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PHYC30012 – Computational Physics Report 3: Monte-Carlo Simulation of a Neutrino Beam

Aim

To study the simulation of probabilistic events with the use of *Monte-Carlo Methods,* which is the use of random number generators in C to simulate probabilistic events.

The methods used to simulate probabilistic events are then employed to study the scattering of neutrino beam formed by the decay of pion, π^+ and kaon, K^+ mesons.

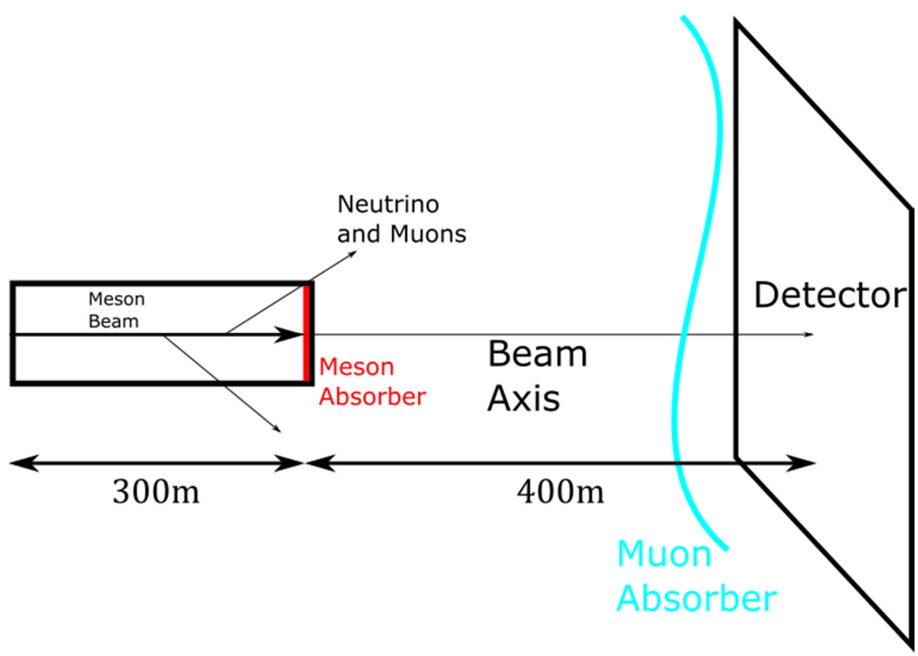
Theory

Neutrinos are a by-product of the decay of both π^+ and K^+ mesons:

$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$
$$K^+ \rightarrow \mu^+ + \nu_\mu$$

Each with a probability of 1.00 and 0.64 respectively. Where μ^+ and ν_μ are the muon and the muonic neutrino produced in the decay of mesons. By producing a beam of mesons, the production of neutrinos and how it scatters can then be studied and simulated using computers.

The experimental setup to be simulated:



The production of meson is done by bombarding protons onto a target, the mesons produced then travels down a $300m$ tunnel, where some of them decay into both muons and neutrinos.

The undecayed mesons are then absorbed by at the end of the tunnel. The distance travelled before decaying down the tunnel, s, is probabilistic and will be discussed further in the next section.

$400m$ away from the tunnel of mesons, a neutrino detector which is circular with radius of $1.5m$ is placed, with a muon absorber in front of it to absorb the muons produced from the decay, to prevent interference with the neutrinos.

For simplicity, the beam of meson produced is treated as infinitely thin and can be treated as the x axis.

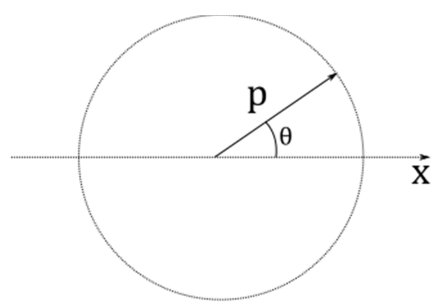
Consider the rest frame of the π^+ meson; by employing conservation of momentum and energy, the momentum of the neutrino, assumed massless, can be shown to be (c can be set to 1 if the momentum's unit is GeV/c):

$$|p_v| = \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right) - (1)$$

For neutrino formed by the decay of kaons, simply replace m_π to m_K. Pion and kaon have masses of $m_\pi = 0.1396GeV/c^2$ and $m_K = 0.4937GeV/c^2$, while the muon produced has a mass of $m_\mu = 0.1057GeV/c^2$.

Furthermore in the rest frame of the meson, the decay is isotropic, meaning there no preferred direction and has equal probability to decay in any direction. However in the lab frame, the rest's frame there is a preferred direction towards the front as the meson is travelling in the forward direction.

Assuming azimuthal symmetry about the beam axis, where the neutrinos hit the detector can be found by first resolving the momentum into longitudinal and transverse components:



In meson's rest frame

$$p_l = p\ cos\theta$$
$$p_t = p \sin\theta$$

To reflect the isotropic nature of the decay, the values of θ is uniformly distributed between 0 to π.

The translation between the two frames' momenergy can be done using Lorentz transformation:

$$\begin{pmatrix} p_l \\ p_t \\ E \end{pmatrix}_{lab} = \begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_l \\ p_t \\ E \end{pmatrix}_{rest}$$

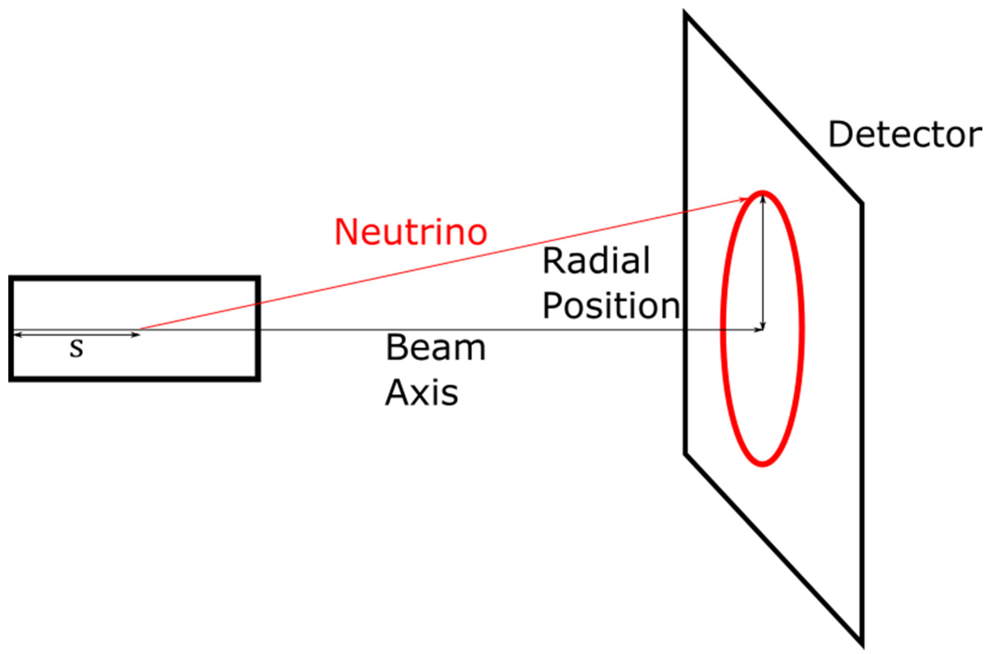
Where the relativistic beta value $\beta = \frac{v}{c}$ and the Lorentz factor γ is:

$$\beta = \frac{|p_\pi|}{\sqrt{p_\pi^2 + m_\pi^2}} \ \& \ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Also $E = |p_\nu|$, which is the energy of the produced neutrino. Writing out the transformation to the lab frame explicitly:

$$p_{l_{lab}} = \gamma\left(p_{l_{rest}} + E_{rest}\beta\right) - (2)$$
$$p_{t_{lab}} = p_{l_{rest}} - (3)$$

Finally the radial position for the neutrino produced for the meson decayed within $300m$ upon production is then is then given by:



$$R = (700 - s)\left|\frac{p_t}{p_l}\right|$$

Hence once s,p and θ is known, the radial distance can be evaluated with ease.

As the process of decay and the direction of neutrino produced are probabilistic in nature, the simulation of this experiment can be done by a computer using *Monte-Carlo Methods*, which refers to a collection of computational techniques to simulate random events.

Implementation of Monte-Carlo Methods

Production of Muons: Normal Distribution

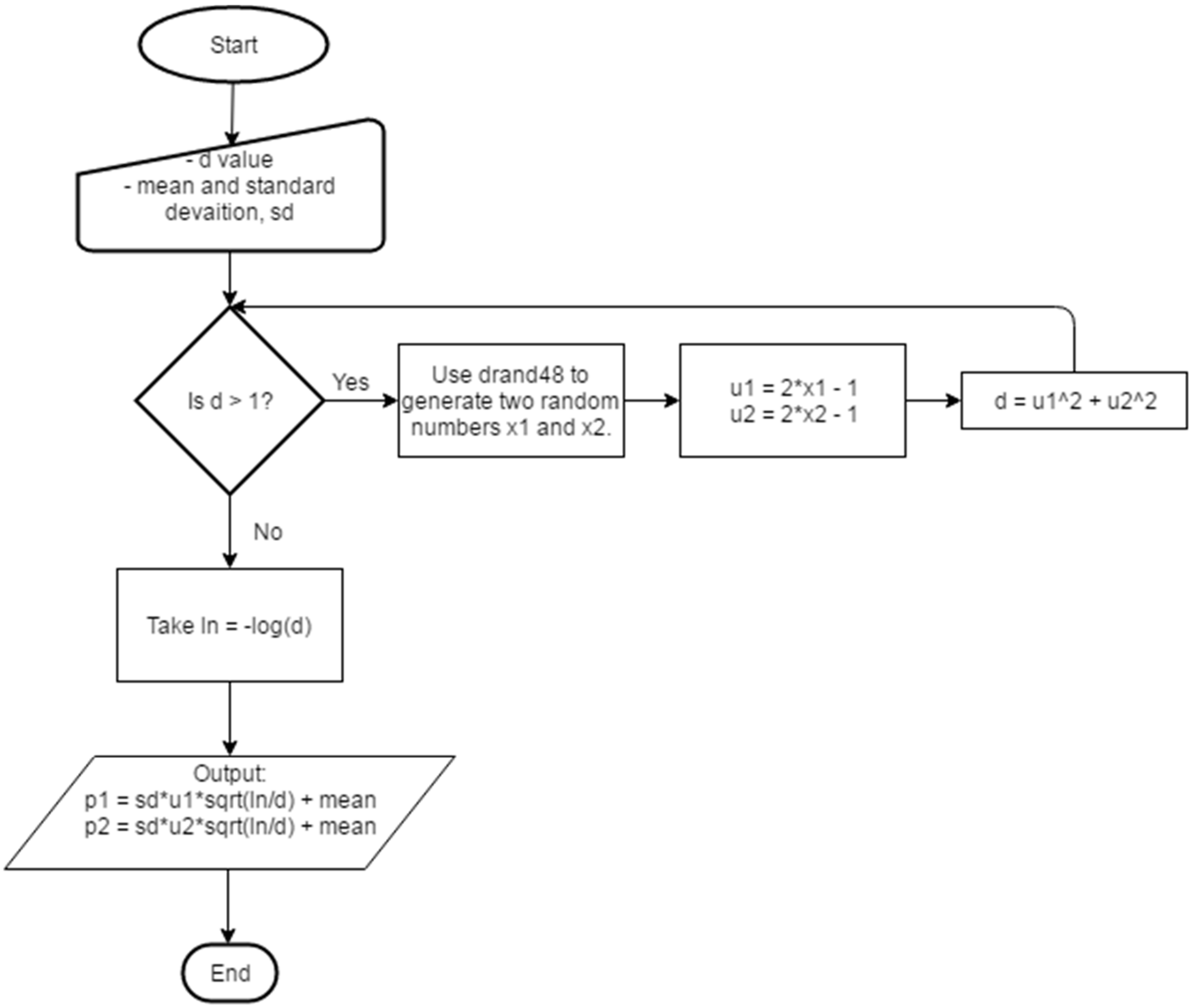
As the momentum of the pion and kaon produced by the bombardment of protons onto the target is normally distributed with a mean of $200GeV/c$ and standard deviation of $10GeV/c$.

The simulation of the generation of the random variable subjected to the normal distribution of mean μ and standard-deviation σ can be done using *Box-Muller Method:*

1. Generate two terms x_1 and x_2 that are both uniformly distributed along the interval $[0,1]$.
2. Define new variables $u_1 = 2x_1 - 1$ and $u_2 = 2x_2 - 1$, and take $d = u_1^2 + u_2^2$.
3. If $d > 1$, return to step (1).
4. Otherwise when $d \leq 1$, take $y_1 = \sigma \times u_1\sqrt{-\frac{\ln(d)}{d}} + \mu$ and $y_2 = \sigma \times u_2\sqrt{-\frac{\ln(d)}{d}} + \mu$.

This generates two numbers y_1 and y_2 that are normally distributed and independent of each other. This provides an advantage of generating the momentum of both the pion and kaon simultaneously. The histogram of the momentum of generated mesons will be plotted shortly alongside with the decay of mesons.

The Box-Muller method subroutine has the following subroutine, which is to be included in the programs later:



Decay of Mesons: Exponential Distribution

The unstable particles when travelling through the tunnel will decay, the probability distribution function for its decay in terms of the time passed through in the particle's rest frame follows an exponential distribution:

$$f(t) = \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}}$$

Where τ_0 is the characteristic time, the time taken for the number of particles to reduce to by a factor of e.

As particles that have decayed no longer exist at later time, the proportion of particles that have decayed at time t found by taking the cumulative distribution of $F(t) = \int_0^t f(t')dt'$ which leads to:

$$F(t) = 1 - e^{-\frac{t}{\tau_0}}$$

The cumulative distribution can be used by C to simulate the time at which the particle have decayed by first generating u which is uniformly distributed between $[0,1]$, and then finding t such that:

$$u = F(t)$$

Hence whenever C generates a random number u, it can be transformed into t by:

$$t = F^{-1}(u)$$

Which for this case:

$$t = -\tau_0 \ln u$$

However since in the experiment, in order to determine the amount of particles that have decayed over the distance travelled s in the lab frame. For a particle with momentum p and mass m given in GeV/c and GeV/c^2 respectively, the distance travelled in time t (in the particle's rest frame) is:

$$s = \frac{p}{m}ct$$

The cumulative distribution $F(t)$ then can be expressed in terms of s:

$$F(s) = 1 - e^{-\frac{m}{pc\tau_0}s}$$

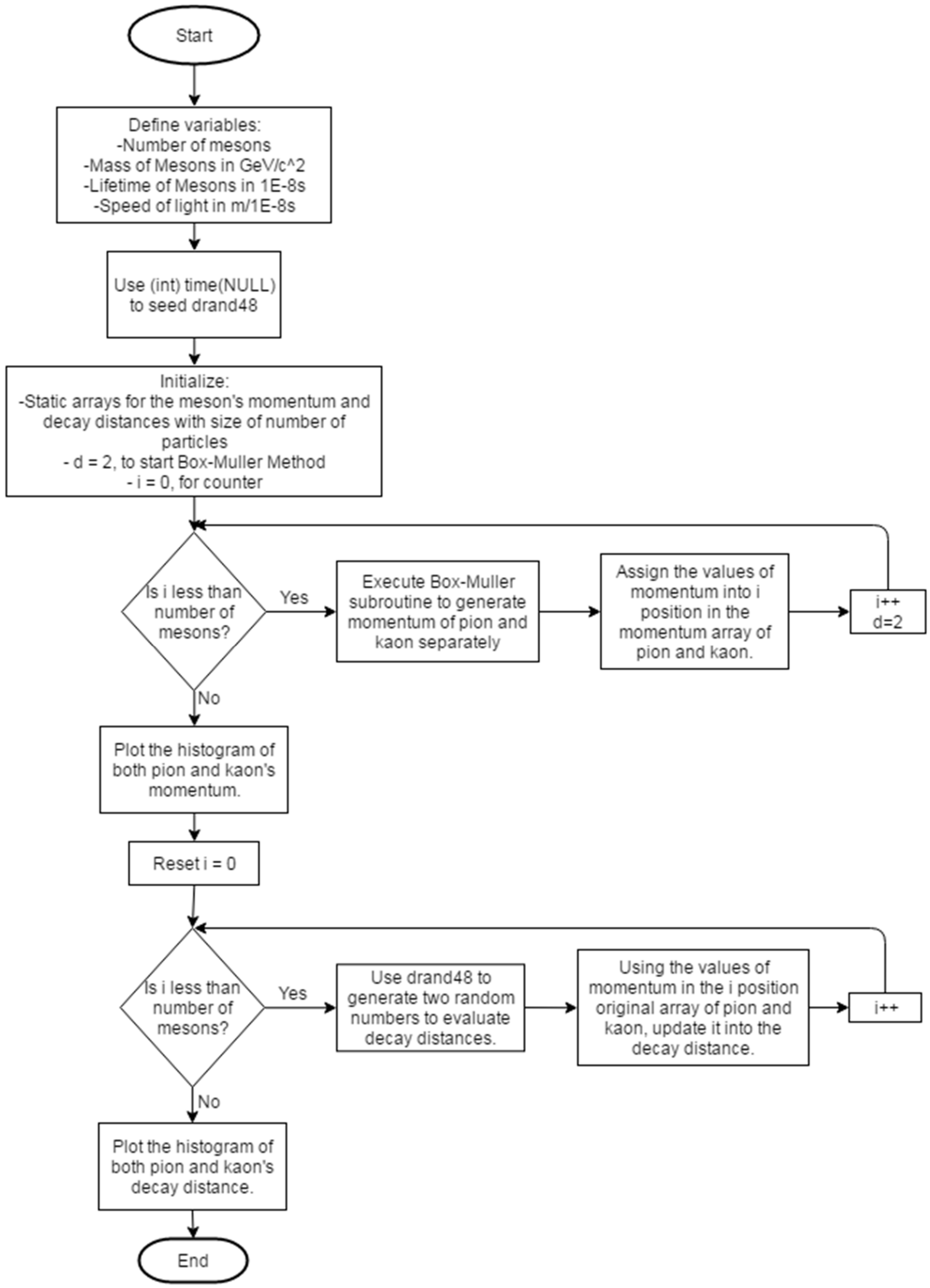
Hence to simulate the distance the particle have travelled before decaying using the uniformly randomly distributed random variable u:

$$s = -\frac{pc\tau_0}{m}\ln u$$

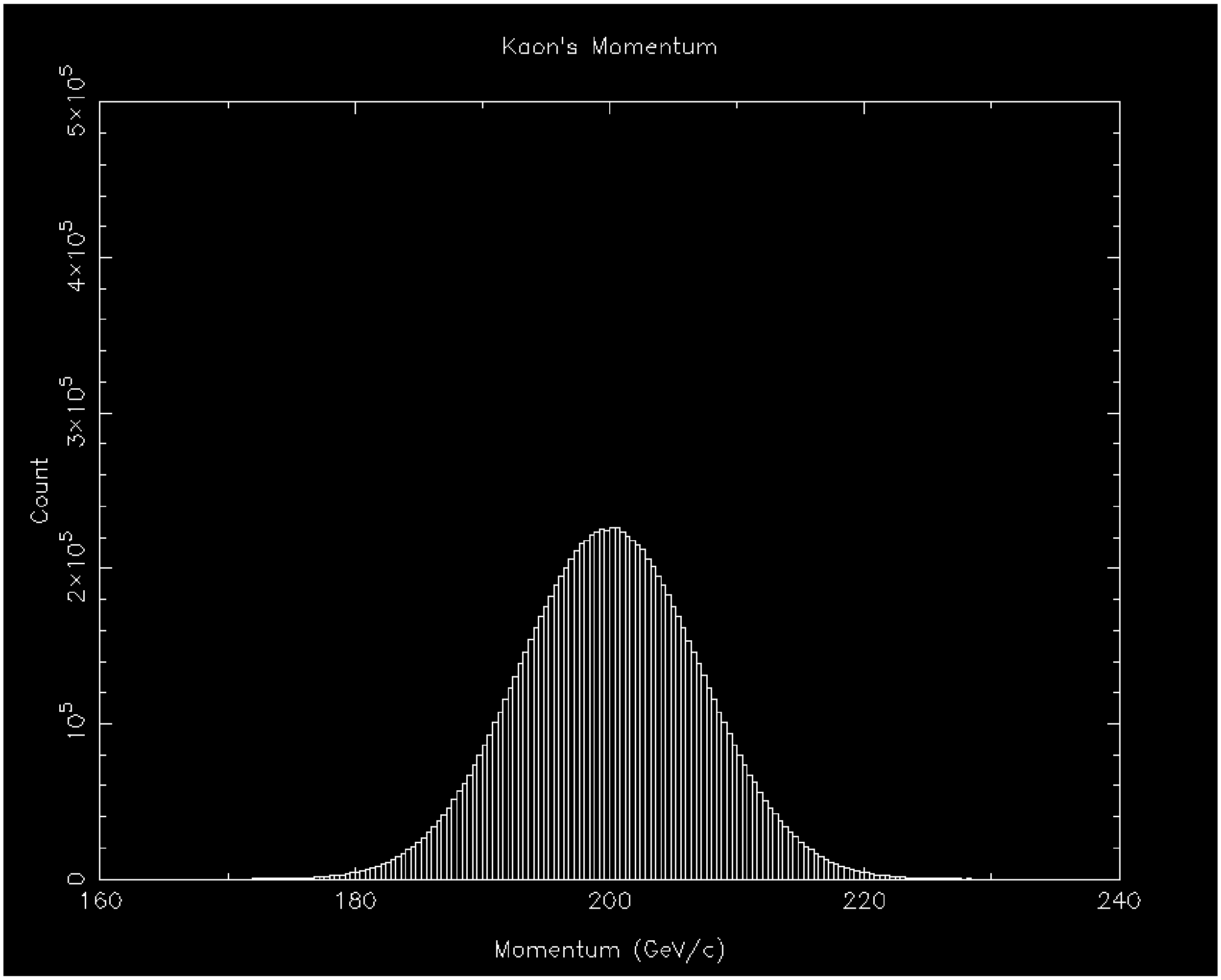
By using the randomly generated value of p and u, the distance the particle have travelled before decaying can be found. This is done in the file *meson_decay.c,* which runs a simulation of 10^7 mesons of each type, and plotting the histogram of both the particle's momentum upon production and the distance travelled before decaying.

Also take note that as in C, there is a limit on the array size that can be taken within a function to prevent interfering other arrays in its memory. Fortunately, this can be circumvented by setting the arrays which records the masons' momentum and decay distance as static rather than automatic.

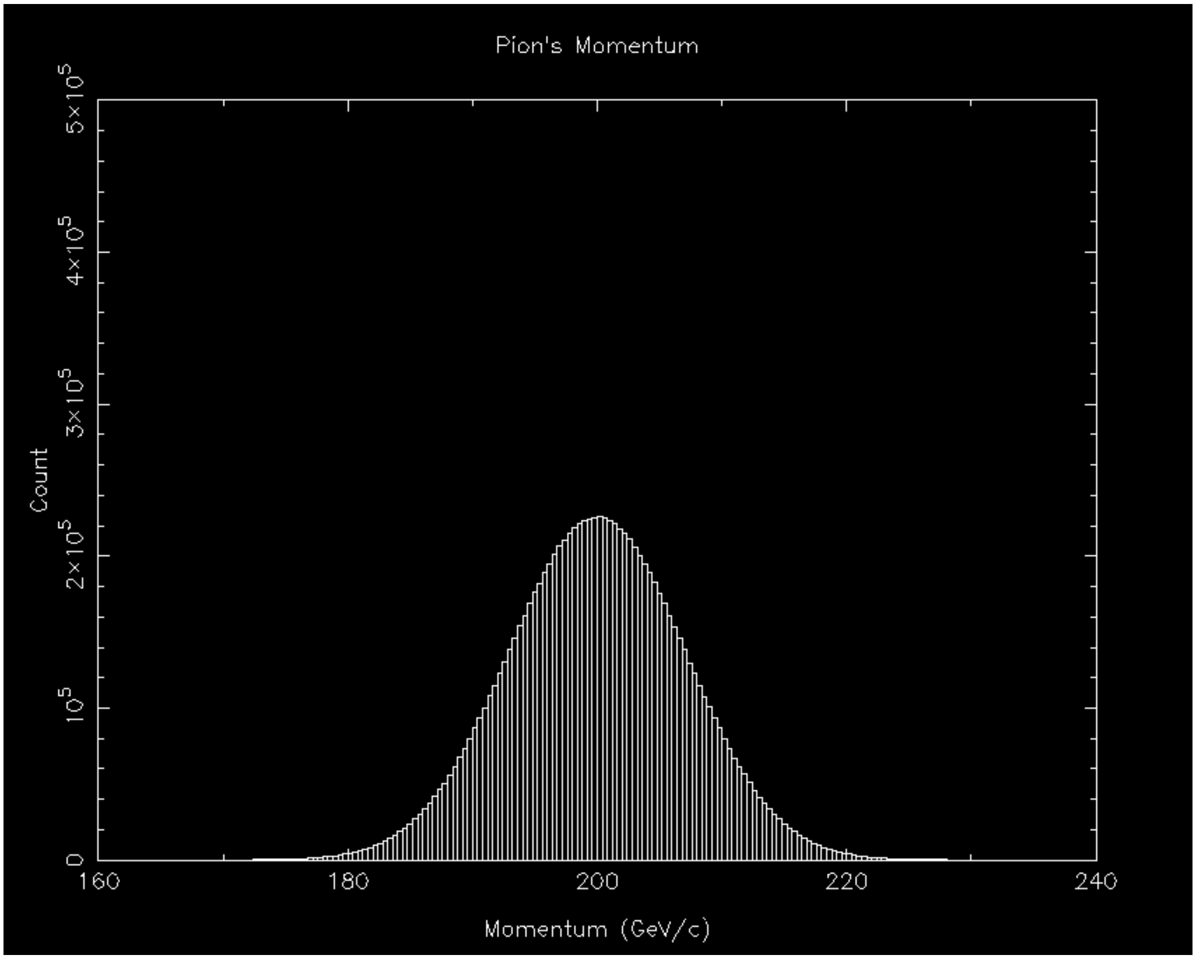
The program *meson_decay.c* has the following flowchart:



The units used in the program are GeV/c for momentum, $10^{-8}s$ for time, and m for distance.
Obtaining the histogram for momentum p:

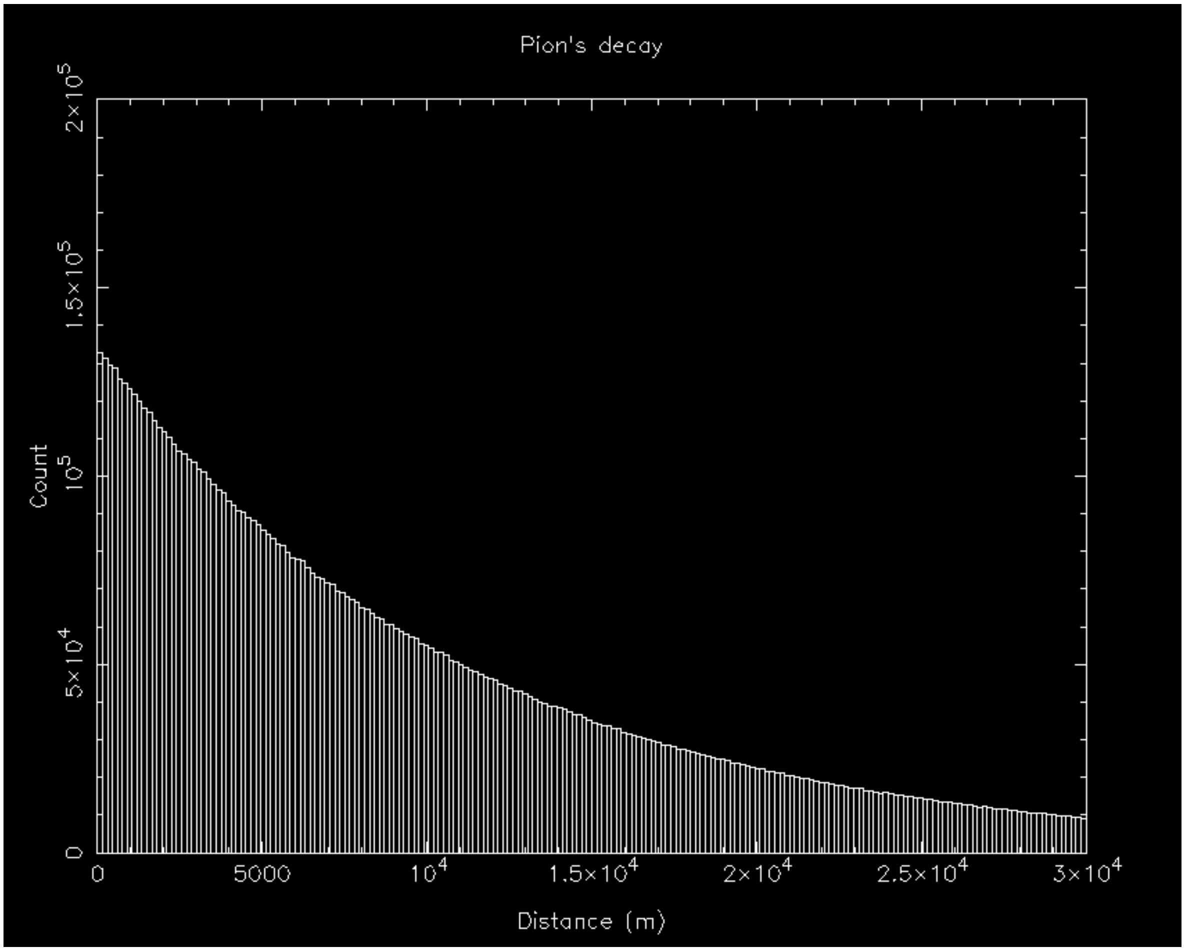


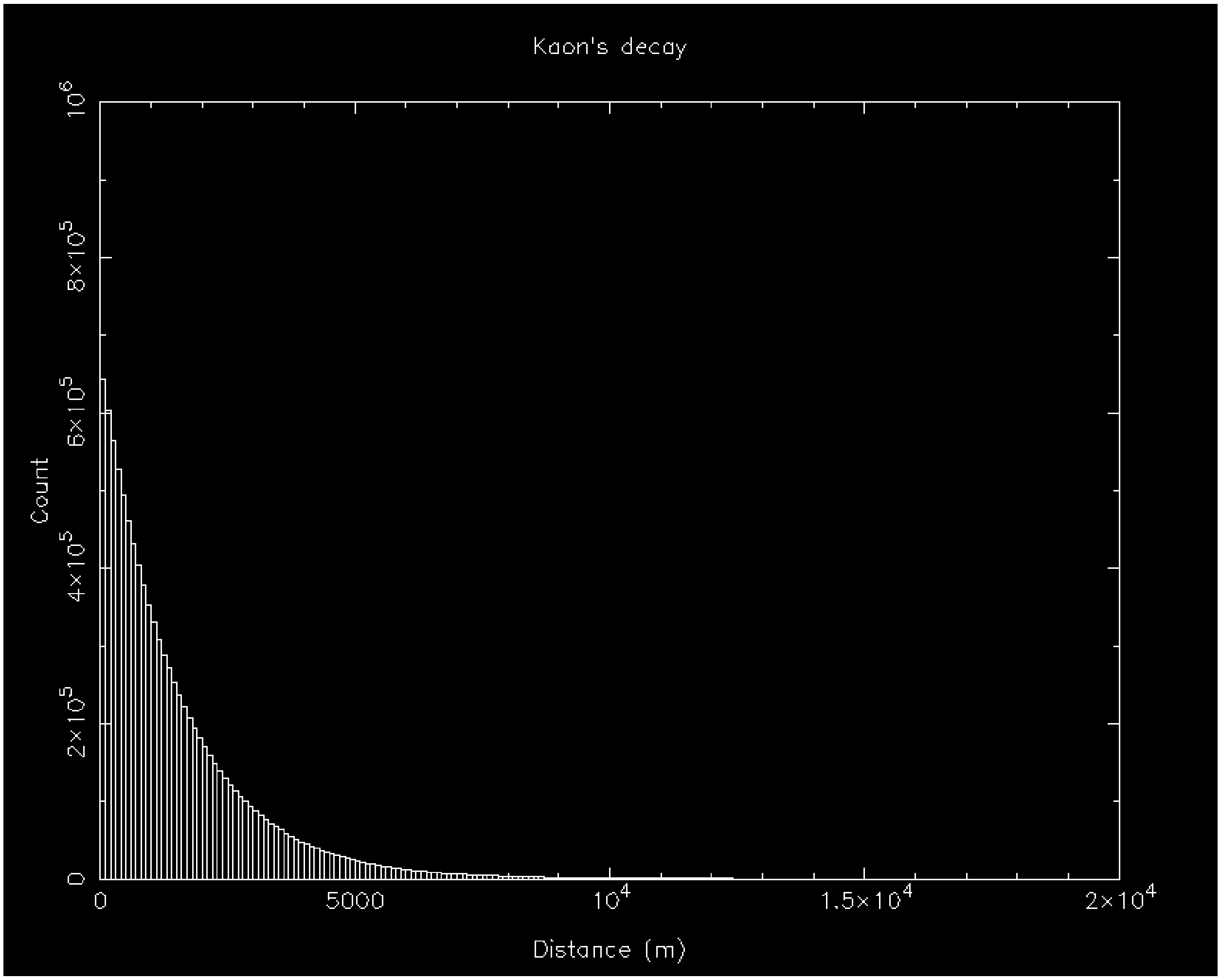
Kaon's Momentum
Count
Momentum (GeV/c)
0
10^5
2×10^5
3×10^5
4×10^5
5×10^5
160
180
200
220
240



As the distribution of momentum for both the π^+ and K^+ meson are both normally distributed with equal mean and standard-deviation, it is not surprising that they have identical histogram.

Obtaining the histogram for decay distance s:


Pion's decay
Count
0
5×10^4
10^5
1.5×10^5
2×10^5
0
5000
10^4
1.5×10^4
2×10^4
2.5×10^4
3×10^4
Distance (m)



For the decay length however, it can be seen that for π^+ fewer of them decay at shorter distances compared to K^+; due to their higher lifetime and lower mass, which results in them travelling faster and longer before decaying. The number of decay particles decayed at each distance is later used to determine the radial position upon detection.

<u>Momentum of Produced Neutrino in Lab Frame</u>

Considering the finite decay length of $300m$ to reflect the experiment, as in the rest frame of the mesons, the neutrino produced have a fixed momentum from equation (1):

$$|p_\nu| = \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)$$

However as there is no preferred direction of decay, and the momentum of the decayed meson is not fixed; the momentum of the neutrino in the lab frame produced is also probabilistic. The simulation of the momentum can be done by considering the arbitrary decay angle θ and translating into the lab frame via Lorentz transformation, where the longitudinal and transverse components are found separately from equation (2) and (3), the magnitude of momentum is then found via:

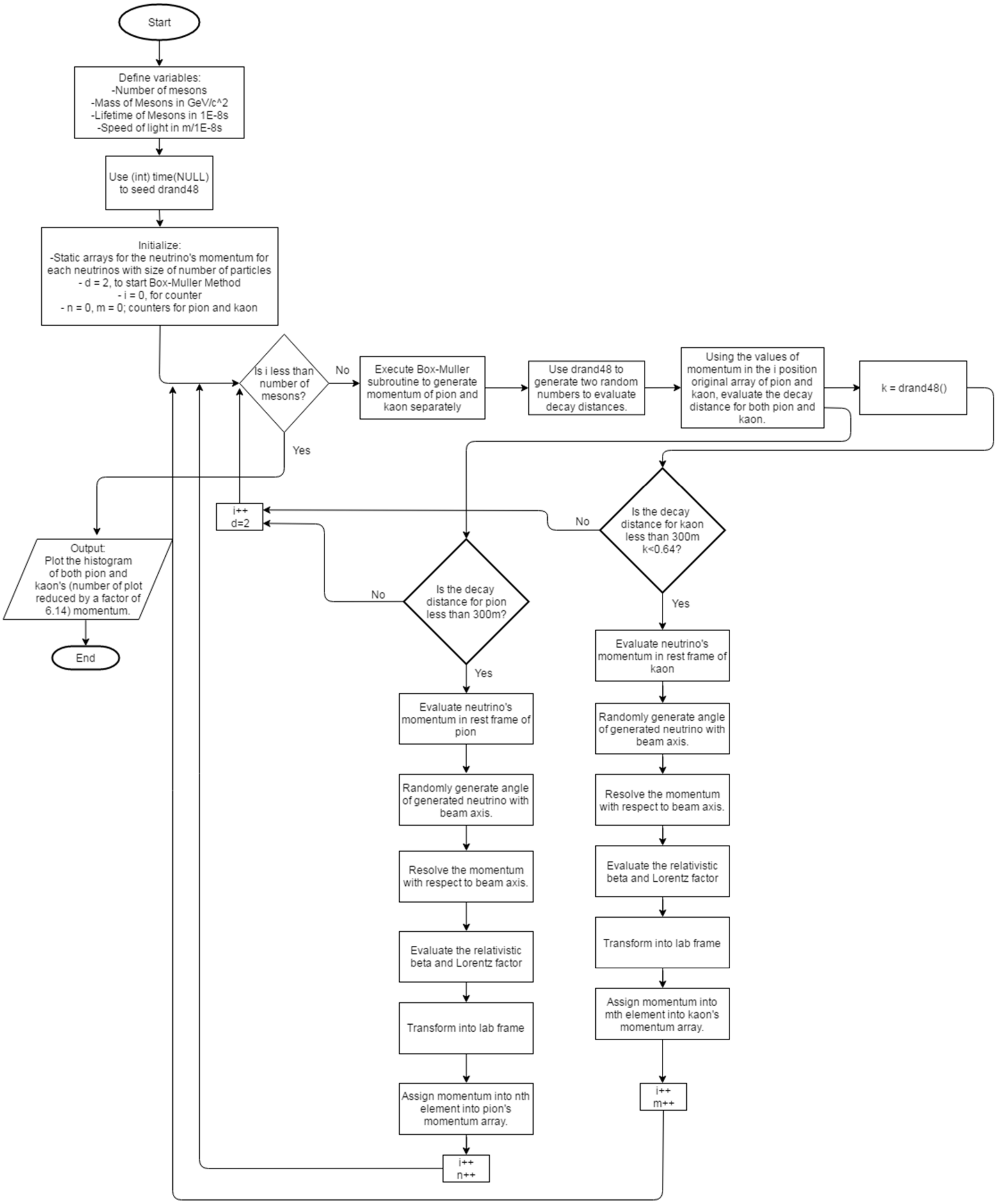
$$|p_{lab}| = \sqrt{p_{l_{lab}}^2 + p_{t_{lab}}^2}$$

Furthermore only mesons that have decayed before 300m will be detected by the neutrino, so for the particles with higher decay distances will be discarded.

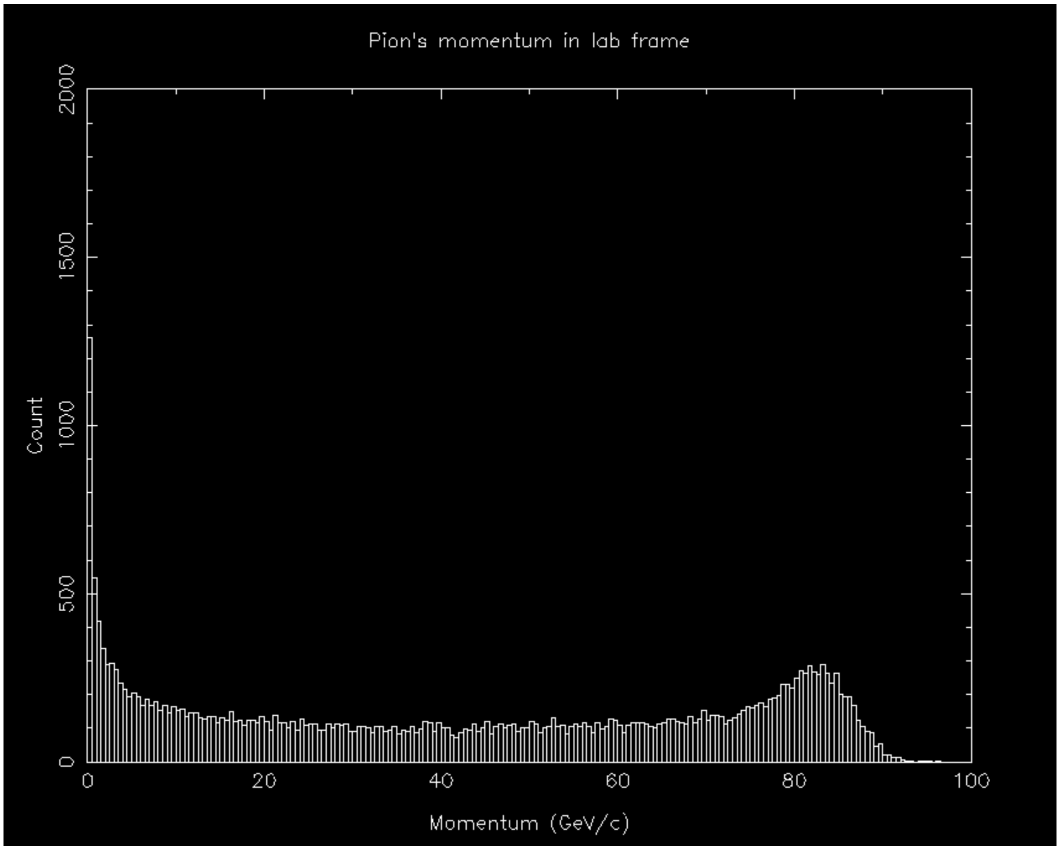
To correctly reflect the ratio of 86% π^+ and 14% K^+ composition in the beam, this can be done easily by plotting fewer points for K^+ by a factor of $\frac{86}{14} = 6.14$. Since all the simulated points are independent of each other, the first reduced number of plots can be taken.

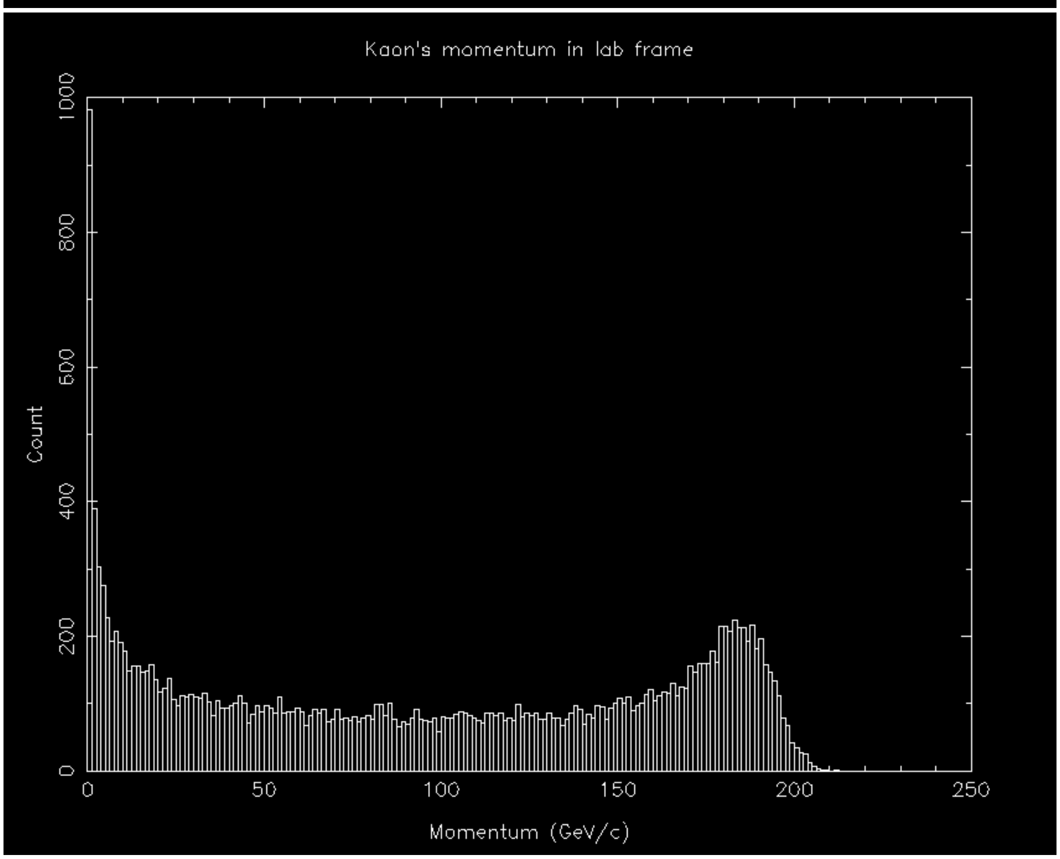
The 64% probability of kaon decaying into neutrino and muon will also be taken into account here.

This is done in the file *neutrino_momentum.c* which simulates the production of 10^7 π^+ mesons, where the histogram of the produced neutrino in the lab frame is plotted for both π^+ and K^+ with the following flowchart:



Obtaining the scatterplot for neutrino's momentum in the lab's frame:


Pion's momentum in lab frame
Count
Momentum (GeV/c)
0
500
1000
1500
2000
0
20
40
60
80
100


Kaon's momentum in lab frame
Count
Momentum (GeV/c)
0
200
400
600
800
1000
0
50
100
150
200
250

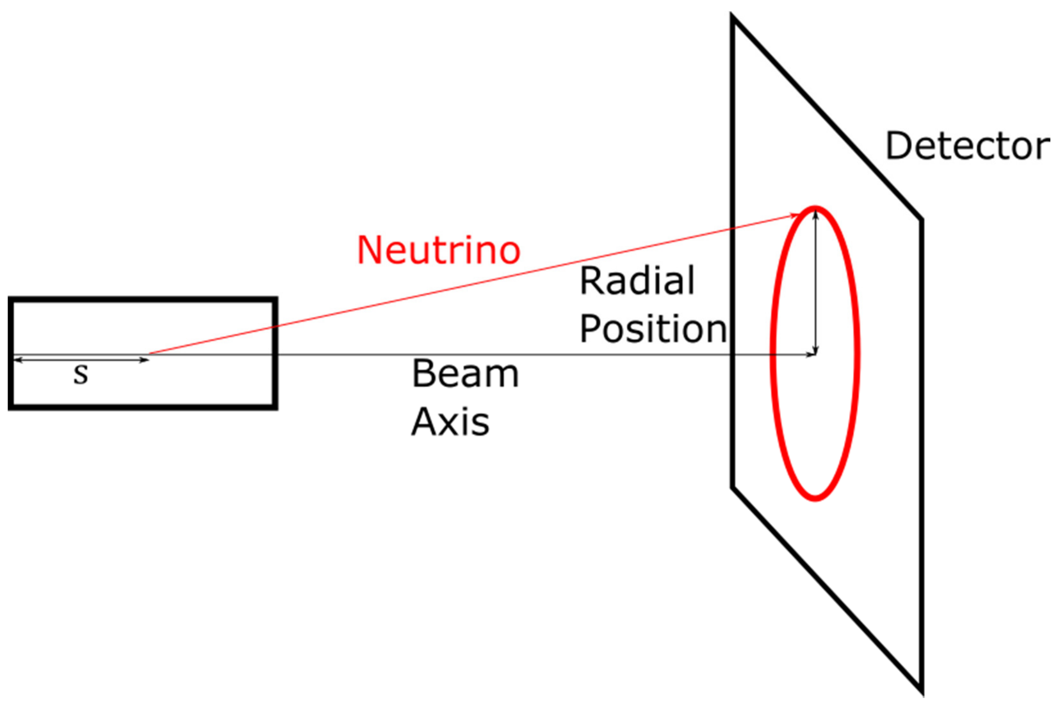
It can be seen that the momentum for the neutrino produced by both the mesons have the same distribution shape; where neutrino's momentum are roughly uniform in the middle, having a peak near its maximum momentum and have a large tendency to have zero momentum.

It can also be seen even though kaon has much higher probability of decaying at shorter distance, due to their lower composition in the beam and smaller probability of producing neutrinos, their count is of similar order of magnitude with the number of neutrinos detected by pions.

The range of momentum of neutrino produced from the decay kaon, from $0GeV/c$ to around $210GeV/c$, is higher than pion, which is from $0GeV/c$ to around $90GeV/c$.

Neutrino Detection

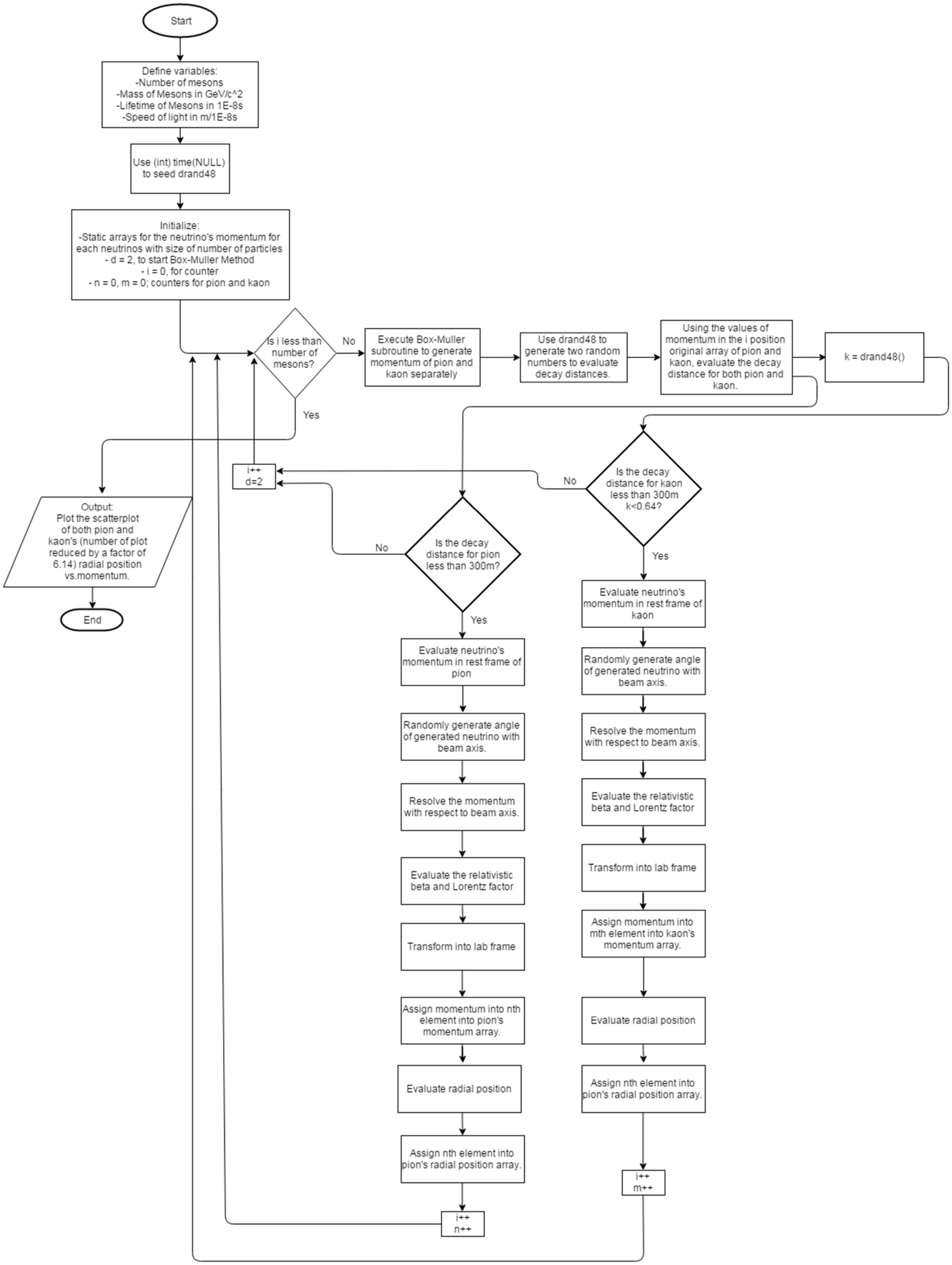
Finally, to simulate the detection of neutrino when it hits the detector. As the experiment has azimuthal symmetry about the beam's axis, neutrino produced at a certain angle will hit the detector about a circle about the center of the detector. The radius of this circle is the radial position:



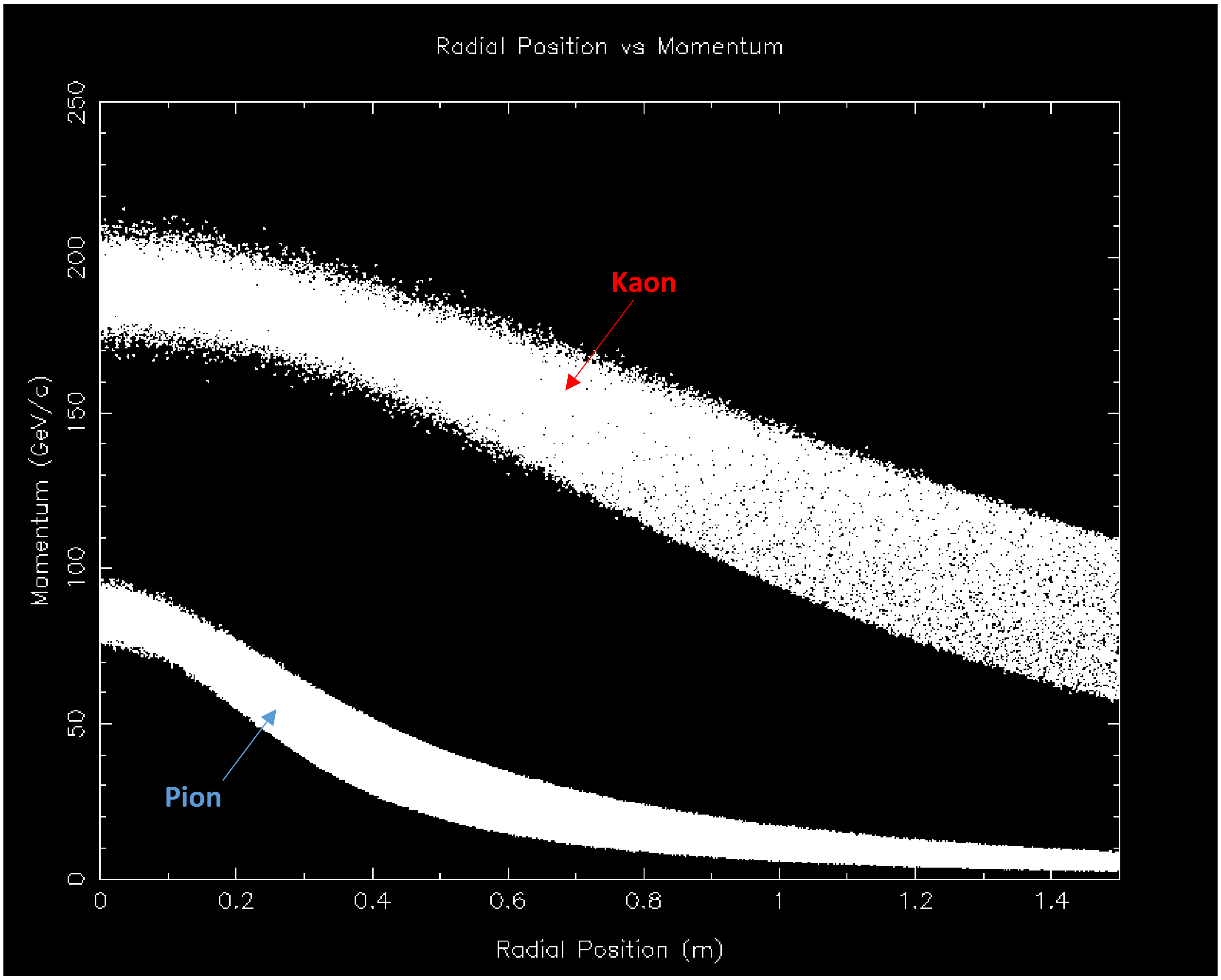
The radial position can then be found by the following equation:

$$R = (700 - s)\left|\frac{p_{t_{lab}}}{p_{l_{lab}}}\right|$$

The simulation of the experiment, and the plotting of a scatter plot of the radial position vs. momentum (or energy) is done in *neutrino.c* with the following flowchart:

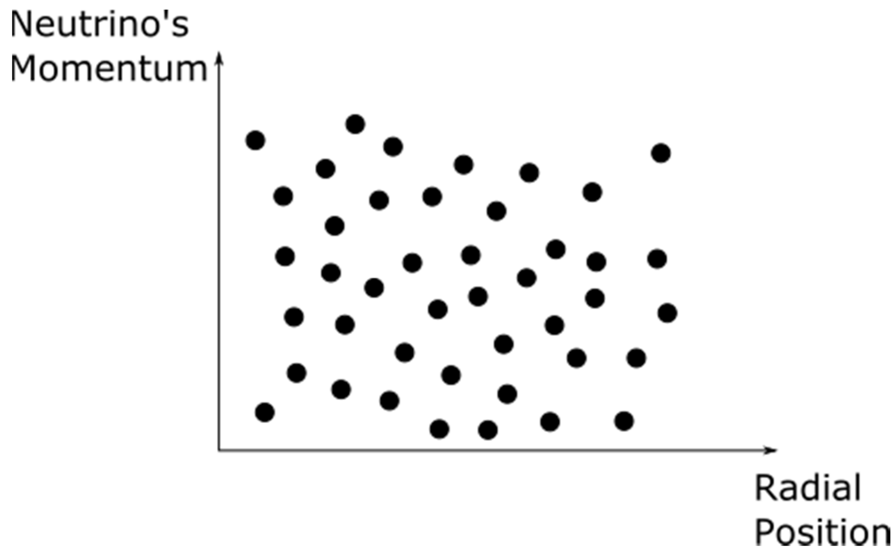

Start
Define variables:
-Number of mesons
-Mass of Mesons in GeV/c^2
-Lifetime of Mesons in 1E-8s
-Speed of light in m/1E-8s
Use (int) time(NULL) to seed drand48
Initialize:
-Static arrays for the neutrino's momentum for each neutrinos with size of number of particles
- d = 2, to start Box-Muller Method
- i = 0, for counter
- n = 0, m = 0; counters for pion and kaon
Is i less than number of mesons?
No
Yes
Execute Box-Muller subroutine to generate momentum of pion and kaon separately
Use drand48 to generate two random numbers to evaluate decay distances.
Using the values of momentum in the i position original array of pion and kaon, evaluate the decay distance for both pion and kaon.
k = drand48()
i++
d=2
Is the decay distance for kaon less than 300m k<0.64?
No
Yes
Output:
Plot the scatterplot of both pion and kaon's (number of plot reduced by a factor of 6.14) radial position vs.momentum.
End
Is the decay distance for pion less than 300m?
No
Yes
Evaluate neutrino's momentum in rest frame of pion
Randomly generate angle of generated neutrino with beam axis.
Resolve the momentum with respect to beam axis.
Evaluate the relativistic beta and Lorentz factor
Transform into lab frame
Assign momentum into nth element into pion's momentum array.
Evaluate radial position
Assign nth element into pion's radial position array.
i++
n++
Evaluate neutrino's momentum in rest frame of kaon
Randomly generate angle of generated neutrino with beam axis.
Resolve the momentum with respect to beam axis.
Evaluate the relativistic beta and Lorentz factor
Transform into lab frame
Assign momentum into mth element into kaon's momentum array.
Evaluate radial position
Assign nth element into pion's radial position array.
i++
m++

Obtaining the scatterplot for radial position vs. energy:



It can be seen that there is a correlation between the radial position and momentum/energy of the produced neutrino, where neutrinos with higher energy has a tendency to have lower radial position.

If there were no correlation, the scatter plot will be expected be look like this:



Where the plots will be randomly scattered with no discernible pattern.

The percentage of neutrino detected can be easily calculated by printing the number of times the radial position is calculated in the terminal. For pions, out of the 10^7 mesons, $263{,}763$ neutrinos are detected, meaning only 2.64% of the neutrino produced is detected. While for Kaon, out of 1.627×10^6 mesons (taking into account the 14% composition), $188{,}911$ neutrinos are detected, meaning that a much higher percentage of 11.2% of kaon's neutrino will be detected, reflecting the shorter decay distance for kaon.

Analysis of Results and Concluding Remarks

In this project, the implementation of Monte Carlo methods is implemented successfully to simulate the experimental result of neutrino detection via the use of random number generators and the construction of probability distributions; producing testable results that can be verified by actual experiments.

However the programs *neutrino_momentum.c* and *neutrino.c* were admittedly inefficient as the size of arrays used to produce the plots is the number of mesons produced; and only a very few of them is detected as the meson that have decay distance below $300m$ are detected have very low probability of occurring, while a vast majority is undetected. This means that the array used to obtain the plot have many unused elements, which unnecessarily consumes memory.

This can be rectified by appending the array, rather than specifying the number of element at the beginning, the array's size initially has a size of zero, and evaluated terms are added it to and increase the size each time. This rectifies the unnecessary usage of memory.

Unfortunately in C, appending arrays are cumbersome, so they are not used in this project as a more simple code is more preferred.

(1998 words)