi}(x) | \psi \rangle$$

$$= -i [-\vec{\nabla}^2 + m^2]^{1/2} \psi(x)$$

$$= -im [1 - \frac{1}{m^2} \vec{\nabla}^2]^{1/2} \psi(x)$$

$$\approx -im [1 - \frac{1}{2m^2} \vec{\nabla}^2] \psi(x)$$

Then

$$\frac{i}{t} \psi \approx \left(m - \frac{1}{2m} \vec{\nabla}^2 \right) \psi$$

$$= (\hat{H}_{\text{nonrelativistic}} + m) \psi$$

The term m (really mc^2) can be dropped. It contributes a constant to the overall Hamiltonian and does not affect the dynamics.

So, we have

$$\frac{i}{t} \psi \approx -\frac{1}{2m} \vec{\nabla}^2 \psi$$

A heuristic derivation:

$$\approx -i \langle 0 | \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} m \left(1 + \frac{\vec{k}^2}{2m^2} \right) [\hat{a}(\vec{k}) e^{-ik \cdot x} - \hat{a}^\dagger(\vec{k}) e^{ik \cdot x}] | \psi \rangle$$

$$= -i \langle 0 | m \left(1 - \frac{1}{2m^2} \vec{\nabla}^2 \right) \hat{\phi}(x) | \psi \rangle$$

$$= -i m \left(1 - \frac{1}{2m^2} \vec{\nabla}^2 \right) \langle 0 | \hat{\phi}(x) | \psi \rangle$$

$$= -i m \left(1 - \frac{1}{2m^2} \vec{\nabla}^2 \right) \phi(x)$$

(9)

in the nonrelativistic limit, which is just Schrödinger's equation for the single free scalar particle in this limit.

Multi-Particle States and Fock Space

Using the creation operators, one can construct multiparticle states.

For example, the following is a 2-particle state

$$\hat{a}^+(\vec{k}_1) \hat{a}^+(\vec{k}_2) |0\rangle = |\vec{k}_1, \vec{k}_2\rangle$$

The possibilities are literally endless

$$|0\rangle, \underbrace{\hat{a}^+(\vec{k}_1)}_{|\vec{k}_1\rangle} |0\rangle, \underbrace{\hat{a}^+(\vec{k}_1) \hat{a}^+(\vec{k}_2)}_{|\vec{k}_1, \vec{k}_2\rangle} |0\rangle, \underbrace{\hat{a}^+(\vec{k}_1) \hat{a}^+(\vec{k}_2) \hat{a}^+(\vec{k}_3)}_{|\vec{k}_1, \vec{k}_2, \vec{k}_3\rangle} |0\rangle, \dots$$

The Hilbert space is a direct sum of n -particle Hilbert spaces ($n=0, 1, 2, \dots$)

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n$$

known as Fock Space.

Let's look at some matrix elements (observables). Consider a two-particle system with one particle in mode \vec{k}_1 and the other in mode \vec{k}_2 .

Let's check that the number operator

(11)

$$\hat{N} = \int \frac{d^3 \vec{k}}{(2\pi)^3} \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k})$$

returns the number 2:

$$\begin{aligned}
 & \langle \vec{k}_1 \vec{k}_2 | \hat{N} | \vec{k}_1 \vec{k}_2 \rangle \\
 &= \langle \vec{k}_1 \vec{k}_2 | \int \frac{d^3 \vec{k}}{(2\pi)^3} \hat{a}^\dagger(\vec{k}) \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}_1) \hat{a}^\dagger(\vec{k}_2) | 0 \rangle \\
 &= \langle \vec{k}_1 \vec{k}_2 | \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[\hat{a}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}_1) \hat{a}^\dagger(\vec{k}_2) \hat{a}^\dagger(\vec{k}_1) \right. \\
 &\quad \left. + (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_1) \hat{a}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}_2) \right] | 0 \rangle \\
 &= \langle \vec{k}_1 \vec{k}_2 | \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[\hat{a}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}_1) \hat{a}^\dagger(\vec{k}_2) \hat{a}^\dagger(\vec{k}_1) \right. \\
 &\quad \left. + (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_2) \hat{a}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}_1) \right. \\
 &\quad \left. + (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_1) \hat{a}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}_2) \right] | 0 \rangle \\
 &= \langle \vec{k}_1 \vec{k}_2 | \hat{a}^\dagger(\vec{k}_2) \hat{a}^\dagger(\vec{k}_1) | 0 \rangle + \langle \vec{k}_1 \vec{k}_2 | \hat{a}^\dagger(\vec{k}_1) \hat{a}^\dagger(\vec{k}_2) | 0 \rangle \\
 &= \langle \vec{k}_1 \vec{k}_2 | \vec{k}_1 \vec{k}_2 \rangle + \langle \vec{k}_1 \vec{k}_2 | \vec{k}_2 \vec{k}_1 \rangle \quad \begin{cases} \langle \vec{k}_1 \vec{k}_2 \rangle = \langle \vec{k}_2 \vec{k}_1 \rangle \\ \text{for } [\hat{a}(\vec{k}_1), \hat{a}^\dagger(\vec{k}_2)] = 0 \end{cases} \\
 &= \langle \vec{k}_1 \vec{k}_2 | \vec{k}_1 \vec{k}_2 \rangle + \langle \vec{k}_1 \vec{k}_2 | \vec{k}_1 \vec{k}_2 \rangle \quad \text{- i.e., for bosons} \\
 &= 2 \langle \vec{k}_1 \vec{k}_2 |
 \end{aligned}$$

What about the total energy?

$$\begin{aligned}
 & \langle \vec{k}_1 \vec{k}_2 | \hat{H} | \vec{k}_1 \vec{k}_2 \rangle \\
 = & \langle \vec{k}_1 \vec{k}_2 | \int \frac{\lambda^2 \vec{k}}{(2\pi)^3} E(\vec{k}) [\text{from page 11}] | 0 \rangle \\
 = & \langle \vec{k}_1 \vec{k}_2 | E(\vec{k}_2) c^+(\vec{k}_2) c^+(\vec{k}_1) | 0 \rangle \\
 + & \langle \vec{k}_1 \vec{k}_2 | E(\vec{k}_1) c^+(\vec{k}_1) c^+(\vec{k}_2) | 0 \rangle \\
 = & E(\vec{k}_2) \langle \vec{k}_1 \vec{k}_2 | \vec{k}_1 \vec{k}_2 \rangle + E(\vec{k}_1) \langle \vec{k}_1 \vec{k}_2 | \vec{k}_2 \vec{k}_1 \rangle \\
 = & [E(\vec{k}_2) + E(\vec{k}_1)] \langle \vec{k}_1 \vec{k}_2 | \vec{k}_1 \vec{k}_2 \rangle \\
 = & E(\vec{k}_1) + E(\vec{k}_2)
 \end{aligned}$$

as expected.

Note that the symmetry/antisymmetry associated with identical particles is built into the commutators.

For bosons:

$$|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$$

$$\langle x_1, x_2 | \vec{k}_1, \vec{k}_2 \rangle$$

$$= \frac{1}{\sqrt{2}} [\psi_{\vec{k}_1}(x_1) \psi_{\vec{k}_2}(x_2) + \psi_{\vec{k}_2}(x_2) \psi_{\vec{k}_1}(x_1)]$$

$$= \frac{1}{\sqrt{2}} [\psi_{\vec{k}_1}(x_1) \psi_{\vec{k}_2}(x_2) + \psi_{\vec{k}_2}(x_1) \psi_{\vec{k}_1}(x_2)]$$

$$= \langle x_1, x_2 | \vec{k}_2, \vec{k}_1 \rangle$$

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^+(\vec{k}_2) \hat{a}^+(\vec{k}_1) |0\rangle$$

$$= \psi(x_1, x_2)$$

$$= \psi(x_2, x_1)$$

$$|\vec{k}_2, \vec{k}_1\rangle = \hat{a}^+(\vec{k}_1) \hat{a}^+(\vec{k}_2) |0\rangle$$

Given that

$$[\hat{a}^+(\vec{k}_1), \hat{a}^+(\vec{k}_2)] = 0$$

for bosons, it is easy to see that

$$|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$$

For fermions, we will see that

$$\{\hat{a}^+(\vec{k}_1), \hat{a}^+(\vec{k}_2)\} = 0$$

which then gives

$$|\vec{k}_1, \vec{k}_2\rangle = - |\vec{k}_2, \vec{k}_1\rangle$$

$$\left\langle x_1, x_2 | \vec{k}_1, \vec{k}_2 \right\rangle$$

$$= \frac{1}{\sqrt{2}} \left[\psi_{\vec{k}_1}(x_1) \psi_{\vec{k}_2}(x_2) - \psi_{\vec{k}_1}(x_2) \psi_{\vec{k}_2}(x_1) \right]$$

$$= -\frac{1}{\sqrt{2}} \left[\psi_{\vec{k}_1}(x_2) \psi_{\vec{k}_2}(x_1) - \psi_{\vec{k}_1}(x_1) \psi_{\vec{k}_2}(x_2) \right]$$

$$= -\langle x_1, x_2 | \vec{k}_2, \vec{k}_1 \rangle$$

$$\psi(x_1, x_2) = -\psi(x_2, x_1)$$