Discrete Probability Distributions

Chapter 6

Learning Objectives

LO6-1 Identify the characteristics of a probability distribution

LO6-2 Distinguish between discrete and continuous random variables

LO6-3 Compute the mean, variance, and standard deviation of a discrete probability distribution

LO6-4 Explain the assumptions of the binomial distribution and apply it to calculate probabilities

LO6-5 Explain the assumptions of the Poisson distribution and apply it to calculate probabilities

What is a Probability Distribution?

PROBABILITY DISTRIBUTION A listing of all the outcomes of an experiment and the probability associated with each outcome.

CHARACTERISTICS OF A PROBABILITY DISTRIBUTION

1. The probability of a particular outcome is between 0 and 1 inclusive.
2. The outcomes are mutually exclusive.
3. The list of outcomes is exhaustive. So the sum of the probabilities of the outcomes is equal to 1.

- Example: Spalding Golf Products, Inc. assembles golf clubs and as a part of statistical control they inspect arriving shipments of components. From experience, they know the probability of receiving a defective shaft is 5%. Therefore, in a shipment of 20 shafts they can expect one to be defective. Further, we can know the probability of none, two, three, or up to 20 are defective.

Probability Distribution Example

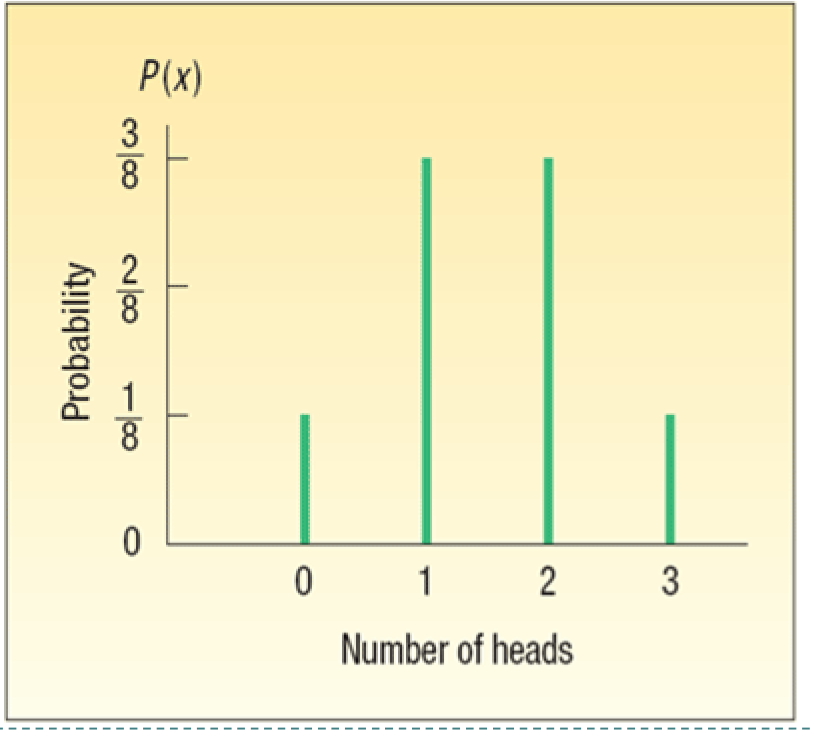
- Suppose we are interested in the number of heads showing face up with 3 tosses of a coin
- The possible outcomes are 0 heads, 1 head, 2 heads, and 3 heads

Possible Result	Coin Toss First	Coin Toss Second	Coin Toss Third	Number of Heads
1	T	T	T	0
2	T	T	H	1
3	T	H	T	1
4	T	H	H	2
5	H	T	T	1
6	H	T	H	2
7	H	H	T	2
8	H	H	H	3

Probability Distribution Table

- Probability distribution table and chart for the events of zero, one, two, and three heads

Number of Heads, x	Probability of Outcome, $P(x)$
0	$\frac{1}{8} = .125$
1	$\frac{3}{8} = .375$
2	$\frac{3}{8} = .375$
3	$\frac{1}{8} = .125$
Total	$\frac{8}{8} = 1.000$



Random Variables

- In any experiment of chance, the outcomes occur randomly, and so are called random variables

RANDOM VARIABLE A variable measured or observed as the result of an experiment. By chance, the variable can have different values.

- Examples
- The number of employees absent from the day shift on Monday, the number might be 0, 1, 2, 3, … The number absent is the random variable
- The grade level (Freshman, Sophomore, Junior, or Senior) of the members of the St. James High School Varsity girls' basketball team. Grade level is the random variable

Two Types of Random Variables

- One type of random variable is the discrete random variable
- Discrete variables are usually the result of counting

> DISCRETE RANDOM VARIABLE A random variable that can assume only certain clearly separated values.

- Examples
- Tossing a coin three times and counting the number of heads
- A bank counting the number of credit cards carried by a group of customers

Discrete Random Variable

- For example, the Bank of the Carolinas counts the number of credit cards carried by a group of customers
- The number of cards carried is the discrete random variable

Number of Credit Cards	Relative Frequency
0	.03
1	.10
2	.18
3	.21
4 or more	.48
Total	1.00

Continuous Random Variables

- Continuous random variables can assume an infinite number of values within a given range
- Continuous variables are usually the result of measuring
- Examples
- The time between flights between Atlanta and LA are 4.67 hours, 5.13 hours, and so on
- The annual snowfall in Minneapolis, MN measured in inches

Mean and Variance of a Probability Distribution

- The mean is a typical value used to represent the central location of the data
- The mean is also referred to as the expected value

MEAN OF A PROBABILITY DISTRIBUTION	$\mu = \Sigma[xP(x)]$	**[6–1]**

- The amount of spread (or variation) in the data is described by the variance

VARIANCE OF A PROBABILITY DISTRIBUTION	$\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$	**[6–2]**

- The standard deviation of the probability distribution is the positive square root of the variance

Probability Distribution Mean Example

John Ragsdale sells new cars for Pelican Ford. John usually sells the most cars on Saturday. He has developed a probability distribution for the number of cars he expects to sell on Saturday.

1. What type of distribution is this?
2. How many cars does John expect to sell on a typical Saturday?
3. What is the variance?

Number of Cars Sold, x	Probability, $P(x)$
0	.1
1	.2
2	.3
3	.3
4	.1
Total	1.0

Number of Cars Sold, x	Probability, $P(x)$	$x \cdot P(x)$
0	.1	0.0
1	.2	0.2
2	.3	0.6
3	.3	0.9
4	.1	0.4
Total	1.0	$\mu =$ 2.1

Probability Distribution Variance Example

- The computational steps for variance
- Subtract the mean from each value of x and square
- Multiply each squared difference by its probability
- Sum the resulting products to arrive at the variance

Number of Cars Sold, x	Probability, $P(x)$	$(x - \mu)$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	.10	0 − 2.1	4.41	0.441
1	.20	1 − 2.1	1.21	0.242
2	.30	2 − 2.1	0.01	0.003
3	.30	3 − 2.1	0.81	0.243
4	.10	4 − 2.1	3.61	0.361
				$\sigma^2 = 1.290$

Binomial Distribution

- There are four requirements of a binomial probability distribution
  1. There are only two possible outcomes and the outcomes are mutually exclusive, identified as either a success or a failure
  2. The number of trials is fixed and known
  3. The probability of a success is the same for each trial
  4. Each trial is independent of any other trial

- Example
- A young family has two children, both boys. The probability of the third birth being a boy is still .50. The gender of the third child is independent of the gender of the other two.

Binomial Probability Experiment

- Use the number of trials, n, and the probability of a success, π, to compute binomial probability

BINOMIAL PROBABILITY EXPERIMENT

1. An outcome on each trial of an experiment is classified into one of two mutually exclusive categories — a success or a failure.
2. The random variable is the number of successes in a fixed number of trials.
3. The probability of success is the same for each trial.
4. The trials are independent, meaning that the outcome of one trial does not affect the outcome of any other trial.

- Note: Do not confuse the symbol π, with the mathematical constant 3.1416

How is a Binomial Probability Computed?

BINOMIAL PROBABILITY FORMULA	$P(x) = {}_nC_x \pi^x (1 - \pi)^{n-x}$	[6–3]

where:
- C denotes a combination.
- n is the number of trials.
- x is the random variable defined as the number of successes.
- π is the probability of a success on each trial.

There are five flights daily from Pittsburgh via US Airways into the Bradford Regional Airport in Bradford, Pennsylvania. Suppose the probability that any flight arrives late is .20.

What is the probability that none of the flights are late today?

$$P(x) = {}_nC_r(\pi)^r(1-\pi)^{n-r}$$
$$P(0) = {}_5C_0(.20)^0(1-.20)^{5-0}$$
$$= (1)(1)(.3277) = .3277$$

What is the probability that exactly one of the flights is late today?

$$P(x) = {}_nC_r(\pi)^r(1-\pi)^{n-r}$$
$$P(1) = {}_5C_1(.20)^1(1-.20)^{5-1}$$
$$= (1)(1)(.4096) = .4096$$

Binomial Probability Distribution

There are five flights daily from Pittsburgh via US Airways into the Bradford Regional Airport in Bradford, Pennsylvania. Suppose the probability that any flight arrives late is .20. What is the probability that none of the flights are late today? What is the probability that exactly 1 of the flights is late today?

Number of Late Flights, x	$P(x)$	$xP(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	0.3277	0.0000	−1	1	0.3277
1	0.4096	0.4096	0	0	0
2	0.2048	0.4096	1	1	0.2048
3	0.0512	0.1536	2	4	0.2048
4	0.0064	0.0256	3	9	0.0576
5	0.0003	0.0015	4	16	0.0048
		$\mu = 1.0000$			$\sigma^2 = 0.7997$

Shortcut Formulas

MEAN OF A BINOMIAL DISTRIBUTION	$\mu = n\pi$	[6–4]

VARIANCE OF A BINOMIAL DISTRIBUTION	$\sigma^2 = n\pi(1 - \pi)$	[6–5]

- Using the preceding example of flights into Bradford Airport; n=5 and $\pi = .20$ and the shortcut formulas

$$\mu = n\,\pi$$
$$\mu = (5)(.20) = 1.00$$

$$\sigma^2 = n\pi(1 - \pi)$$
$$\sigma^2 = (5)(.20)(1 - .20) = .80$$

Binomial Probability Tables

- Tables are already constructed for use as well

In the Southwest, 5% of all cell phone calls are dropped. What is the probability that out of six randomly selected calls, none was dropped? Exactly one? Exactly two? Exactly three? Exactly four? Exactly five? Exactly six out of six? See the table below for the answers.

TABLE 6–2 Binomial Probabilities for $n = 6$ and Selected Values of π

$n = 6$ Probability											
$x \backslash \pi$	.05	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95
0	$\boxed{.735}$	.531	.262	.118	.047	.016	.004	.001	.000	.000	.000
1	.232	.354	.393	.303	.187	.094	.037	.010	.002	.000	.000
2	.031	.098	.246	.324	.311	.234	.138	.060	.015	.001	.000
3	.002	.015	.082	.185	.276	.313	.276	.185	.082	.015	.002
4	.000	.001	.015	.060	.138	.234	.311	.324	.246	.098	.031
5	.000	.000	.002	.010	.037	.094	.187	.303	.393	.354	.232
6	.000	.000	.000	.001	.004	.016	.047	.118	.262	.531	.735

Cumulative Binomial Probability Distributions

A study by the Illinois Department of Transportation concluded that 76.2% of front seat occupants wore seat belts. That is, both occupants of the front seat were using their seat belts. Suppose we decide to compare that information with current usage. We select a sample of 12 vehicles.

1. What is the probability that the front seat occupants in exactly 7 of the 12 vehicles are wearing seat belts?

$$P(x) = {}_nC_r(\pi)^r(1-\pi)^{n-r}$$

$$P(x=7) = {}_{12}C_7(.762)^7(1-.762)^{12-7}$$
$$= 792(.149171)(.000764) = .0902$$

2. What is the probability that at least 7 of the 12 front seat occupants are wearing seat belts?

$$P(x \geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10) + P(x=11) + P(x=12)$$
$$= .0902 + .1805 + .2569 + .2467 + .1436 + .0383$$
$$= .9562$$

Poisson Probability Distribution

- This describes the number of times some event occurs during a specified interval
- The interval can be time, distance, area, or volume
- Two assumptions
  - The probability is proportional to the length of the interval
  - The intervals are independent
- The Poisson has many applications like describing
  - The distribution of errors in data entry
  - The number of accidents on I-75 during a three-month period

Poisson Distribution

POISSON PROBABILITY EXPERIMENT

1. The random variable is the number of times some event occurs during a defined interval.
2. The probability of the event is proportional to the size of the interval.
3. The intervals do not overlap and are independent.

POISSON PROBABILITY DISTRIBUTION $$P(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad (6\text{–}6)$$

where:

μ (mu) is the mean number of occurrences (successes) in a particular interval.
e is the constant 2.71828 (base of the Napierian logarithmic system).
x is the number of occurrences (successes).
$P(x)$ is the probability for a specified value of x.

MEAN OF A POISSON DISTRIBUTION $$\mu = n\pi \qquad (6\text{–}7)$$

Poisson Distribution Example

Budget Airlines is a seasonal airline that operates flights from Myrtle Beach, South Carolina, to various cities in the northeast. Recently Budget has been concerned about the number of lost bags. Ann Poston from the Analytics Department was asked to study the issue. She randomly selected a sample of 500 flights and found that a total of twenty bags were lost on the sampled flights.

The mean number of bags lost, μ, is found by 20/500 = .04

The probability that no bags are lost is found using formula 6-6.

$$P(0) = \frac{\mu^x e^{-\mu}}{x!} = \frac{.04^0 e^{-0.04}}{0!} = .9608$$

Then calculate the probability that one or more bags is lost.

$$P(x \geq 1) = 1 - P(0) = 1 - \frac{\mu^x e^{-\mu}}{x!} = 1 - \frac{.04^0 e^{-0.04}}{0!} = 1 - .9608 = .0392$$

Poisson Probability Distribution Tables

New York-LA Trucking company finds the mean number of breakdowns on the New York to Los Angeles route is 0.30. From the table, we can locate the probability of no breakdowns on a particular run. Find the column 0.3, then read down that column to the row labeled 0; the value is .7408. The probability of 1 breakdown is .2222

	μ								
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.9048	0.8187	$\boxed{0.7408}$	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000