Noether's Theorem

If ϕ satisfies the EL EOM ($\delta S = 0$), for **any** variation we have

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

REMINDER

$$\delta \mathcal{L} = \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$\delta S = \int d^4x \, \delta \mathcal{L} = \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = 0$$

Assume the action also has the symmetry

$$\delta_\Delta S = 0$$

under the **specific** transformation

$$\phi \to \phi + \Delta$$

Given the action is invariant under the transformation even though the Lagrangian density may not be $\left[\delta_\Delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \right) \neq 0 \right]$, (1)

(2) $$\delta_\Delta \mathcal{L} = \partial_\mu K^\mu \qquad \delta_\Delta S = \int d^4x \, \delta_\Delta \mathcal{L} = \int d^4x \, \partial_\mu K^\mu = 0$$

– i.e. $\delta_\Delta \mathcal{L}$ must be a total divergence. Equating (1) and (2), we get

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu \right) = 0$$

– i.e., we have a **conserved** **current** ($\partial_\mu j^\mu = 0$)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

so, with a symmetry there is a conserved current

How do we find K^μ?

Consider the **translation** **invariance** of the action under the transformation

A conserved current will be associated with a conserved "charge"

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_i j^i$$

Integrate over all of space

$$\int d^3x \left(\partial_0 j^0 + \partial_i j^i \right) = 0$$

Then

$$\partial_0 \int d^3x \, j^0 = - \int d^3x \, \partial_i j^i$$

$$= 0$$

and we identify

$$Q \equiv \int d^3x \, j^0$$

with the conserved "charge."

$$x^\mu \longrightarrow x'^\mu = x^\mu - a^\mu$$

Then

(2a-b) > $$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + \frac{\partial\phi}{\partial x^\mu} a^\mu$$

and

$$\delta_\Delta \phi = \frac{\partial\phi}{\partial x^\mu} a^\mu$$

Then

$$\delta_\Delta(\partial_\mu \phi) = \partial_\mu(\delta_\Delta \phi)$$

$$= \partial_\mu\left(\frac{\partial\phi}{\partial x^\alpha} a^\alpha\right)$$

$$= \frac{\partial^2\phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

Now we can compute

$$\delta_\Delta \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\Delta(\partial_\mu\phi)$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha}a^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial^2\phi}{\partial x^\mu\partial x^\alpha}a^\alpha$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial}{\partial x^\alpha}(\partial_\mu\phi)\right)a^\alpha$$

$$= \frac{\partial\mathcal{L}}{\partial x^\alpha}a^\alpha$$

$$= \partial_\alpha(\mathcal{L}a^\alpha)$$

Then

Consider a coordinate transformation corresponding to a spacetime translation

$$x \longrightarrow x' = x - a$$

- i.e.,

$$x^\mu \longrightarrow x'^\mu = x^\mu - a^\mu$$

where

$$a^\mu = (a^0, a^1, a^2, a^3)$$

and $a^{0,1,2,3}$ are all constants (they can be different).

Consider three points P, Q, and R, and their coordinates before and after the transformation:

P	Q	R		P	Q	R
$x-a$	x	$x+a$	before	$\phi(x-a)$	$\phi(x)$	$\phi(x+a)$
$x'-a$ $= x-2a$	x' $= x-a$	$x'+a$ $= x$	after	$\phi'(x'-a)$	$\phi'(x')$	$\phi'(x'+a)$ $= \phi'(x)$

$\longrightarrow$

WE HAVE SLID THE COORDINATES OVER

$\longleftarrow$

WE HAVE SLID THE FIELDS OVER SINCE THE NEW FIELD AT x, $\phi'(x)$, IS THE OLD FIELD AT $x+a$, $\phi(x+a)$

So, at $\underline{x}$, we have replaced the field $\phi(\underline{x})$ by $\phi(\underline{x+a})$, and

$$\delta\phi(x) = \phi'(x) - \phi(x) = \phi(x+a) - \phi(x)$$

$$= \phi(x) + \partial_\mu \phi(x)\, a^\mu - \phi(x)$$

$$= \partial_\mu \phi(x)\, a^\mu$$

It is in this sense that we mean we have moved the fields rather than the coordinates, which is a viewpoint that emerges when we focus on a given x.

This is the ACTIVE viewpoint.

The PASSIVE viewpoint emerges when we focus on what happens at a specific point (e.g., Q):

@ Q : $\phi'(x') = \phi(x)$ $\qquad\qquad \phi'(Q) = \phi(Q)$

@ R : $\phi'(x'+a) = \phi(x+a)$ $\qquad \phi'(R) = \phi(R)$

– i.e., the value of the scalar field at a point does not change as the coordinates of the point change

$$\delta\phi(Q) = \phi'(x') - \phi(x) = 0$$

$$K^\mu = \mathcal{L} a^\mu$$

and

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} a^\alpha - \mathcal{L} a^\mu$$

$$= \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) a^\alpha$$

Then

$$\partial_\mu j^\mu = a^\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Since a^α is arbitrary, for the above to be true always, each "coefficient" of a^α must be zero:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Contract with $\eta^{\nu\alpha}$ $\leftarrow \eta^{\nu\alpha} \partial_\mu (\quad) = 0$

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x_\nu} - \mathcal{L} \eta^{\mu\nu} \right)}_{= T^{\mu\nu}} = 0 \qquad \Rightarrow \partial_\mu [\eta^{\nu\alpha}(\quad)] = 0$$

$= T^{\mu\nu}$ The energy-momentum tensor for the scalar field ϕ

What about Lorentz invariance?

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\underline{\nu = 0}$

$$\partial_0 T^{00} + \partial_i T^{i0} = 0 \qquad \text{conservation of energy}$$

$\underline{\nu = j}$

$$\partial_0 T^{0j} + \partial_i T^{ij} = 0 \qquad \text{conservation of momentum}$$

0 $\quad \partial_0 \int d^3x\, T^{00} = \partial_0 \int d^3x\, \mathcal{H} = -\int d^3x\, \partial_i T^{i0}$

$\Rightarrow H = \int d^3x\, \mathcal{H}$ is the conserved "charge"

$\nu = j$ $\quad \partial_0 \int d^3x\, T^{0j} = -\int d^3x\, \partial_i \underbrace{T^{ij}}_{\text{stress tensor}} = 0$

$\Rightarrow P^i = \int d^3x\, \underbrace{T^{0i}}_{\text{momentum density}}$ is the conserved charge

$$x^\beta \to \Lambda^\beta{}_\nu x^\nu = x^\beta - \underbrace{\varepsilon^\beta{}_\nu x^\nu}_{\equiv \delta x^\beta} = x^\beta - \delta x^\beta$$

$$(\delta^\beta{}_\nu - \varepsilon^\beta{}_\nu) x^\nu$$

4a) >

Then

$$\phi(x) \to \phi(\Lambda^{-1} x) = \phi(x) + \frac{\partial \phi}{\partial x^\alpha} \delta x^\alpha$$

and

$$\delta_\Delta \phi = \frac{\partial \phi}{\partial x^\alpha} \delta x^\alpha$$

$$= \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu$$

Then

$$\delta_\Delta (\partial_\mu \phi) = \partial_\mu (\delta_\Delta \phi)$$

$$= \partial_\mu (\partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu)$$

$$= (\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu \delta^\nu{}_\mu$$

$$= (\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

Now

$$\delta_\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_\Delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\Delta (\partial_\mu \phi)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left[(\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu \right]$$

$$= \frac{\partial \mathcal{L}}{\partial x^\alpha} \varepsilon^\alpha{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

For the case where there is relative motion in the x-direction only we know that the (t, x) and (t', x') coordinates are related by

$$t' = \gamma(t - \beta x)$$

$$x' = \gamma(x - \beta t)$$

where

$$\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}$$

$$\beta = \frac{v}{c}$$

and v is the relative velocity.

Then

$$x'^{\mu} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \equiv \Lambda^{\mu}{}_{\nu} x^{\nu}$$

We can write

$$\frac{\partial \mathcal{L}}{\partial x^\alpha}\,\varepsilon^\alpha{}_\nu\, x^\nu = \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L}\varepsilon^\alpha{}_\nu\,\delta^\nu{}_\alpha$$

$$= \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L}\,\cancel{\varepsilon^\alpha{}_\alpha}^{\;\to 0}$$ since $\varepsilon^\mu{}_\nu = -\varepsilon^\nu{}_\mu$

5a-b >

So

$$\delta_\Delta \mathcal{L} = \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\alpha\phi\,\varepsilon^\alpha{}_\mu$$

5c-d >

$$= \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial_\mu\phi\right]\varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[T^\mu{}_\alpha + \mathcal{L}\delta^\mu{}_\alpha - T^\alpha{}_\mu - \mathcal{L}\delta^\alpha{}_\mu\right]\varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}(T^\mu{}_\alpha - T^\alpha{}_\mu)$$

$$= \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu)$$ since $T^\mu{}_\alpha = T^\alpha{}_\mu$ for the scalar field

$$= \partial_\alpha \mathcal{L}$$ since $\mathcal{L}\varepsilon^\alpha{}_\nu\partial_\alpha x^\nu = \mathcal{L}\varepsilon^\nu{}_\nu = 0$

But

$$\delta_\Delta \mathcal{L} = \partial_\alpha(\mathcal{L}\varepsilon^\alpha{}_\nu x^\nu) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\right)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi\,\varepsilon^\alpha{}_\nu x^\nu\right)$$

For an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}$$

We know that the Lorentz transformation leaves the proper time (or proper distance) the same

$$d\tau^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu}$$

$$= dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\Rightarrow dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$$

$$\Rightarrow \eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} dx^{\alpha} \Lambda^{\nu}{}_{\beta} dx^{\beta}$$

$$= \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

$$\Rightarrow \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$$

Then

$$\eta_{\alpha\beta} = (\delta^{\mu}{}_{\alpha} + \varepsilon^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \varepsilon^{\nu}{}_{\beta})\eta_{\mu\nu}$$

$$\cong (\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \delta^{\mu}{}_{\alpha}\varepsilon^{\nu}{}_{\beta} + \varepsilon^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta})\eta_{\mu\nu}$$

$$= \eta_{\alpha\beta} + \eta_{\alpha\nu}\varepsilon^{\nu}{}_{\beta} + \varepsilon^{\mu}{}_{\alpha}\eta_{\mu\beta}$$

$$= \eta_{\alpha\beta} + \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha}$$

Then

$$\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0$$

The quantity

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi$$

is a tensor of rank $\binom{1}{1}$ defined as

$$T^\mu{}_\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi$$

I can decompose $T^\mu{}_\alpha$ into its symmetric and antisymmetric components as follows

$$T^\mu{}_\alpha = \tfrac{1}{2}\left(T^\mu{}_\alpha + T^\alpha{}_\mu + T^\mu{}_\alpha - T^\alpha{}_\mu\right)$$

$$= \underbrace{\tfrac{1}{2}\left(T^\mu{}_\alpha + T^\alpha{}_\mu\right)}_{\text{symmetric under } \alpha \leftrightarrow \mu} + \underbrace{\tfrac{1}{2}\left(T^\mu{}_\alpha - T^\alpha{}_\mu\right)}_{\text{antisymmetric under } \alpha \leftrightarrow \mu}$$

If $T^\mu{}_\alpha$ is contracted with an antisymmetric tensor $\varepsilon^\alpha{}_\mu$, only the antisymmetric component of $T^\mu{}_\alpha$ will contribute.

$$T^\mu{}_\alpha \, \varepsilon^\alpha{}_\mu = \left(S^\mu{}_\alpha + A^\mu{}_\alpha\right) \varepsilon^\alpha{}_\mu$$

$$= A^\mu{}_\alpha \, \varepsilon^\alpha{}_\mu$$

since

$$S^{\mu}{}_{\alpha} \, \varepsilon^{\alpha}{}_{\mu} = - S^{\alpha}{}_{\mu} \, \varepsilon^{\mu}{}_{\alpha}$$

which implies

$$S^{\mu}{}_{\alpha} \, \varepsilon^{\alpha}{}_{\mu} = 0$$

So

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu - \mathcal{L} \varepsilon^\mu{}_\nu x^\nu \right) = 0$$

or

$$\varepsilon^\alpha{}_\nu \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, x^\nu - \mathcal{L} \delta^\mu{}_\alpha x^\nu \right) = 0$$

Rewriting

$$\varepsilon^\alpha{}_\nu \, \partial_\mu \Big[\underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \mathcal{L} \delta^\mu{}_\alpha \right)}_{T^\mu{}_\alpha} x^\nu \Big] = 0$$

6a) Since $\varepsilon^\alpha{}_\nu$ is antisymmetric, we can conclude that the anti-symmetric component of the term in [ ] must vanish - i.e.,

$$\partial_\mu \left[T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha \right] = 0$$

So

$$M^\mu{}_\alpha{}^\nu \equiv T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha$$

is the conserved current.

For $\alpha, \nu = 1, 2, 3$, the Lorentz transformation corresponds to a

We can decompose the tensor

$$T^{\mu}{}_{\alpha} x^{\nu}$$

into its symmetric and antisymmetric components

$$T^{\mu}{}_{\alpha} x^{\nu} = \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} + T^{\mu}{}_{\nu} x^{\alpha} + T^{\mu}{}_{\alpha} x^{\nu} - T^{\mu}{}_{\nu} x^{\alpha}\right]$$

$$= \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} + T^{\mu}{}_{\nu} x^{\alpha}\right] \leftarrow \text{SYMMETRIC}$$

$$+ \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} - T^{\mu}{}_{\nu} x^{\alpha}\right] \leftarrow \text{ANTISYMMETRIC}$$

spatial rotation. In this case

$$\partial_0 \left[T^0{}_i x^j - T^0{}_j x^i \right] + \partial_k \left[T^k{}_i x^j - T^k{}_j x^i \right] = 0$$

and the conserved charge is

$$\int d^3x \left(T^0{}_i x^j - T^0{}_j x^i \right)$$

which is the total angular momentum tensor $Q_i{}^j$ with $Q_i{}^i = 0$ and $Q_i{}^j = -Q_j{}^i$ – i.e., $Q^i{}_j$ has 3 independent components.

Commutation of ∂_μ and δ

$F = F(x, y(x), y'(x))$ a functional

Change $y(x)$ in the following manner

$$y(x) \longrightarrow y(x) + \epsilon\eta(x)$$

The variation δy is defined to be

$$\delta y \equiv \epsilon\eta(x) \qquad (1)$$

The variation is a change in a function.

At fixed x,

$$F(x, y, y') \longrightarrow F(x, y + \epsilon\eta, y' + \epsilon\eta')$$

$$= \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

$$\equiv \delta F$$

If we let $F = y'$, we can see that generally

$$\delta F = \delta y' = \epsilon\eta' \qquad (2)$$

So, at fixed x, from (1) and (2)

$$\delta y' = (\delta y)'$$

- i.e., the derivative w.r.t. the independent variable x and the variation of a function of x commute.