Experimental Approach to the Hankel Transform of Catalan Number Combinations

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Brief

1. Introductions
2. Experiments
3. Conjectures

Review: Fibonacci Numbers

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$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

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$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

More precisely,

$$F_0 = 0, \;\; F_1 = 1;$$

$$F_n = F_{n-1} + F_{n-2}, n \geqslant 2.$$

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For instance, a recurrence relation for the sequence

$$X = \{x^2, x^4, x^6, x^8, ...\}$$

can be defined as $x_n = x^2 x_{n-1}$.

Catalan Numbers

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By definition, $C_0 = 1$;

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0, n \geq 1.$$

Examples in Combinatorics

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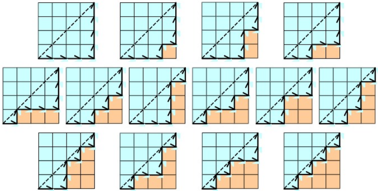
Monotonic Path:



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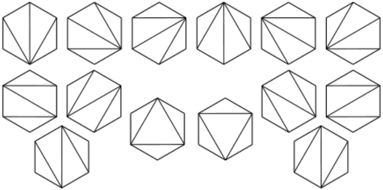
Convex Polygon:



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$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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$$h_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-1} \end{vmatrix}$$

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$$h_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3;$$

$$\ldots\ldots$$

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$$H\{C_n\} = \{1, 1, 1, \cdots\}$$

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Another example, the Hankel transform of the sum of adjacent Catalan numbers,
$C_n + C_{n+1} = \{1, 1, 2, 5, 14, 42, 132, ...\} + \{1, 2, 5, 14, 42, 132, 429, ...\} = \{2, 3, 7, 19, 56, 174, 561, ...\}$:

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$$H\{C_n + C_{n+1}\} = \{(0), (1), (1), \mathbf{2}, (3), \mathbf{5}, (8), \mathbf{13}, (21), \mathbf{34}, ...\} = F_{2n+3}$$

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Conjecture

The recurrence relation for $H\{C_n + C_{n+1}\}$ is, $h_{n+1} = 3h_n - h_{n-1}$.

Goals

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Today, we focus on the conjectures/experimental part of the project.

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Conjecture

The recurrence relation for $H\{aC_n\}$ is $h_{n+1} = ah_n$.

$aC_n + bC_{n+1}$

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Basic Methods?

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Basic Methods? List and Compare!

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Tool: Wolfram Mathematica 7.0

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$aC_n + bC_{n+1}$ Continue[a]

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$H\{aC_n + bC_{n+1}\}$ Comparison Table

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Simply notation: Rewrite the recurrence relation as a coefficient column vector and a variable row vector whose product is 0.

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Conjecture of the Coefficient Column Vector:

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The general recurrence relation for $H\{aC_n + bC_{n+1}\}$ is,

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But, is this method efficient?

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Think: What happens if we replace M for the Hankel matrix of our desired sequence?

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Advantage/Disadvantage...

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$D_0 * h_{n+1} + D_1 * h_n + D_2 * h_{n-1} + \cdots + D_7 * h_{n-6} + D_8 * h_{n-7} = 0$, *where*

$$D_0 = 1$$

$$D_1 = -a - 2b - 4c - 8d$$

$$D_2 = b^2 + c(-2a + 4b + 6c) + d(-12a + 4b + 24c + 28d)$$

$$D_3 = c^2(-a - 2b - 4c) + d(2ab + 4b^2 - 4ac - 24c^2 - 7ad + 2bd - 60cd - 56d^2)$$

$$D_4 = c^4 + d(-4bc^2 + 8c^3) + d^2(a^2 + 4ab + 6b^2 + 12ac - 8bc + 36c^2)$$
$$+ d^3(40a - 8b + 80c) + 70d^4$$

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$$
\begin{aligned}
E_0 &= 1 \\
E_1 &= -a - 2b - 4c - 8d - 16e \\
E_2 &= b^2 + c(-2a + 4b + 6c) + d(-12a + 4b + 24c + 28d) + \\
&\quad e(-52a - 8b + 40c + 112d + 120e) \\
\ldots &= \ldots \\
E_{14} &= e^{12} * (b^2 + c(-2a + 4b + 6c) + d(-12a + 4b + 24c + 28d) + \\
&\quad e(-52a - 8b + 40c + 112d + 120e)) \\
E_{15} &= e^{14} * (-a - 2b - 4c - 8d - 16e) \\
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More Results

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Conjecture

Tthe recurrence relations for q terms of adjacent Catalan numbers must satisfy the relation,

$$Q_{n+2m} = q^{2m} * Q_n.$$

Next Step: Proof

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Conjectures $\Rightarrow$ Theorems

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$\Uparrow$

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Reference

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1. J. W. Layman, "The Hankel Transform and Some of its Properties", Journal of Integer Sequences, Article 01.1.5, Volume 4, 2001.

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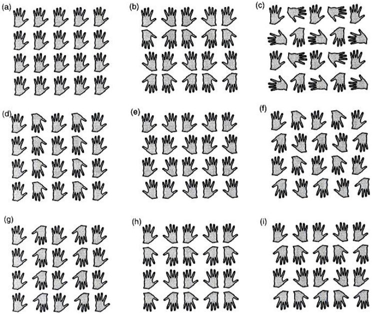

Thank you for coming!