Origin Problem (P1)

The objective is to maximize the total fare.

$$\max \sum_{a} \sum_{(i,j)\in L} \pi_{i,j} \times x_{i,j}^{a} \tag{1}$$

The capacity of each arc is bounded:

$$\sum_{a} x_{i,j}^{a} \le Cap_{i,j}, \quad \forall (i,j) \in L \tag{2}$$

Flow conservation constraints:

$$\sum_{i:(i,j)\in L} x_{i,j}^{a} - \sum_{i:(j,i)\in L} x_{i,j}^{a} = \begin{cases} -1 & j = o\left(a\right) \\ 1 & j = d\left(a\right) \\ 0 & otherwise \end{cases}, \quad \forall a \tag{3}$$

BRUE constratins:

$$\sum_{(i,j)\in L} \pi_{i,j} \times x_{i,j}^{a} \le \sum_{(i,j)\in \phi(w_a,k)} \pi_{i,j} + \varepsilon\left(a\right), \quad \forall a, k \in \Omega\left(w_a\right) \tag{4}$$

The fare on each arc should be bounded: (We assume $0 \le \underline{\pi}_{i,j} \le \bar{\pi}_{i,j}$)

$$\pi_{i,j} \in \left[\underline{\pi}_{i,j}, \bar{\pi}_{i,j}\right], \ \forall (i,j) \in L \tag{5}$$

Other constraints:

$$x_{i,j}^{a} = \{0,1\}, \quad \forall a, (i,j) \in L \tag{6}$$

Notifications:

- $x_{i,j}^{a}$ is an indicator of whether agent a pass through link (i,j).
- $\Omega\left(w_a\right)$ is the set of possible paths of OD pair $w(a)$ of agent a.
- $\phi\left(w_a, k\right)$ is the set of links in the k th path of OD pair $w(a)$ of agent a.

Linearization

Proposion : $Z_{i,j}^{a} \Leftrightarrow \left(\pi_{i,j}, x_{i,j}^{a}\right)$ when $0 \le Z_{i,j}^{a} \le \pi_{i,j} \times x_{i,j}^{a}$ and $Z_{i,j}^{a} \le \bar{\pi}_{i,j} \times x_{i,j}^{a}$

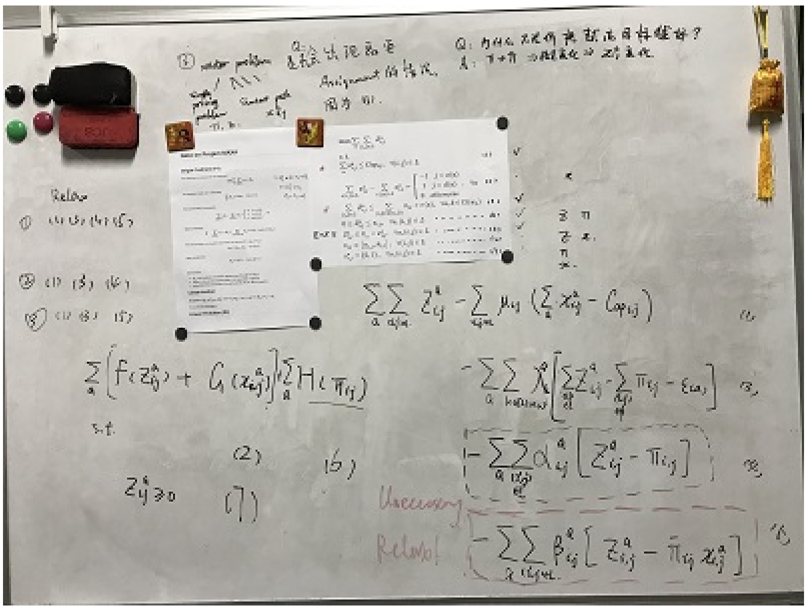
P1 and P2 are equivalent.

Linear Problem (P2)

$$
\begin{aligned}
&\max \sum_{a} \sum_{(i,j)\in L} Z_{i,j}^{a} \\
&s.t. \\
&\sum_{a} x_{i,j}^{a} \leq Cap_{i,j}, \quad \forall (i,j)\in L \\
&\sum_{i:(i,j)\in L} x_{i,j}^{a} - \sum_{i:(j,i)\in L} x_{i,j}^{a} = \begin{cases} -1 & j=o(a) \\ 1 & j=d(a) \quad , \quad \forall a \\ 0 & otherwise \end{cases} \\
&\sum_{(i,j)\in L} Z_{i,j}^{a} \leq \sum_{(i,j)\in \phi(w_a,k)} \pi_{i,j} + \varepsilon(a), \quad \forall a, k \in \Omega(w_a) \\
&0 \leq Z_{i,j}^{a} \leq \pi_{i,j}, \quad \forall a, (i,j)\in L \\
&Z_{i,j}^{a} \leq \bar{\pi}_{i,j} \times x_{i,j}^{a}, \quad \forall a, (i,j)\in L \\
&\pi_{i,j} \in \left[\underline{\pi}_{i,j}, \bar{\pi}_{i,j}\right], \quad \forall (i,j)\in L \\
&x_{i,j}^{a} = \{0,1\}, \quad \forall a, (i,j)\in L
\end{aligned}
\tag{7}
$$

Lagrangian Decomposition

$$
\begin{aligned}
&\max \sum_{a} \sum_{(i,j)\in L} Z_{i,j}^{a} - \lambda \times \left(\sum_{(i,j)\in L} Z_{i,j}^{a} - \sum_{(i,j)\in \phi(w_a,k)} \pi_{i,j} - \varepsilon(a) \right) - \sum_{(i,j)\in L} \mu_{i,j} \times \left(\sum_{a} x_{i,j}^{a} - Cap_{i,j} \right) \\
&\sum_{i:(i,j)\in L} x_{i,j}^{a} - \sum_{i:(j,i)\in L} x_{i,j}^{a} = \begin{cases} -1 & j=o(a) \\ 1 & j=d(a) \\ 0 & otherwise \end{cases} \\
&0 \leq Z_{i,j}^{a} \leq \pi_{i,j} \\
&Z_{i,j}^{a} \leq \bar{\pi}_{i,j} \times x_{i,j}^{a} \\
&\pi_{i,j} \in \left[\underline{\pi}_{i,j}, \bar{\pi}_{i,j}\right] \\
&x_{i,j}^{a} = \{0,1\}
\end{aligned}
\tag{8}
$$



$$
\max \sum_{a} \sum_{(i,j)\in L} \beta_{i,j} \times \pi_{i,j} \times x_{i,j}^{a}
$$