Appendix

Proof of Table 5 When Enterprise A makes decisions first, taking the cross-brand trade-in of both enterprises (Strategy Combination CC) as an example, we obtain the market share of consumers according to the utilities of first-time and repeat consumers of the two enterprises and Table 4. (1) The threshold of first-time consumers who purchase from Enterprise A is $x_A^1 = \frac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t}$ and those who purchase from Enterprise B is $x_B^1 = \frac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t}$. (2) The threshold of repeat consumers of Enterprise A who purchase from Enterprise A is $x_{AN}^1 = \frac{(1-\lambda)v - p_A^1 + r_A}{t}$ and those who purchase from Enterprise B is $x_{BN}^1 = \frac{(\lambda-\beta)v + p_B^1 + t - \delta_1 r_A}{t}$. (3) The threshold of repeat consumers of Enterprise B who purchase from Enterprise A is $x_{AN}^2 = \frac{(1-\lambda\beta)v - p_A^1 + r_A}{t}$ and those who purchase from Enterprise B is $x_{BN}^2 = \frac{(\lambda-1)\beta v + p_B^1 + t - \delta_1 r_A}{t}$. Submit these thresholds into the profit functions in the Strategy Combination CC:

$$\Pi_A^1 = \tau_0 p_A^1 \frac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t} + \tau_1 (p_A^1 - r_A + s)\frac{(1-\lambda)v - p_A^1 + r_A}{t} + \tau_2 (p_A^1 - r_A + s)\frac{(1-\lambda\beta)v - p_A^1 + r_A}{t}$$

$$\Pi_B^1 = \tau_0 p_B^1 (1 - \frac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t}) + \tau_1 (p_B^1 - \delta_1 r_A + s)(1 - \frac{(\lambda-\beta)v + p_B^1 + t - \delta_1 r_A}{t})$$
$$+\tau_2 (p_B^1 - \delta_1 r_A + s)(1 - \frac{(\lambda-1)\beta v + p_B^1 + t - \delta_1 r_A}{t})$$

According to inverse induction, the Hessian matrix of Π_B^1 with respect to p_B^1 and δ_1 is $\begin{bmatrix} -\frac{\tau_0 + 2\tau_1 + 2\tau_2}{t} & \frac{2r_A(\tau_1+\tau_2)}{t} \\ \frac{2r_A(\tau_1+\tau_2)}{t} & -\frac{2r_A^2(\tau_1+\tau_2)}{t} \end{bmatrix}$, $|H_1| = -\frac{\tau_0 + 2\tau_1 + 2\tau_2}{t} < 0$, $|H_2| = \frac{2r_A^2(\tau_1+\tau_2)\tau_0}{t^2} > 0$. The Hessian matrix is negative definite, so Π_B^1 is a joint concave function with respect to p_B^1 and δ_1. Find the first order derivative of Π_B^1 with respect to p_B^1 and δ_1 respectively and setting them to 0, we have $p_B^1 = \frac{(\beta-1)v + t + p_A^1}{2}$ (1), $\delta_1 = \frac{((\lambda-1)v + s + t + p_A^1)\tau_1 + ((\lambda\beta-1)v + s + t + p_A^1)\tau_2}{2r_A(\tau_1+\tau_2)}$ (2).

Substitute (1) and (2) into Π_A^1 and find its Hessian matrix about p_A^1 and r_A: $\begin{bmatrix} -\frac{\tau_0 + 4\tau_1 + 4\tau_2}{2t} & \frac{2(\tau_1+\tau_2)}{t} \\ \frac{2(\tau_1+\tau_2)}{t} & -\frac{2(\tau_1+\tau_2)}{t} \end{bmatrix}$, $|H_1| = -\frac{\tau_0 + 4\tau_1 + 4\tau_2}{2t} < 0$, $|H_2| = \frac{(\tau_1+\tau_2)\tau_0}{t^2} > 0$. The Hessian matrix is negative definite, i.e., Π_A^1 is a joint concave function with respect to p_A^1 and r_A, i.e., there is a (p_A^1, r_A) such that Π_A^1 has a maximal point. We can obtain the optimal solution without constraints by solving the first derivatives and setting them to 0: $p_A^1 = \frac{(1-\beta)v + 3t}{2}$ (3), $r_A = \frac{((\lambda-\beta)v + s + 3t)\tau_1 + ((\lambda-1)\beta v + s + 3t)\tau_2}{2(\tau_1+\tau_2)}$ (4). Substitute (3) and (4) into (1) and (2) to get the values of p_B^1 and δ_1, we have $p_B^1=\frac{(\beta-1)v+5t}{4}$ (5), $\delta_1=\frac{2\lambda v(\tau_1+\beta\tau_2)+(\tau_1+\tau_2)(2s+5t-(1+\beta)v)}{2\lambda v(\tau_1+\beta\tau_2)+2(\tau_1+\tau_2)(s+3t-\beta v)}$ (6).

According to p_A^1 , p_B^1 , δ_1 , r_A, we obtain the demand of different types of consumers and the profits of the two enterprises under the Strategy Combination CC.

The process of solving the equilibrium solution for Strategy Combination WW, WC and CW is similar to that for Strategy Combination CC and we will not repeat. Table 5 summarizes the optimal solutions of the four strategy combinations.

Proof of Lemma 1. When Enterprise A makes decisions first, we take Strategy Combination CC as an example. The quantity constraint of first-time consumers is $0<x_A^1<1$. The quantity constraint of repeat consumers of Enterprise A is $0<x_{AN}^1<x<x_{BN}^1<1$. The quantity constraint of repeat consumers of Enterprise B is $0<x_{AN}^2<x<x_{BN}^2<1$. By submitting the optimal solutions into the quantity constraints, we obtain the constraints on product durability, transportation costs per unit distance, and brand difference coefficients of the four trade-in strategy combinations.

Proof of Proposition 1. As for the prices of the two enterprises, $p_A^1-p_B^1=\frac{(1-\beta)3v+t}{4}>0$. As for the rebate, $r_A^W-r_A^C=\frac{\lambda v\tau_2(1-\beta)}{2(\tau_1+\tau_2)}>0$. Moreover, $\frac{\partial r_A}{\partial\beta}=-\frac{v}{2}<0$, $\frac{\partial r_A}{\partial\lambda}=\frac{v}{2}>0$. As for the rebate correlation coefficient, $\delta_1^{WW}-\delta_1^{CW}=\frac{\lambda v\tau_1(\beta-1)}{(\tau_1+\tau_2)((\lambda-\beta)v+s+3t)}<0$, $\delta_1^{CW}-\delta_1^{WC}=\frac{\left\{\begin{matrix}\lambda v\tau_1(1-\beta)(((\lambda-\beta)v+s+3t)2\tau_1^2+((1-\beta)v\\+t)\tau_1\tau_2-(((2\lambda-1)\beta-1)v+2s+5t)\tau_2^2)\end{matrix}\right\}}{\left\{\begin{matrix}2(\tau_1+\tau_2)((\lambda-\beta)v+s+3t)(((\lambda-\beta)v\\+s+3t)\tau_1+((\lambda-1)\beta v+s+3t)\tau_2)\end{matrix}\right\}}<0$,

$\delta_1^{WC}-\delta_1^{CC}=\frac{\lambda v\tau_1(\beta-1)}{(((\lambda-\beta)v+s+3t)\tau_1+((\lambda-1)\beta v+s+3t)\tau_2)}<0$, i.e., $\delta_1^{WW}<\delta_1^{CW}<\delta_1^{WC}<\delta_1^{CC}$ .We derive the sensitivity analysis of the discount correlation coefficient to brand difference coefficient and product durability by taking the first-order derivatives of the coefficient and find $\frac{\partial\delta_1}{\partial\beta}>0$, $\frac{\partial\delta_1}{\partial\lambda}>0$. For repeat consumers of the two enterprise, we have $x_{AN}^{1W}-x_{AN}^{1C}=\frac{\lambda v\tau_2(1-\beta)}{2(\tau_1+\tau_2)}>0$, $x_{BN}^{2WW}-x_{BN}^{2WC}=\frac{\lambda v\tau_1(1-\beta)}{2(\tau_1+\tau_2)}>0$, i.e., $x_{AN}^{1WW}=x_{AN}^{1WC}>x_{AN}^{1CW}=x_{AN}^{1CC}$, $1-x_{BN}^{2WW}=1-x_{BN}^{2CW}<1-x_{BN}^{2WC}=1-x_{BN}^{2CC}$. Other findings can be obtained from the equilibrium solutions in Table 5.

Proof of Proposition 2. We can easily obtain Lemma 4 by submitting p_A^1 , p_B^1 , δ_1 , r_A into Π_A^1, Π_B^1 to get the profits of the two enterprises. $\Pi_A^{1W}=\frac{\left\{\begin{matrix}((\beta-1)^2\tau_0+4(\lambda-1)^2\tau_1)v^2+((1\\-\beta)6t\tau_0+(1-\lambda)8s\tau_1)v+4s^2\tau_1+9t^2\tau_0\end{matrix}\right\}}{16t}$,

$$\Pi_A^{1C}=\frac{\left\{\begin{matrix}(4(\lambda-1)^2\tau_1^2+((\beta-1)^2\tau_0+8(\lambda-1)(\lambda\beta-1)\tau_2)\tau_1+\tau_2(4(\lambda\beta-1)^2\tau_2+(\beta-1)^2\tau_0))v^2\\-8(\tau_1+\tau_2)((\beta-1)6t\tau_0+(\lambda-1)8s\tau_1+(\lambda\beta-1)8s\tau_2)v+(\tau_1+\tau_2)(9t^2\tau_0+4s^2(\tau_1+\tau_2))\end{matrix}\right\}}{16t(\tau_1+\tau_2)},$$

$\Pi_A^{1W}-\Pi_A^{1C}=-\frac{\left\{\begin{matrix}(((1+(2\beta-1)\lambda^2-2\lambda\beta)\tau_1+(\lambda\beta-1)^2\tau_2)v^2\\+2s(\tau_1+\tau_2)(1-\lambda\beta)v+(\tau_1+\tau_2)s^2)\tau_2\end{matrix}\right\}}{4t(\tau_1+\tau_2)}<0$, i.e., for the profit of Enterprise A ,we have $\Pi_A^{1WW}=\Pi_A^{1WC}\le\Pi_A^{1CW}=\Pi_A^{1CC}$. $\Pi_B^{1W}=\frac{\left\{\begin{matrix}((\tau_0+(\lambda-1)^2 8\tau_2)\beta^2+(1-2\beta)\tau_0)v^2+((10t\tau_0\\+(1-\lambda)16s\tau_2)\beta-10t\tau_0)v+8s^2\tau_2+25t^2\tau_0\end{matrix}\right\}}{32t}$,

$$\Pi_B^{1C} = \frac{\begin{Bmatrix} (8(\beta-\lambda)^2\tau_1^2 + ((\beta-1)^2\tau_0 + 16(1-\lambda)(\beta-\lambda)\tau_2)\tau_1 + \tau_2(8\beta^2(\lambda-1)^2\tau_2 + (\beta-1)^2\tau_0))v^2 \\ -2(\tau_1+\tau_2)((1-\beta)5t\tau_0 + (\lambda-\beta)8s\tau_1 + (\lambda-1)8\beta s\tau_2)v + (\tau_1+\tau_2)(25t^2\tau_0 + 8s^2(\tau_1+\tau_2)) \end{Bmatrix}}{32t(\tau_1+\tau_2)},$$

$\Pi_B^{1W} - \Pi_B^{1C} = -\frac{\begin{Bmatrix} (((\beta-\lambda)^2\tau_1 + ((\lambda+1)\beta-2\lambda)(1-\lambda)\beta\tau_2)v^2 \\ +2s(\tau_1+\tau_2)(\beta-\lambda)v + (\tau_1+\tau_2)s^2)\tau_1 \end{Bmatrix}}{4t(\tau_1+\tau_2)} < 0$, i.e., for the profit of Enterprise B, we have $\Pi_B^{1WW} = \Pi_B^{1CW} \le \Pi_B^{1WC} = \Pi_B^{1CC}$. Moreover, $\frac{\partial \Pi_A^{1W}}{\partial \beta} = \frac{((\beta-1)v-3t)\tau_0 v}{8t} < 0$, $\frac{\partial \Pi_A^{1W}}{\partial \lambda} = -\frac{((1-\lambda)v+s)\tau_1 v}{2t} < 0$, $\frac{\partial \Pi_B^{1W}}{\partial \beta} = \frac{(((\lambda-1)^2 8\beta\tau_2 + (\beta-1)\tau_0)v + (1-\lambda)8s\tau_2 + 5t\tau_0)v}{16t} > 0$, $\frac{\partial \Pi_B^{1W}}{\partial \lambda} = -\frac{\beta v\tau_2((1-\lambda)\beta v + s)}{2t} < 0$;

$\frac{\partial \Pi_A^{1C}}{\partial \beta} = -\frac{\begin{Bmatrix} (((1-\beta\lambda)v+s)4\lambda\tau_2^2 + (((1-\lambda)4\lambda\tau_1 + (1-\beta)\tau_0)v \\ +4\lambda s\tau_1 + 3t\tau_0)\tau_2 + \tau_0\tau_1((1-\beta)v+3t))v \end{Bmatrix}}{8t(\tau_1+\tau_2)} < 0$, $\frac{\partial \Pi_A^{1C}}{\partial \lambda} = -\frac{(\beta\tau_2+\tau_1)(((1-\lambda)v+s)\tau_1 + ((1-\beta\lambda)v+s)\tau_2)v}{2t(\tau_1+\tau_2)} < 0$,

$\frac{\partial \Pi_B^{1C}}{\partial \beta} = -\frac{\begin{Bmatrix} (((1-\lambda)\beta v+s)(\lambda-1)8\tau_2^2 + (((\lambda-1)(2\beta \\ -\lambda)8\tau_1 + (1-\beta)\tau_0)v + 8(\lambda-2)s\tau_1 - 5t\tau_0)\tau_2 \\ -\tau_1(((\beta-\lambda)8\tau_1 + (\beta-1)\tau_0)v + 8s\tau_1 + 5t\tau_0)v \end{Bmatrix}}{16t(\tau_1+\tau_2)} > 0$, $\frac{\partial \Pi_B^{1C}}{\partial \lambda} = -\frac{(\beta\tau_2+\tau_1)(((\beta-\lambda)v+s)\tau_1 + ((1-\lambda)\beta v+s)\tau_2)v}{2t(\tau_1+\tau_2)} < 0$.

That is, $\frac{\partial \Pi_A^1}{\partial \beta} < 0$, $\frac{\partial \Pi_B^1}{\partial \beta} > 0$, $\frac{\partial \Pi_A^1}{\partial \lambda} < 0$, $\frac{\partial \Pi_B^1}{\partial \lambda} < 0$.

Proof of Table 7. When Enterprise B makes decisions first, taking the cross-brand trade-in of both enterprises (Strategy Combination CC) as an example, we obtain the market share of consumers according to the utilities of first-time and repeat consumers of the two enterprises and Table 4. (1) The threshold of first-time consumers who purchase from Enterprise A is $x_A^1 = \frac{(1-\beta)v - p_A^2 + p_B^2 + t}{2t}$ and those who purchase from Enterprise B is $x_B^1 = \frac{(1-\beta)v - p_A^2 + p_B^2 + t}{2t}$. (2) The threshold of repeat consumers of Enterprise A who purchase from Enterprise A is $x_{AN}^1 = \frac{(1-\lambda)v - p_A^2 + \delta_2 r_B}{t}$ and those who purchase from Enterprise B is $x_{BN}^1 = \frac{(\lambda-\beta)v + p_B^2 + t - r_B}{t}$. (3) The threshold of repeat consumers of Enterprise B who purchase from Enterprise A is $x_{AN}^2 = \frac{(1-\lambda\beta)v - p_A^1 + \delta_2 r_B}{t}$ and those who purchase from Enterprise B is $x_{BN}^2 = \frac{(\lambda-1)\beta v + p_B^2 + t - r_B}{t}$. Submit these thresholds into the profit functions in the Strategy Combination CC:

$$\Pi_A^2 = \tau_0 p_A^2 \frac{(1-\beta)v - p_A^2 + p_B^2 + t}{2t} + \tau_1(p_A^2 - \delta_2 r_B + s)\frac{(1-\lambda)v - p_A^2 + \delta_2 r_B}{t} + \tau_2(p_A^2 - \delta_2 r_B + s)\frac{(1-\lambda\beta)v - p_A^1 + \delta_2 r_B}{t}$$

$$\Pi_B^2 = \tau_0 p_B^2\left(1 - \frac{(1-\beta)v - p_A^2 + p_B^2 + t}{2t}\right) + \tau_1(p_B^2 - r_B + s)\left(1 - \frac{(\lambda-\beta)v + p_B^2 + t - r_B}{t}\right) + \tau_2(p_B^2 - r_B + s)\left(1 - \frac{(\lambda-1)\beta v + p_B^2 + t - r_B}{t}\right)$$

According to inverse induction, the Hessian matrix of Π_A^2 with respect to p_A^2 and δ_2 is

$\begin{bmatrix} -\frac{\tau_0+2\tau_1+2\tau_2}{t} & \frac{2r_B(\tau_1+\tau_2)}{t} \\ \frac{2r_B(\tau_1+\tau_2)}{t} & -\frac{2r_B^2(\tau_1+\tau_2)}{t} \end{bmatrix}$, $|H_1| = -\frac{\tau_0+2\tau_1+2\tau_2}{t} < 0$, $|H_2| = \frac{2r_B^2(\tau_1+\tau_2)\tau_0}{t^2} > 0$. The Hessian matrix is negative definite, so Π_A^2 is a joint concave function with respect to p_A^2 and δ_2. Find the first order derivative of Π_A^2 with respect to p_A^2 and δ_2 respectively and setting them to 0, we have $p_A^2 = \frac{(1-\beta)v+t+p_B^2}{2}$ (7), $\delta_2 = \frac{((\lambda-\beta)v+s+t+p_B^2)\tau_1+((\lambda-1)\beta v+s+t+p_B^2)\tau_2}{2r_B(\tau_1+\tau_2)}$ (8).

Substitute (7) and (8) into Π_B^2 and find its Hessian matrix about p_B^2 and r_B: $\begin{bmatrix} -\frac{\tau_0+4\tau_1+4\tau_2}{2t} & \frac{2(\tau_1+\tau_2)}{t} \\ \frac{2(\tau_1+\tau_2)}{t} & -\frac{2(\tau_1+\tau_2)}{t} \end{bmatrix}$, $|H_1| = -\frac{\tau_0+4\tau_1+4\tau_2}{2t} < 0$, $|H_2| = \frac{(\tau_1+\tau_2)\tau_0}{t^2} > 0$. The Hessian matrix is negative definite, i.e., Π_B^2 is a joint concave function about p_B^2 and r_B, i.e., there is a (p_B^2, r_B) such that Π_B^2 has a maximal point. We can obtain the optimal solution without constraints by solving the first derivatives and setting them to 0: $p_B^2 = \frac{(\beta-1)v+3t}{2}$ (9), $\delta_2 = \frac{2\lambda v(\tau_1+\beta\tau_2)+(\tau_1+\tau_2)(2s+5t-(1+\beta)v)}{2\lambda v(\tau_1+\beta\tau_2)+2(\tau_1+\tau_2)(s+3t-v)}$ (10). Substitute (9) and (10) into (7) and (8) to get the values of p_A^2 and δ_2, we have $p_A^2 = \frac{(1-\beta)v+5t}{4}$ (11), $r_B = \frac{((\lambda-1)v+s+3t)\tau_1+((\lambda\beta-1)v+s+3t)\tau_2}{2(\tau_1+\tau_2)}$ (12).

According to p_A^2, p_B^2, δ_2, r_B, we obtain the demand of different types of consumers and the profits of the two enterprises in the Strategy Combination CC.

The process of solving the equilibrium solution for Strategy Combination WW, WC and CW is similar to that for Strategy Combination CC and we will not repeat. Table 7 summarizes the optimal solutions of the four strategy combinations.

Proof of Lemma 2. The proof is the same as when Enterprise A makes decisions first.

Proof of Proposition 3. As for the rebate, $r_B^W - r_B^C = \frac{\lambda v\tau_1(1-\beta)}{2(\tau_1+\tau_2)} > 0$. Moreover, $\frac{\partial r_A}{\partial \beta} = -\frac{v}{2} < 0$, $\frac{\partial r_A}{\partial \lambda} = \frac{v}{2} > 0$. As for the rebate correlation coefficient, $\delta_2^{WW} - \delta_2^{CW} = \frac{\lambda v\tau_1(1-\beta)((2\lambda-\beta-1)v+2s+5t)}{\left\{\begin{array}{l}2((\lambda\beta-1)v+s+3t)((\tau_1+\beta\tau_2)\lambda \\ -\tau_1-\tau_2)v+(\tau_1+\tau_2)(s+3t))\end{array}\right\}} > 0$,

$\delta_2^{CW} - \delta_2^{WC} = \frac{\left\{\begin{array}{l}\lambda v\tau_1(\beta-1)(((2\lambda-\beta-1)v+2s+5t)\tau_1^2 \\ +((1-\beta)v-t)\tau_1\tau_2-(((\lambda\beta-1)v+s+3t)2\tau_2^2)\end{array}\right\}}{\left\{\begin{array}{l}2(\tau_1+\tau_2)((\lambda\beta-1)v+s+3t)(((\lambda-1)v \\ +s+3t)\tau_1+((\lambda\beta-1)v+s+3t)\tau_2)\end{array}\right\}} > 0$, $\delta_2^{WC} - \delta_2^{CC} = \frac{\left\{\begin{array}{l}\lambda v\tau_1(1-\beta)(((2\lambda-\beta-1)v+2s+5t)\tau_1 \\ +(((2\lambda\beta-\beta-1)v+2s+5t)\tau_2)\end{array}\right\}}{\left\{\begin{array}{l}2(\tau_1+\tau_2)((\lambda\beta-1)v+s+3t)(((\lambda-1)v \\ +s+3t)\tau_1+((\lambda\beta-1)v+s+3t)\tau_2)\end{array}\right\}} > 0$,

i.e., $\delta_2^{CC} < \delta_2^{CW} < \delta_2^{WC} < \delta_2^{WW}$. We derive the sensitivity analysis of the discount correlation coefficient to brand difference coefficient and product durability by taking the first-order derivatives of the coefficient and find $\frac{\partial \delta_2}{\partial \beta} < 0$, $\frac{\partial \delta_2}{\partial \lambda} > 0$. The market share of repeat consumers of the two enterprises of the four strategy combinations under two decision sequences are equal, respectively. And we will not repeat. Other findings can be obtained from the equilibrium solutions in Table 7.

Proof of Proposition 4. We can easily obtain Lemma 8 by submitting p_A^2, p_B^2, δ_2, r_B into Π_A^2,

Π_B^2 to get the profits of the two enterprises. $\Pi_A^{2W} = \dfrac{\begin{Bmatrix} ((\beta-1)^2\tau_0 + 8(\lambda-1)^2\tau_1)v^2 + ((1-\beta)10t\tau_0 \\ +(1-\lambda)16s\tau_1)v + 8s^2\tau_1 + 25t^2\tau_0 \end{Bmatrix}}{32t}$,

$$\Pi_A^{2C} = \frac{\begin{Bmatrix} (8(\lambda-1)^2\tau_1^2 + ((\beta-1)^2\tau_0 + 16(\lambda-1)(\lambda\beta-1)\tau_2)\tau_1 + \tau_2(8(\lambda\beta-1)^2\tau_2 + (\beta-1)^2\tau_0))v^2 \\ -2(\tau_1+\tau_2)((\beta-1)5t\tau_0 + (\lambda-1)8s\tau_1 + (\lambda\beta-1)8s\tau_2)v + (\tau_1+\tau_2)(25t^2\tau_0 + 8s^2(\tau_1+\tau_2)) \end{Bmatrix}}{32t(\tau_1+\tau_2)},$$

$\Pi_A^{2W} - \Pi_A^{2C} = -\dfrac{\begin{Bmatrix} (((1+(2\beta-1)\lambda^2 - 2\lambda\beta)\tau_1 + (\lambda\beta-1)^2\tau_2)v^2 \\ +2s(\tau_1+\tau_2)(1-\lambda\beta)v + (\tau_1+\tau_2)s^2)\tau_2 \end{Bmatrix}}{4t(\tau_1+\tau_2)} < 0$, i.e., for the profit of Enterprise A ,we have $\Pi_A^{2WW} = \Pi_A^{2WC} \le \Pi_A^{2CW} = \Pi_A^{2CC}$. $\Pi_B^{2W} = \dfrac{\begin{Bmatrix} ((\tau_0 + (\lambda-1)^2 4\tau_2)\beta^2 + (1-2\beta)\tau_0)v^2 + ((6t\tau_0 \\ +(1-\lambda)8s\tau_2)\beta - 6t\tau_0)v + 4s^2\tau_2 + 9t^2\tau_0 \end{Bmatrix}}{16t}$,

$$\Pi_B^{2C} = \frac{\begin{Bmatrix} (4(\beta-\lambda)^2\tau_1^2 + ((\beta-1)^2\tau_0 + 8(1-\lambda)(\beta-\lambda)\tau_2)\tau_1 + (4\beta^2(\lambda-1)^2\tau_2 + (\beta-1)^2\tau_0)\tau_2)v^2 \\ -2(\tau_1+\tau_2)((1-\beta)3t\tau_0 + (\lambda-\beta)4s\tau_1 + (\lambda-1)4\beta s\tau_2)v + (\tau_1+\tau_2)(9t^2\tau_0 + 4s^2(\tau_1+\tau_2)) \end{Bmatrix}}{16t(\tau_1+\tau_2)},$$

$\Pi_B^{2W} - \Pi_B^{2C} = -\dfrac{\begin{Bmatrix} (((\beta-\lambda)^2\tau_1 + ((\lambda+1)\beta - 2\lambda)(1-\lambda)\beta\tau_2)v^2 \\ +2s(\tau_1+\tau_2)(\beta-\lambda)v + (\tau_1+\tau_2)s^2)\tau_1 \end{Bmatrix}}{4t(\tau_1+\tau_2)} < 0$, i.e., for the profit of Enterprise B ,we have $\Pi_B^{2WW} = \Pi_B^{2CW} \le \Pi_B^{2WC} = \Pi_B^{2CC}$. Moreover, $\dfrac{\partial\Pi_A^{2W}}{\partial\beta} = \dfrac{((\beta-1)v - 5t)\tau_0 v}{16t} < 0$, $\dfrac{\partial\Pi_A^{2W}}{\partial\lambda} = -\dfrac{((1-\lambda)v + s)\tau_1 v}{2t} < 0$, $\dfrac{\partial\Pi_B^{2W}}{\partial\beta} = \dfrac{(((\lambda-1)^2 4\beta\tau_2 + (\beta-1)\tau_0)v + (1-\lambda)4s\tau_2 + 3t\tau_0)v}{8t} > 0$, $\dfrac{\partial\Pi_B^{1W}}{\partial\lambda} = -\dfrac{\beta v\tau_2((1-\lambda)\beta v + s)}{2t} < 0$;

$\dfrac{\partial\Pi_A^{2C}}{\partial\beta} = -\dfrac{\begin{Bmatrix} (((1-\beta\lambda)v + s)8\lambda\tau_2^2 + (((1-\lambda)8\lambda\tau_1 + (1-\beta)\tau_0)v \\ +8\lambda s\tau_1 + 5t\tau_0)\tau_2 + \tau_0\tau_1((1-\beta)v + 5t))v \end{Bmatrix}}{16t(\tau_1+\tau_2)} < 0$, $\dfrac{\partial\Pi_A^{2C}}{\partial\lambda} = -\dfrac{(\beta\tau_2 + \tau_1)(((1-\lambda)v + s)\tau_1 + ((1-\beta\lambda)v + s)\tau_2)v}{2t(\tau_1+\tau_2)} < 0$,

$$\frac{\partial\Pi_B^{2C}}{\partial\beta} = -\frac{\begin{Bmatrix} (((1-\lambda)\beta v + s)(\lambda-1)4\tau_2^2 + (((\lambda-1)(2\beta-\lambda)4\tau_1 + (1-\beta)\tau_0)v \\ +4(\lambda-2)s\tau_1 - 3t\tau_0)\tau_2 - \tau_1(((\beta-\lambda)4\tau_1 + (\beta-1)\tau_0)v + 4s\tau_1 + 3t\tau_0)v \end{Bmatrix}}{8t(\tau_1+\tau_2)} > 0,$$

$\dfrac{\partial\Pi_B^{2C}}{\partial\lambda} = -\dfrac{(\beta\tau_2 + \tau_1)(((\beta-\lambda)v + s)\tau_1 + ((1-\lambda)\beta v + s)\tau_2)v}{2t(\tau_1+\tau_2)} < 0$. That is, $\dfrac{\partial\Pi_A^2}{\partial\beta} < 0$, $\dfrac{\partial\Pi_B^2}{\partial\beta} > 0$, $\dfrac{\partial\Pi_A^2}{\partial\lambda} < 0$, $\dfrac{\partial\Pi_B^2}{\partial\lambda} < 0$.

Proof of Table 8. When Enterprise A makes decisions first, we discuss cases where the repeat consumer market is partially or fully covered as both enterprises adopt the cross-brand trade-in. The Case PP is consistent with Strategy Combination CC. Take Case FF as an example. (1) The threshold of first-time consumers who purchase from Enterprise A is $x_A^1 = \dfrac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t}$ and those who purchase from Enterprise B is $x_B^1 = \dfrac{(1-\beta)v - p_A^1 + p_B^1 + t}{2t}$. (2) The threshold of repeat consumers of Enterprise A who purchase from Enterprise A is $x_{AN}^1 = \dfrac{(1-\beta)v + (1-\delta_1)r_A - p_A^1 + p_B^1 + t}{2t}$ and those who purchase from Enterprise B is $x_{BN}^1 = \dfrac{(1-\beta)v + (1-\delta_1)r_A - p_A^1 + p_B^1 + t}{2t}$. (3) The threshold of repeat consumers of Enterprise B who purchase from Enterprise A is $x_{AN}^2=\frac{(1-\beta)v+(1-\delta_1)r_A-p_A^1+p_B^1+t}{2t}$ and those who purchase from Enterprise B is $x_{BN}^2=\frac{(1-\beta)v+(1-\delta_1)r_A-p_A^1+p_B^1+t}{2t}$. Submit these thresholds into the profit functions in the Case FF. The subsequent steps are similar to those of Proposition 1 and we will not repeat. The optimal solutions are summarized in Table 8.

Proof of Lemma 3. The proof is the same as that of the basic models.

Proof of Proposition 5. Compare the rebates pairwise under different market competition situations. $r_A^{FF}-r_A^{FP}=\frac{\tau_2(((1-\lambda)3\beta v+3s-10t)\tau_1+((1-\lambda)\beta v+s-3t)4\tau_2)}{\tau_1^2+8\tau_1\tau_2+8\tau_2^2}<0$, $r_A^{FP}-r_A^{PF}<0$, $r_A^{PF}-r_A^{PP}<0$, i.e., $r_A^{PP}>r_A^{PF}>r_A^{FP}>r_A^{FF}=s$. Compare the rebate correlation coefficient pairwise, the magnitude of δ_1^{PP} and δ_1^{PF} is influenced by the durability of the product. When the repeat consumer market of Enterprise B is fully covered, the rebate correlation coefficient is higher, and we have $\delta_1^{PF}<\delta_1^{FF}=1$. Submit the equilibrium solutions into the profit functions and compare the profits pairwise under different market competition situations. For both enterprises, profits are higher when repeat consumer market of Enterprise A is fully covered. Moreover, $\Pi_A^{1FF}-\Pi_A^{1FP}>0$, $\Pi_B^{1FF}-\Pi_B^{1FP}>0$.

Proof of Table 9. The calculation process is similar to that of when Enterprise B makes decisions first. The optimal solutions are summarized in Table 9.

Proof of Lemma 4. The proof is the same as that of the basic models.

Proof of Proposition 6. Compare the rebates pairwise under different market competition situations. $r_B^{FF}-r_B^{FP}=\frac{\tau_2(((1-\lambda\beta)3v+3s-10t)\tau_1+((1-\lambda\beta)v+s-3t)4\tau_2)}{\tau_1^2+8\tau_1\tau_2+8\tau_2^2}<0$, $r_B^{FP}-r_B^{PF}<0$, $r_B^{PF}-r_B^{PP}<0$, i.e., $r_B^{PP}>r_B^{PF}>r_B^{FP}>r_B^{FF}=s$. Compare the rebate correlation coefficient pairwise, and we have $\delta_2^{FP}<\delta_2^{PF}<\delta_2^{PP}<\delta_2^{FF}=1$. Submit the equilibrium solutions into the profit functions and compare the profits pairwise under different market competition situations. For both enterprises, profits are higher when repeat consumer market of Enterprise B is fully covered. Moreover, $\Pi_A^{2FF}-\Pi_A^{2FP}>0$, $\Pi_B^{2FF}-\Pi_B^{2FP}>0$.

Proof of Proposition 7. We take the difference of the profits of the two enterprises under different decision sequences: $p_A^1-p_A^2=\frac{(1-\beta)v+t}{4}>0$; $p_B^1-p_B^2=\frac{(1-\beta)v-t}{4}$, if $\beta<\frac{v-t}{v}$, $p_B^1>p_B^2$, and if $\beta\geq\frac{v-t}{v}$, $p_B^1\leq p_B^2$. $r_A^W-r_B^W=\frac{(1-\beta)(1+\lambda)v}{2}>0$, $r_A^C-r_B^C=\frac{(1-\beta)v}{2}>0$; $r_A^{PP}>r_B^{PP}$, $r_A^{PF}>r_B^{PF}$, $r_A^{FP}>r_B^{FP}$, $r_A^{FF}=r_B^{FF}$; $\delta_1^{WW}<\delta_2^{WW}$, $\delta_1^{WC}<\delta_2^{WC}$, $\delta_1^{CW}<\delta_2^{CW}$, $\delta_1^{CC}<\delta_2^{CC}$; $\delta_1^{PP}<\delta_2^{PP}$, $\delta_1^{PF}<\delta_2^{PF}$, $\delta_1^{FP}<\delta_2^{FP}$, $\delta_1^{FF}=\delta_2^{FF}$.

Proof of Proposition 8. Take the difference of the profits of the two enterprises under different decision sequences. In the basic model, we have $\Pi_A^{1W}-\Pi_A^{2W}=\frac{((\beta-1)^2v^2+(1-\beta)2tv-7t^2)\tau_0}{32t}<0$, $\Pi_B^{1W}-\Pi_B^{2W}=\frac{(-(\beta-1)^2v^2+(1-\beta)2tv+7t^2)\tau_0}{32t}>0$, $\Pi_A^{1C}-\Pi_A^{2C}=\frac{((\beta-1)^2v^2+(1-\beta)2tv-7t^2)\tau_0}{32t}<0$, $\Pi_B^{1C}-\Pi_B^{2C}=\frac{(-(\beta-1)^2v^2+(1-\beta)2tv+7t^2)\tau_0}{32t}>0$, i.e., $\Pi_A^1<\Pi_A^2$, $\Pi_B^1>\Pi_B^2$. The solution process of the extended model is consistent with that of the basic model. So, we have $\Pi_A^1<\Pi_A^2$, $\Pi_B^1>\Pi_B^2$.