## Causality

$$[\hat{O}_1(x), \hat{O}_2(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \qquad c^2(x_0-y_0)^2 - (x_1-y_1)^2 - \dots < 0$$

Measurements associated with operators $\hat{O}_{1,2}$ will commute for spacelike separations in spacetime.

Consider (for a real scalar field the fields are observable)

$$\Delta(x,y) \equiv [\hat{\phi}(x), \hat{\phi}(y)]$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{q})}}\frac{1}{\sqrt{2E(\vec{k})}}\left[e^{ik_\mu x^\mu}\hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu}\hat{a}(\vec{k}),\ e^{iq_\mu y^\mu}\hat{a}^\dagger(\vec{q}) + e^{-iq_\mu x^\mu}\hat{a}(\vec{q})\right]$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{q})}}\frac{1}{\sqrt{2E(\vec{k})}}\left\{e^{i(k_\mu x^\mu + q_\mu y^\mu)}[\hat{a}^\dagger(\vec{k})\,\hat{a}^\dagger(\vec{q})] + e^{i(k_\mu x^\mu - q_\mu y^\mu)}[\hat{a}^\dagger(\vec{k}), \hat{a}(\vec{q})]\right.$$

$$\left. + e^{-i(k_\mu x^\mu - q_\mu y^\mu)}[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{q})] + e^{-i(k_\mu x^\mu + q_\mu x^\mu)}[\hat{a}(\vec{k}), \hat{a}(\vec{q})]\right\}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})}\left\{-e^{ik_\mu(x^\mu - y^\mu)} + e^{-ik_\mu(x^\mu-y^\mu)}\right\}$$

$$\equiv D(x-y) - D(y-x)$$

where

$$D(x-y) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})}e^{-ik\cdot(x-y)}$$

is the <u>propagator</u>.

$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 < 0$

$\underbrace{\Delta t}_{x^0-y^0}$ $\underbrace{\Delta x}_{x^1-y^1}$ …

spatial distance is larger than $c\Delta t$

Proof that $\Delta(x-y) = 0$ for $(x-y)^2 < 0$

$$\Delta(x-y) = D(x-y) - D(y-x)$$

Let's compute $D(x-y)$ for $(x-y)^2 < 0$ – specifically for equal time (i.e., $x^0 = y^0$):

$$(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$$

Recall

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\vec{k})} e^{-ik_\mu(x^\mu - y^\mu)}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\vec{k})} e^{i\vec{k}\cdot\vec{r}}$$

where $\vec{r} = \vec{x} - \vec{y}$. Choosing spherical $k$ coordinates

$$D(x-y) = \frac{1}{(2\pi)^3} \int k^2 dk \sin\theta_k \, d\theta_k \, d\phi_k \frac{1}{2E(\vec{k})} e^{ikr\cos\theta_k}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^2} \int_0^\infty \int_{-1}^{+1} k^2 dk \, d\cos\theta_k \frac{1}{E(\vec{k})} e^{ikr\cos\theta_k}$$

$$= \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_0^\infty dk \frac{k^2}{[k^2+m^2]^{1/2}} \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$= \frac{-i}{2(2\pi)^2 r} \left\{ \int_0^\infty dk \frac{k}{[k^2+m^2]^{1/2}} e^{ikr} - \int_0^{-\infty} dk \frac{k}{[k^2+m^2]^{1/2}} e^{-ikr} \right\}$$

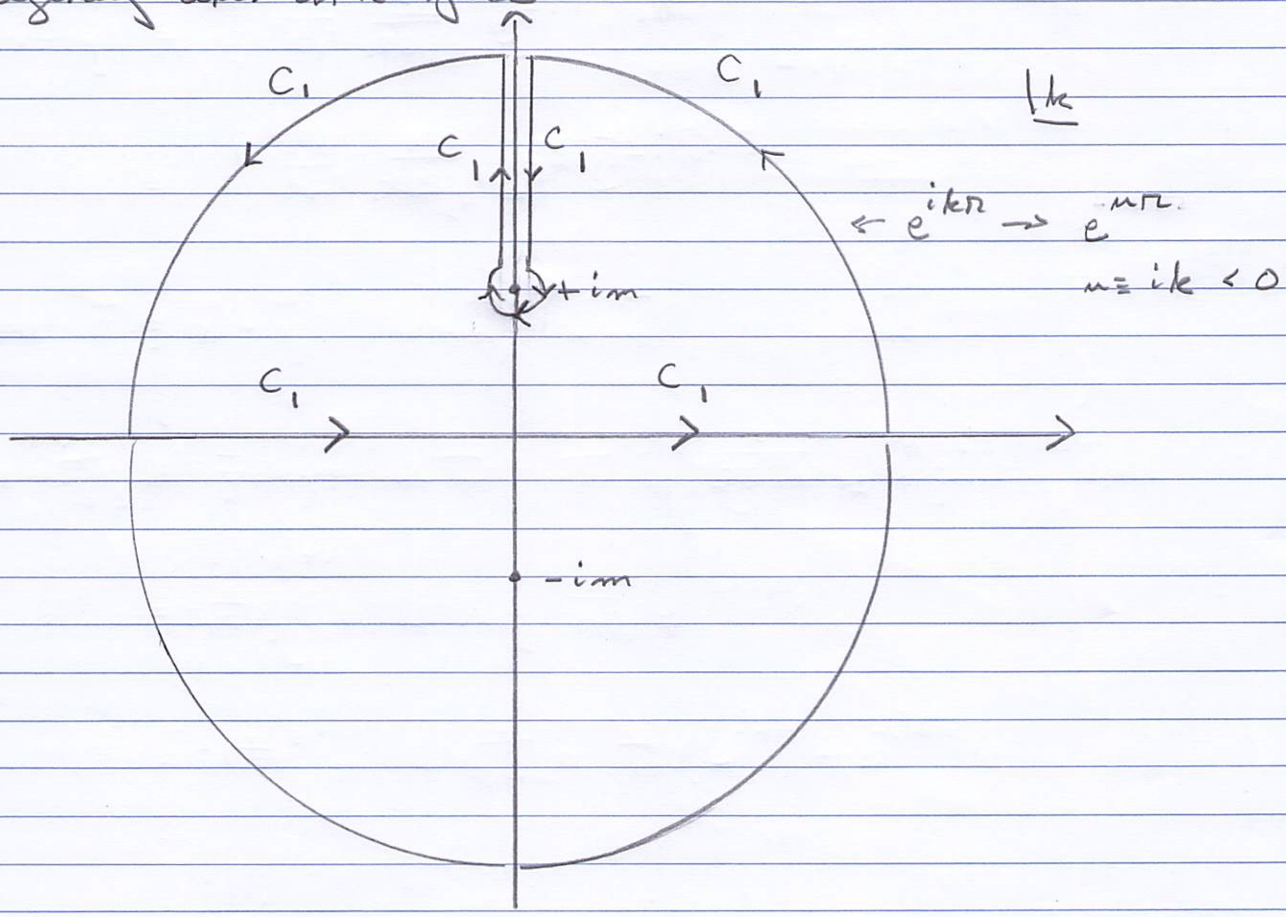
$$\int_0^{\infty} dk \frac{k}{k^2+m^2} e^{-ikr} = \int_0^{-\infty} d(-k) \frac{-k}{[(-k)^2+m^2]} e^{i(-k)r}$$

$$= -\int_{-\infty}^{0} dk \frac{k}{k^2+m^2} e^{ikr}$$

Then

$$D(x-y) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dk \frac{k}{[k^2+m^2]^{1/2}} e^{ikr}$$

This integral possesses two cuts along the upper and lower imaginary axis in k-space




Because of the sign of the exponential ($e^{ikr}$), we use the upper contour to evaluate the integral (the lower contour will be

## Branch Points / Branch Cuts

We are working in the complex plane and we are working with the function $\sqrt{z}$.

At $z = 0$, $\sqrt{z}$ is singular (it does not have a derivative).

$\Rightarrow$ $\sqrt{z}$ will be discontinuous across contours that encircle $z=0$.

Consider the unit circle around $z = 0$ in the complex $z$ plane.

In general, $z$ can be written as

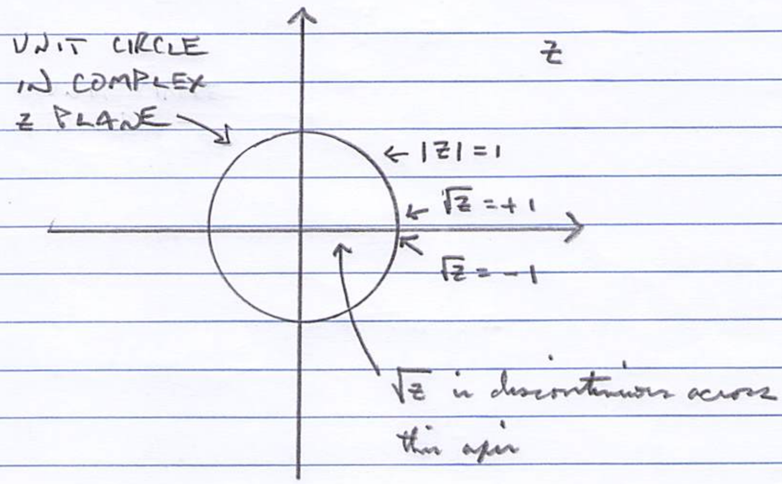
$$z = a + ib$$

or

$$= A e^{i\theta} \qquad a = A\cos\theta \qquad b = A\sin\theta$$

On the unit circle, $A = 1$ and

$$\sqrt{z} = e^{i\theta/2}$$



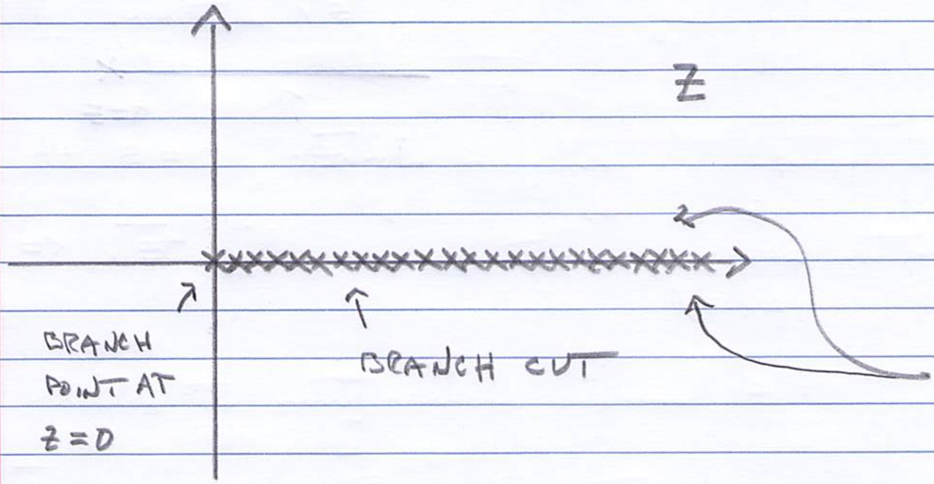

At $\theta = 0$,

$$\sqrt{z} = +1$$

At $\theta = 2\pi$

$$\sqrt{z} = -1$$

So, $\sqrt{z}$ is discontinuous across the $x > 0$ line. $z = 0$ is known as a branch point, and we introduce a branch cut to prevent our contour from crossing it.




The values of $z$ differ by $e^{i\pi} = -1$ on either side of the branch cut!

$$\sqrt{z}\Big|_{\theta = 2\pi} = -\sqrt{z}\Big|_{\theta = 0}$$

used for $\Delta(y-x)$.

Cauchy's Theorem

If $f(z)$ is analytic on and inside $C$

$$\oint_C f(z)\, dz = 0$$

Applying Cauchy's Theorem to our case, this tells us that the integral along the real $k$ axis must be opposite of the integral along the branch cut (the integral along the semicircle vanishes by virtue of the exponential).

The integral along the branch cut is

$$\int_{i\infty}^{im} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr} - \int_{im}^{i\infty} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr}$$

when passing from the right side of the branch cut to the left side of the branch cut, we pick up a factor of $e^{i\pi} = -1$

Transform coordinates

$$\mu \equiv i k$$

$$\int_{i\infty}^{im} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr} \longrightarrow -i \int_{-\infty}^{-m} d\mu \frac{-i\mu}{\sqrt{-\mu^2+m^2}} e^{\mu r}$$

$$= i \int_{-\infty}^{-m} d\mu \frac{\mu}{\sqrt{\mu^2-m^2}} e^{\mu r}$$

Transform coordinates again

$$y \equiv -\mu$$

Then

$$i\int_{-\infty}^{-m} d\mu \, \frac{\mu}{\sqrt{\mu^2-m^2}} \, e^{\mu r} \underset{y=-\mu}{=} -i\int_{\infty}^{m} dy \, \frac{-y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

$$= -i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

This leaves us with

$$\int_{i\infty}^{im} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} = -i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

Similarly

$$\int_{im}^{i\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} \underset{\mu \equiv ik}{=} -i\int_{-m}^{-\infty} d\mu \, \frac{-i\mu}{\sqrt{-\mu^2+m^2}} \, e^{\mu r}$$

$$= +i\int_{-m}^{-\infty} d\mu \, \frac{\mu}{\sqrt{\mu^2-m^2}} \, e^{\mu r}$$

$$\underset{y=-\mu}{=} +i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

Pulling it all together, we have

$$\Delta(x-y) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr}$$

$$= \frac{-i}{2(2\pi)^2 r} (-1)$$

$$\times \left\{ \int_{i\infty}^{im} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} - \int_{im}^{i\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} \right\}$$

$$= \frac{-i}{2(2\pi)^2 r} (-1)$$

$$\times \left\{ -i \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr} - i \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr} \right\}$$

$$= \frac{1}{4\pi^2 r} \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

$$= \frac{1}{4\pi^2} \frac{m}{r} K_1(mr)$$

↖ modified Bessel Function of the first kind

$$\xrightarrow[r\to\infty]{} \frac{m}{r} e^{-mr}$$

Performing the calculation for $D(y-x)$ yields the same answer, so

$$\Delta(x,y) = D(x-y) - D(y-x) = 0$$

for this case - i.e., equal times, $x^0 = y^0$. Lorentz invariance generalizes this to all $(x, y) \ni (x-y)^2 < 0$.

Now look at

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

$$= \langle 0|\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\left[e^{ik_\mu x^\mu}\hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu}\hat{a}(\vec{k})\right]$$

$$\times\int\frac{d^3g}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}\left[e^{ig_\mu y^\mu}\hat{a}^\dagger(\vec{g}) + e^{-ig_\mu y^\mu}\hat{a}(\vec{g})\right]|0\rangle$$

$$= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\int\frac{d^3\vec{g}}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}\, e^{-ik_\mu x^\mu}e^{-ig_\mu y^\mu}\underbrace{\langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{g})|0\rangle}_{(2\pi)^3\delta^{(3)}(\vec{k}-\vec{g})}$$

$$= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E(\vec{k})}\, e^{-ik_\mu(x^\mu - y^\mu)}$$

$$= D(x-y)$$

= AMPLITUDE FOR A PARTICLE TO BE CREATED AT $y$, PROPAGATE TO $x$, AND BE ANNIHILATED AT $x$

Feynman Propagator

Consider the time-ordered product

$$T\{\hat{\phi}(x)\hat{\phi}(y)\} \equiv \begin{cases}\hat{\phi}(x)\hat{\phi}(y) & x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x) & x^0 < y^0\end{cases}$$

The Feynman propagator is defined as

$$D_F(x-y) = \langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle$$

$D_F(x-y)$ can be written as

$$D_F(x-y) = i \int_C \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik_\mu (x^\mu - y^\mu)}$$

PROOF The denominator in the integrand can be expressed as

$$k^2 - m^2 + i\epsilon$$

$$= k_\mu k^\mu - m^2 + i\epsilon$$

$$= \eta_{00}(k^0)^2 + \eta_{ii}(k^i)^2 - m^2 + i\epsilon$$

$$= (k^0)^2 - (k^i)^2 - m^2 + i\epsilon$$

$$= (k^0)^2 - \left[(k^i)^2 + m^2\right] + i\epsilon$$

$$= (k^0)^2 - E_k^2 + i\epsilon$$

$$= \left[k^0 - (E_k - i\epsilon)\right]\left[k^0 - (-E_k + i\epsilon)\right]$$

N.B.

$$\left[k^0 - (E_k - i\epsilon)\right]\left[k^0 - (-E_k + i\epsilon)\right]$$

$$= (k^0)^2 - E_k^2 + 2i\epsilon E_k + \mathcal{O}(\epsilon^2)$$

$$\equiv (k^0)^2 - E_k^2 + i\epsilon'$$

where

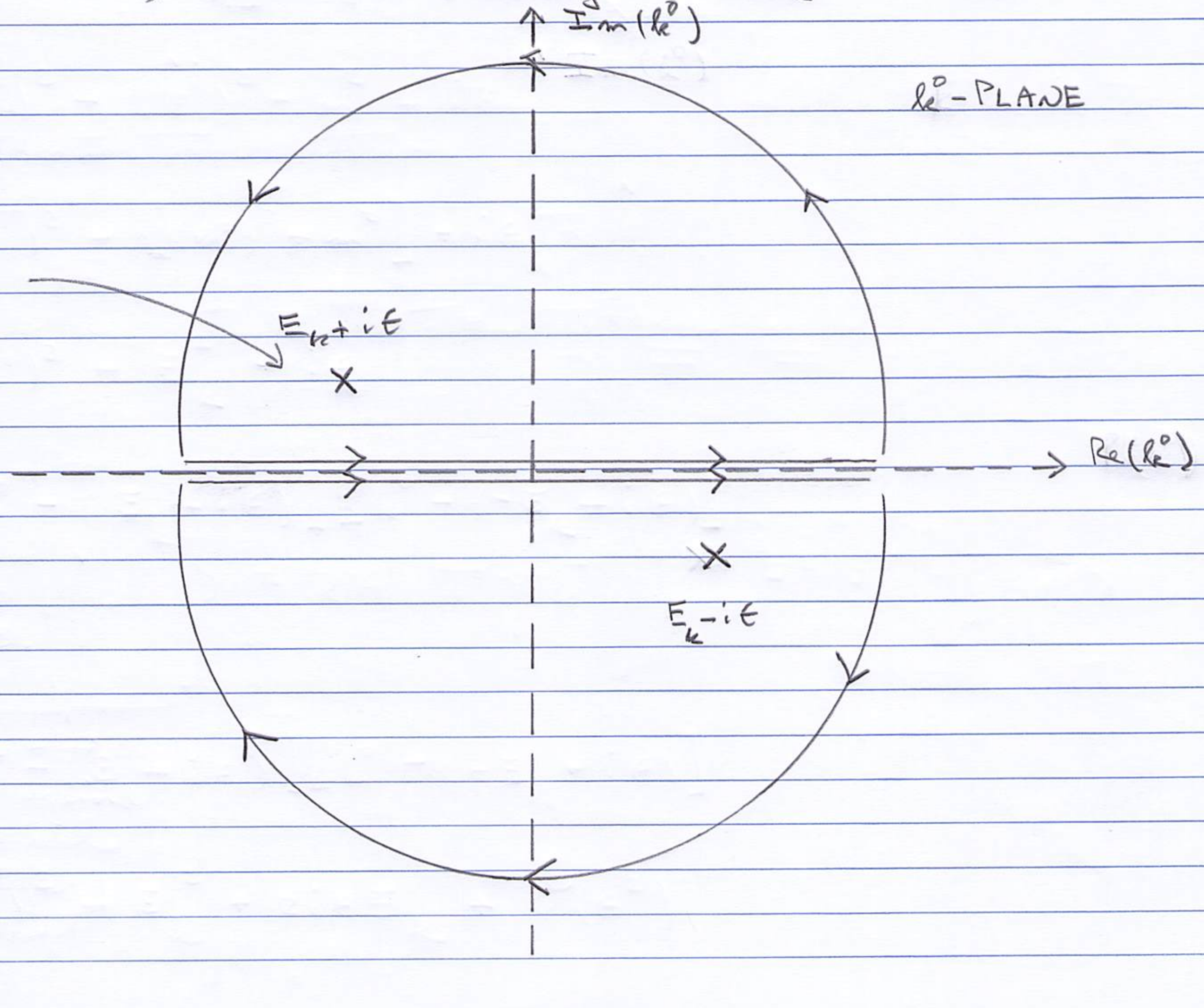
$$\epsilon' \equiv 2 i E_k \epsilon$$

Since we are going to take the limit $\epsilon \longrightarrow 0$, the distinction between $\epsilon'$ and $\epsilon$ can't matter in the final result.

Now perform the integral over $k^0$.

To do so, we'll use Cauchy's Integral Formula.

We have 2 poles in our integrand.



# Cauchy's Integral Formula

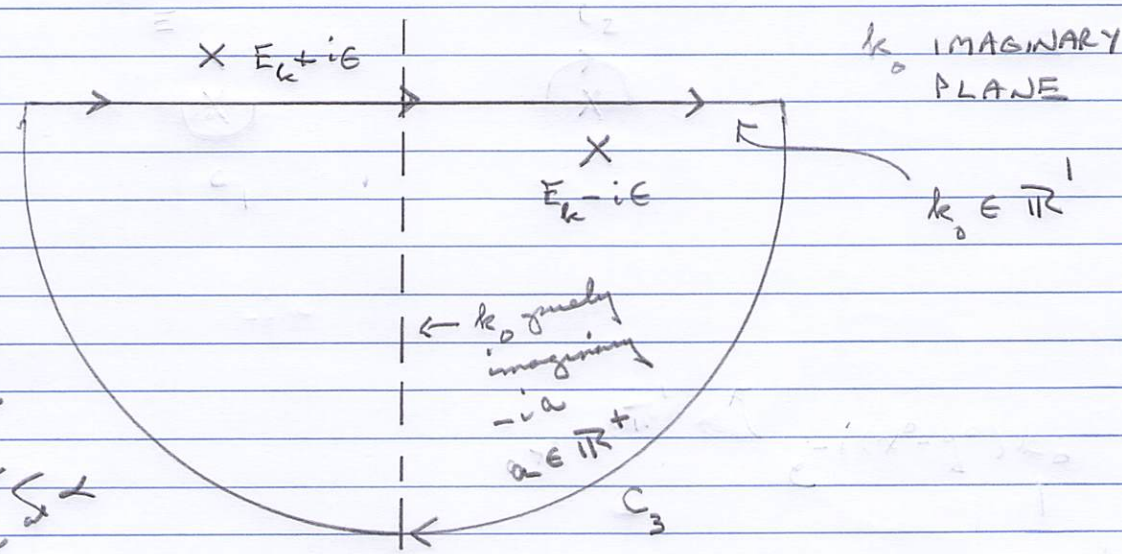
This is an excellent approach to evaluating real integrals that have singularities in the integrand.

**Example**

N.B. $k^0 = k_0$

$$\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{1}{(k^0 - E_k)(k^0 + E_k)} e^{-ik_0(x^0 - y^0)}$$

$x^0 - y^0 > 0$




closed contour $C$ that encloses the pole at $\alpha$

$$\oint \frac{f(z)}{z-\alpha}\, dz = \pm 2\pi i\, f(\alpha)$$

$+$: counterclockwise around $C$; $-$: clockwise around $C$

$$\Rightarrow \underbrace{\int_{-\infty}^{+\infty} \frac{f(x)}{x-\alpha}\, dx}_{} + \underbrace{\int_{C_3} \frac{f(z)}{z-\alpha}\, dz}_{} = \pm 2\pi i\, f(\alpha)$$

as $\epsilon \to 0$ we regain the real integral of interest

$\to 0$ as the $C_3$ contour is extended to $\infty$ in order to cover the entire real line

For $x^0 > y^0$, the contour must extend into the lower half plane since

$$\lim_{k_0 \to -i\infty} e^{-ik_0(x^0-y^0)} = 0$$

whereas

$$\lim_{k_0 \to +i\infty} e^{-ik_0(x^0-y^0)} = \infty$$

As the radius of the $C_3$ contour is extended to infinity, the integral along $C_3$ goes to 0 and does not contribute, and we are left with an evaluation of the integral along the real line (real $k^0$), which is the only part of the contour integral that contributes.

For us

$$\oint \frac{f(z)}{z-\alpha}\, dz \longrightarrow \oint dk_0 \frac{1}{k_0-(E_k - i\epsilon)} f(k_0)$$

where

$$f(k_0) = \frac{1}{2\pi} \frac{1}{k^0+E_k} e^{-ik_0(x^0-y^0)}$$

which gives

for counterclockwise rotation

$$-2\pi i\, f(\alpha) \longrightarrow -2\pi i\, f(E_k - i\epsilon) \xrightarrow[\epsilon\to 0]{} \frac{-i}{2E_k} e^{-iE_k(x^0-y^0)}$$

$$D_F(x-y) = i\int_{C_F} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}\, e^{-ik_\mu(x^\mu-y^\mu)}$$

$$= i\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\pi}\,\underbrace{\frac{-2\pi i}{2E_k}\, e^{-iE_k(x^0-y^0)}}_{-2\pi i\, f(E_k)}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2E_k}\, e^{-ik\cdot(x-y)}$$

$$= D(x-y)$$

Similarly, for $y^0 > x^0$, we close the contour in the upper half plane, enclosing the pole at $-E_k$ and

$$D_F(x-y) = D(y-x)$$

Now consider

$$(\Box + m^2)\, D_F(x-y)$$

$$= i(\Box + m^2)\int_{C_F} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}\, e^{-ik\cdot(x-y)}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}\,(\partial_\mu\partial^\mu + m^2)\, e^{-ik_\mu(x^\mu-y^\mu)}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}\,(\partial_0\partial^0 - \partial_i\partial^i + m^2)\, e^{-ik_0(x^0-y^0)}\, e^{-ik_i(x^i-y^i)}$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}\left[\eta^{00}(-ik_0)^2 - \eta^{ii}(-ik_i)^2 + m^2\right] e^{-ik\cdot(x-y)}$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}\left[-\underbrace{(k_0^2 + k_i^2)}_{} + m^2\right] e^{-ik\cdot(x-y)}$$

$$k_0 k^0 - k_i k^i = k_\mu k^\mu = k^2$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{-k^2+m^2}{k^2-m^2} e^{-ik\cdot(x-y)}$$

$$= -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

$$= -i\, \delta^{(4)}(x-y)$$

That is

$$(\Box + m^2)\, \Delta_F(x-y) = -i\, \delta^{(4)}(x-y)$$

So $\Delta_F(x-y)$ is the <u>Green's</u> <u>Function</u> for the Klein-Gordon equation.