Algebra and geometry of Triphos

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Abstract

Triphos is a number system motivated by the idea of using three axes (equally spaced 120° apart) on the two-dimensional coordinate plane rather than two. The Triphosian coordinate system has some similarities to the RGB additive coloring model of light. More importantly, Triphos has interesting algebraic and geometric properties. In this paper, we detail the construction of the Triphosian numbers, which has done previously, and we additionally provide a handful of new results about Triphos. Namely, regarding the algebraic structure of Triphos, we show that the field of Triphosian numbers has characteristic 0. We also provide novel results in Triphosian geometry: we construct the unit circle in Triphos, calculate the Triphosian equivalent of π, and finally, we derive a form of the Pythagorean theorem in Triphos.

1 Background

In Euclidean geometry, points in two-dimensional space are plotted on a plane with two axes (usually called x and y), placed perpendicularly (specifically, 90° apart)–one axis specifying each dimension of the point. A consequence of this definition of the plane is that the plane can be subdivided into tiny squares. However, this definition of the plane is not the only definition allowing the visualization of two-dimensional points. Consider instead a world where the plane is defined by three axes, spaced 120° apart. The world of Triphos is defined by these axes, usually called r, g, and b. In this world, two-dimensional points are specified by three coordinates, one for each axis. One interesting consequence in this world is that we do not need negative numbers–one can reach any point on the plane from any other point by only moving in positive directions parallel to the three axes.

There are currently two major examinations of the properties of Triphos. One paper focuses primarily on explicit derivations of the algebra of Triphos ([1]), while the other focuses on the motivation behind Triphos and many of its interesting properties ([2]). The inspiration behind Triphos is the RGB model for additive mixing of light [2]. If equal quantities of red, blue, and green light are mixed, the result is white light, and this is exactly how addition in Triphos works. While the analogy to light begins to break down when we consider multiplication of Triphosian numbers, the world of Triphos still provides an interesting system to study.

We now offer a brief recap of what these two papers have already found, along with the definitions which will be necessary for the remainder of the article. We take the following definitions of Triphosian numbers with addition, multiplication, and the hexa-metric distance from [2].

Definition (Triple relation)**.** Let $t = (r, g, b)$ be a triple with $r, g, b \in \mathbb{R}$ and also let $c \in \mathbb{R}$. We define the "triple relation" as $(r_1, g_1, b_1) \sim (r_2, g_2, b_2)$ if $(r_2, g_2, b_2) = (r_1 + c, g_1 + c, b_1 + c$.

This is an equivalence relation [1].

Definition (Triphosian numbers)**.** The set of Triphosian real numbers, $\mathbb{R}_T$, is the set of all equivalence classes of real-valued triples $t = [(r, g, b)] \in \mathbb{R}^3$ under the triple relation. That is,

$$\mathbb{R}_T = \mathbb{R}^3 / \sim .$$

Addition of two Triphosian real numbers is defined as

$$[(r_1, g_1, b_1)] + [(r_2, g_2, b_2)] = [(r_1 + r_2, g_1 + g_2, b_1 + b_2)].$$

Multiplication of two Triphosian real numbers is defined as

$$[(r_1, g_1, b_1)] \times [(r_2, g_2, b_2)] = [(r_1r_2 + g_1b_2 + b_1g_2, r_1g_2 + g_1r_2 + b_1b_2, r_1b_2 + g_1g_2 + b_1r_2)].$$

The distance between two Triphosian real numbers is defined by the hexa-metric distance,

$$\begin{aligned} H([(r_1, g_1, b_1)], [(r_2, g_2, b_2)]) = \\ \min\{&|(r_1 - b_1) - (r_2 - b_2)| + |(g_1 - b_1) - (g_2 - b_2)|, \\ &|(r_1 - g_1) - (r_2 - g_2)| + |(b_1 - g_1) - (b_2 - g_2)|, \\ &|(g_1 - r_1) - (g_2 - r_2)| + |(b_1 - r_1) - (b_2 - r_2)|\}. \end{aligned}$$

Addition and multiplication with these definitions are well-defined, and the set of Triphosian numbers equipped with these two operations forms a field , and this field equipped with the hexa-metric distance is a metric space [1].

Finally, since the set of Triphosian numbers is a set of equivalence classes, it remains for us to discuss which member of each equivalence class is most useful for manipulation. Both [1] and [2] define a "standard" or "reduced" form of a Triphosian number to have all nonnegative coordinates, with one coordinate being zero. (Indeed, [2] defines Triphosian numbers as only having nonnegative coordinates, but this restriction was not imposed by [1], where it is noted that this restriction is plausible but not necessary.) The transformation used by both previous works is

$$t - \min\{r, g, b\} = [(r - \min\{r, g, b\}, g - \min\{r, g, b\}, b - \min\{r, g, b\})].$$

While this definition of standard form is quite useful as it allows for easy determination of where a point should lie on the plane, it will be less useful for some of the proofs we discuss in the next section. So, we will refer to the previous definition as "standard form," and we define the "reduced form" of a Triphosian number as follows.

Definition (Reduced form)**.** A Triphosian number t is in reduced form if $b = 0$. Any Triphosian number can be converted to reduced form, which we denote $\bar{t}$ by the transformation

$$\bar{t} = [(r - b, g - b, 0)].$$

2 Isomorphism to $\mathbb{C}$

Since Triphosian numbers are plotted on the two-dimensional plane, our intuition tells us that Triphosian numbers should, in some way, be equivalent to the Cartesian two-dimensional plane. However, the transformation which converts between a Triphosian number $t = (r, g, b)$ and its Cartesian equivalent $z = (x, y)$ given by [1],

$$(x, y) = (r, g, b) \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix},$$

is not injective—multiple Triphosian numbers will be mapped to the same pair of Cartesian coordinates.

However, by examining the equivalence classes of triples and making use of the standard form transformation we defined previously, we can construct a bijection between the Triphosian real numbers and $\mathbb{C}$. We note that we construct a bijection between $\mathbb{R}_T$ and $\mathbb{C}$ rather than $\mathbb{R}_T$ and $\mathbb{R}^2$ for an important reason: multiplication in Triphos obeys the same rules as multiplication of complex numbers (see the discussion in both [1] and [2]). This suggests an isomorphism to $\mathbb{C}$.

Theorem 1. *$\mathbb{R}_T$ is isomorphic to $\mathbb{C}$.*

Proof. Define the mapping $\lambda : \mathbb{R}_T \to \mathbb{C}]$ such that

$$\lambda([(r, g, b)]) = (r - b) + (g - b)i.$$

First we will show that λ is a bijection.

1. The mapping λ is injective: to prove this, let $t_1, t_2 \in \mathbb{R}_T$ such that $\lambda(t_1) = \lambda(t_2)$ but $t_1 \neq t_2$. It follows that

$$(r_1 - b_1) + (g_1 - b_1)i = (r_2 - b_2) + (g_2 - b_2)i,$$

which is only true if the equations $r_1 - b_1 = r_2 - b_2 \implies r_2 = r_1 + (b_2 - b_1)$ and $g_1 - b_1 = g_2 - b_2 \implies g_2 = g_1 + (b_2 - b_1)$ are simultaneously true. However, it is also trivially true that $b_2 = b_1 + (b_2 - b_1)$, which means that $t_2 = t_1 + (b_2 - b_1)$, i.e. t_1 and t_2 represent the same equivalence class, which is a contradiction. Thus, if $\lambda(t_1) = \lambda(t_2)$, then $t_1 = t_2$, as desired.

2. The mapping λ is surjective: to prove this, let $t \in \mathbb{R}_T$. We will show that there exists some $z \in \mathbb{C}$ such that $\lambda(t) = z$. We have that $\lambda(t) = (r-b) + (g-b)i$. Since $(r-b), (g-b) \in \mathbb{R}$, $\lambda(t) = z \in \mathbb{C}$, as desired.

$\square$

3 Isometric geometry of Triphos

* brief explanation of hexa-metric

4 Computer programs

References

[1] Paula Egging and Erica Johnson. Triphos: An alternative coordinate system. *The Pentagon*, 4(1):16–45, 2015.

[2] Keely Grossnickle, Brian Hollenbeck, Jeana Johnson, and Zhihao Sun. Triphos: A world without subtraction. *Mathematics Magazine*, 92(4):272–285, 2019.