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PHYC30012 – Computational Physics Report 2: Structure of White Dwarf Stars

Aim

To study the distribution of mass in a white dwarf star caused by gravity and degeneracy pressure, and the numerical methods to solve coupled ordinary differential equations, or ODE for short.

The model will then be tested with a real star to determine its validity.

Theory

A white dwarf star is a star that have exhausted all of its nuclear fusion reaction fuel. Hence a white dwarf star usually consists of heavy atoms, typically carbon-12 or iron-56. Due to their high temperature, they emit faint white light.

The mass of a white dwarf star will collapse into a small volume due to gravity, but it does not collapse completely, as there is another factor: the *degeneracy pressure* due to the Pauli-Exclusion principle. The constituents of the white dwarf star cannot collapse into a single point, as they have to occupy a single quantum state that way.

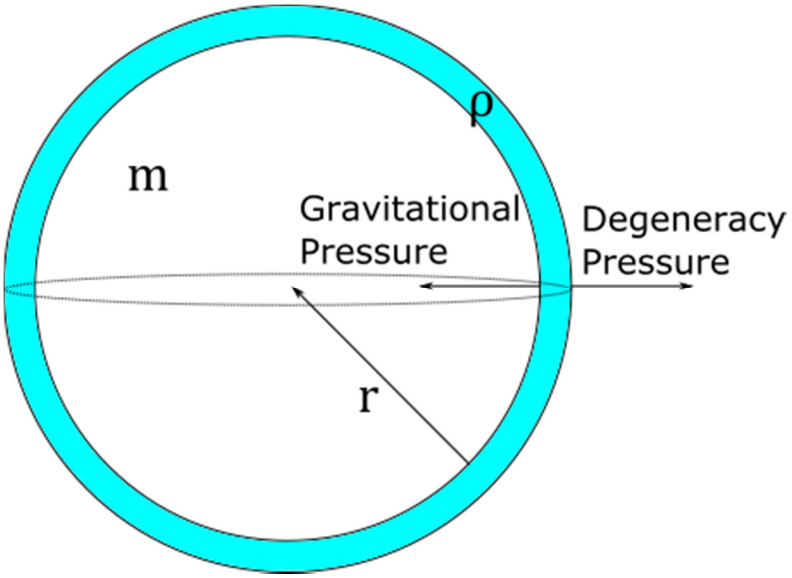
The wave function of electrons confined in a sphere can be shown to be [1]:

$$\psi_{nml}(r,\theta,\phi) = A_{nl} j_l\left(\frac{\beta_{nl} r}{a}\right) Y_l^m(\theta,\phi)$$

This shows that within a sphere, an electron can be characterized by 4 quantum numbers: n, l, m and also the spin s which is not included in the wave function. In a sphere containing multiple electrons, they all cannot have the same set of quantum numbers.

This causes degeneracy pressure, which is due to the electron gas (stripped away from the nucleus at high temperature) in the white dwarf star. For simplicity, the nucleons can be assumed immobile, providing only the gravitational attraction.

Consider a sphere with a spherically symmetric, where $m(r)$ and $\rho(r)$ refers to the interior mass and density at r respectively:



Due to gravity alone, it can be shown that:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$
$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1} \frac{Gm(r)\rho(r)}{r^2}$$

Which is a coupled ODE, to be solved. Unfortunately it cannot be solve analytically as they are non-linear, so numerical methods will be employed.

At hydrostatic equilibrium, the pressure caused by gravity is balanced by the degeneracy pressure, hence $\frac{dP}{d\rho}$ in the equation above denotes the degeneracy pressure.

The term $\frac{dP}{d\rho}$ can be found by treating the electrons as a Fermi gas providing outward pressure. The heavy nuclei present do not contribute to the degeneracy pressure. It can be shown that for multiple electrons in a sphere:

$$\frac{dP}{d\rho} = \frac{Y_e m_e}{M_p} \gamma(x)$$

Where Y_e is the number of electrons per nucleon. Also:

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{\frac{1}{2}}} \; and \; x = \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3}}$$

The coupled ODE can be simplified by scaling them:

$$r = R_0 \bar{r}, \rho = \rho_0 \bar{\rho} \; and \; m = M_0 \bar{m}$$

Where $R_0 = 7.72 \times 10^6 Y_e\ m$, $\rho_0 = 9.79 \times 10^8 Y_e^{-1}\ kg\ m^{-3}$ and $M_0 = 5.67 \times 10^{30} Y_e^2\ kg$ respectively. So the coupled differential equation can be reduced to:

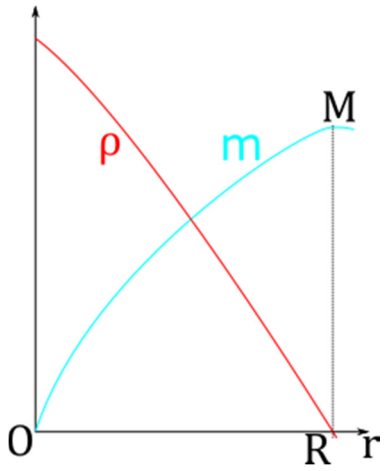
$$\frac{d\bar{\rho}}{d\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma \bar{r}^2}$$
$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2\bar{\rho}$$

From here on the bar on top quantities will be omitted, and they will mean the scaled quantity unless otherwise stated [2].

To specify the initial conditions; at the center, it has no interior mass so $m(0) = 0$; also $\rho(0) = \rho_c$, which is the star's central density.

As $\frac{d\rho}{dr}$ is negative, ρ will fall off, eventually reaching zero. To evaluate the radius R and mass M of the star; as at the surface, the density necessarily falls to zero so $\rho(R) = 0$, and this R value can be used to evaluate the star's mass at $M = m(R)$, which is also its maximum point.

The expected sketch of the graph of $m(r)$ and $\rho(r)$ is as shown:



Solving Coupled ODE Numerically

Before tackling the issue of the structure of white dwarf stars, as the coupled ODE that arise are analytically unsolvable, means to solve coupled ODE numerically is needed.

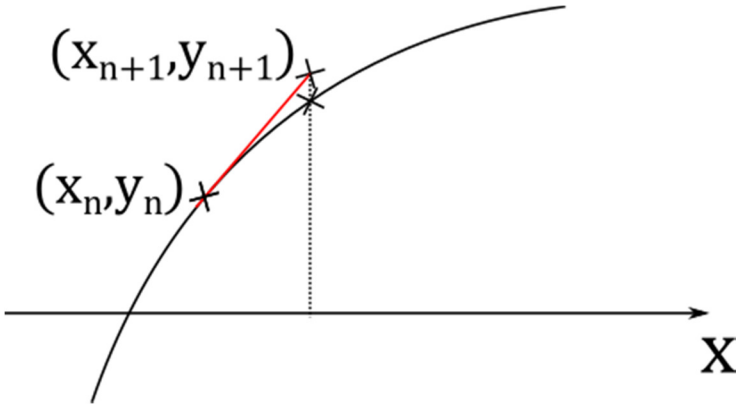
The easiest method is the *Euler's Method.* However *Runge-Kutta's* method will be employed, which is an improved version of Euler's method.

Consider a general coupled ODE for x and y given with respect to t:

$$\frac{dx}{dt} = f(x, y, t)$$
$$\frac{dy}{dt} = g(x, y, t)$$

To solve for x and y in terms of t, suppose that for some value of t_0, the values of x and y are x_0 and y_0 respectively; the values for x and y at an incremental step of h for t, x_1 and y_1 at $t_1 = t_0 + h$ can be approximated by:

Euler's method

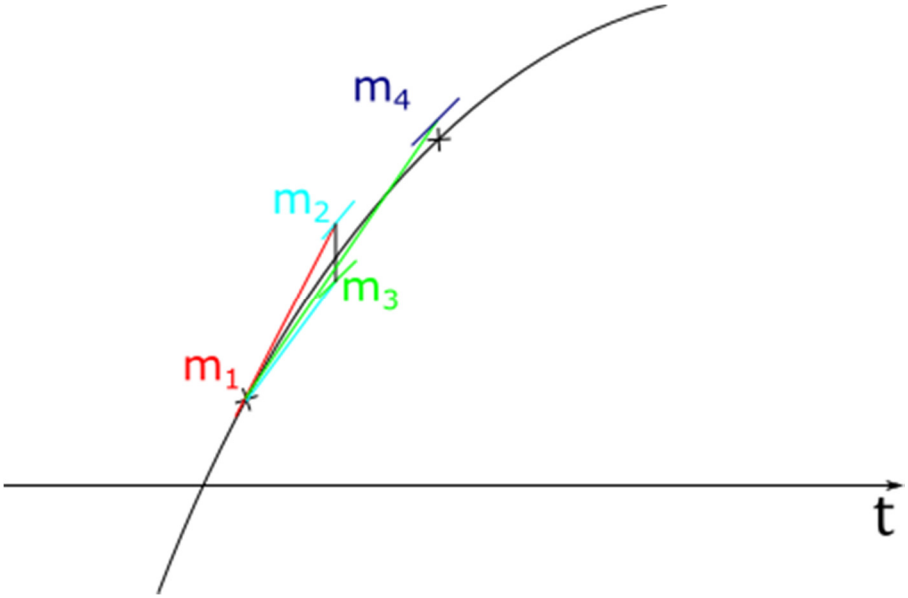


$$x_1 \approx x_0 + hf$$
$$y_1 \approx y_0 + hg$$

Where,

$$f = f(x_0, y_0, t_0)$$
$$g = g(x_0, y_0, t_0)$$

Runge-Kutta method



$$x_1 \approx x_0 + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4)$$
$$y_1 \approx y_0 + \frac{h}{6}(g_1 + 2g_2 + 2g_3 + g_4)$$

Where f_i and g_i are the gradients for x and y:

$$
\begin{aligned}
f_1 &= f(x_0, y_0, t_0) \\
g_1 &= g(x_0, y_0, t_0) \\
f_2 &= f\left(x_0 + \frac{h}{2} f_1, y_0 + \frac{h}{2} g_1, t_0 + \frac{h}{2}\right) \\
g_2 &= g\left(x_0 + \frac{h}{2} f_1, y_0 + \frac{h}{2} g_1, t_0 + \frac{h}{2}\right) \\
f_3 &= f\left(x_0 + \frac{h}{2} f_2, y_0 + \frac{h}{2} g_2, t_0 + \frac{h}{2}\right) \\
g_3 &= g\left(x_0 + \frac{h}{2} f_2, y_0 + \frac{h}{2} g_2, t_0 + \frac{h}{2}\right) \\
f_4 &= f(x_0 + f_3 h, y_0 + g_3 h, t_0 + h) \\
g_4 &= g(x_0 + f_3 h, y_0 + g_3 h, t_0 + h)
\end{aligned}
$$

By repeated use of both methods over small step sizes over an interval, approximate numerical solution to the coupled ODE can be obtained.

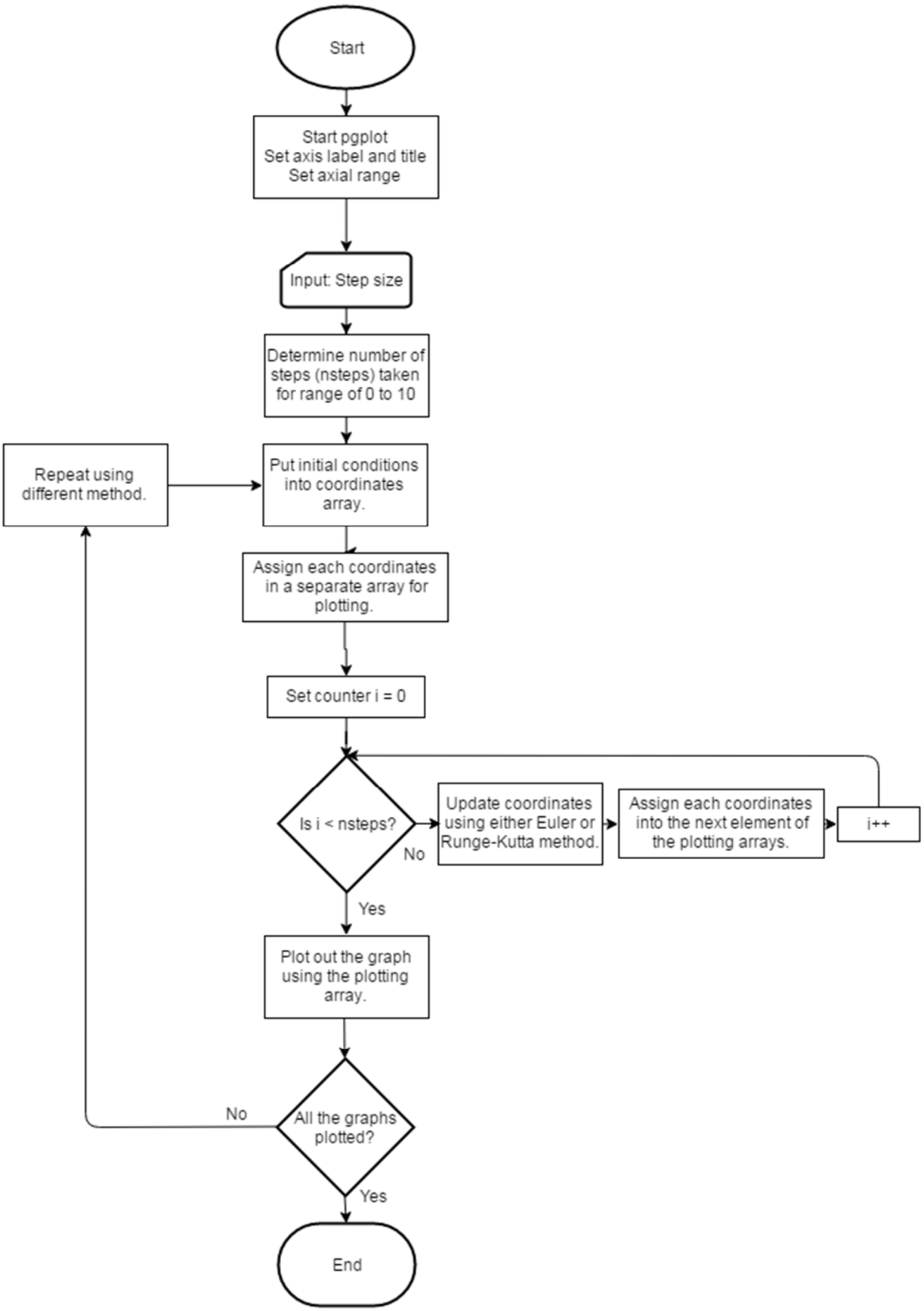
In the implementation of both methods for coupled ODE, they require to iterate both x and y; however functions in C forbid having two outputs, and arrays as an output.

Even though it is possible to have two separate function to iterate x and y, it is inefficient as there will be many repeated calculations of the gradients which will affect its performance.

Fortunately a workaround can be done by instead of outputting the new values x and y in the function, the initial values of x, y and t can be stored in an array outside of the function. Both the functions can take the pointer of the initial condition array as the parameter, and update the values in the array.

The function then can be repeated any number of times till the desired range of t is obtained. The implementation of the above method can be seen in *euler_coupled* and *rk_coupled* function in the file *coupled_test.c*.

To obtain the plot, the values of x, y and t are all stored separated into another array every time their values are updated by both the *euler_coupled* and *rk_coupled* function, then using pgplot to plot them out. This is illustrated in the flow diagram:


Start
Start pgplot
Set axis label and title
Set axial range
Input: Step size
Determine number of steps (nsteps) taken for range of 0 to 10
Repeat using different method.
Put initial conditions into coordinates array.
Assign each coordinates in a separate array for plotting.
Set counter i = 0
Is i < nsteps?
No
Update coordinates using either Euler or Runge-Kutta method.
Assign each coordinates into the next element of the plotting arrays.
i++
Yes
Plot out the graph using the plotting array.
No
All the graphs plotted?
Yes
End

The accuracy of the solution obtained with the Runge-Kutta method will be more accurate than the solution obtained from Euler's method; their accuracies can be compared by tested on a simple coupled ODE with a known solution:

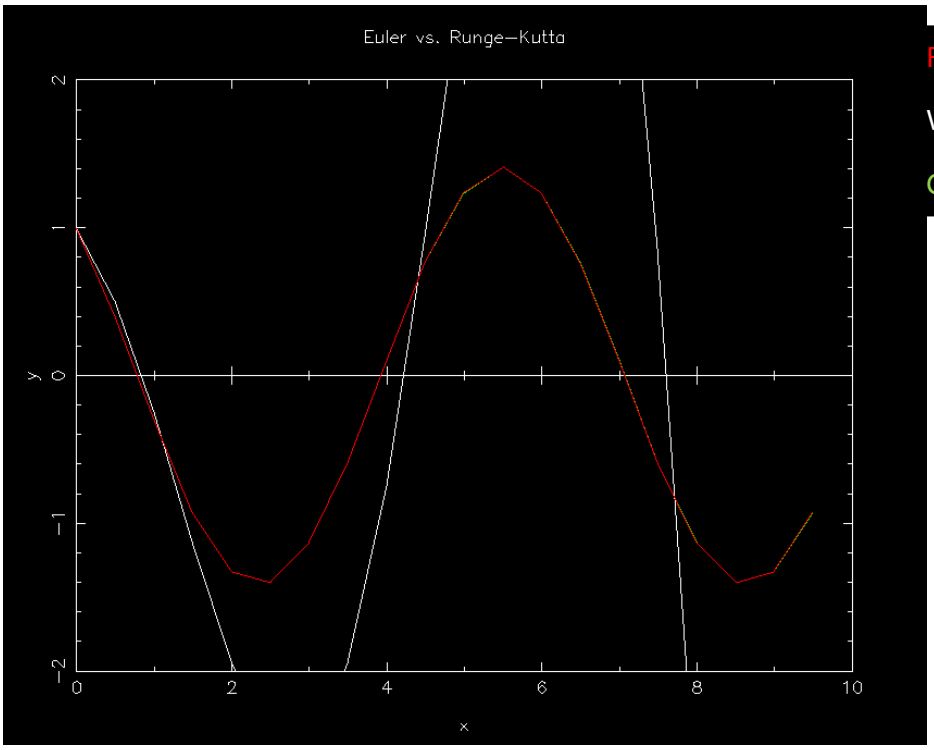
$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dy} = x$$

Subjected to the initial condition $x_0 = 1$, $y_0 = 1$ and $t_0 = 0$. The solution for x and y are:

$$x(t) = -\sin(t) + \cos(t)$$
$$y(t) = -\sin(t) - \cos(t)$$

The testing of the Euler's and Runge-Kutta method on this simple coupled ODE is performed in *coupled_test.c* over various step sizes, h, which is manually inputted in the terminal. The actual graphs will be plotted alongside for comparison. For convenience, only the solutions for $x(t)$ will be plotted.

$h = 0.5$:

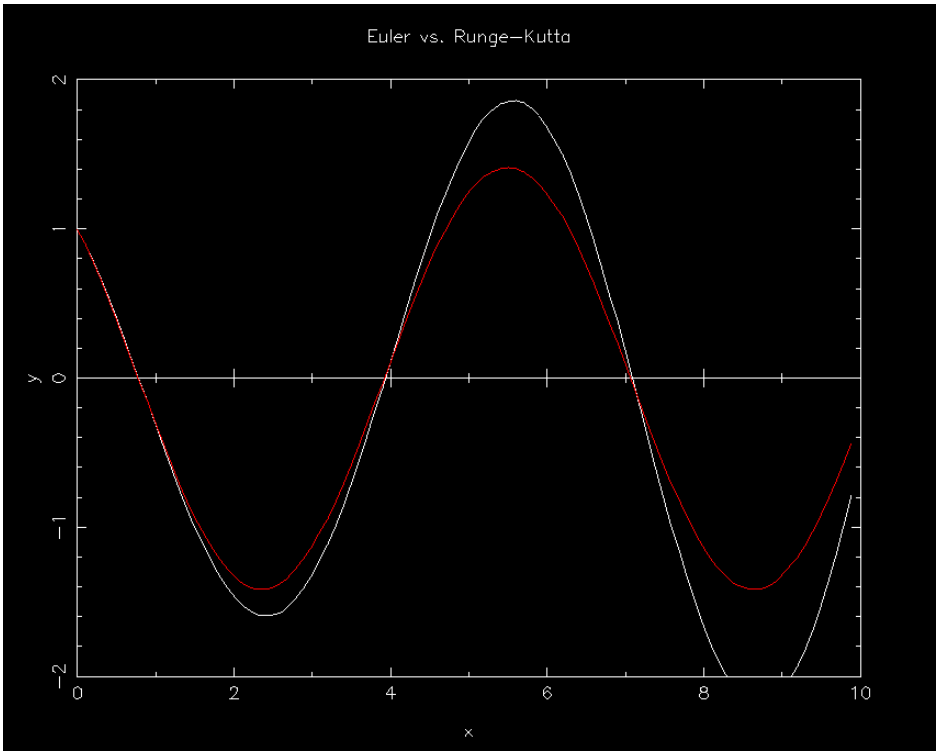


Red: Actual plot

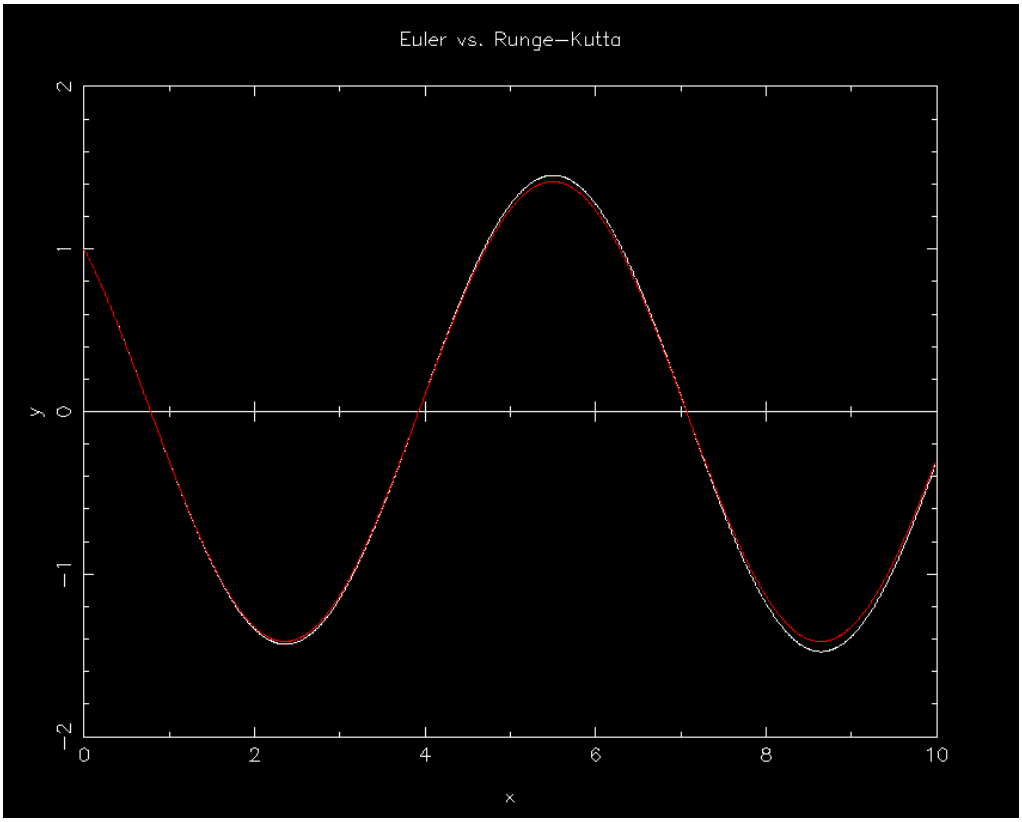
White: Euler's Method

Green: Runge-Kutta Method

$h = 0.1$:


Euler vs. Runge–Kutta
y
x
2
1
0
−1
−2
0
2
4
6
8
10

$h = 0.01$


Euler vs. Runge–Kutta
y
x
2
1
0
−1
−2
0
2
4
6
8
10

It can be seen that the accuracy of Runge-Kutta method is unparalleled to Euler's. At a large step size of 0.5, there are virtually no errors for Runge-Kutta; while even at a very tiny step size of 0.01, the numerical solution obtained from Euler's method deviates significantly, which can be seen by its growing amplitude.

Solving the Coupled Differential Equations for White Dwarf Stars

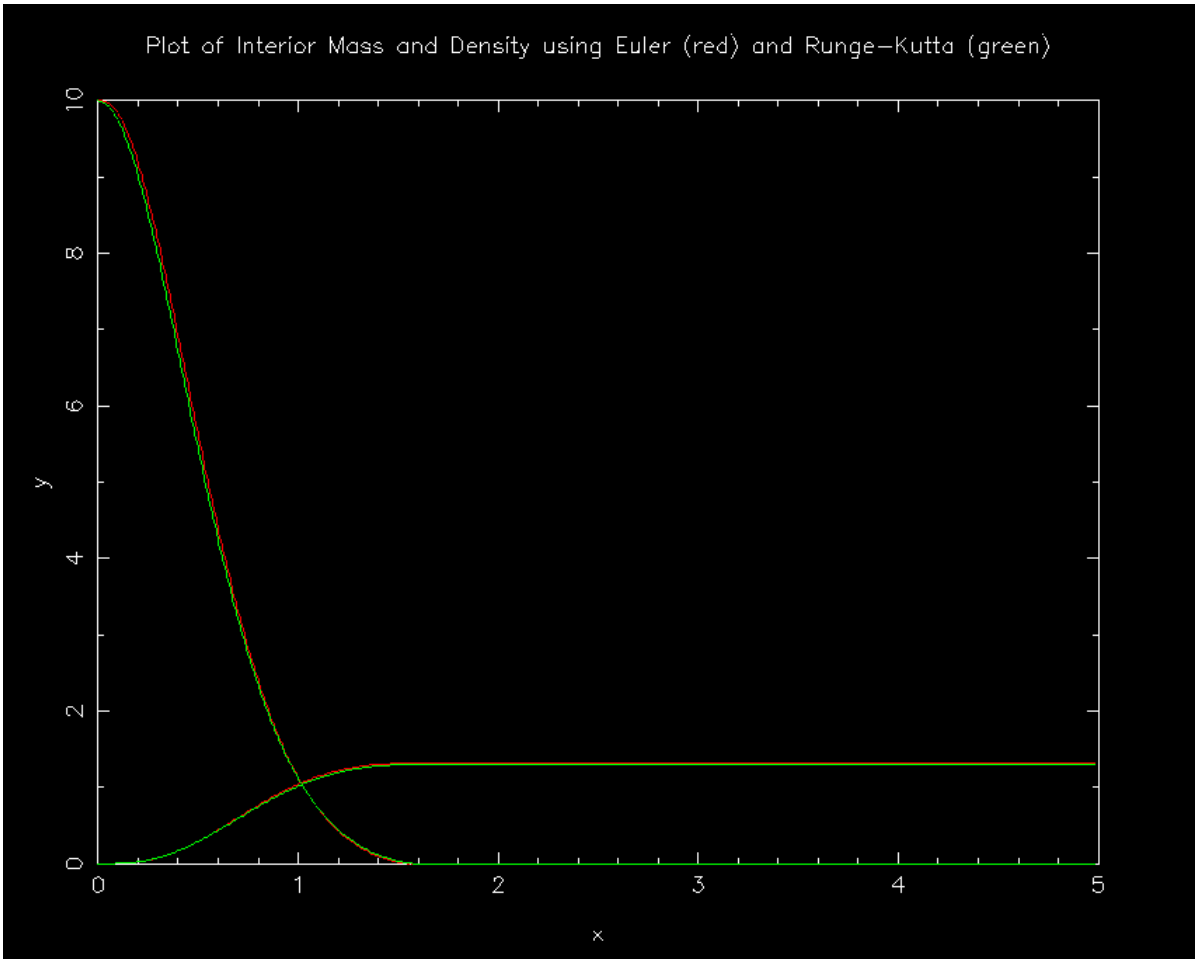
Tackling the coupled ODE involving white dwarf stars involving its density and interior mass:

$$\frac{d\rho}{dr} = -\frac{m\rho}{\gamma r^2}$$
$$\frac{dm}{dr} = r^2\rho$$

Subjected to initial condition $m(0) = 0$ and $\rho(0) = \rho_c$.

For different values of ρ_c different values of star's mass M and radius R will be obtained. This not only allows the testing of the model employed, it also calculates the central density of actual observed white dwarf stars, which will be done shortly.

Implementing both the Euler and Runge-Kutta algorithm with step size of 0.01 and central density $\rho_c = 10$, the following plot is obtained from the file *star_plot.c*:

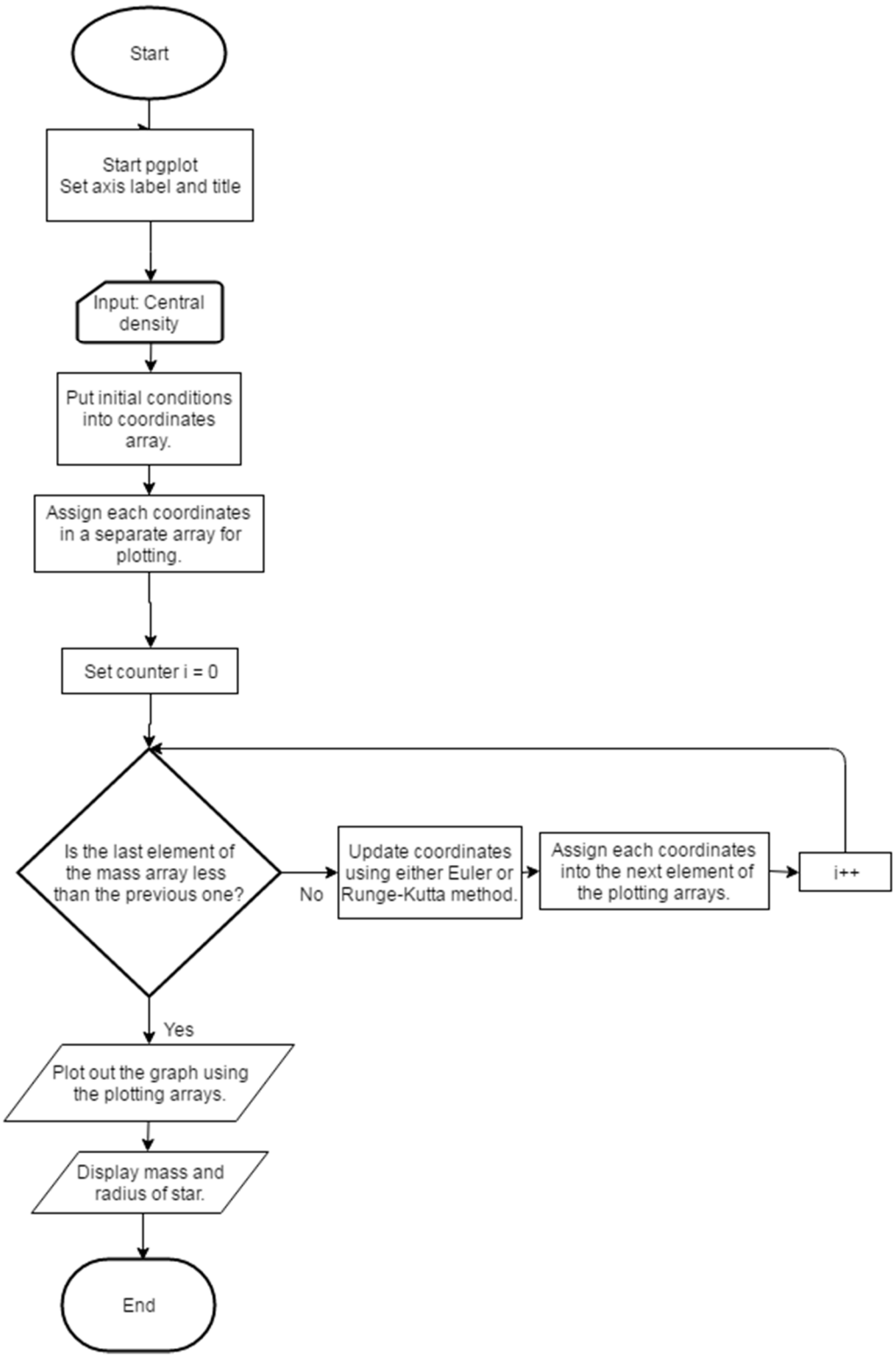


By inspection, both Euler's and Runge-Kutta method predicts that the star's density to approach zero asymptotically, while the mass approaches an upper limit but with a slight different value of 1.313 and 1.298 respectively. Unfortunately this is problematic as the star's radius is not well-defined as the particles from the star will spread itself out to infinity.

Printing out the values of m and ρ, and upon closer examination, it can be seen that m does increase to some maximum value before starting to decrease as predicted. The maximum point of m and zero of ρ is masked by the scales of the axis.

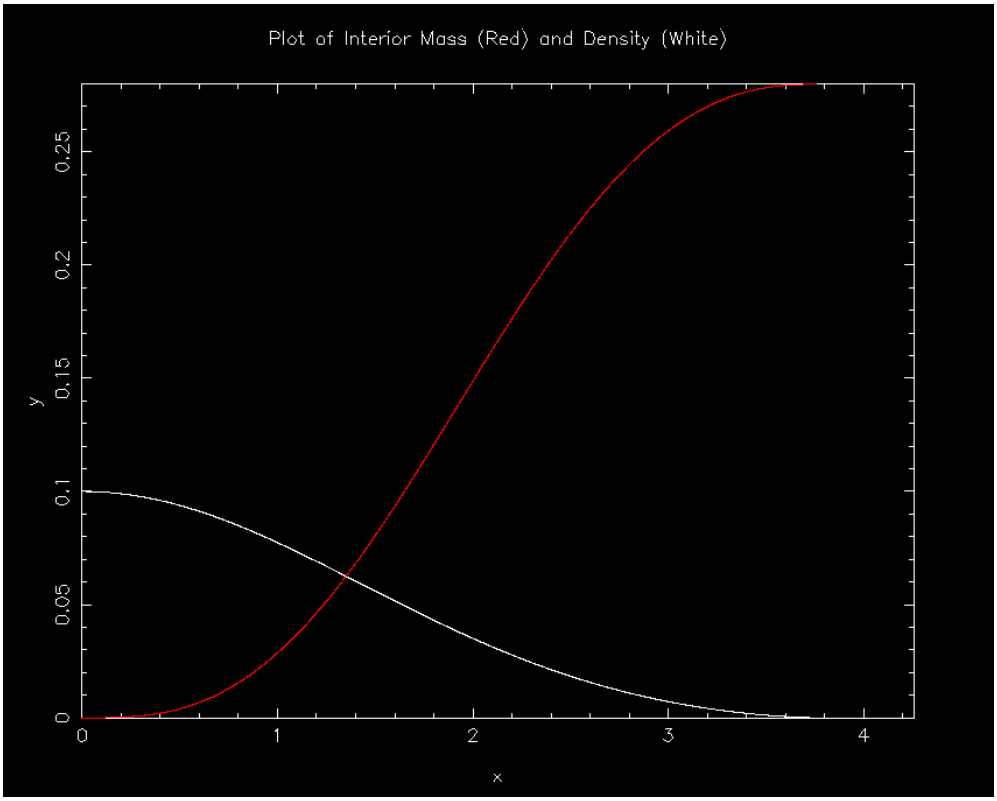
Fortunately this means that as density approaches zero, it does crosses over to the negative side slightly before approaching to zero. This is critical as it allows the evaluation of radius and mass by finding the maximum point of $m(r)$.

The implementation of this method to obtain the both the plot of $m(r)$ and $\rho(r)$ is done in the program *star_mass.c* .The step size is reduced to 10^{-3} and the ρ_c value is to be manually keyed into the terminal. The following flow diagram illustrates the structure of the program.

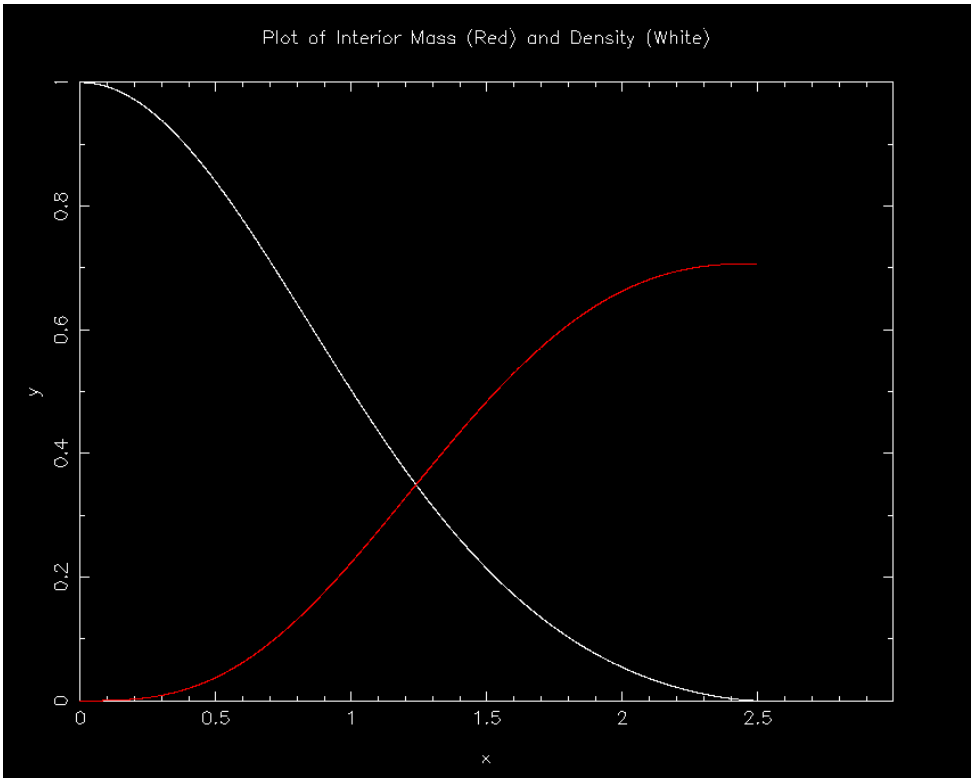


Obtaining the plot for various values of ρ_c and their corresponding value of M and R. Since R's precision is limited to 3 decimal places, the value of M will also be corrected to 3 decimal places.

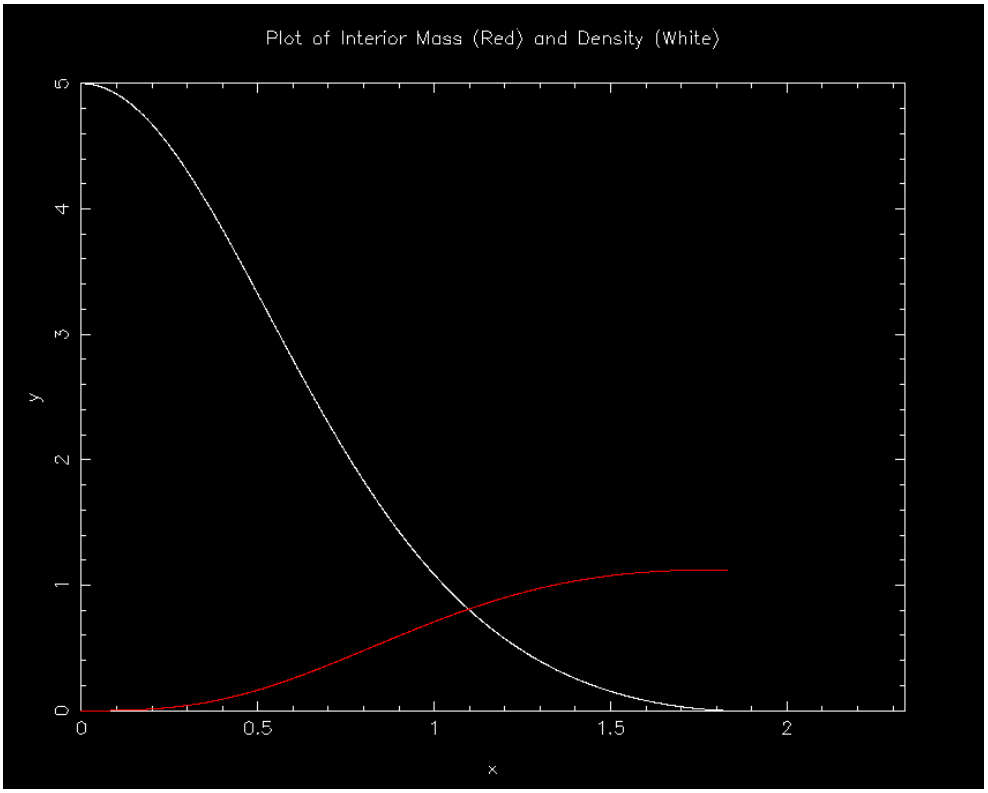
$\rho_c = 0.1$:



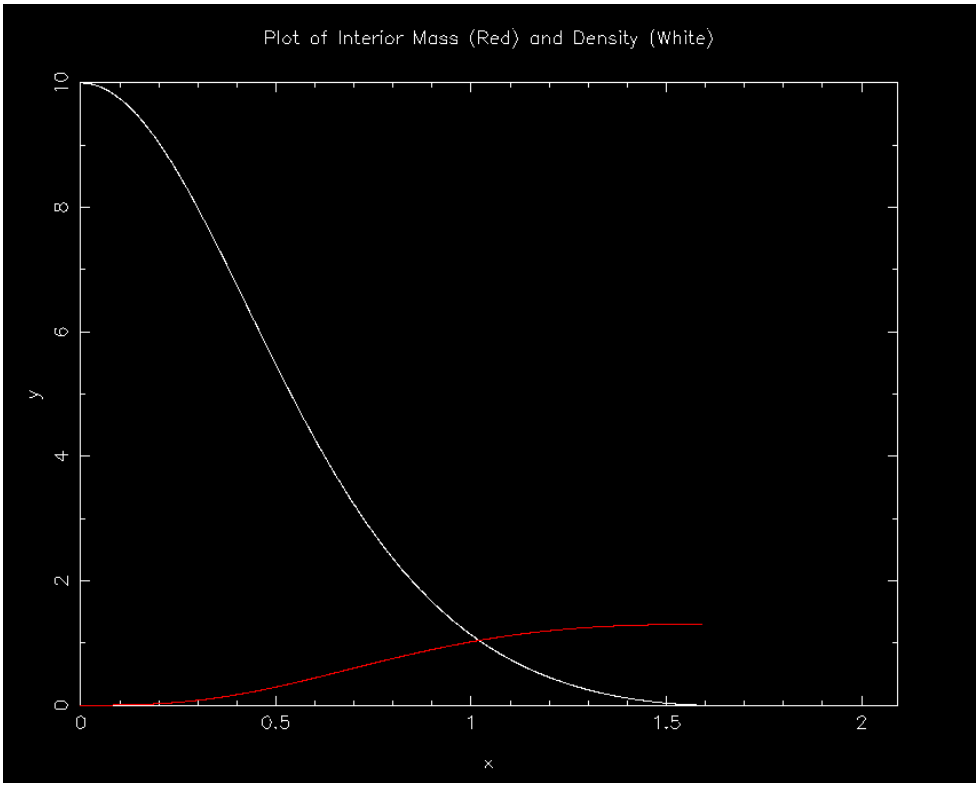
$\rho_c = 1.0$:



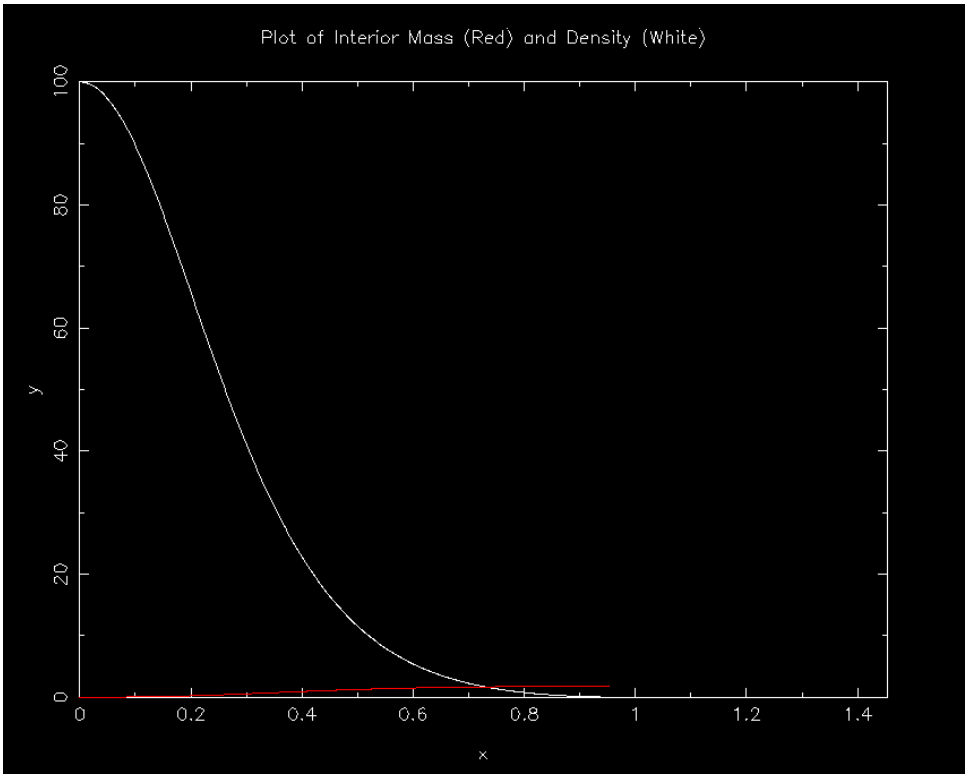
$\rho_c = 5$:



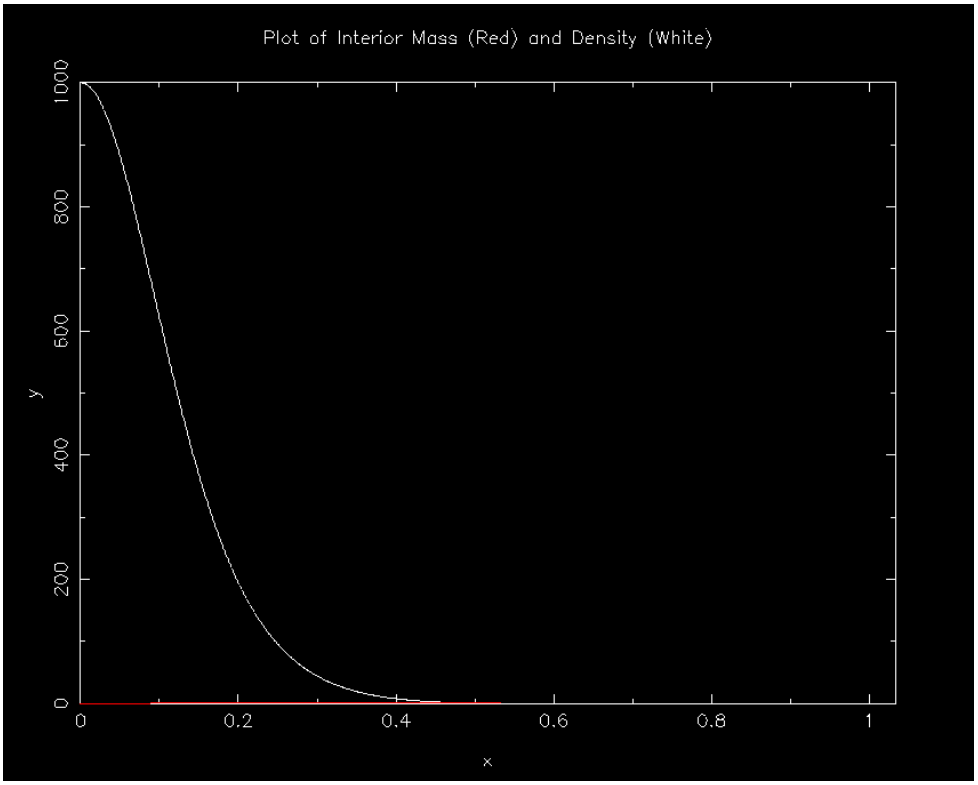
$\rho_c = 10$:



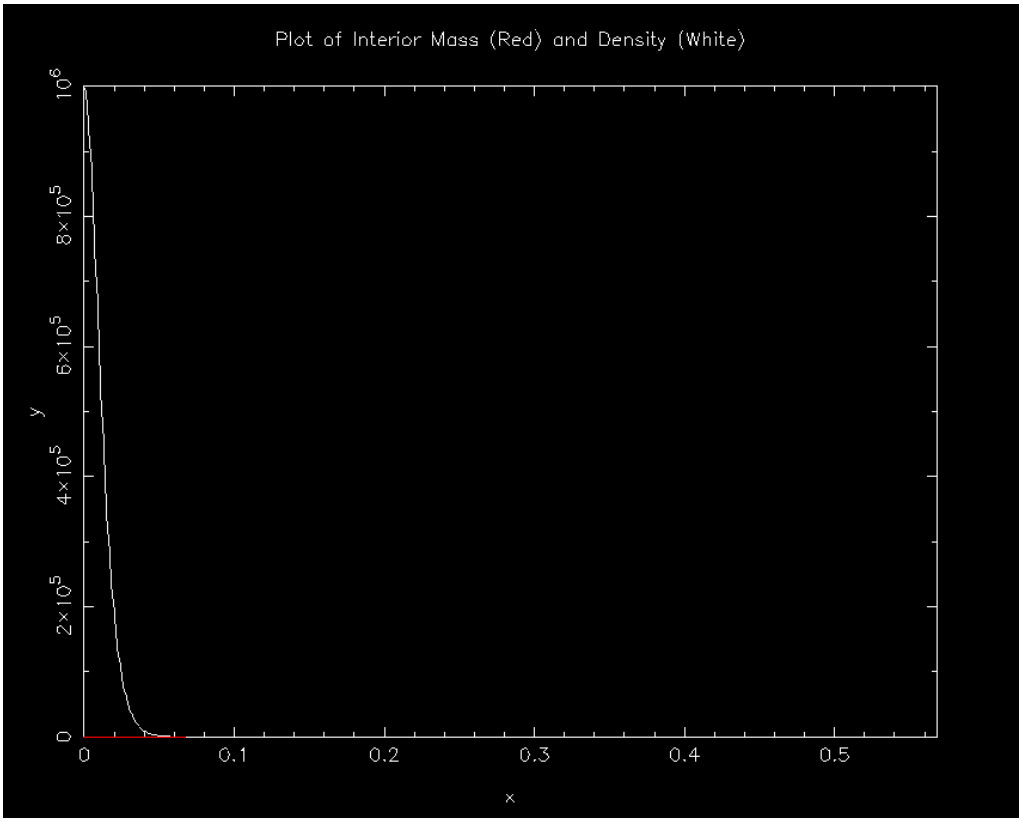
$\rho_c = 100$:



$\rho_c = 1000$:



$\rho_c = 10^6$:



For various ρ_c values, tabulating the corresponding mass and radius:

ρ_c	M	R
0.1	0.280	3.756
1.0	0.707	2.497
5.0	1.123	1.833
10.0	1.298	1.591
100.0	1.736	0.953
1000	1.933	0.533
10^6	2.017	0.068

It can be seen that at higher central density, the radius collapses down to zero, while the mass increases very slowly, suggesting a logarithmic relationship, which is subjected to further verification.

Comparing to Real White Dwarf Stars

As the radius and mass values obtained above are in terms of the scaled quantity where distance, density and mass are expressed in multiples of $R_0 = 7.72 \times 10^6 Y_e\ m$, $\rho_0 = 9.79 \times 10^8 Y_e^{-1}\ kg\ m^{-3}$ and $M_0 = 5.67 \times 10^{30} Y_e^2\ kg$ respectively.

As mentioned, the composition of white dwarf stars are either carbon-12 or iron-56 with Y_e value of 0.500 and 0.4643 respectively. Other than ρ_c, the mass and radius of the star also depend on the Y_e value.

The following real white dwarf stars observed will be used to test out the results:

Name	Mass (Solar units)	Radius (Solar Units)
Sirius B	1.053	0.0074
40 Eri B	0.48	0.0124

Since the mass of radius is given in solar unit, it is necessary to convert the scaled mass and radius into solar units.

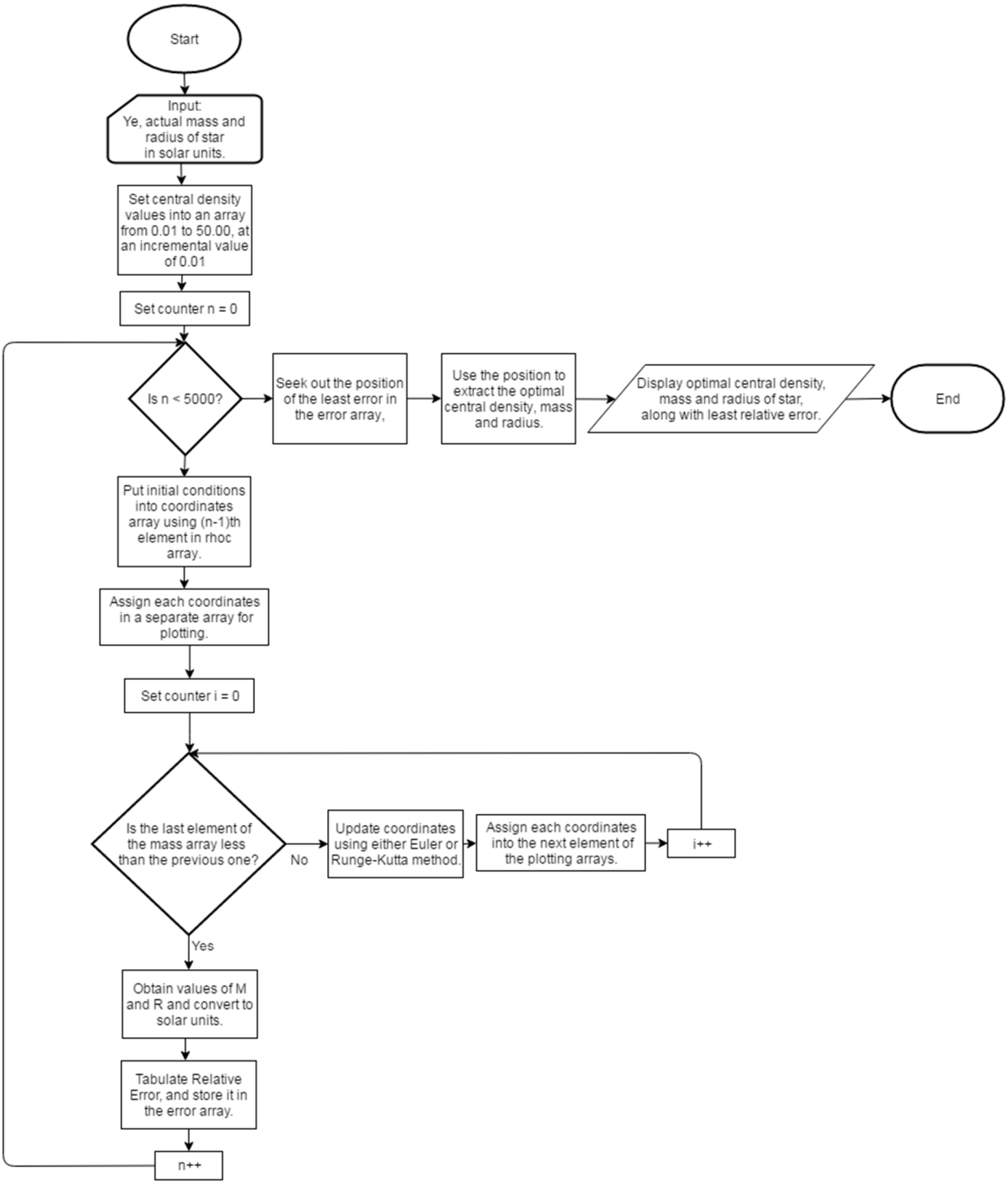
To find the optimal ρ_c that fits best the mass and radius of the actual white dwarf stars; first scan through the values of ρ_c from 0.01 to 50.00, obtain the values of mass and radius, converting them to solar unit and finding the relative error:

$$Relative\ error = \left|\frac{M - M_A}{M_A}\right| + \left|\frac{R - R_A}{R_A}\right|$$

Where M_A and R_A is the actual mass and radius respectively. The errors will be stored in an array, and the optimal value of ρ_c, M and R will then be extracted.

The program will be repeated for the two different values of Y_e, allowing the comparison of the M and R value that has the lowest relative error. The values of Y_e, actual mass and radius is prompted in the terminal.

The implementation is done in the file *star_optimize.c*, with the following flow diagram:



Tabulating the optimal value of ρ_c (in scaled units), and their corresponding M, R (both in solar units) and error, for $Y_e = 0.500$ (Carbon-12):

Star	ρ_c	M	R	Relative Error (%)
Sirius B	22.84	1.058	0.0074	0.50
40 Eri B	0.870	0.480	0.1468	14.68

$Y_e = 0.4656$:

Star	ρ_c	M	R	Relative Error (%)
Sirius B	16.28	0.867	0.0074	17.63
40 Eri B	1.350	0.4800	0.0122	1.903

Analysis

To solve a coupled ODE, the Runge-Kutta method is found to be a lot more accurate than Euler's method, and it has been successfully applied to the study of the structure of a white dwarf star.

From the relative errors shown above for different Y_e values, it can be seen that Sirius B contains Carbon-12, while 40 Eri B contains Iron-56, due to their much smaller relative error. What is really impressive is the calculated mass and radius fits with the data well, with only a slight deviation. This provides strong evidence for the existence of degeneracy pressure.

This deviation can be attributed to the assumption that the star either contains only Carbon-12 or Iron-56. In reality it might contain a mixture of both of them. The programs can be improved to also scan through various values of Y_e and optimizing for both central density and Y_e. Doing so allows a more accurate determination of the Y_e value, and hence to determine the composition of the white dwarf star.

However only two stars were tested with the model presented, and the models can be further investigated and refined by applying to more white dwarf stars.

(2000 words)

References

1. Griffiths, D. J. (2005). *Introduction to quantum mechanics Second Edition*. Pearson Education.
2. Koonin, S. E., & Meredith, D. (1990). *Computational Physics (FORTRAN Version); Disk Enclosed*. Addison-Wesley Longman Publishing Co., Inc..