Renormalization - In Words

In an interacting theory, the fundamental properties of a particle, such as its mass, will change. They are renormalized.

Renormalization is the process by which a particle dresses itself in interactions.

Familiar example from condensed matter physics:

Positive ions in a metal dress themselves with clouds of electrons around them. Distant electrons see a reduced positive charge.

While it is possible to remove an electron from a condensed matter system (e.g., a metal or crystal) in order to measure its properties, such as its mass and charge (as opposed to its effective mass and charge in medium), in vacuum, it is not possible to remove an electron from the QED vacuum to measure its "bare" mass, etc.

Thus far, we have been using the bare mass and bare coupling constants in writing down our Lagrangians, as well as the bare fields.

As we will see, this is the source of the infinities in QFT.

A theory is renormalizable if it has a finite number of divergent Feynman diagrams. Divergences appear at all orders of perturbation theory.

A finite number of renormalizations renders all the infinities finite at all orders of perturbation theory.

A theory is non-renormalizable if it has an infinite number of divergent Feynman diagrams.

If we put a test particle into an interacting system, it will pull particle-antiparticle pairs out of the vacuum, which themselves in turn will interact with the vacuum and other particles, etc., etc.

Q1: Can we even identify single particles in this system?

Q2: Will we have to abandon QFT, which is based on creating and annihilating single particles?

Q3: Are there single-particle excitations in such a system?

A1: Yes.

A2: No.

A3: Yes – specifically, dressed particles or quasiparticles

Quasiparticle ≡ the excitations in interacting systems, resembling free particles but having different masses and interactions

$$\text{FREE PARTICLES} \xrightarrow[\text{RENORMALIZATION}]{} \text{QUASIPARTICLES}$$

Two Important Scalar Field Theories

Scalar Yukawa Theory

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - M^2 \psi^\dagger \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g \psi^\dagger \psi \phi$$

This serves as a full-scalar model of the strong interactions between nucleons via pion exchange.

$\lambda \phi^4$ Theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

This Lagrangian will play a crucial role in Electroweak Theory in describing the Higgs field, spontaneous symmetry breaking, and the origin of mass for all particles in nature.

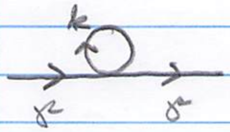
$\lambda \phi^4$ Theory is the canonical theory used to illustrate the issues of renormalization and will be the model we use.

Renormalization of $\lambda\phi^4$ Theory

$$\hat{\mathcal{L}} = \frac{1}{2}\partial^\mu\hat{\phi}\,\partial_\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2 - \frac{1}{4!}\lambda\hat{\phi}^4$$

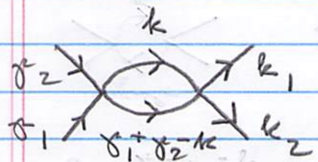
The infinities arise when we have loops. Consider all of the divergent one-loop Feynman diagrams: arise in the conditions we just discussed

(1)



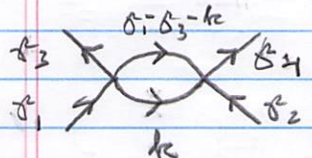
$$\mathcal{M} = -i\lambda\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}$$

(2a) "s-channel"



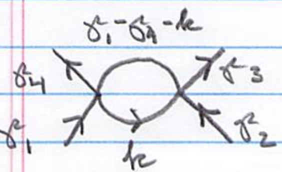
$$\mathcal{M}_s = \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(\underbrace{p_1+p_2}_{\equiv\sqrt{s}}-k)^2-m^2+i\epsilon}$$

(2b) "t-channel"



$$\mathcal{M}_t = \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(\underbrace{p_1-p_3}_{\equiv\sqrt{t}}-k)^2-m^2+i\epsilon}$$

(2c) "u-channel"



$$\mathcal{M}_u = \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(\underbrace{p_1-p_4}_{\equiv\sqrt{u}}-k)^2-m^2+i\epsilon}$$

Let's evaluate $\mathcal{M}_s$:

We will introduce Feynman's formula

$$\frac{1}{xy} = \int_0^1 d\alpha\,\frac{1}{[\alpha x+(1-\alpha)y]^2}$$

From

$x = k^2 - m^2$

$y = (p-k)^2 - m^2$

Then

$$\eta_{s} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \, \frac{1}{\{\alpha(k^2-m^2) + (1-\alpha)[(p-k)^2 - m^2]\}^2}$$

$$\alpha k^2 - \cancel{\alpha m^2} + (p-k)^2 - m^2 - \alpha(p-k)^2 + \cancel{\alpha m^2}$$

$$\alpha \cancel{k^2} + (p-k)^2 - m^2 - \alpha p^2 + 2\alpha p k - \cancel{\alpha k^2}$$

$$(k-p)^2 - m^2 - \alpha p^2 + 2\alpha p k$$

$$k^2 - 2pk + p^2 - m^2 - \alpha p^2 + 2\alpha p k$$

$$k^2 - 2(1-\alpha)pk + (1-\alpha)p^2 - m^2$$

$$[k - (1-\alpha)p]^2 \underbrace{- (1-\alpha)^2 p^2 + (1-\alpha)p^2 - m^2}$$

$$- (1 - 2\alpha + \alpha^2)p^2$$
$$+ (1-\alpha)p^2$$
$$- m^2$$

$$= -p^2 + 2\alpha p^2 - \alpha^2 p^2$$
$$+ p^2 - \alpha p^2$$
$$- m^2$$

$$= \alpha p^2 - \alpha^2 p^2 - m^2$$

$$= \underbrace{\alpha(1-\alpha)p^2 - m^2}_{= -M^2}$$

So

$$\mathcal{M}_s = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \, \frac{1}{\{[k-(1-\alpha)s]^2 - M^2\}^2}$$

Let

$$k' \equiv k - (1-\alpha)s$$

Then

$$\mathcal{M}_s = \frac{(-i\lambda)^2}{2} \int \frac{d^4k'}{(2\pi)^4} \int_0^1 d\alpha \, \frac{1}{[(k')^2 - M^2]^2}$$

Now, drop the prime on k and write

$$d^4k = d^3k \, dk^0$$

Perform a Wick Rotation

$$k^0 \longrightarrow ik^0 \equiv k^0_E \implies dk^0_E = i\,dk^0 \text{ and } dk^0 = -i\,dk^0_E$$

and

$$k^2 = (k^0)^2 - \vec{k}^2 \longrightarrow (-ik^0_E)^2 - \vec{k}_E^2 = -[(k^0_E)^2 + \vec{k}_E^2] \equiv -k_E^2$$

Then

We have introduced a cutoff.

$$\mathcal{M}_s \to -i\frac{(-i\lambda)^2}{2} \frac{1}{(2\pi)^4} \underbrace{\int d\Omega_4}_{2\pi^2} \int^{\Lambda} k_E^3 \, dk_E \int_0^1 d\alpha \, \frac{1}{(k_E^2 + M^2)^2}$$

$$= \frac{i\lambda^2}{2}\frac{1}{8\pi^2}\underbrace{\int^{\Lambda} k_E^3\, dk_E}_{\frac{1}{2}\int^{\Lambda^2} k_E^2\, d(k_E^2)} \int_0^1 d\alpha \frac{1}{(k_E^2+M^2)^2}$$

$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \underbrace{\int^{\Lambda^2} d(k_E^2)\frac{k_E^2}{(k_E^2+M^2)^2}}_{1+\ln\frac{\Lambda^2}{M^2}+\mathcal{O}\left(\frac{M^2}{\Lambda^2}\right)}$$

$$M^2 \equiv m^2 - \alpha(1-\alpha)s^2$$

$$\underset{\Lambda^2 \gg M^2}{\approx} \frac{i\lambda^2}{32\pi^2}\ln\Lambda^2$$

$$\xrightarrow[\Lambda\to\infty]{} \infty \text{ as } \ln\Lambda$$

This is a logarithmic $\underline{\underline{\text{UV}}}$ divergence (i.e., high-energy divergence)

$\mathcal{M}_{t,u}$ have the same large-Λ term.

Then

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u = \frac{3i\lambda^2}{32\pi^2}\ln\Lambda$$

To eliminate this UV divergence, a counter term is added to the Lagrangian density

$$\mathcal{L} \longrightarrow \mathcal{L} - \frac{1}{4!}\frac{3\lambda^2}{32\pi^2}\ln\Lambda\,\hat{\phi}^4 \equiv \mathcal{L} + \frac{1}{4!}C^{(2)}\hat{\phi}^4$$

we'll need to calculate $C^{(n)}$ for each (n^{th}) order in perturbation theory

The S-Matrix operator then becomes

$$\hat{S} = T\left\{ e^{-i\int d^4x \frac{1}{4!}\left(\lambda\hat{\phi}^4 - C\hat{\phi}^4\right)} \right\}$$

We redo the calculation to second order in λ and the counter term leads to a finite result for diagrams (2a) − (2c).

We will discuss the physical interpretation of the counter term later.

Now we have to deal with the one-loop diagram (1). The UV divergence from this graph is even more severe, diverging as Λ^2 rather than $\ln\Lambda$.

$$-i\lambda\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \xrightarrow[k^0\to ik^0\equiv k^0_E]{} -i\lambda\frac{1}{(2\pi)^4}\underbrace{\int d\Omega_4}_{2\pi^2}\underbrace{\int^\Lambda k_E^3\,dk_E\frac{1}{k_E^2+m^2}}_{\frac{1}{2}\int^\Lambda d(k_E^2)\frac{k_E^2}{k_E^2+m^2}}$$

$$= \frac{-i\lambda}{16\pi^2}\frac{1}{2}\left(\Lambda^2 - m^2\ln\frac{\Lambda^2+m^2}{m^2}\right)$$

$$\equiv -iB^{(1)}$$

To see the physical meaning of this correction from one loop at first order, consider the entire class of such diagrams at all orders

This corresponds to the series

$$\frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}\left(-iB^{(1)}\right)\frac{i}{k^2-m^2}$$

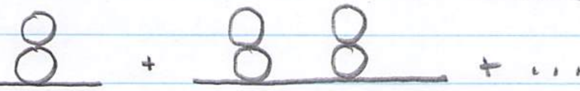
$$+ \frac{i}{k^2-m^2}\left(-iB^{(1)}\right)\frac{i}{k^2-m^2}\left(-iB^{(1)}\right)\frac{i}{k^2-m^2}$$

$$+ \dots$$

$$= \frac{i}{k^2-m^2}\left(1 + \left(-iB^{(1)}\right)\frac{i}{k^2-m^2} + \left(-iB^{(1)}\right)\frac{i}{k^2-m^2}\left(-iB^{(1)}\right)\frac{i}{k^2-m^2} + \dots\right)$$

$$= \frac{i}{k^2-m^2}\,\frac{1}{1 + \frac{i(iB^{(1)})}{k^2-m^2}}$$

$$= \frac{i}{k^2-m^2-B^{(1)}}$$



$B^{(2)}$

$$\frac{i}{k^2-m^2-B^{(1)}} \rightarrow \frac{i}{k^2-m^2-\left(B^{(1)}+B^{(2)}\right)}$$

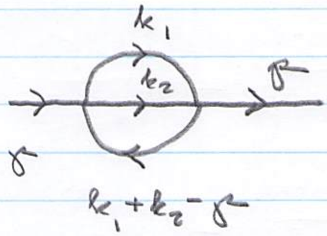
$$\dots \quad i/k^2-m^2-\left(\sum_n B^{(n)}\right)$$

That is, this class of diagrams gives rise to a correction of the bare mass, m, in the Lagrangian, giving rise to the physical mass, m_P, of the particle.

There is one other class of UV divergences that arise in $\lambda\phi^4$ theory, but they arise at two loops.

The class of two-loop diagrams is

Consider the single "oyster" diagram



$$= \frac{i\lambda^2}{6}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{1}{(p-k_1-k_2)^2-m^2+i\epsilon}\;\frac{1}{k_1^2-m^2+i\epsilon}\;\frac{1}{k_2^2-m^2+i\epsilon}$$

$$= \lambda^2\left(C_1\ln\frac{\Lambda^2}{p^2}+C_2\right)p^2 \equiv A^{(2)}p^2$$

The full sum of all the diagrams in the class corresponds to:

$$\frac{i}{p^2-m^2}+\frac{i}{p^2-m^2}\,iA^{(2)}p^2\,\frac{i}{p^2-m^2}+\frac{i}{p^2-m^2}\,iA^{(2)}p^2\,\frac{i}{p^2-m^2}\,iA^{(2)}p^2\,\frac{i}{p^2-m^2}+\ldots$$

$$= \frac{i}{p^2-m^2}\left(1+iA^{(2)}p^2\frac{i}{p^2-m^2}+iA^{(2)}p^2\frac{i}{p^2-m^2}\,iA^{(2)}p^2\frac{i}{p^2-m^2}+\ldots\right)$$

$$= \frac{i}{p^2-m^2}\;\frac{1}{1-\dfrac{iA^{(2)}p^2 i}{p^2-m^2}}$$

$$= \frac{i}{p^2-m^2}\;\frac{1}{1+\dfrac{A^{(2)}p^2}{p^2-m^2}}$$

$$= \frac{1}{(1+A^{(2)})p^2-m^2}$$

Combining the one- and two-loop corrections to the propagator, we get

$$\frac{i}{p^2 - m^2} \longrightarrow \frac{i}{(1+A^{(2)})p^2 - m^2 - B^{(1)}}$$

with

$$A^{(2)} = \lambda^2 \left(c_1 \ln \frac{\Lambda^2}{p^2} + c_2 \right), \quad B^{(1)} = \frac{\lambda}{16\pi^2} \frac{1}{2} \left(\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2} \right)$$

Now let's consider the physical interpretation of the factors $A^{(2)}$ and $B^{(1)}$:

Clearly $B^{(1)}$ is a mass correction

$$m_p^2 = m^2 + B^{(1)}$$

When all corrections are included, not just $B^{(1)}$, we have m_p^2. This equation is just illustrative.

$$= m^2 - \frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2 + m^2}{m^2} \right) \right]$$

where m_p is the physical mass of the particle – i.e., where the propagator has its pole:

$$\frac{i}{p^2 - m_p^2 + i\epsilon}$$

Note, then, that we can invert the above relationship between m_p and m, Λ to express m as a function of Λ

$$m = m(\Lambda) \xrightarrow[\Lambda \to \infty]{} \infty \qquad m^2(\Lambda) \underset{\Lambda \to \infty}{\sim} \frac{\lambda}{32\pi^2} \Lambda^2$$

m, the "bare" mass we have been using, is a freely specifiable parameter. As $\Lambda \to \infty$ (corresponding to $k \to \infty$), we can adjust $m(\Lambda)$ such that the physical mass, m_p, remains constant. The physical mass is of course the measured mass of the particle.

Considering now the fully corrected propagator

$$\frac{i}{(1+A)p^2 - m^2 - B}$$

We keep the pole at $p^2 = m_p^2$. That is,

$$\left[1 + A(\Lambda, p^2)\right]p^2 - m^2(\Lambda) - B(\Lambda)\Big|_{p^2 = m_p^2} \equiv 0$$

Expand about $p^2 = m_p^2$

$$\begin{aligned}\left[1 + A(\Lambda, p^2)\right]p^2 - m_p^2 &= \left[1 + A(\Lambda, m_p^2)\right]m_p^2 - m_p^2 \\ &\quad + \frac{d}{d(p^2)}\left[A(\Lambda, p^2)p^2 - m_p^2\right]\Big|_{p^2 = m_p^2}(p^2 - m_p^2) \\ &= A(\Lambda, m_p^2)m_p^2 + Z^{-1}(p^2 - m_p^2)\end{aligned}$$

where

$$Z^{-1} \equiv \frac{d}{d(p^2)}\left[A(\Lambda, p^2)p^2\right]\Big|_{p^2 = m_p^2}$$

That is,

near the pole

$$\frac{i}{A p^2 - m^2 - B} = \frac{iZ}{p^2 - m_p^2}$$

Includes all corrections, not just $A^{(2)}$ corrections.

Includes all corrections, not just $B^{(1)}$.

We know that the physical field, $\phi_p(x)$, would have the physical propagator defined as

$$\int d^4x\, e^{ip\cdot x} \langle 0 | T\{\hat{\phi}_p(x)\hat{\phi}_p(y)\} | 0 \rangle = \frac{i}{p^2 - m_p^2}$$

Then, we can consider bare fields related to the physical fields via

$$\hat{\phi}(x,\Lambda) = \sqrt{Z}\,\hat{\phi}_p(x) \qquad \text{recall that } Z = Z(\Lambda)$$

Then

$$\int d^4x\, e^{ip\cdot x} \langle 0 | T\{\hat{\phi}(x,\Lambda)\hat{\phi}(y,\Lambda)\} | 0 \rangle = \frac{iZ}{p^2 - m_p^2}$$

Thus, A corresponds to a renormalization of the field.

10a >

Last, but not least, obviously C corresponds to a renormalization of the bare coupling constant

$$\lambda_p = \lambda - C(\Lambda)$$

which gives $\lambda = \lambda(\Lambda)$.

Consider the free-field generator

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}\left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^{\dagger}(\vec{k})\, e^{ik\cdot x}\right)$$

Consider its action on the vacuum of the interacting theory, $|\Omega\rangle$:

$$\hat{a}^{\dagger}(\vec{k})\,|\Omega\rangle = \underbrace{\langle \vec{k}_\lambda|\, \hat{a}^{\dagger}(\vec{k})\,|\Omega\rangle}\ \underbrace{|\vec{k}_\lambda\rangle} + \text{multiparticle states}$$

amplitude to create a single particle state

$\equiv \sqrt{Z}$

single-particle state

λ indicates we are in the interacting theory

$E(\vec{k}_\lambda) = \left[|\vec{k}_\lambda|^2 + m_\lambda^2\right]^{1/2}$

physical mass

$\equiv m_p$

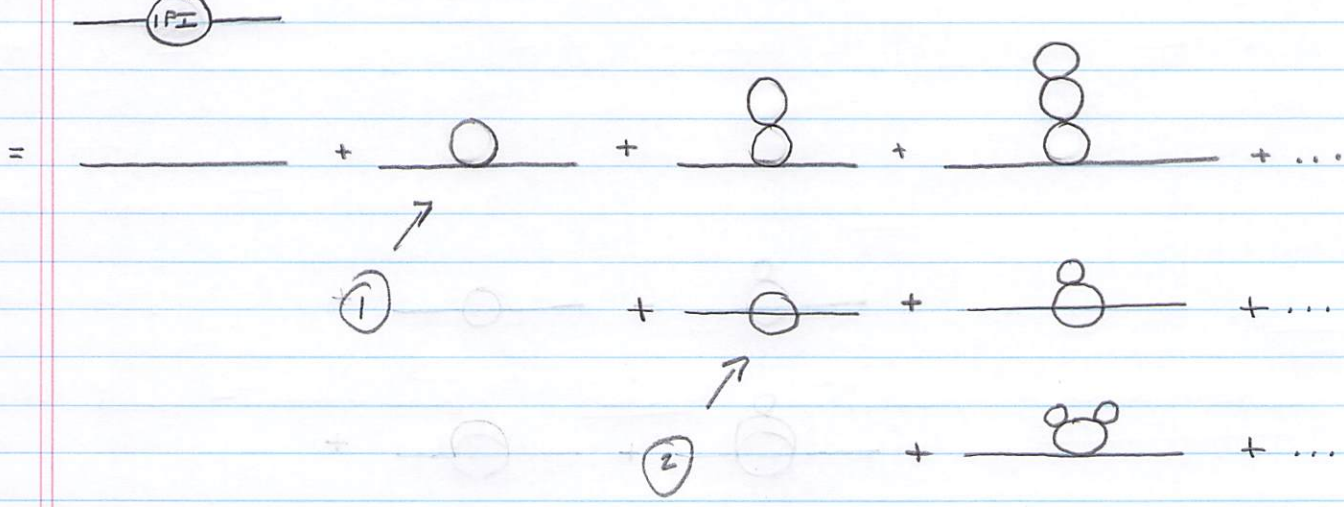
this will yield a renormalization of the field

$$\hat{\phi} \longrightarrow \hat{\phi}_p = \frac{1}{\sqrt{Z}}\,\hat{\phi}$$

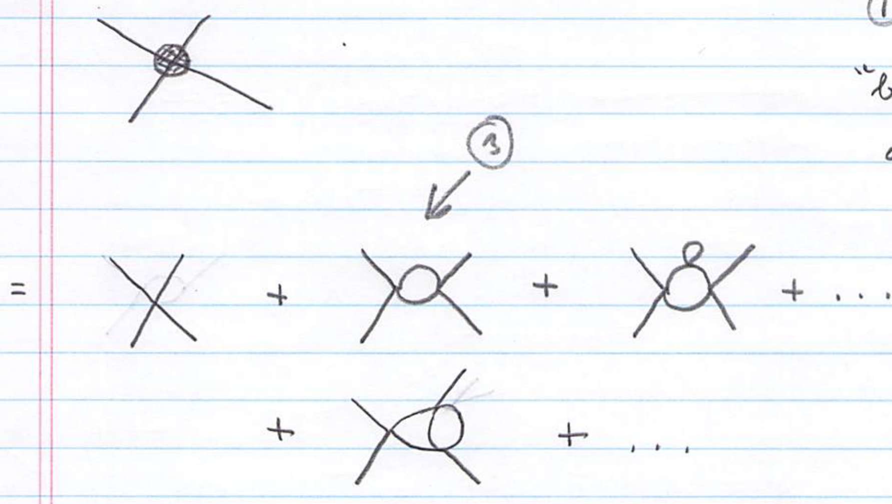
to scale the field in such a way that we can discuss the creation and annihilation of single particles with it

Given these three types of corrections, let's look at arbitrary order in perturbation theory

EXTERNAL LEGS

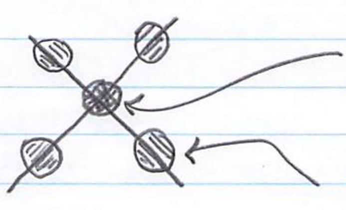


VERTEX



1, 2, and 3 are the "basis" diagrams we just considered.

So, in general we have



Vertex Correction (associated with λ)

Self-Energy Correction (associated with m and $\hat{\phi}$)

 $= -i\tilde{\Sigma}(p^2)$

 $=$ 

$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2}$$

$$+ \frac{i}{p^2-m^2}\left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2}\left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2}$$

$$+ \ldots$$

$$= \frac{i}{p^2-m^2}\Big[1 + \left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2}$$

$$+ \left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2}\left(-i\tilde{\Sigma}(p^2)\right)\frac{i}{p^2-m^2} + \ldots\Big]$$

$$= \frac{i}{p^2-m^2}\left[\frac{1}{1-\frac{\tilde{\Sigma}(p^2)}{p^2-m^2}}\right]$$

$$= \frac{i}{p^2-m^2-\tilde{\Sigma}(p^2)+i\epsilon}$$

Then m_p is defined by the position of the pole

$$p^2 - m^2 - \mathrm{Re}\left(\tilde{\Sigma}(p^2)\right)\Big|_{m=m_p} = 0$$

That is

$$m_p^2 = m^2 + \mathrm{Re}\left(\tilde{\Sigma}(p^2 = m_p^2)\right)$$

As in the case of adding a counter term with coefficient $C(\Lambda)$ to the Lagrangian density, there are counter terms involving $A(\Lambda)$ and $B(\Lambda)$:

Begin with the original Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\,\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2 - \frac{1}{4!}\lambda\hat{\phi}^4$$

$$= \frac{Z}{2}\partial_\mu\hat{\phi}_p\,\partial^\mu\hat{\phi}_p - \frac{Zm^2}{2}\hat{\phi}_p^2 - \frac{1}{4!}Z^2\lambda\hat{\phi}_p^4$$

Then

$$Z m^2 = m_p^2 + \delta m^2 \qquad -B = \delta m^2$$

$$Z^2\lambda = \lambda_p + \delta\lambda \qquad -C = \delta\lambda$$

$$Z = 1 + \delta Z \qquad A = \delta Z$$

and

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}_p\,\partial^\mu\hat{\phi}_p - \frac{1}{2}m_p^2\hat{\phi}_p^2 - \frac{1}{4!}\lambda_p\hat{\phi}_p^4$$

$$+ \frac{\delta Z}{2}\partial_\mu\hat{\phi}_p\,\partial^\mu\hat{\phi}_p - \frac{1}{2}\delta m^2\hat{\phi}_p^2 - \frac{1}{4!}\delta\lambda\,\hat{\phi}_p^4$$

A renormalizable theory has a finite number of counter terms that cancel <u>all</u> divergences at <u>all</u> orders in perturbation theory.

Given the structure of the counter terms, the structure of the EOM for $\hat{\phi}$ does not change, just the parameters (m, λ) in $\dot{\pi}$.