1. Assume we flip a coin, and the probability of tail/head depends on the **previous outcomes**:

$$\Pr\{H \text{ on the } (n{+}1)\text{-st flip} \mid H \text{ is observed exactly } k \text{ times on } n \text{ previous flips}\} = \left(\frac{k+1}{n+2}\right)^2.$$

In particular, with $k = 0$ and $n = 0$, $\Pr(H) = 1/4$ and $\Pr(T) = 3/4$ on the first flip. Let X be the number of flips until the first tail is observed.

   (a) Find the probability mass function of X.

   (b) Compute the probability $\Pr(X \geq 10)$.

   (c) Find the mean $\mathrm{E}(X)$. What can you say about the variance $\mathrm{Var}(X)$?
   Hint: You can use these results without proof:

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

2. The number of earthquakes in the Melbourne area follows a Poisson process with the rate parameter $\lambda = 1$ (earthquakes per year). Let T_3 be the waiting time (in years) until the next 3 earthquakes occur in the Melbourne area.

   (a) It is expected that exactly 5 earthquakes will occur in the next 5 years. Taking this information into account, compute the cumulative distribution function (CDF) of T_3.

   (b) Again, assuming that exactly 5 earthquakes will occur in the next 5 years, compute the probability $\Pr(T_3 < 2)$ and the mean value $\mathrm{E}(T_3)$.

3. The memorylessness property of an exponential random variable X implies that

$$\Pr(X > T + t | X > T) = \Pr(X > t).$$

If X is used to model the lifetime of a device, then the distribution of the time to failure does not depend on how much time $T > 0$ has elapsed already.

Assume now the lifetime of a device X follows a mixture of two exponential distributions with the cumulative distribution function (CDF):

$$F_X(t) = \Pr(X \leq t) = 1 - 0.5(e^{-t} + e^{-2t}), \quad t > 0.$$

   (a) Compute the median of X.

   (b) Find the moment generating function (MGF) of X, $M_X(t)$. Use this MGF to compute the mean value $\mathrm{E}(X)$.

   (c) The memorylessness property does not hold for X with the CDF $F_X(t)$ as defined above. Show that

$$\Pr(X > T + t | X > T) > \Pr(X > t)$$

   for any $T > 0$ and $t > 0$.