Hamiltonian Connectivity in k-Ary n-Cubes Under a Region-Based Fault Model (Student Abstract)

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Abstract

The resilience of large-scale computing networks, such as k-ary n-cubes, is often compromised by spatially correlated failures, a reality that creates performance bottlenecks in distributed AI workloads (Dean et al. 2012). This paper introduces the Region-Based Fault (RBF) model, which explicitly represents failures as geographically clustered entities. We establish a novel theoretical condition for maintaining Hamiltonian connectivity under the RBF model, leveraging a combinatorial bound based on fault separation. We also present an adaptive, recursive algorithm for path construction. Experiments show our model and algorithm vastly outperform traditional approaches, tolerating orders of magnitude more faults and providing a robust framework for designing next-generation fault-tolerant systems with predictable, high-bandwidth communication paths essential for deadlock-free routing.

Code — https://github.com/wyqmath/Hamiltonian_Path

Introduction

The k-ary n-cube (Q_n^k) topology is foundational to modern high-performance computing (Jouppi et al. 2017). In such systems, failures are inevitable and rarely random, often occurring as spatially correlated clusters (Vishwanath and Nagappan 2010). Traditional fault models, which assume independent faults, fail to capture this reality.

To bridge this gap, we introduce the Region-Based Fault (RBF) model, which represents failures as "fault clusters" with defined size, shape, and, crucially, a minimum separation distance.

Definition 1 (Region-Based Fault (RBF) Model)**.** *A faulty edge set F is a valid RBF set characterized by $(k_{\max}, s_{\max}, d_{sep}, \mathcal{S})$ if its fault-induced graph partitions into m clusters satisfying:*

1. ***Count:*** $m \leq k_{\max}$.
2. ***Size:*** *For any cluster C_i, $|E(C_i)| \leq s_{\max}$.*
3. ***Separation:*** *The minimum Hamming distance between any two distinct clusters is at least d_{sep}.*
4. ***Shape:*** *The topology of each cluster is in a predefined set $\mathcal{S}$.*

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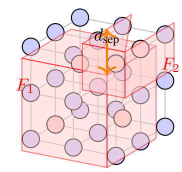

Figure 1: Illustration of the RBF model applied to a Q_3^3 topology. The figure demonstrates two fault cluster regions (F_1, F_2) constrained by a minimum separation distance (d_{sep}).

Guarantees, Algorithm, and Validation

Our central theoretical result establishes a sufficient condition for Hamiltonicity, leveraging a key combinatorial argument that uses d_{sep} to bound the number of faulty edges crossing between any two sub-networks.

Theorem 1 (RBF Hamiltonian Connectivity)**.** *For odd $k \geq 3$ and $n \geq 2$, a Q_n^k remains Hamiltonian-connected if an RBF fault set F satisfies:*

$$\min\left(k_{\max}, K_{bound}\right) \cdot s_{\max} < \frac{k^{n-1}}{4}$$

where $K_{bound} = \frac{k^{n-1}}{\sum_{i=0}^{\lfloor (d_{sep}-1)/2 \rfloor} \binom{n-1}{i}(k-1)^i}$.

Our constructive algorithm, outlined in Algorithm 1, is based on a recursive divide-and-conquer strategy. Its key innovation is the adaptive dimension selection (line 4), a heuristic that analyzes the geometric distribution of all fault clusters to choose a decomposition dimension d^* that best isolates them. This proactive strategy is the primary reason for the algorithm's high practical performance.

Experiments validate the superior fault tolerance of our RBF-aware algorithm (Table 1). A sensitivity analysis (Figure 2) further reveals a super-linear relationship between fault tolerance and d_{sep}, providing empirical validation that spatial isolation is more critical than total fault count.

This robustness is further evidenced by the algorithm's performance under stress. Figure 3 shows that it maintains a 100% success rate far beyond the theoretical boundary defined by Theorem 1. This graceful degradation stems from the adaptive dimension selection, which consistently finds practical decompositions that are far better than the theoretical worst-case.

Algorithm 1 RBF Hamiltonian Path Construction

Input: Q_n^k, RBF fault set F, source s, target t
Output: Hamiltonian path from s to t

if $n = 2$ **then**
  return HamiltonianPath2D(Q_2^k, F, s, t)
end if
$\mathcal{C} \leftarrow$ AnalyzeFaultClusters(F)
$d^* \leftarrow$ SelectOptimalDimension$(\mathcal{C}, n)$
$\{Q[0..k-1]\} \leftarrow$ Decompose(Q_n^k, d^*)
SubPaths $\leftarrow []$
for $i = 0$ to $k-1$ **do**
  $(u_i, v_i) \leftarrow$ SelectEndpoints$(s, t, i, \text{SubPaths})$
  $P_i \leftarrow$ RBF_HPath$(Q[i], F_i, u_i, v_i)$
  if $P_i =$ NULL **then**
    return NULL
  end if
  SubPaths.append(P_i)
end for
return StitchPaths$(\text{SubPaths}, d^*)$

Table 1: Max tolerated faulty edges for RBF, PEF, and FT models.

Network Config.	RBF Tolerance	PEF Tolerance	FT Tolerance
Q_5^3 (3-ary 5-cube)	41	24	3
Q_4^4 (4-ary 4-cube)	134	77	5
Q_5^5 (5-ary 5-cube)	1,607	770	7
Q_4^6 (6-ary 4-cube)	2,648	1,353	9

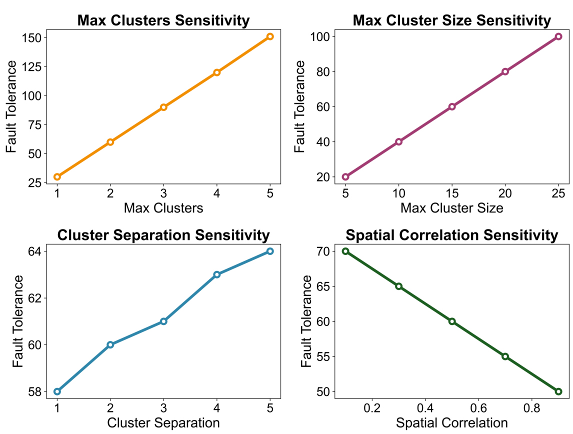

Figure 2: Sensitivity analysis shows the strong positive impact of separation distance (d_{sep}) on fault tolerance in a Q_5^6 network.

Conclusion

This research introduces the RBF model, aligning fault-tolerance theory with the physical reality of clustered failures. We provided a formal proof of Hamiltonian connectivity and a robust, adaptive algorithm that demonstrates significantly enhanced resilience. This work bridges theoretical graph theory, algorithm design, and the physical architecture

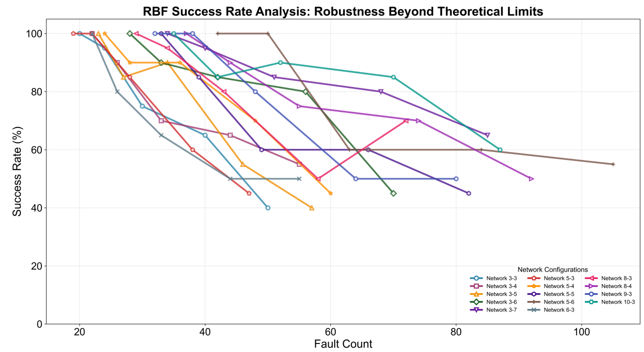

Figure 3: Success rate analysis shows graceful degradation. The algorithm maintains 100% success far beyond the theoretical boundary.

of AI systems.

The impact on large-scale AI is direct. Distributed training of large models is often bottlenecked by "stragglers"—nodes slowed by network issues. Clustered faults can exacerbate this, creating communication blackouts. By guaranteeing Hamiltonian connectivity under these realistic fault scenarios, our work ensures the persistence of predictable, high-bandwidth communication paths. This is a prerequisite for implementing efficient, deadlock-free routing protocols (Dally and Seitz 1987), which are critical for minimizing latency in communication-intensive AI workloads. Ultimately, this research provides AI system architects with an actionable principle: investing in physical fault isolation (increasing d_{sep}) yields a quantifiable, super-linear return in logical network resilience, paving the way for more robust and efficient supercomputers for AI.

Acknowledgments

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