Triphos: Algebra and Geometry of an Isometric World

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Abstract

Triphos is a number system motivated by the idea of using three axes (equally spaced 120° apart) on the two-dimensional coordinate plane rather than two. The Triphosian coordinate system has some similarities to the RGB additive coloring model of light. More importantly, Triphos has interesting algebraic and geometric properties. In this paper, we detail the construction of the Triphosian numbers, which has done previously, and we additionally provide a handful of new results about Triphos. Namely, regarding the algebraic structure of Triphos, we show that the field of Triphosian numbers has characteristic 0. We also provide novel results in Triphosian geometry: we construct the unit circle in Triphos, calculate the Triphosian equivalent of π, and finally, we derive a form of the Pythagorean theorem in Triphos.

1 Introduction

In Euclidean geometry, points in two-dimensional space are plotted on a plane with two axes (usually called x and y), placed perpendicularly (specifically, 90° apart)–one axis specifying each dimension of the point. A consequence of this definition of the plane is that the plane can be subdivided into tiny squares. However, this definition of the plane is not the only definition allowing the visualization of two-dimensional points.

Consider instead a world where the plane is defined by three axes, spaced 120° apart. The world of Triphos is defined by these axes, usually called r, g, and b. In this world, two-dimensional points are specified by three coordinates, one for each axis. One interesting consequence in this world is that we do not need negative numbers–one can reach any point on the plane from any other point by only moving in positive directions parallel to the three axes. Examining Figure 1, one of the most interesting and motivating consequences of this definition can be observed: if we place ourselves at the origin, denoted $(0, 0, 0)$, and move one unit along the red axis, one unit parallel to the green axis, and finally one unit parallel to the blue axis, we have ended up back at the origin! This observation, that $(1, 1, 1) = (0, 0, 0)$, and consequently, that $(c, c, c) = (0, 0, 0)$ for any real number c, is the major motivation behind the study of Triphos.

There are currently two major examinations of the properties of Triphos. One paper focuses primarily on explicit derivations of the algebra of Triphos ([**?**]), while the other focuses on the motivation behind Triphos and many of its interesting properties ([**?**]). The inspiration behind Triphos is the RGB model for additive mixing of light [**?**]. If equal quantities of red, blue, and green light are mixed, the result is white light, and this is exactly how addition in Triphos works. While the analogy to light begins to break down when we consider multiplication of Triphosian numbers, the world of Triphos still provides an interesting system to study.

Here, we will cover the formal definition of the Triphosian numbers, briefly show that Triphos is both a field and a metric space (all of these results were previously proven in the former literature ([**?**, **?**]), but they are worth repeating), before deriving several new results. We will show that the field of Triphosian real numbers has characteristic zero, as well as show some interesting results in Triphosian geometry: the Triphosian unit circle is a hexagon, the Triphosian equivalent of π is equal

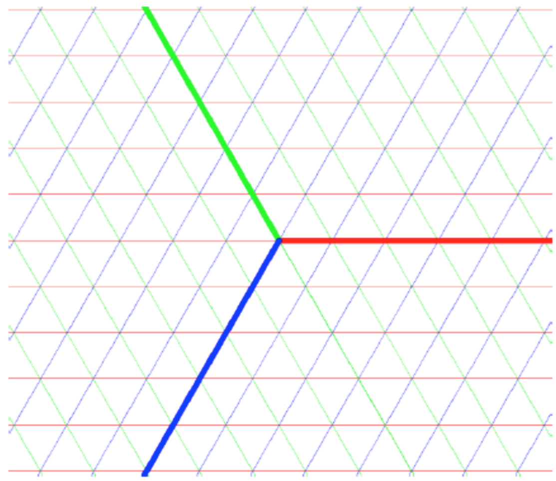

Figure 1: This figure, from [**?**], is an excellent visualization of the three Triphosian axes.

to three, and finally, we derive a version of the Pythagorean theorem which applies to Triphosian triangles.

2 Algebra of Triphos

We first concern ourselves with the formal algebraic definition of the Triphosian numbers. While the existing literature deals with Triphosian triples directly for the construction of Triphos, we define Triphos as a set of equivalence classes.

2.1 Definition of Triphos

Consider the set of all real-valued triples. We define the following equivalence relation on these triples, representing the observation that (c, c, c) represents the origin for any $c \in \mathbb{R}$.

Definition 1. Triple relation [**?**]. Let (r, g, b) be a triple, where $r, g, b \in \mathbb{R}$. Also, suppose $c \in \mathbb{R}$. We define the relation $(r_1, g_1, b_1) \sim (r_2, g_2, b_2)$ if $(r_2, g_2, b_2) = (r_1 + c, r_2 + c, r_3 + c)$.

This relation is actually an equivalence relation on the set of real-valued triples, which lends itself well to the definition of the set of Triphosian numbers.

Definition 2. Equivalence relation [**?**]. An equivalence relation is any relation between two sets which preserves the reflexive, symmetric, and transitive properties. That is, given a relation, R, we have that:

1. R is reflexive: xRx for all x.
2. R is symmetric: if xRy, then yRx.
3. R is transitive: if xRy and yRz, then xRz.

Theorem 1. *The relation defined by* $(r, g, b) \sim (r+c, g+c, b+c),\ c \in \mathbb{R}$ *is an equivalence relation. [**?**]*

Proof. In order to show that $\sim$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. A proof is provided in [**?**], but reproduced in greater detail here.

1. Reflexive: let $(r, g, b) \in \mathbb{R}_T$. By the definition of $\sim$, $(r, g, b) \sim (r + 0, g + 0, b + 0) = (r, g, b)$, so $(r, g, b) \sim (r, g, b)$ and the reflexive property holds.

2. Symmetric: let $(r_1, g_1, b_1), (r_2, g_2, b_2) \in \mathbb{R}_T$, and suppose that $(r_1, g_1, b_1) \sim (r_2, g_2, b_2)$. By definition of $\sim$, there exists some $c \in \mathbb{R}$ such that $(r_2, g_2, b_2) = (r_1 + c, g_1 + c, b_1 + c)$. This implies that $(r_2, g_2, b_2) \sim (r_1 + c + (-c), g_1 + c + (-c), b_1 + c + (-c)) = (r_1, g_1, b_1)$. So $(r_2, g_2, b_2) \sim (r_1, g_1, b_1)$ and thus the symmetric property holds.

3. Transitive: let $(r_1, g_1, b_1, (r_2, g_2, b_2), (r_3, g_3, b_3) \in \mathbb{R}_T$ such that $(r_1, g_1, b_1) \sim (r_2, g_2, b_2)$ and $(r_2, g_2, b_2) \sim (r_3, g_3, b_3)$. By definition of $\sim$, we have that there exist $c, d \in \mathbb{R}$ such that $(r_2, g_2, b_2) = (r_1 + c, g_1 + c, b_1 + c)$ and $(r_3, g_3, b_3) = (r_2 + d, g_2 + d, b_2 + d)$. So we have that $(r_2, g_2, b_2) \sim (r_2 + d, g_2 + d, b_2 + d) = (r_1 + c + d, g_1 + c + d, b_1 + c + d)$. Thus, $(r_1, g_1, b_1) \sim (r_1 + c + d, g_1 + c + d, b_1 + c + d) = (r_3, g_3, b_3)$. Thus, $(r_1, g_1, b_1) \sim (r_3, g_3, b_3)$ and the transitive property holds.

Since the reflexive, symmetric, and transitive properties hold, $\sim$ is an equivalence relation on the set of real-valued triples. $\square$

This relation has an interesting geometric meaning in Triphos: any points which are equivalent under $\sim$ will be plotted in the same location on the Triphosian plane. Given that $\sim$ is an equivalence relation, we can construct equivalence classes for each triple: these are an easy way to "combine" all of the triples which lie on the same point in Triphos into one compact notation.

Definition 3. Equivalence classes of triples. We define the equivalence class of a triple, (r, g, b) under $\sim$, denoted $[(r, g, b)]$, to be the set of all triples related to (r, g, b). Formally,

$$[(r, g, b)] = \{(x, y, z) \mid (x, y, z) \sim (r, g, b)\}.$$

This is a specific case of the general definition of an equivalence class [**?**]. Now that we have a notion of Triphosian equivalence classes, we are finally ready to give a precise definition of the Triphosian real numbers. In the two major papers to date about the Triphos system ([**?**, **?**]), the set of Triphosian reals is considered as the set of all real-valued triples. I feel that this definition is confusing and as such have chosen to define the set of Triphosian real numbers as the set of all equivalence classes.

Definition 4. Triphosian real numbers. The set of Triphosian real numbers (or simply, the Triphosian reals), denoted $\mathbb{R}_T$, is defined as the set of all equivalence classes defined by the relation $\sim$ on the set of real-valued triples. That is, in the typical formal notation,

$$\mathbb{R}_T = \mathbb{R}^3 / \sim .$$

Noting that if we have a set S and an equivalence relation R, the equivalence classes under R form a partition of S [**?**], we can say that every real-valued triple is in one, and only one, equivalence class.

Since each equivalence class contains an uncountable number of triples (each equivalence class contains as many triples as there are real numbers c, from the definition of our classes), we are faced with a few important questions about Triphosian numbers. Firstly, which triple should we use to represent each equivalence class? And secondly, how can we quickly tell if two triples are in the same equivalence class without plotting them? Both of these questions can be answered by defining a standard form for Triphosian numbers.

Definition 5. Standard form [?]. A Triphosian number $[(r, g, b)]$ is said to be in standard form if all entries are non-negative $(r, g, b > 0)$, and at least one entry is zero ($r = 0$, or $b = 0$, or $g = 0$).

Importantly, each standard form Triphosian number uniquely represents a point on the plane [?]. A consequence of this fact is that if two Triphosian numbers are both in standard form, they can only be equivalent if each of their components is equivalent.

Definition 6. Triphosian equality. [?] Let $[(r_1, g_1, b_1)], [(r_2, g_2, b_2)]$ be two Triphosian numbers in standard form. We say that $[(r_1, g_1, b_1)] = [(r_2, g_2, b_2)]$ if and only if $r_1 = r_2, g_1 = g_2$, and $b_1 = b_2$.

Now with a formal definition of the Triphosian reals in our toolbox, we can examine the algebra of Triphos: as luck and the spirits of mathematics would have it, we can define operations on the Triphosian reals that give them an incredibly similar structure to the numbers we are used to working with.

2.2 Field Structure of Triphos

In order to examine the algebraic structure of Triphos, we first define notions of addition and multiplication of Triphosian numbers. Addition on the Triphosian numbers is equivalent to addition over vectors in $\mathbb{R}^3$.

Definition 7. Triphosian addition [?]. The sum of two Triphosian numbers, $[(r_1, g_1, b_1)], [(r_2, g_2, b_2)]$ is defined as:

$$[(r_1, g_1, b_1)] + [(r_2, g_2, b_2)] = [(r_1 + r_2, g_1 + g_2, b_1 + b_2)].$$

Theorem 2. *Triphosian addition is well-defined (i.e. unambiguous) over the Triphosian reals. [?]*

While Triphosian multiplication has a messy formula for direct computations, this definition of multiplication satisfies a similar geometric property of complex multiplication: if two Triphosian numbers are plotted as vectors on the Triphosian plane, their angles (from the red axis) add, and their magnitudes multiply (using the Hexa-Metric distance, which is defined later).

Definition 8. Triphosian multiplication. [?] The product of two Triphosian numbers, $[(r_1, g_1, b_1)] \cdot [(r_2, b_2, g_2)]$ is defined as:

$$[(r_1, g_1, b_1)] \cdot [(r_2, b_2, g_2)] = [(r_1r_2 + g_1b_2 + b_1g_2, r_1g_2 + g_1r_2 + b_1b_2, r_1b_2 + g_1g_2 + b_1r_2)].$$

Theorem 3. *Triphosian multiplication is well-defined over the Triphosian reals. [?]*

The proofs that Triphosian addition and multiplication are well-defined can be found in [?]. Now, in order to discuss the properties of these operations, we must first define several types of algebraic structures.

Definition 9. Group [?]. A group, $G = (S, *)$, is a set S equipped with a binary operation $*$ satisfying the following properties:

1. G is closed under $*$: if $x, y \in G$, then $x * y \in S$.
2. G is associative under $*$: if $x, y, z \in G$, then $x * (y * z) = (x * y) * z$.
3. G has an identity element, e, under the operation: $x * e = x = e * x$ for all $x \in G$.
4. For every $x \in G$, there is an inverse element: if $x \in G$, then $y \in G$ such that $x * y = e = y * x$.

Definition 10. Abelian group [**?**]. A group, $G(S,+)$, is said to be Abelian if G is commutative under $+$; that is, if $x, y \in G$, then $x + y = y + x$.

Definition 11. Ring [**?**]. A ring, $R = (S, +, \cdot)$, is a set S equipped with two binary operations, typically called addition $(+)$ and multiplication $(\cdot)$ satisfying the following properties:

1. R forms an Abelian group equipped with addition; that is, $(S,+)$ is a group (note that in a ring, the additive identity is often written as 0).

2. R is closed under multplication: if $x, y \in R$, then $x \cdot y \in R$.

3. R contains a multiplicative identity, 1: if $x \in R$, then $1 \cdot x = x = x \cdot 1$.

4. R is associative under multplication: if $x, y, z \in R$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

5. R obeys the distributive property with respect to $+$ and $\cdot$; that is, given $x, y, z \in R$ we have that $x \cdot (y + z) = x \cdot y + x \cdot z$.

Definition 12. Field [**?**]. A field, $F = (S, +, \cdot)$, is a set S with two operations, typically called addition $(+)$ and multiplication $(\cdot)$, satisfying the following properties:

1. F is an Abelian group under $+$.

2. $F - \{0\}$ is an Abelian group under $\cdot$ (i.e. division by zero is not allowed; the additive identity is excluded as it has no multiplicative inverse).

3. Addition distributes over multiplication in F: if $x, y, z \in F$, then $x \cdot (y + z) = x \cdot y + x \cdot z$.

This definition is equivalent to the 11 field axioms listed in many sources. Finally, note that every field is also a ring by definition.

These definitions serve to illustrate the properties of the Triphosian reals, as well as provide the Triphosian reals with a common structure: by showing that the Triphosian reals have a well-studied structure, we immediately become aware of several facts that are true about the Triphosian reals.

Theorem 4. *The set of Triphosian reals, equipped with Triphosian addition and multiplication,* $(\mathbb{R}_T, +, \cdot)$*, forms a field. [**?**]*

Proof. In order to show that the Triphosian reals are a field, we will first show that the Triphosian reals, equipped with Triphosian addition, form an Abelian group. The proof of all of these statements can be found in [**?**].

- The Triphosian reals are closed under addition.

- The associative and commutative properties hold for Triphosian reals under addition.

- The additive identity is $[(0, 0, 0)]$.

- Let $[(r, g, b)] \in \mathbb{R}_T$. The additive inverse of $[(r, g, b)]$ is $[(-r, -g, -b)] \in \mathbb{R}_T$. When written in standard form this is $[(g + b, r + b, r + g)]$ [**?**].

Furthermore, the following properties hold for Triphosian multiplication [**?**].

- The Triphosian reals are closed under multiplication.

- The associative and commutative properties hold for Triphosian reals under multiplication.

- The multiplicative identity is $[(1,0,0)]$.

Additionally the multiplicative inverse is calculated as $[(r/n, g/n, b/n)]$ where $n = r^2 + b^2 + g^2 - rb - rg - gb$; this is not defined for $[(0,0,0)]$, but it is defined everywhere else [**?**]. So, the above properties are sufficient to show that the set $\mathbb{R}_T - \{[(0,0,0)]\}$ is an Abelian group under Triphosian multiplication.

Finally, the distributive property is verified in [**?**] as well. So, the set of Triphosian reals equipped with Triphosian addition and multiplication forms a field. □

With this knowledge in mind, our next goal is to define a notion of distance in Triphos. A way to measure distance is crucial for our results in the geometry of Triphos, so our next goal is to show that distance can be measured with all of the familiar properties of Euclidean distance in Triphos.

2.3 The Hexa-Metric

The distance in Triphos is known as the Hexa-Metric function. The Hexa-Metric function is similar to the Taxicab metric function in Euclidean space, as it only allows movement parallel to the axes. Interestingly, there is no unique shortest path between points in Triphos: between two points there can be many paths, which stay parallel to the axes, yet all have the same minimum distance. Visual demonstrations of this metric can be found in [**?**], just like many other results covered here.

Definition 13. Hexa-Metric function [**?**]. The Hexa-Metric function is defined as $H : \mathbb{R}_T \times \mathbb{R}_T \to \mathbb{R}$ such that

$$\begin{aligned} H([(r_1, g_1, b_1)], [(r_2, g_2, b_2)]) = \min\{&|(r_1 - b_1) - (r_2 - b_2)| + |(g_1 - b_1) - (g_2 - b_2)|, \\ &|(r_1 - g_1) - (r_2 - g_2)| + |(b_1 - g_1) - (b_2 - g_2)|, \\ &|(g_1 - r_1) - (g_2 - r_2)| + |(b_1 - r_1) - (b_2 - r_2)|\}. \end{aligned}$$

In addition to forming a field, if we equip the Triphosian reals with the Hexa-Metric function, we obtain another well-studied structure known as a metric space.

Definition 14. Metric space [**?**]. A metric space, $M = (S, d)$, is a set S with a function $d : S \times S \to \mathbb{R}$ defined for all $a, b \in S$ such that:

1. $d(a, a) = 0$ for all a.
2. $d(a, b) > 0$ for all $a \neq b$.
3. $d(a, b) = d(b, a)$.
4. $d(a, c) \leq d(a, b) + d(b, c)$ (the triangle inequality holds).

Additionally, the function d of a metric space is called a metric.

Intuitively, a metric space is a coordinate space where distance behaves the way we are used to–Euclidean space (the set of real-valued doubles equipped with the Euclidean distance function) is a metric space as well, and the notion of a generalized metric space generalizes many of the properties we know and love.

Theorem 5. *The Triphosian reals equipped with the Hexa-Metric function, $(\mathbb{R}_T, H)$, forms a metric space. [**?**]*

We again cite the proof in [**?**] for this result. Now, knowing that Triphos is a metric space, we have a notion of distance between Triphosian numbers–this result is important to the geometry of Triphos, and we will use the Hexa-Metric function to prove a handful of results in geometry.

3 Results

3.1 Characteristic of the Field of Triphosian Reals

The first result we will derive is the characteristic of the field of Triphosian reals. In order to define the characteristic of a ring, we need the following definitions from algebra.

Definition 15. Subgroup [?]. A subgroup, H, of a group $G = (S, *)$, is a subset of G which is a group under $*$ itself.

Definition 16. Subring [?]. A subring A of a ring $R = (S, +, \cdot)$ is a subset of R which is a ring under $+$ and $\cdot$ itself. By definition, any subring A of R must be a subgroup of R under $+$ as well.

Definition 17. Ring homomorphism [?]. Let A and B be rings, and define a mapping $f : A \to B$. Furthermore, let 0 represent the additive identity and let 1 represent the multiplicative identity. We say f is a ring homomorphism if f satisfies the following properties for all $a_1, a_2 \in A$.

- $f(a_1 + a_2) = f(a_1) + f(a_2)$.
- $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$.
- $f(0) = 0$.
- $f(1) = 1$.

Definition 18. Isomorphism [?]. Any homomorphism f is an isomorphism if and only if f is also a bijection. Specifically, an isomorphism which is a group homomorphism is called a group isomorphism, an isomorphism which is a ring homomorphism is called a ring isomorphism, etc.

Definition 19. Characteristic of a ring [?]. The characteristic of a ring, $R = (S, +, \cdot)$, denoted $\mathrm{char}(R)$, is the minimum positive integer $n > 0$ such that there is a subring A of R which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. If R has a subring which is isomorphic to $\mathbb{Z}$, then R has characteristic zero.

Now that we have the characteristic of a ring defined, and all of the definitions we need for our proof, we can state and prove the result.

Result 1. *The field of Triphosian reals, $\mathbb{R}_T$, has characteristic zero.*

Proof. Consider the Triphosian reals, $\mathbb{R}_T$. In order to show that $\mathbb{R}_T$ has characteristic 0, we will construct a subring of $\mathbb{R}_T$ and show it is isomorphic to $\mathbb{Z}$.

Let S be a subset of $\mathbb{R}_T$ such that $S = \{[(k, 0, 0)] \mid k \in \mathbb{Z}\}$. First, we will show that S is an additive subgroup of $\mathbb{R}_T$.

1. Closure: Let $[(m, 0, 0)], [(n, 0, 0)] \in S$. Then, $[(m, 0, 0)] + [(n, 0, 0)] = [(m + n, 0, 0)]$. Since $m + n \in \mathbb{Z}$, we have that $[(m + n, 0, 0)] \in S$, so S is closed under addition.

2. Associativity: Associativity and commutativity are true for all elements of $\mathbb{R}_T$, so both hold in S.

3. Identity: Since $0 \in \mathbb{Z}$, $[(0, 0, 0)] \in S$, thus S contains the additive identity of R.

4. Inverse: Given $x \in S$, we have $y \in S$ such that $x + y = [(0, 0, 0)]$. Let $x = [(k, 0, 0)] \in S$. Then the additive inverse of x is $y = [(-k, 0, 0)] \in S$ (we see that $x+y = [(k+(-k), 0, 0)] = [(0, 0, 0)]$), so S contains an additive inverse of every element.

This proves that S is a subgroup under addition, so it remains to show that S is also a ring.

1. Identity: since $1 \in \mathbb{Z}$, we have that $[(1,0,0)] \in S$ and thus S contains the multiplicative identity of R.

2. Associativity and distributivity: the associative property under $\cdot$ and the distributive property are inherited from R.

3. Closure: let $[(m,0,0)], [(n,0,0)] \in S$. Then, $[(m,0,0)] \cdot [(n,0,0)] = [(m \cdot n + 0 + 0, m \cdot 0 + 0 \cdot n + 0, m \cdot 0 + 0 + 0 \cdot n)] = [(m \cdot n, 0, 0)]$. Since $m \cdot n \in \mathbb{Z}$, we have that $[(m \cdot n, 0, 0)] \in S$, so S is closed under multiplication.

So, S is a ring, and in particular, a subring of $\mathbb{R}_T$.

Now we will define a mapping $\lambda : \mathbb{Z} \to S$ and prove that this mapping is an isomorphism. Let $\lambda : \mathbb{Z} \to S$ be defined by $\lambda(n) = [(n,0,0)]$.

First we will show that λ is a bijection.

1. The mapping λ is injective: to prove this, let $m, n \in \mathbb{Z}$ such that $f(m) = f(n)$ but $m \neq n$. Then, $\lambda(m) = [(m,0,0)] = [(n,0,0)] = \lambda(n)$. But since both of these Triphosian numbers are in standard form, for this to be true we must have that $m = n$, which is a contradiction. Hence, if $f(m) = f(n)$, then $m = n$ and λ is an injection.

2. The mapping λ is surjective: let $y = [(x,0,0)] \in S$. We will show that there exists some $x \in \mathbb{Z}$ such that $f(x) = y$. Let $x \in \mathbb{Z}$. Then, $f(x) = [(x,0,0)]$ and hence $f(x) = y$. So, λ is surjective.

So, λ is a bijection. Now we will show that λ is a ring homomorphism.

1. The mapping λ correctly maps identities: $\lambda(0) = [(0,0,0)]$ and $\lambda(1) = [(1,0,0)]$ by definition.

2. The mapping λ preserves addition: let $m, n \in \mathbb{Z}$. We have that $\lambda(n+m) = [(m+n,0,0)] = [(m,0,0)] + [(n,0,0)] = \lambda(m) + \lambda(n)$, so addition is preserved.

3. The mapping λ preserves multiplication: let $m, n \in \mathbb{Z}$. Then, $\lambda(m \cdot n) = [(m \cdot n, 0, 0)]$. We have that $[(m,0,0)] \cdot [(n,0,0)] = [(m \cdot n + 0 + 0, m \cdot 0 + 0 \cdot n + 0, m \cdot 0 + 0 + 0 \cdot n)] = [(m \cdot n, 0, 0)]$, so we see that $\lambda(m \cdot n) = [(m,0,0)] \cdot [(n,0,0)] = \lambda(m) \cdot \lambda(n)$. So, multiplication is preserved.

Thus, λ is a ring homomorphism. Since λ is both a ring homomorphism and a bijection, λ is a ring isomorphism. To recap, we have that S is a subring of $\mathbb{R}_T$, and we have established that λ is a ring isomorphism between S and $\mathbb{Z}$.

This establishes that $\mathbb{R}_T$ has characteristic 0. □

3.2 The Triphosian Unit Circle and Pi

Result 2. *The Triphosian pi, π_T, defined as the ratio of the circumference of a circle to its diameter in Triphos, using the Hexa-Metric distance, is $\pi_T = 3$.*

In order to calculate the Triphosian equivalent of π, which we will denote π_T, we must first construct a circle, specifically the unit circle.

Definition 20. Unit circle. The unit circle is the set of all points which are distance 1 away from the origin.

Figure 2: The Triphosian unit circle, generated in `MATLAB`.

So, in order to construct the Triphosian unit circle we define the set of points

$$\{[(r, g, b)] \mid H([(r, g, b)], [(0, 0, 0)]) = 1\},$$

where H is the Hexa-Metric function. A visual plot of the Triphosian unit circle can be seen in Figure 2.

The unit circle in Triphos forms a hexagon, whose vertices are the set

$$\{[(1, 0, 0)], [(0, 1, 1)], [(0, 1, 0)], [(1, 0, 1)], [(0, 0, 1)], [(1, 1, 0)]\}.$$

Verifying that these points are in the set requires one calculation using the Hexa-Metric function as described in the definition of the set above, and so it is not included here. Now, given the set of vertex points of the hexagon, we can compute the side lengths of the hexagon. We show the calculation of the length of the side formed by the points $[(1, 0, 0)]$ and $[(1, 1, 0)]$ below.

$$\begin{aligned} H([(1, 0, 0)], [(1, 0, 1)]) &= \min\{|(1 - 0) - (1 - 1)| + |(0 - 0) - (0 - 1)|, \\ &\qquad |(1 - 0) - (1 - 0)| + |(0 - 0) - (1 - 0)|, \\ &\qquad |(0 - 1) - (0 - 1)| + |(0 - 1) - (1 - 1)|\} \\ &= \min\{2, 1, 1\} = 1 \end{aligned}$$

Note that the length of each side is the same (we will not show each of the computations, as they are all equivalent to the computation above), so the "circumference", or more accurately, the perimeter, of our unit circle, is $6 \cdot 1 = 6$.

Now, by visual inspection of Figure 2, we see that the diameter of this regular hexagon can be found by taking the distance between $[(1, 0, 0)]$ and $[(0, 1, 1)]$. (Or, equivalently, we could take the distance between $[(0, 1, 0)]$ and $[(1, 0, 1)]$ or between $[(0, 0, 1)]$ and $[(1, 1, 0)]$, but all of these lengths are equal.)

$$
\begin{aligned}
H([(1,0,0)],[(0,1,1)]) = \min\{&|(1-0)-(0-1)|+|(0-0)-(1-1)|,\\
&|(1-0)-(0-1)|+|(0-0)-(1-1)|,\\
&|(0-1)-(1-0)|+|(0-1)-(1-0)|\}\\
= \min\,&(2,2,4) = 2.
\end{aligned}
$$

So the perimeter of the unit circle in Triphos is 6 and the diameter is 2, and thus the Triphosian equivalent of π is $\pi_T = 6/2 = 3$.

3.3 A Triphosian Pythagorean Theorem

Result 3. *In Triphosian coordinates, if a triangle with a* 60° *angle has leg lengths* a *and* b*, then the length,* c *of the third side (which is opposite from a* 60° *angle) is*

$$c = \min(|a-b|+|a|\,, |a|+|b|\,, |a-b|+|b|).$$

Assume we have a triangle with a 60° angle. First, we will impose Triphosian coordinates onto the triangle. Place the origin of the coordinate system at the vertex of the 60° angle, as shown in Figure 3.

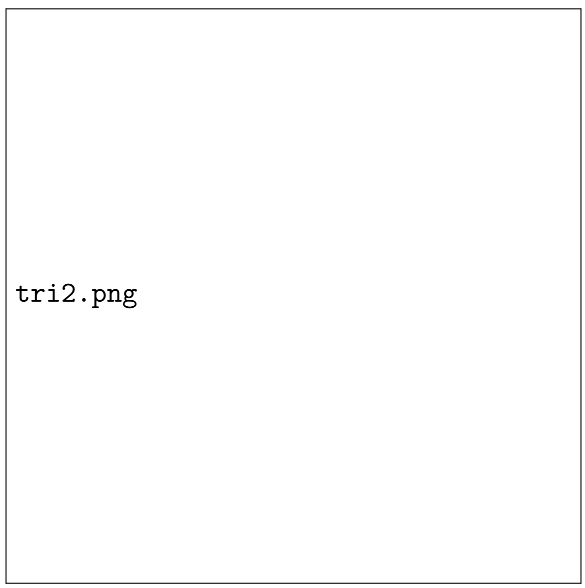

Figure 3: A triangle with a 60°: the origin is imposed on the vertex of the 60° angle. Note that once the origin is imposed, the triangle can be rotated to a similar position as this triangle without changing any of its properties.

Now, we can assign a length to each of the legs (the sides forming the 60° angle). We assign one leg the arbitrary length a and the other leg the arbitrary length b, as shown in Figure 4. Since we have imposed the origin of the coordinate system on our triangle in a specific way, we now know the three points that define the triangle. Note that one of the legs is aligned with the blue axis, and one of the legs is aligned with the red axis.

Without loss of generality, we can assign the arbitrary length a to the side aligned with the blue axis. Note (as can be seen in either Figure 3 or 4) that this side goes in the negative direction along the blue axis, so we can say that this leg of the triangle terminates at the point $[(0, 0, -a)]$, which when converted to standard form is $[(a, a, 0)]$.

Similarly, we notice that the other leg is aligned in the positive direction on the red axis. We can assign this leg the arbitrary length b, and then this leg terminates at the point $[(b, 0, 0)]$.

Then, we can calculate the length of the remaining side, c, as

$$c = H([(a, a, 0)], [(b, 0, 0)]),$$

using the Hexa-Metric function.

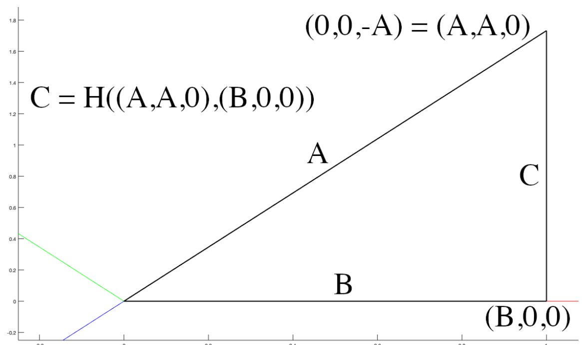

Figure 4: Because of the way we have imposed coordinates, we can use the arbitrary lengths of the legs to determine the coordinates which define the third side of the triangle. Then, the length of the third side is the length between these two points.

We then compute the length c as

$$\begin{aligned} H([(a, a, 0)], [(b, 0, 0)]) = \min\{&|(a-0)-(b-0)| + |(a-0)-(0-0)|, \\ &|(a-a)-(b-0)| + |(0-a)-(0-0)|, \\ &|(a-a)-(0-b)| + |(0-a)-(0-b)|\}, \end{aligned}$$

which simplifies to

$$c = \min\left(|a-b| + |a|, |a| + |b|, |a-b| + |b|\right).$$

3.4 Computer Codes

In order to derive many of these results, a suite of `MATLAB` functions was created to do math in Triphos. While this suite of functions is currently relatively small and needs to be expanded, it is still novel in that no function suite currently exists for Triphosian math. We hope to expand this suite of functions in `MATLAB`, or potentially port the functions into Python and expand the suite there.

Finally, when the Triphosian Pythagorean theorem was explored, a Mathematica notebook was created and used to verify this geometric theorem, and to create a more complex version of the theorem defined in terms of Triphosian coordinates.

4 Conclusion and Future Work

In addition to the results we have defined, we have a set of conjectures about the algebraic structure of Triphos.

Conjecture. The field of Triphosian reals, $\mathbb{R}_T$, contains subfields isomorphic to $\mathbb{Q}$ and $\mathbb{R}$.

We believe that this conjecture can be proven by using a similar mapping to the mapping λ we defined previously.

Conjecture. The field of Triphosian reals contains a subgroup isomorphic to $\mathbb{Z}^3/\sim$ (potentially a subring), and a subgroup isomorphic to $\mathbb{Q}^3/\sim$ (potentially a subfield).

Conjecture. There is an isomorphism between $\mathbb{R}_T$ and $\mathbb{R}^2$.

In addition to showing these facts about the algebraic structure of Triphos, we are additionally interested in defining a further notion of trigonometry in Triphos. Our first goal in this regard would be to define a notion of trigonometric functions on Triphosian numbers.

Finally, we have not presented some of the additional interesting results seen in [**?**]–notably, notions of Triphosian integers and primes, and a different place value system which may be more appropriate for describing Triphosian numbers. We are also interested in exploring properties and distribution of Triphosian primes, as well as other sequences of Triphosian integers.

In conclusion, we have constructed the Triphosian real numbers, and defined their algebraic structure and notion of distance. Furthermore, as unique results, we have proven that the field of Triphosian integers has characteristic zero, and regarding Triphosian geometry, we have constructed the Triphosian unit circle and shown that $\pi_T = 3$. We have also found an analog of the Pythagorean theorem in Triphosian geometry, and defined several more interesting problems in Triphos.