MAST90105 Methods of Mathematical Statistics

Assignment 4, Semester 1, 2024

Due: Sunday May 26, end of day.

Please submit a scanned or other electronic copy of your work via the Canvas Learning Management System

1. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with the cumulative distribution function (CDF)

$$F_X(x;\theta) = \begin{cases} 1 - \left(\frac{3}{x}\right)^\theta, & x \geq 3, \\ 0, & \text{otherwise} \end{cases}$$

Here, $\theta > 0$ is an unknown parameter.

(a) Find the maximum likelihood estimator of θ, $\widehat{\theta}_{ML}$. Show that $\widehat{\theta}_{ML}$ maximizes the likelihood function.

(b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ.

(c) Now assume the sample size is $n = 100$ and the observed data are such that the maximum likelihood estimate $\widehat{\theta} = 2.5$. Using these data and asymptotic normality of the maximum likelihood estimator $\widehat{\theta}_{ML}$, construct an approximate 95% confidence interval for θ.

(d) Consider the exponential prior distribution for the parameter θ with the probability density function (PDF):

$$f_\Theta(\theta) = \begin{cases} e^{-\theta}, & \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the posterior distribution of θ using the same observed data as in (c). Find the posterior mean of θ. Is it close to the maximum likelihood estimate $\widehat{\theta} = 2.5$? Explain why or why not. *Hint:* $a^x = e^{x \ln a}$ for $a > 0$.

(e) We would like to test the null hypothesis $H_0 : \theta = 2$ against the alternative $H_a : \theta > 2$. Construct the test using asymptotic normality of $\widehat{\theta}_{MLE}$. Compute the p-value of this test using the same observed data as in (c). Do we reject H_0 at the 1% significance level?

2. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with the cumulative distribution function (CDF)

$$F_X(x;\theta) = \begin{cases} 1 - \left(\frac{\theta}{x}\right)^3, & x \geq \theta, \\ 0, & \text{otherwise} \end{cases}$$

Here, $\theta > 0$ is an unknown parameter.

(a) Find the maximum likelihood estimator of θ, $\widehat{\theta}_{ML}$. Show that $\widehat{\theta}_{ML}$ maximizes the likelihood function.

(b) Show that $\widehat{\theta}_{ML}/\theta$ is a pivot and use it to construct the 95% confidence interval for θ.

(c) We want to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_a : \theta = 2$. We reject the null hypothesis if $\widehat{\theta}_{ML} > c$ where $c > 0$ is some constant. Find c that minimizes $\alpha + \beta$ where α and β are the type 1 and type 2 errors of this test, respectively.

(d) Now assume that the prior distribution of θ is a continuous distribution with the cumulative distribution function (CDF):

$$F_\Theta(\theta) = \begin{cases} 1 - 1/\theta^5, & \theta > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Assume the sample size is $n = 5$ and the observed data are $1.68, 2.52, 1.79, 3.07, 3.84$. Find the posterior distribution of θ using these data. Find the posterior mean of θ.