# Solution for One-Variable Lasso Problem

Consider two vectors:

$$\mathbf{v}_{n\times 1} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}, \quad \mathbf{z}_{n\times 1} = \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{pmatrix},$$

where  $\mathbf{v}_{n\times 1}$  is the response vector and  $\mathbf{z}_{n\times 1}$  is the feature vector. We are interested in solving the following one-variable Lasso optimization problem:

$$\min_{b} \frac{1}{2n} \|\mathbf{v} - b \cdot \mathbf{z}\|^2 + \lambda |b|$$
 (1)

where  $\lambda > 0$  and b is the coefficient we seek to determine.

### 1. Relationship with Earlier Optimization Problem

In a previous class, we discussed how to minimize the function

$$f(x) = (x - a)^2 + \eta |x|.$$

The minimizer, denoted by  $x^*$ , is:

$$x^* = \arg\min_{x} f(x) = \operatorname{sign}(a)(|a| - \eta/2)_{+} = \begin{cases} a - \eta/2, & \text{if } a > \eta/2\\ 0, & \text{if } |a| \le \eta/2\\ a + \eta/2, & \text{if } a < -\eta/2 \end{cases}$$
(2)

Next we will write (1) in the form of f(x) and then use the solution above.

## 2. Connection to Least Squares Estimate

Let's first acknowledge that the term  $\|\mathbf{v} - b \cdot \mathbf{z}\|^2$  in (1) is the Residual Sum of Squares (RSS) for a simple regression model without intercept, where the response  $\mathbf{v}$  is regressed against the feature  $\mathbf{z}$ , and b is the coefficient for this feature.

The Least Squares (LS) estimate for b is then given by

$$\hat{b} = \frac{\mathbf{v}^t \mathbf{z}}{\|\mathbf{z}\|^2}.$$

### 3. Expanding and Simplifying the Objective

We can rewrite and simplify the RSS term as follows:

$$\|\mathbf{v} - b \cdot \mathbf{z}\|^2 = \|\mathbf{v} - \hat{b} \cdot \mathbf{z}\|^2 + \|(\hat{b} - b) \cdot \mathbf{z}\|^2. \tag{3}$$

The detailed derivation for this simplification is provided below.

$$\|\mathbf{v} - b \cdot \mathbf{z}\|^{2} = \|\mathbf{v} - \hat{b} \cdot \mathbf{z} + \hat{b} \cdot \mathbf{z} - b \cdot \mathbf{z}\|^{2}$$

$$= \|\mathbf{v} - \hat{b} \cdot \mathbf{z} + (\hat{b} - b) \cdot \mathbf{z}\|^{2}$$

$$= \|\mathbf{v} - \hat{b} \cdot \mathbf{z}\|^{2} + \|(\hat{b} - b) \cdot \mathbf{z}\|^{2} +$$

$$2 \times \text{inner-product-between } (\mathbf{v} - \hat{b} \cdot \mathbf{z}) \text{ and } (\hat{b} - b) \cdot \mathbf{z}$$

where the inner-product term disappears due to the orthogonality between the residual and the feature vector  $\mathbf{z}^1$ .

### 4. Reduced Optimization Problem

Since the first term in (3) does not depend on b, it doesn't affect the minimization. Our problem is then equivalent to minimizing:

$$(b-\hat{b})^2 + \frac{2n\lambda}{\|\mathbf{z}\|^2}|b|.$$

This is because

$$\frac{1}{2n} \|(\hat{b} - b) \cdot \mathbf{z}\|^2 + \lambda |b| = \frac{\|\mathbf{z}\|^2}{2n} (\hat{b} - b)^2 + \lambda |b|$$

$$= \frac{\|\mathbf{z}\|^2}{2n} \left( (b - \hat{b})^2 + \frac{2n\lambda}{\|\mathbf{z}\|^2} |b| \right)$$

$$\propto (b - \hat{b})^2 + \frac{2n\lambda}{\|\mathbf{z}\|^2} |b|.$$

Now, we can utilize (2), our earlier solution for f(x), by setting:

$$a = \frac{\mathbf{v}^t \mathbf{z}}{\|\mathbf{z}\|^2}, \quad \eta = \frac{2n\lambda}{\|\mathbf{z}\|^2}.$$

By doing so, we can immediately find the minimizer b for our original one-variable Lasso problem.

<sup>&</sup>lt;sup>1</sup>Recall that the residual vector is orthogonal to each column of the design matrix. Here,  $(\mathbf{v} - \hat{b} \cdot \mathbf{z})$  represents the residual vector from a regression model with  $\mathbf{z}$  being a column (actually the only column) of the design matrix, and therefore  $(\mathbf{v} - \hat{b} \cdot \mathbf{z})$  is orthogonal to  $\mathbf{z}$ .