COMAP MCM 2019, Problem A: Solution

Contents

1 Introduction

1.1 Characteristics of Dragons

1.1.1 Physiological Assumptions

In order to model the caloric requirements, energy requirements and ecological impact of dragons, we first need to make some assumptions about the physiology and ecology of dragons. Unfortunately dragons are not real so for most assumptions there is a lack of corresponding data, but some informed decisions can be made.

We will begin our analysis of the physiology of dragons by assuming that dragons are basal non-passerine birds; that is, dragons are evolutionarily somewhere between crocodiles and modern birds. Dragons share powered flight and endothermic temperature regulations, both synapomorphies of the birds (and thus apomorphic to the crocodilians), but also have scales and clawed hands, which are plesiomorphic to modern birds. Thus, for our assumptions about metabolic rate, we will assume that dragons are most similar to the non-passerine birds.

Supporting evidence about the endothermic temperature regulation of dragons can be found in *A Clash of Kings*: dragons are said to be so hot, they steam during cold nights. With this in mind, we assume that dragons have a regular internal temperature of at least 100 degrees Celsius, the boiling point of water. We assume that the internal temperature of dragons is 135°C, due to the mechanism we provide for dragons' fire breathing abilities.

From the problem statement, we assume dragons are born at a mass of about ten kilograms and reach a mass of 30 – 40 kilograms after a year (assumed from the problem statement); we used 35 kilograms for all of our assumptions. We assume that dragons live around 200 years, and the largest possible mass they should obtain during that time is about 5400 kilograms. Balerion, the Black Dread, the largest reported dragon in the *A Song of Ice and Fire* series, lived to almost two centuries [1] before dying of old age [2]. While dragon size is difficult to estimate from the amount of available data, Balerion's smaller relative Vhagar was said to be about the size "of five warhorses" [3]. Assuming a warhorse is roughly the size of a shire draft horse, we can assume a weight of 2000 pounds [4]. Now, noting that Balerion was larger than Vhagar, we assume a weight of about six warhorses (roughly), or 12000 pounds, or 5400 kilograms. We also note that dragons continously grow throughout throughout their lives, an assumption of the problem statement.

Finally, we assume that dragons are solitary apex predators, not subject to predation, the dragons are not capable of reproducing (which we assume due to the lack of knowledge about the dragons' sexes and dragon reproduction in general in the source material), and dragons are facultatively migratory. That is, dragons will only migrate between habitats should they exhaust the resources of their current habitat.

1.1.2 Observation of Caloric Needs of Dragons

While we assume that dragons are genetically closer to non-Passerine birds, we assume dragons follow the behavioral trends of many reptiles, which are mainly sedentary except when hunting or migrating [5]. Since dragons exhibit endothermic temperature regulation, the amount of energy required to sustain homeostasis is dependent on external temperature;

i.e. less energy is required to sustain temperature regulation at a high external temperature. Hence the caloric needs of an individual dragon are dependent on both a basal metabolic rate (a function of body mass, following Kleiber's Law), as well as the temperature.

The basal metabolic rate is calculated under the assumptions that dragons follow Kleiber's Law for non-Passerine, non-migratory birds which states:

$$\text{Caloric Need (kCal a day)} = k * (\text{Body mass in kg})^{0.75}.$$

The k constant in the equation is equal to 78 for non-Passerine birds. This is coupled with an "upkeep" constant of 1.5, multiplied by the BMR to allow for calories needed to recover from any assumed injuries or other unforeseen circumstances that may impact the dragons, such as illness [6].

The effect of the average temperature of the environmenty additionally has a characterized effect on the metabolic rate of non-Passerine birds. Assuming that non-Passerine birds are well-suited to a specific temperature, a change in 20 degrees Celsius corresponds with a metabolic rate change (in the same direction) of 50 % [7]. As a unit rate, a temperature change of one degree Celsius will increase or decrease the metabolic rate by 2.5%. We assume that dragons are well adapted to a temperate climate and deviations from the average climate of a temperate biome will affect the dragon's metabolic rate.

1.2 Observation of Fire Breathing

We predict that dragons breathe fire through a biochemical reaction between chemicals produced in specialized throat glands of the dragons. Ethanol and hydrochloric acid are both products with known biological pathways: ethanol can be produced in glands with specialized microbiomes, likely consisting of species of anaerobic yeast or bacteria which use ethanol fermentation to produce energy from sugars, and hydrochloric acid is secreted by parietal cells [8] in the stomach, so we assume the dragon has another throat gland containing a lumenal surface composed of parietal cells.

The dragon has voluntary control over the two glands. When ethanol and hydrochloric acid are secreted from the glands simultaneously, they spontaneously react to form diethyl ether gas [9]. This reaction must occur above a temperature of 130 degrees, which is part of the reason why we assume the internal temperature of dragons is 135 degrees Celsius. Dietheyl ether can combust readily in the presence of oxygen, but can also form peroxide polymers through spontaneous oxidation [10]. The peroxide compounds are extremely shock-sensitive and will ignite/explode readily with minimal friction. We assume the dragons have some piece of specialized oral anatomy, such as a hard plate on the roof of the rear mouth, which could be struck to generate sparks. This mechanism would allow for the quick production and ignition of flammable gas, and due to the mechanism involving microbiome and cell based secretion of fluids, we assume the calories needed for the production of fire do not have a significant impact on the basal metabolic rate, and should be covered by the upkeep coefficient.

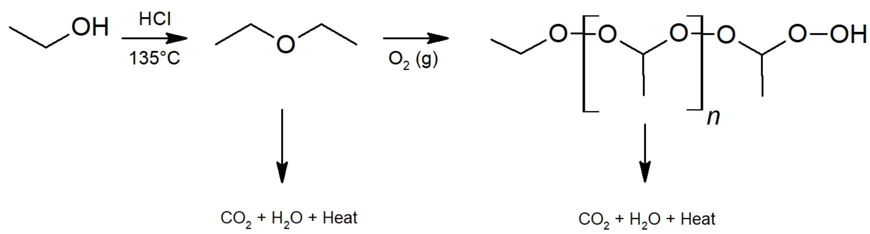

Figure 1: Reaction depicting how liquid ethanol and liquid hydrochloric acid react to produce the diethyl-ether needed for a dragon to breath fire.

1.3 Environmental Interactions and Seasonality

It will be important to showcase the changes in a dragon's needs depending on their environment. To do this we can observe the different biomes in the world of George R. R. Martin and compare them to various real world biomes. For the purposes of this paper, we will observe 4 major biomes, more specifically the arctic tundras, temperate forests, dry and arid plains and hills, and hot and dry deserts. Each of these biomes will have an effect on the caloric needs of a dragon based on temperature discrepancies as well as population density and biomass density of the prey that they consume.

Ambient temperature will play an important role in the caloric needs of a dragon. Since we are assuming that dragons are effectively birds in regards to their regulation of body temperature, we will need to take into account the difference in amount of food they will need to consume to survive between various regions. According to the University of California Museum of Paleontology Berkeley, each biome will have the following ranges of temperatures throughout the year.

1. **Arctic Tundra:** We can expect a range of $-34°C \to [3, 12]°C$ in a cold biome. We will average the maximum temperature within the given interval to get a range of $-34°C \to 7.5°C$.

2. **Temperate Forests:** We will treat any temperate biome with a density of trees as a temperate forest. We can expect a range of $-30°C \to 30°C$

3. **Deserts:** We will treat any biome that is typically extremely hot and dry as a desert and can expect a temperature range of $-18°C \to [43.5, 49]°C$. We will assume these biomes will have temperatures ranging from $-18°C \to 46.25°C$ with a posted average of $20°C \to 25°C$, we will assume this is $22.5°C$.

4. **Plains:** We will treat any non-forest/non-tundra location with a temperate climate as well as any dry and arid grasslands as a plain, with temperatures ranging from $-40°C \to 38°C$.

For the purposes of this section of our analysis, we will be dividing each year into two elongated Summer and Winter seasons to mimic the changes in the book. We will assume the longer seasons each last around 10 years. By using one period of the function $\frac{max-min}{2}(sin(x)+1)+min$, where we assume the Summer and Winter averages are averaged to be at $x = 45°$ and $x = -45°$ respectively, we estimate the seasonal temperatures listed below.

	Desert	Forest	Plains	Tundra
Summer	$36.84°C$	$17.57°C$	$26.58°C$	$1.42°C$
Winter	$-8.59°C$	$-17.57°C$	$-28.58°C$	$-27.92°C$

1.3.1 Difference in Caloric Needs Depending on Biome

As we have noted, a dragon's diet is going to be forced to change depending on the ambient temperature. The dragon will need to burn through extra energy when it is colder and less energy when warmer. Naturally in a tundra biome, as well as colder seasons, there will be an positive effect on calories needed and vice versa when residing in a hotter desert region and during the summer months.

There are also discrepancies in the amount of food available depending on the region. Desert and tundra biomes will naturally have fewer available food sources, so in those regions a dragon will need to make up for the decrease in availability of prey, while in more temperate environments a dragon will not need to hunt as long to find enough food. We can account for this by observing population density over a given region to determine how well each dragon will eat. We will also assume that food available in these regions

1.3.2 Other Ecological Assumptions

We assume that while the lengths of the seasons vary significantly in the present day in the world of A Song of Ice and Fire, there is periodic fluctuation in the length of the seasons that reasons reasonably constant over geologic time. Hence we will assume an average length of ten years for each season. As for the biomes in which the dragons reside, for simplicity's sake we will assume that once a dragon has selected a biome, it will remain their until it dies. We will evaluate various scenarios predetermining where each of the three dragons reside to give a basic idea of what the world will need to support them.

Since dragons grow over time and never stop growing (assumed in the problem statement), we also assume dragons have a base growth rate that is linear in time. From the provided information that dragons are born at a weight of 10 kilograms and are about 35 kilograms after a year, we can use these two points to define a linear function for growth. We obtain the function

$$w(t) = 25t + 10,$$

where $w(t)$ is the weight of a given dragon at year t. Furthermore, the weight of a dragon at a given time is subject to the amount of food the dragon has been able to procure over its lifetime. As mentioned earlier, the main considerations for caloric need come from assumptions derived from Kleiber's law.

2 An Ecological Model

2.1 Modelling Strategies

There are many approaches to modelling the attributes of these dragons that, when combined, will allow us to generate a model for their lives and the interactions with their environment. We can observe the amount of food they need to eat, how they will travel between biomes, and how the different biomes will affect their growth. First let us discuss a general model that can be used to create various scenarios. To do this we will need to abstract each component into a system of several differential equations. These components will model functions of the mass of a dragon M, caloric needs C, food consumed F, regional location L of the dragon, and availability of food in a biome A.

2.1.1 Individual Models for the Dragons

For each dragon, we assume dragon size is the main contributor to the amount of energy needed for a dragon to survive. We assume that dragon size has some base rate of increase every year, but that the availability of food also affects the rate of growth of each dragon. We assume that the dragons are free-roaming and there is no enclosure to limit growth (such as the effect of the dragonpits in the series).

We have established the following system of equations to determine the size of each dragon at year t. Let M_p be the mass of dragon p, C_p be the caloric requirement of the dragon in kilocalories, $F_{n,p}$ be the caloric intake of the dragon, A_n be the availability of kilocalories in the region, and L_n be a function representing the regional temperature, where n is the biome in which the dragon resides.

The function for the mass of the dragon has two components: a time-based rate component, modelling the dragon's growth as a linear function of its age, $\gamma_1 t + M_0$, where γ_1 is the annual growth rate and $M_0 = \gamma_2$ is the initial mass at birth. We will also need an additional component describing how much the dragon grows based on food intake, called σ, which will represent the rate of growth based on the difference between caloric needs and food consumed.

For the caloric requirement function, let α be a term representing the calories needed per unit of mass of the dragon, k be our Kleiber constant over an entire year of 365 days, and β be a ratio of food needed per degree Celsius of the difference between some baseline temperature T_0 and the temperature of the region L_n. We can then use this combined value to construct a difference in caloric needs based on current body mass and temperature.

For a function of food consumption, we will need to break this down into a few components based on availability of food depending on the season, the dragon's body mass, and caloric needs. To do this we will represent the food consumed $F_{n,p}$ as a function of the minimum of either some scaled value κC_p and a slightly randomized ratio of available food A_n to basal caloric needs $(\alpha k M)$, where r_p will represent a random component for hunting and $e_{n,p}$ will be a control constant for the encounter rate.

We will represent the amount of food available as a piece-wise function A_n based on the season, where the percentage difference of available food between summer and winter seasons is represented by $sin(45°) + 1$ and $sin(-45°) + 1$ respectively. This value will then scale a given baseline value by determining kilocalorie per square meter based on biome (δ_n) and the total area of the region, we will call this value ϕ_n. To this value we will add some ratio μ_n of the current availability A_n and we will subtract the total biomass consumed by the dragons in the region. A_n also needs to be scaled to accommodate the amount of energy available to the dragons.

Energy is brought into an ecosystem by primary autotrophs, which harvest their energy needs from the sun. From there, primary consumers eat primary producers, secondary consumers eat primary consumers, and so on. We assume the exsitence of four trophic levels (up to tertiary consumers) other than dragons. Between each trophic level, only at most 10% of energy is transferred [11], i.e., primary consumers will only obtain at most 10% of the energy stored in autotrophs, and secondary consumers will only obtain at most 10% of that energy, which is 1% of the total energy from the plants. So, if we assume dragons can eat any consumers, dragons have access to 10% of the ecosystem energy from primary consumers, 1% of energy from secondary consumers, and 0.1% of energy from tertiary consumers. However, dragons can only obtain 10% of the energy from any of these trophic levels, so summing and multiplying by 10%, we get that dragons can access $1.11\% = 0.0111$ of the total energy obtained in the ecosystem.

For the regional temperature we will utilize a piece-wise function to calculate L_n to accounts for the average temperature of the region in both summer and winter. This will use a function $sin(45°)+1$ to estimate the current temperature based on season and temperature minimums and maximums for the type of biome. This value can then be used to determine the caloric needs for each dragon.

2.2 A System of Equations

2.2.1 Modelling Dragon Behavior

The mathematical model listed below displays all functions, time derivatives, and constants we can use to develop some rudimentary data on the behavior of our dragons. While this system of differential equations is abstracted to allow for variation on constants, the constants used to develop our visual and topical data will be expressed in the following section.

$$
\begin{aligned}
\dot{M}_p &= \gamma_1 + \sigma(C_p - F_{n,p}) \\
\dot{C}_p &= 2 \cdot \frac{3}{4}\alpha k M_p^{-.25} + \beta(T_0 - L_n)M_p \\
\dot{F}_{n,p} &= \min(e_{n,p} \cdot A_n, \kappa C_p) \\
\dot{A}_n &= \begin{cases} t(\text{mod } 20) < 10, & \varepsilon(\mu_n A_n - \sum F_{n,p}) \\ t(\text{mod } 20) \geq 10, & \varepsilon(\mu_n A_n - \sum F_{n,p}) \end{cases} \\
L_n &= \begin{cases} t(\text{mod } 20) < 10, & \psi_n \\ t(\text{mod } 20) \geq 10, & \upsilon_n \end{cases} \\
\psi_n &= \frac{max_n - min_n}{2}(sin(45^\circ) + 1) + min_n \\
\upsilon_n &= \frac{max_n - min_n}{2}(sin(-45^\circ) + 1) + min_n \\
\phi_n &= \delta_n \cdot \text{Area of region } n \\
M_p(0) &= \gamma_2 \\
C_n(0) &= 240148 \ kCal \\
F_{n,p}(0) &= 0 \\
A_n(0) &= \epsilon \cdot \phi_n \cdot (sin(45^\circ) + 1)
\end{aligned}
$$

2.2.2 Defining Constants

We will be applying the constants in the table below to generate data from our model. We will then utilize the results to discuss how the dragons will interact with their environment, as well as what could be altered, improved upon, and the strengths of the model within the given parameters.

Constant	**Assumed Value**	**Why?**
k	$365 \cdot 78$	Kleiber constant for non-Passerine birds [6] scaled to years.
α	1.5	Kleiber law "upkeep" constant [6]
r_p	$N(0, 0.5)$	Normally distributed random component for food aquisition.
$A_n(0)$	$.0111 \cdot \phi_n \cdot (sin(45) + 1)$	The initial condition for A, calculated using summertime values.
γ_1	25	Fitting the line to known data points results in this slope.
$M_0 = \gamma_2$	10	Assumed weight at birth.
σ	0.000005	For an assumed best fit.
n	$\{1, 2, 3, 4\}$	Numerical assignments to each of the 4 biomes.
p	$\{1, 2, 3\}$	Numerical assignments to each of the 3 dragons.
β	0.025	Metabolic rate change due to temperature. [7]
δ_n (kCal per year per m^2)	$\delta_1 = 28$ $\delta_2 = 11650$ $\delta_3 = 4110$ $\delta_4 = 1300$	Primary production from desert [12] Primary production from forest [13] Primary production from plains [14] Primary production from tundra [13]
Area	$A_1 = 1.3 \times 10^6$ $A_2 = 1 \times 10^7$ $A_3 = 4 \times 10^6$ $A_4 = 9.28 \times 10^5$	Assumed area of Desert biome. Assumed area of Forest biome. Assumed area of Plains biome. Assumed area of Tundra biome.
ϵ	.0111	Percent energy available to dragons
ψ_n	$\psi_1 = 36.84°C$ $\psi_2 = 17.57°C$ $\psi_3 = 26.58°C$ $\psi_4 = 1.42°C$	Desert summer temperature Forest summer temperature Plains summer temperature Tundra summer temperature
υ_n	$\upsilon_1 = -8.59°C$ $\upsilon_2 = -17.57°C$ $\upsilon_3 = -28.58°C$ $\upsilon_4 = -27.92°C$	Desert winter temperature Forest winter temperature Plains winter temperature Tundra winter temperature
μ	0	We assume that population of food is not replenishing to observe how quickly the dragons can deplete the local abundance of food.

3 Conclusion

We will conclude by observing some rudimentary results determined by placing a dragon into each of the four regions. After running our model through an Ordinary Differential Equations solving algorithm, we have been able to generate a few plots for each region to outline what effects are immediately noticeable given our assumptions.

3.1 Results for each Biome

3.1.1 Desert Results

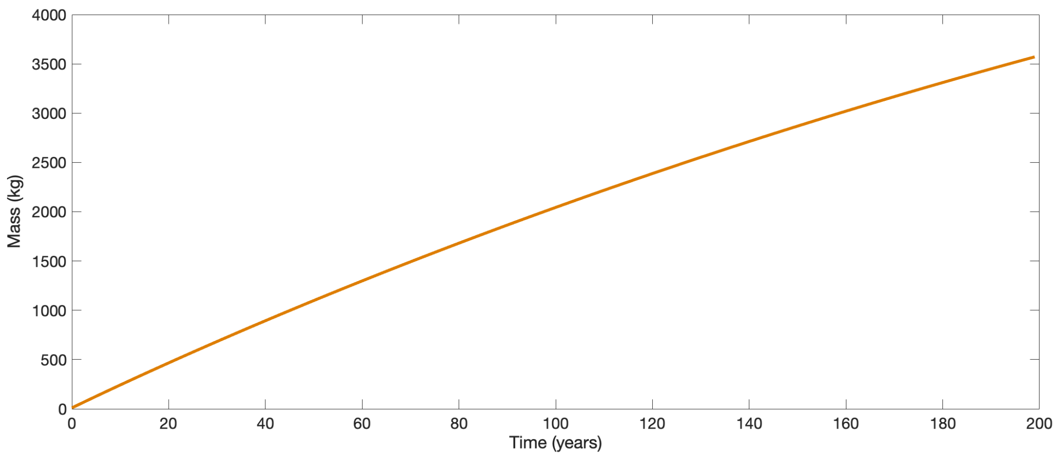

Figure 2: Mass model for dragon during 200 year lifespan.

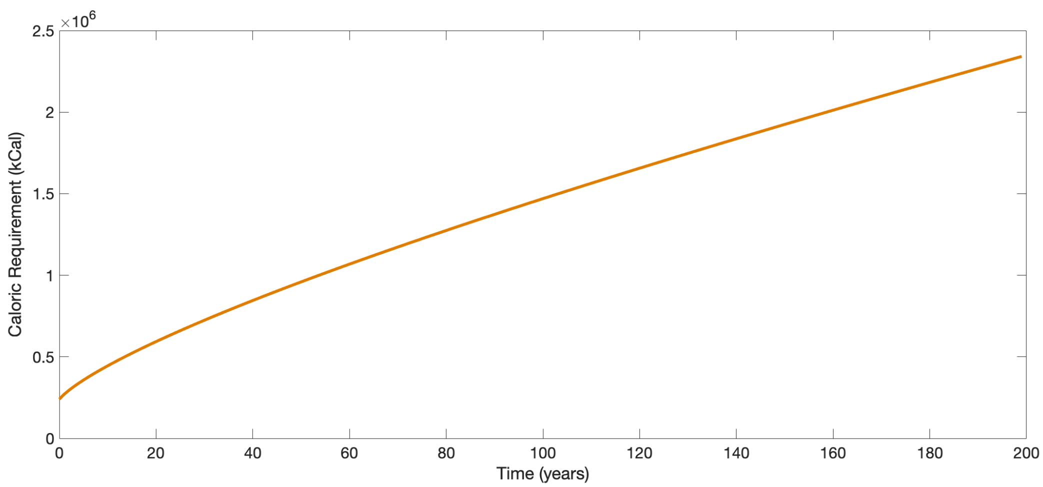

Figure 3: Caloric need of dragon derived from Kleiber's law and environmental impact.

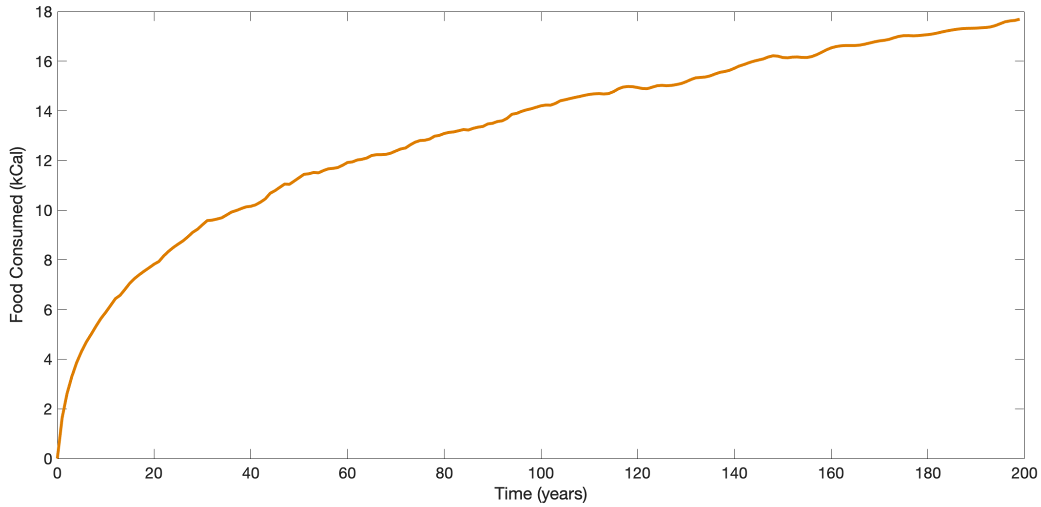

Figure 4: Amount of food a singular dragon takes from the environment per year.

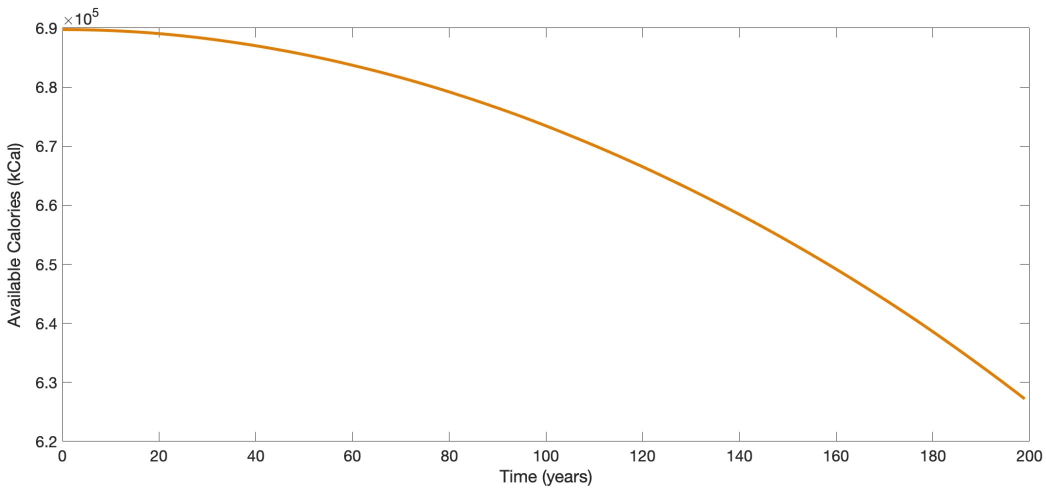

Figure 5: Amount available food after a single dragon consumes food per year.

3.1.2 Forest Results

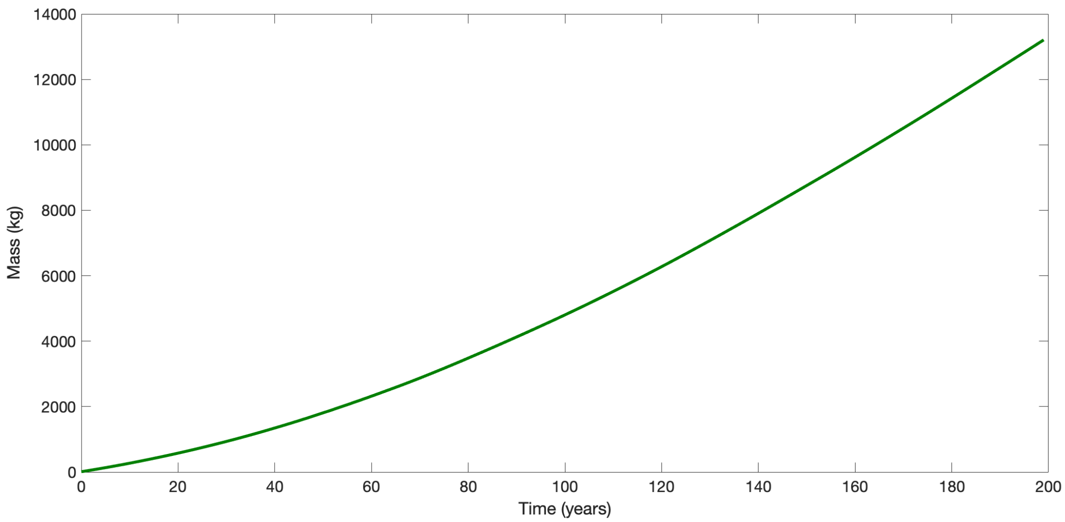

Figure 6: Mass model for dragon during 200 year lifespan.

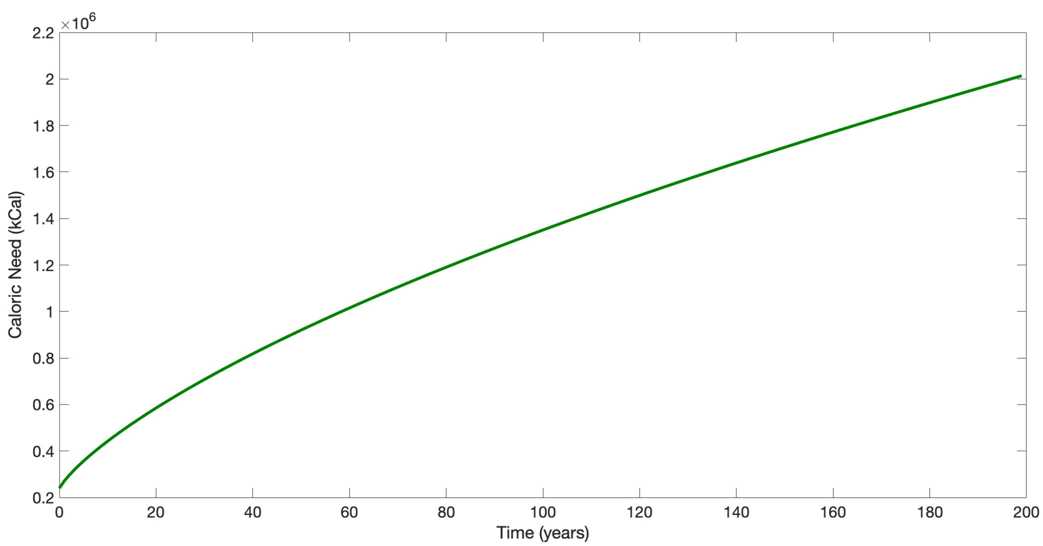

Figure 7: Caloric need of dragon derived from Kleiber's law and environmental impact.

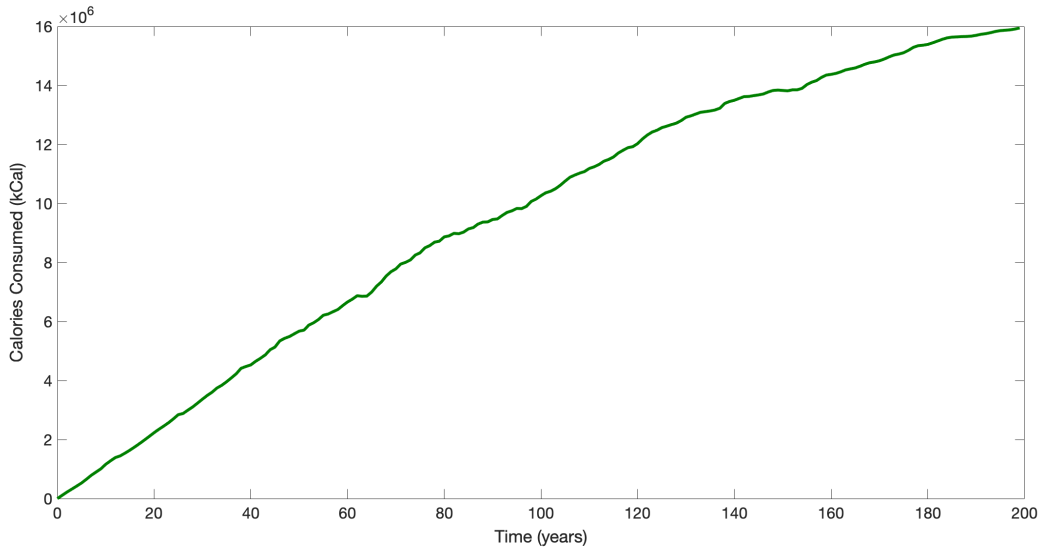

Figure 8: Amount of food a singular dragon takes from the environment per year.

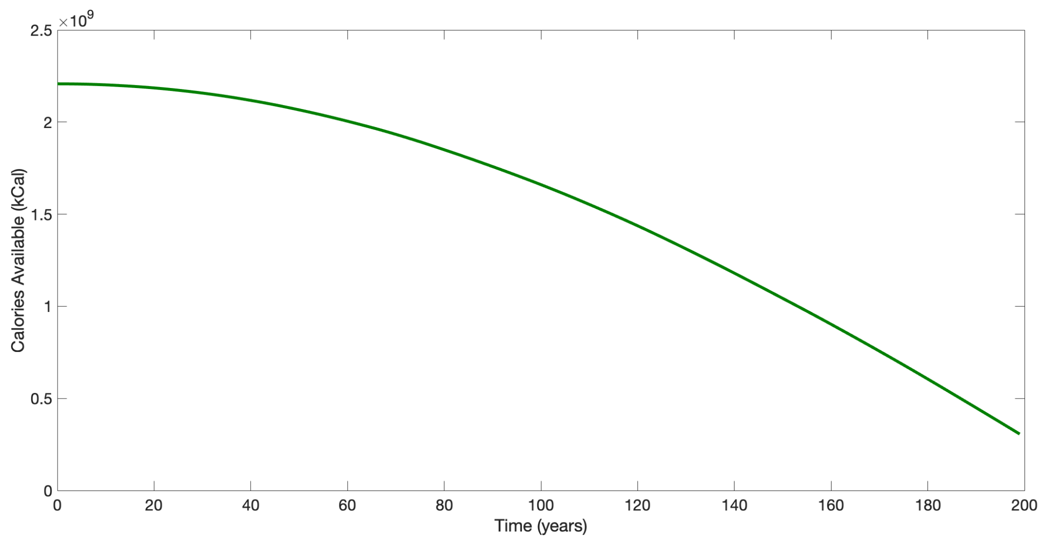

Figure 9: Amount available food after a single dragon consumes food per year.

3.1.3 Plains Results

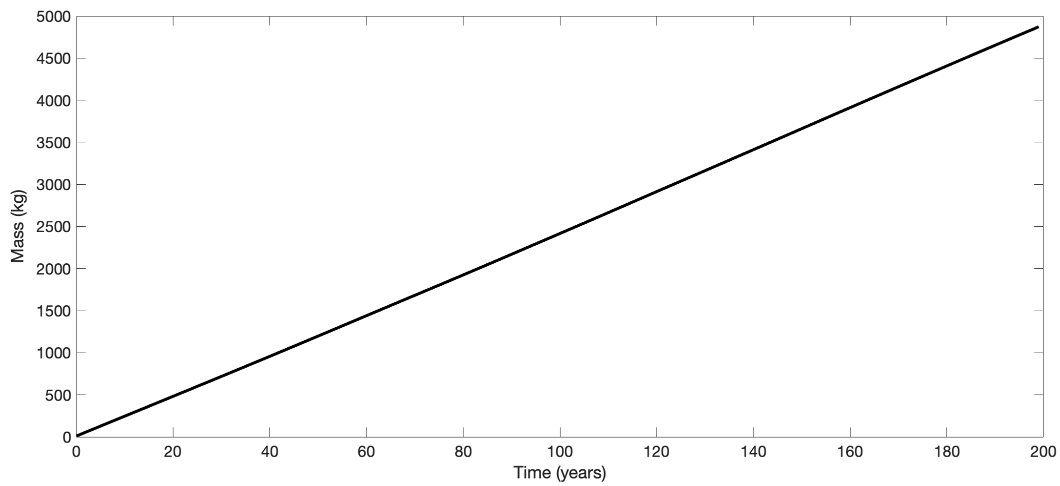

Figure 10: Mass model for dragon during 200 year lifespan.

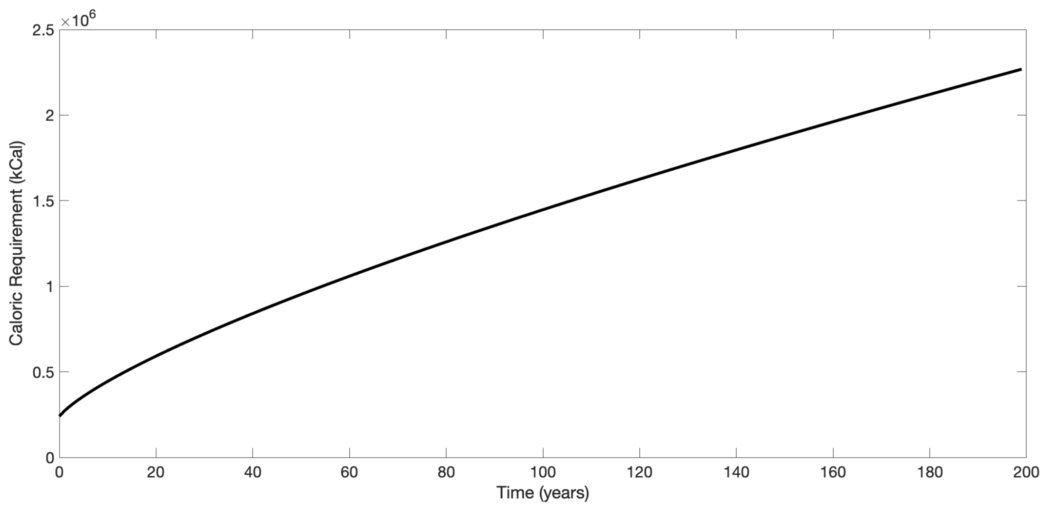

Figure 11: Caloric need of dragon derived from Kleiber's law and environmental impact.

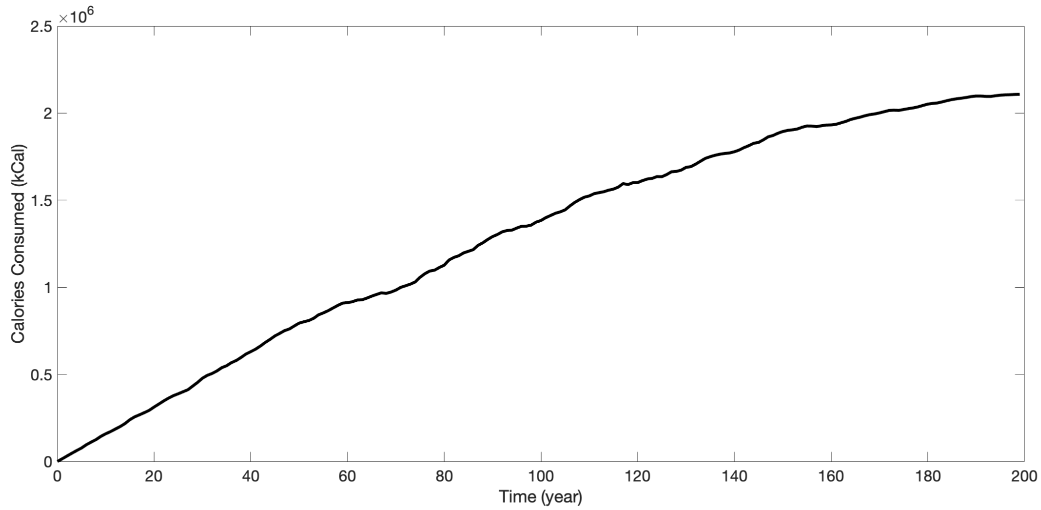

Figure 12: Amount of food a singular dragon takes from the environment per year.

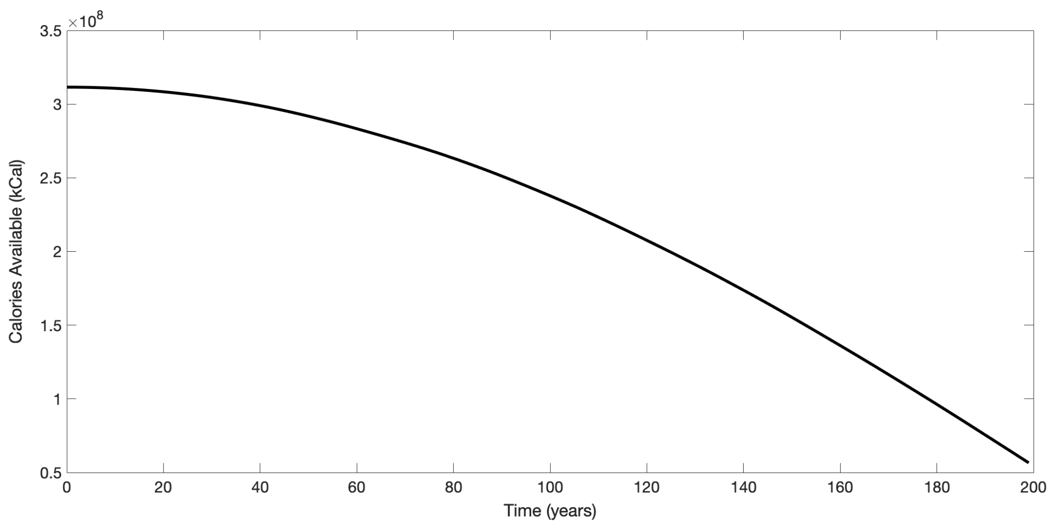

Figure 13: Amount available food after a single dragon consumes food per year.

3.1.4 Tundra Results

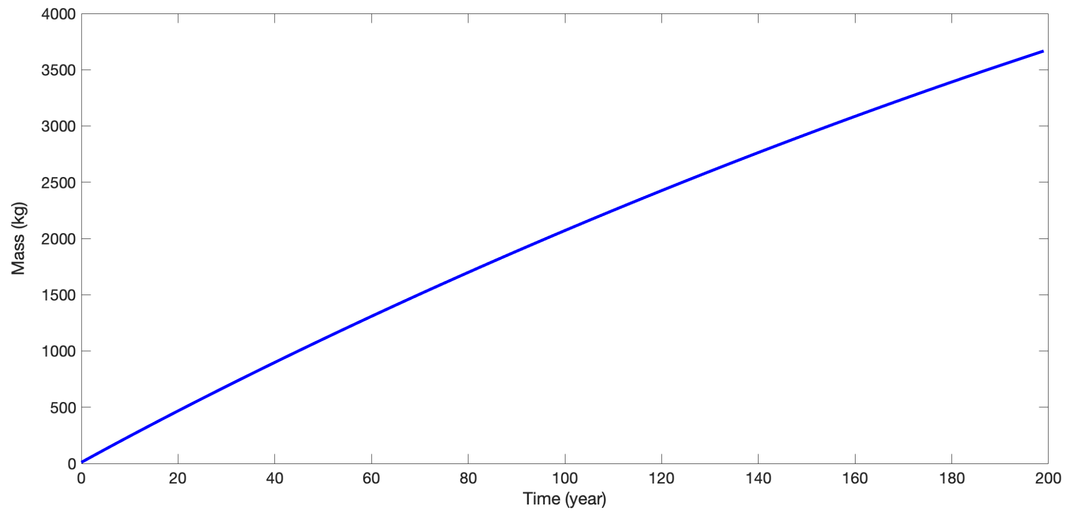

Figure 14: Mass model for dragon during 200 year lifespan.

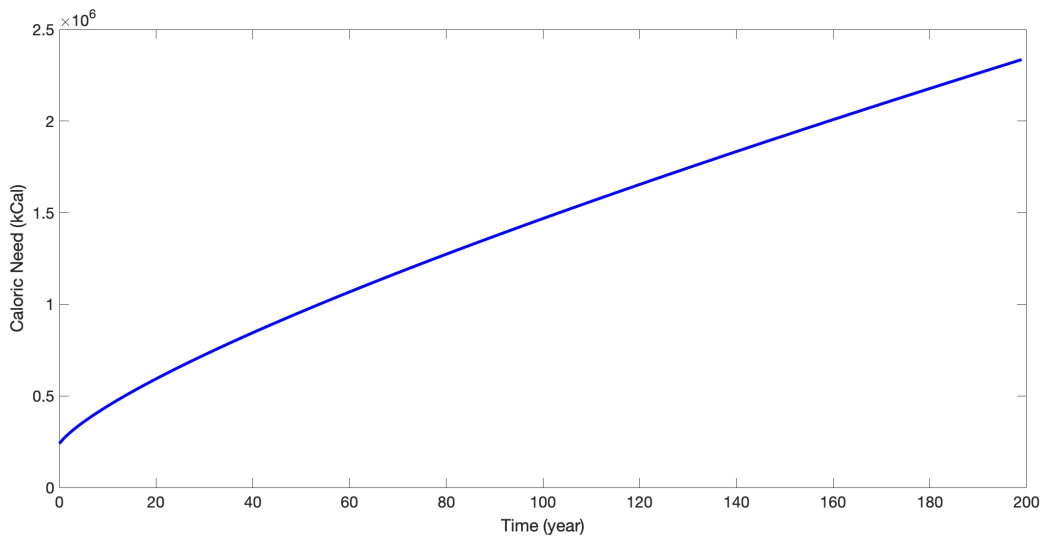

Figure 15: Caloric need of dragon derived from Kleiber's law and environmental impact.

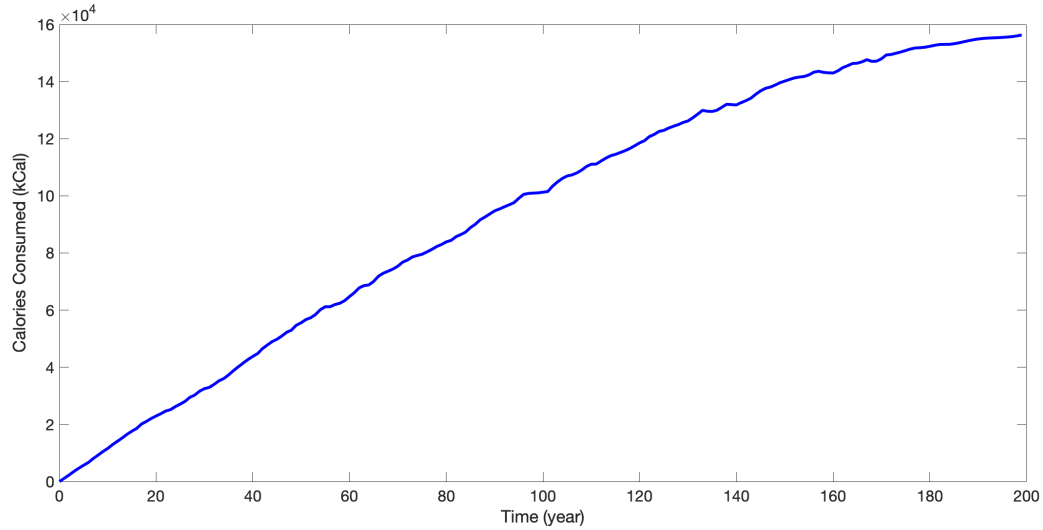

Figure 16: Amount of food a singular dragon takes from the environment per year.

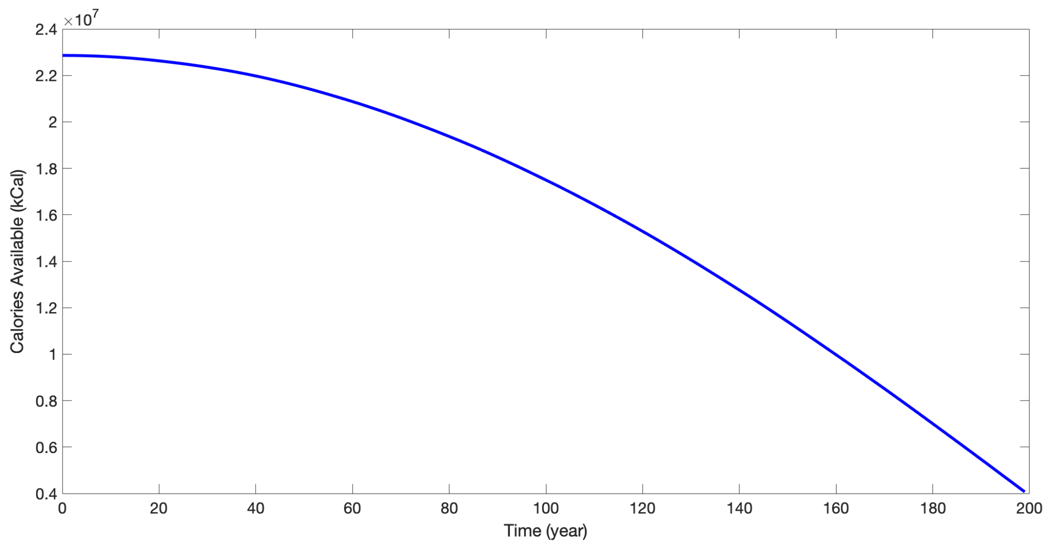

Figure 17: Amount available food after a single dragon consumes food per year.

3.2 Analysis

While in the future we would like to include a gravity model for migration and a possible Leslie or Lefkovich system (or another alternate model for population dynamics) to model reproduction and population models, we can obtain valuable conclusions about the trends incorporated and the potential for survival of dragons in each biome.

Mainly, better parameter estimation based on data is important component that would benefit our model. While our results show what we believe to be the true general trends,

3.2.1 Model Testing and Sensitivity Analysis

While we tried to eliminate as many overly sensitive terms as possible in our model, there are a few that stick out as sensitive components when their values are changed.

3.2.2 Strengths and Weaknesses of the Current Model

The main strength of our model is that we can easily generate trends and conclusions about individual dragons over time in a given biome. For the 4 biomes we assume dragons live in we can easily generate code in software such as `MATLAB`, Mathematica to visually represent an individual due to the simplicity of the equations used to define our parameters. Another strength of our model is that the constants given were derived based on ecological models that originated from other, real species. Our model takes trends seen in non-passerine birds and reptiles. Because both of these animal groups are highly studied by ecologists today, it is hypothetically simple to take these trends and integrate them into a dragon model based off of the assumption that dragons are a hybrid of these phylogenetic group.

While the integration of other species data was a strength of our model, one of its weaknesses is the fact that this piecing together was due to an obvious lack of data to support the dragon model itself. Dragons are fictional creatures, obviously, and many of the larger assumptions made simply don't apply to species that are living today. Mathematically this made setting constants and other parameters very difficult. Values such as our encounter rate, sigma and other symbolic values mentioned above were adjusted during the visualization testing, due to no data being available for the hunting efficiency of dragons for example. Biologically other assumptions had to be made as well. While the number of dragons was stated to be constant and thus made reproduction irrelevant, interaction factors were assumed to be non-existent to make modeling simpler. Another example of this migration. We assumed that dragons didn't migrate unless their resources were exhausted (which we didn't think would happen) but a function wasn't included due to elements such as a gravity model being required for accurate representation. Within the model itself there are weaknesses that simply come assumptions such as linearity. Our growth model (M) doesn't take into consideration the fact that if a dragon isn't getting all of it's required calories then it would start to decrease in weight and eventually die. Death isn't considered in our model at all since we assume lack of predation and protection from trauma related accidents. This is obviously a detrimental blow to accuracy of our model when resources are sparse, as in the tundra and desert environments. On the opposite side, we poorly limit the amount of food our dragon eats and make fail to make growth not accelerate past a certain rate, which is why the forest model shows our dragon growing faster than assumed.

While these omissions are more evident when compared to real species, the fact that our model can easily be implemented into a code based form is extremely powerful and allows for the current working model to be easily modified given more thought and time.

3.2.3 Modeling Stability of Large Apex Predators

While dragons are obviously fictional creatures, the idea of large apex predators existing in low population numbers is one that is observed in real ecological models. Instead of dragons, imagine if animals such as dinosaurs, large mammals or large aquatic predators were being modeled. While difficult in practice, being able to create models for animals that are now extinct is absolutely beneficial for ecological models which take into consideration the impact from animals that are no-longer present. In this case our model is looking at large animals that have no natural predators and are capable of accessing their habitat's available energy at multiple tropic levels. These kinds of assumptions are applicable to a wide range of apex predator models and allows for the predicted ecological impact of apex predators. Take for example a large predator like the Megaladon shark. These now extinct animals are similar to today's Great white sharks but are considerably larger. Using some assumptions such as they are not predated on by other animals once achieving a certain mass and have generalized caloric needs are all things that need to be considered when evaluating the ecological impact. Of course in the real world, living creatures must reproduce and are capable of migration as well so these would be important parameters to include in a model such as this to accurately measure impact.

References

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