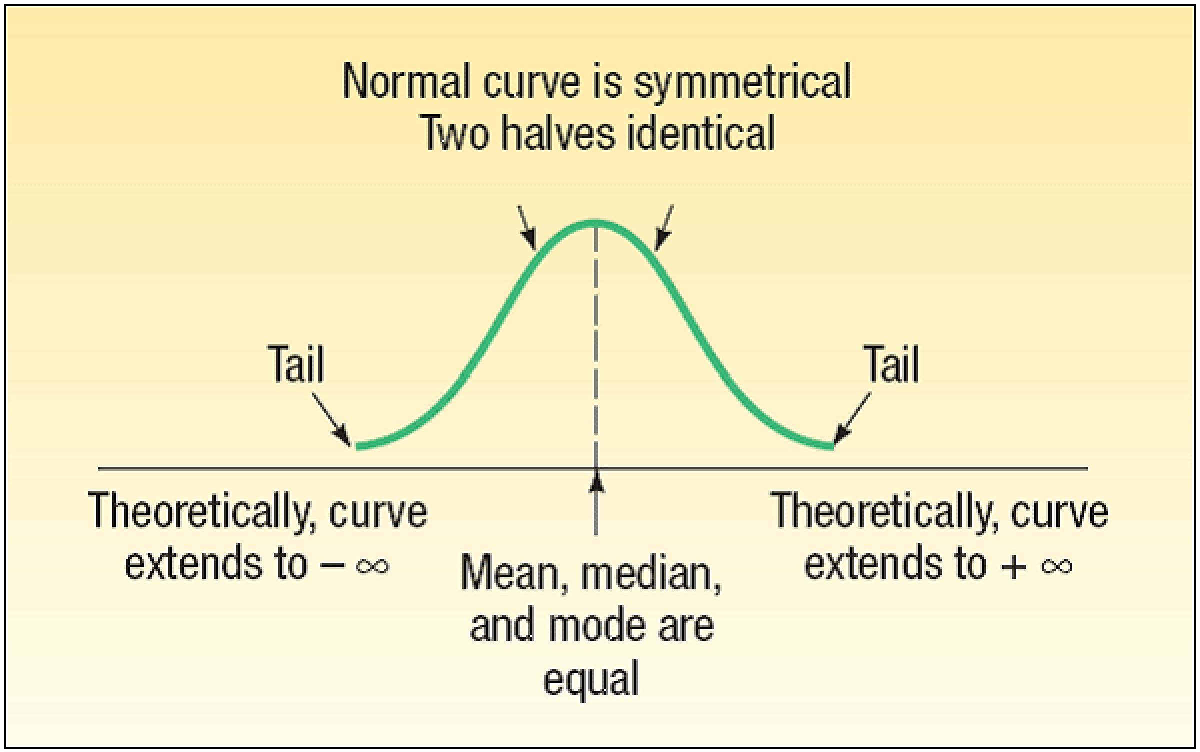
Continuous Probability Distributions

Week 5

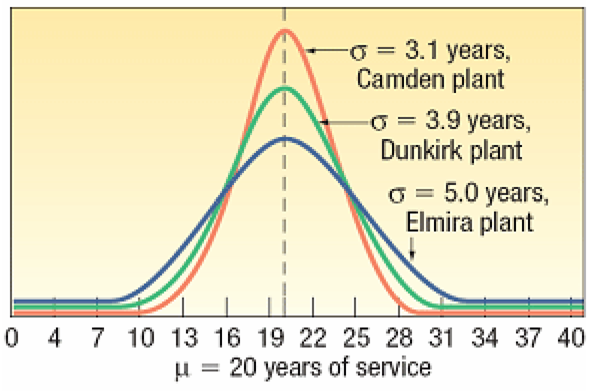
The Normal Probability Distribution

- The normal probability distribution is a continuous distribution with the following characteristics
  - It is bell-shaped and has a single peak at the center of the distribution
  - The distribution is symmetrical about the mean
  - It is asymptotic, meaning the curve approaches but never touches the X-axis
  - It is completely described by its mean and standard deviation
- There is a family of normal probability distributions
  - Another normal probability distribution is created when either the mean or the standard deviation changes

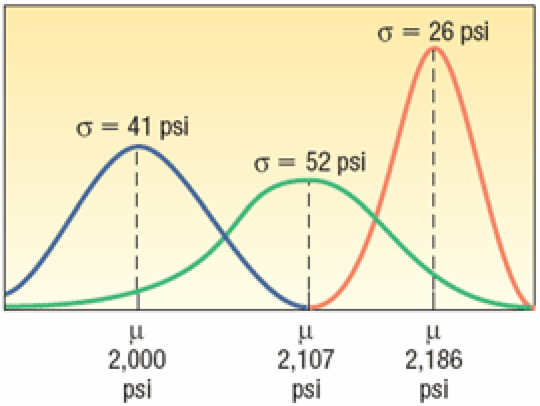
The Normal Curve



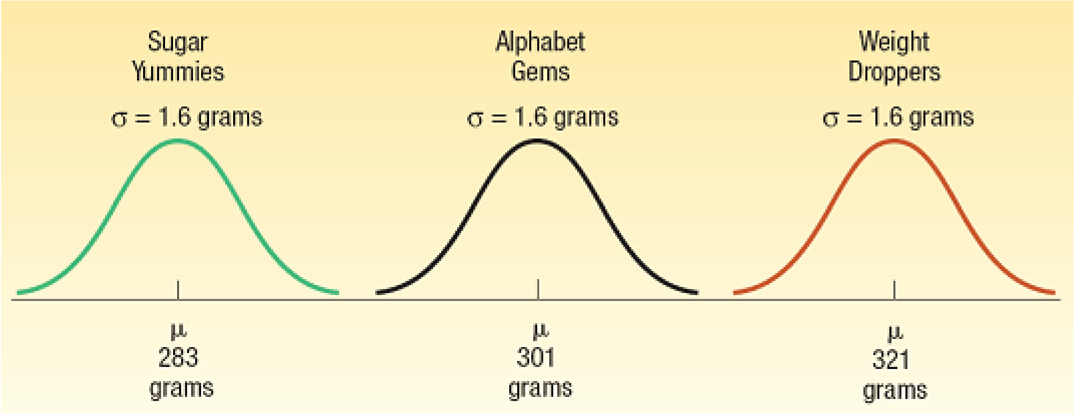
Family of Normal Probability Distributions



Equal Means and Different Standard Deviations



Different Means and Standard Deviations



Different Means and Equal Standard Deviations

Standard Normal Probability Distribution

- The standard normal probability distribution is a particular normal distribution
  - It has a mean of 0 and a standard deviation of 1

z VALUE The signed distance between a selected value, designated x, and the mean, μ, divided by the standard deviation, σ.

- Any normal probability distribution can be converted to the standard normal probability distribution with the following formula

STANDARD NORMAL VALUE $$z = \frac{x - \mu}{\sigma} \quad (7\text{–}5)$$

Areas Under the Normal Curve

- Here is a portion of the "z" Table

z	0.00	0.01	0.02	0.03	0.04	0.05	. . .
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	
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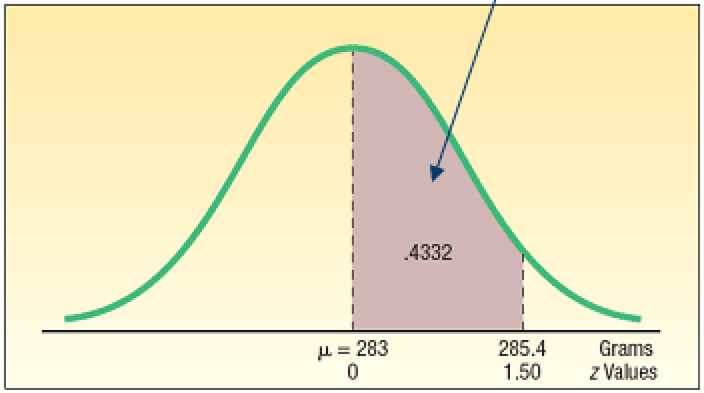


Standard Normal Probability Example

- Calculate the normal probabilities.

(a) $P(Z \leq 1.47)$

(b) $P(Z \geq 1.47)$

(c) $P(-2.25 \leq Z \leq 2.25)$

(d) $P(0.65 \leq Z \leq 1.36)$

Answer : (a) 0.9292 (b) 0.0708 (c) 0.9556 (d) 0.1709

Standard Normal Probability Example

- An investment company believes that the rate of return (X) on an investment portfolio is normally distributed with the mean 30% and standard deviation 10%.

(a) Determine the probability that the rate of return will be more than 55%.

(b) Determine the probability that the rate of return will not exceed 22%.

Answer : (a) 0.0062 (b) 0.2119

Standard Normal Probability Example

In recent years a new type of taxi service has evolved in more than 300 cities world-wide, where the customer is connected directly with a driver via a smartphone. It uses the Uber mobile app, which allows customers with a smartphone to submit a trip request which is then routed to a Uber driver who picks up the customer and takes the customer to the desired location. No cash is involved, the payment for the transaction is handled via a digital payment. Suppose the weekly income of Uber drivers follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100.

What is the z-value of income for a driver who earns $1,100?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$1{,}100 - \$1{,}000}{\$100} = 1.00$$

What is the z-value of income for a driver who earns $900?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$900 - \$1{,}000}{\$100} = -1.00$$

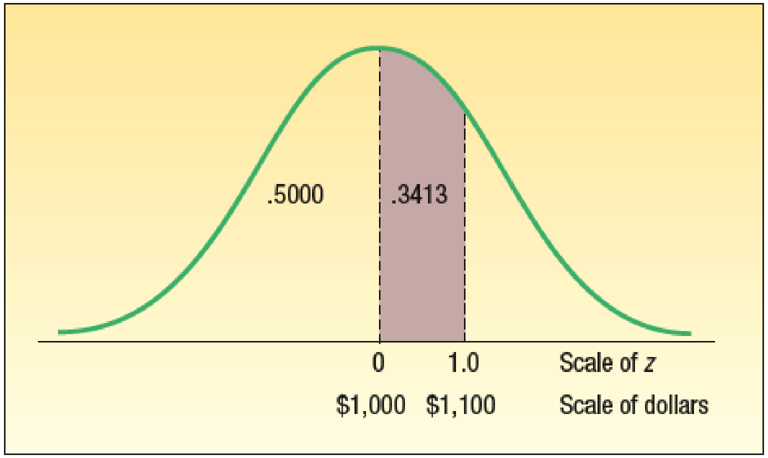
Regardless of whether z is +1or -1, the area under the curve is .3413

Finding Areas under the Normal Curve

What is the z-value of income for a driver who earns $1,100?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$1{,}100-\$1{,}000}{\$100} = 1.00$$

z	0.00	0.01	0.02
⋮	⋮	⋮	⋮
0.7	.2580	.2611	.2642
0.8	.2881	.2910	.2939
0.9	.3159	.3186	.3212
1.0	.3413	.3438	.3461
1.1	.3643	.3665	.3686
⋮	⋮	⋮	⋮



Using the weekly incomes of Uber drivers:

P($1,000 < weekly income < $1,100) = .3413

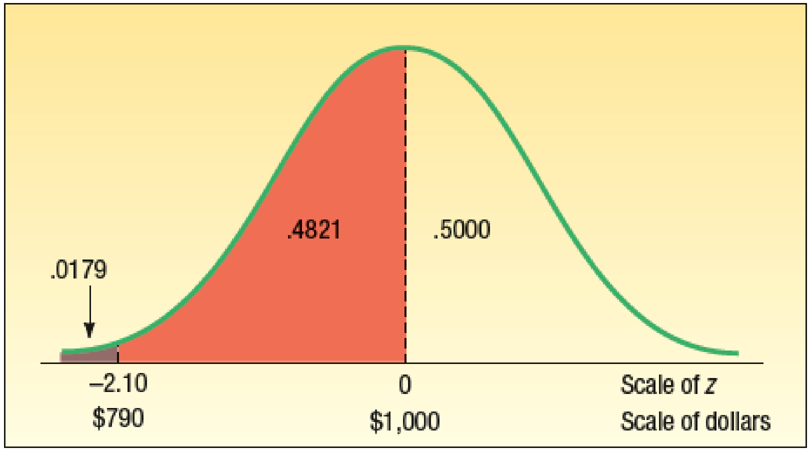
P(weekly income < $1,100) = .3413 + .5000 =.8413

Finding Areas under the Normal Curve

What is the z-value of income for a driver who earns $790?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$790-\$1{,}000}{\$100} = -2.10$$

z	0.00	0.01	0.02
⋮	⋮	⋮	⋮
2.0	.4772	.4778	.4783
2.1	.4821	.4826	.4830
2.2	.4861	.4864	.4868
2.3	.4893	.4896	.4898
⋮	⋮	⋮	⋮



Using the weekly incomes of Uber drivers:

P($790 <weekly income < $1,000) = .4821

P(weekly income < $790) = .5000 - .4821 = .0179

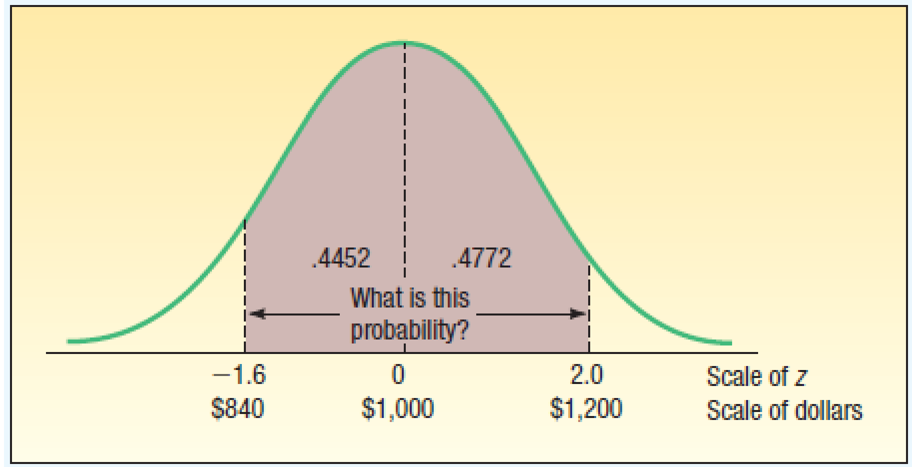
Finding Areas Under the Normal Curve

What is the z-value of income for a driver who earns $840?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$840-\$1{,}000}{\$100} = -1.60$$

What is the z-value of income for a driver who earns $1,200?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$1200-\$1{,}000}{\$100} = 2.00$$



Using the weekly incomes of Uber drivers:

P($840 <weekly income < $1,000) = .4452

P($1,000 <weekly income < $1,200) = .4772

P($840 < weekly income < $1,200) = .4452 + .4772 = .9224

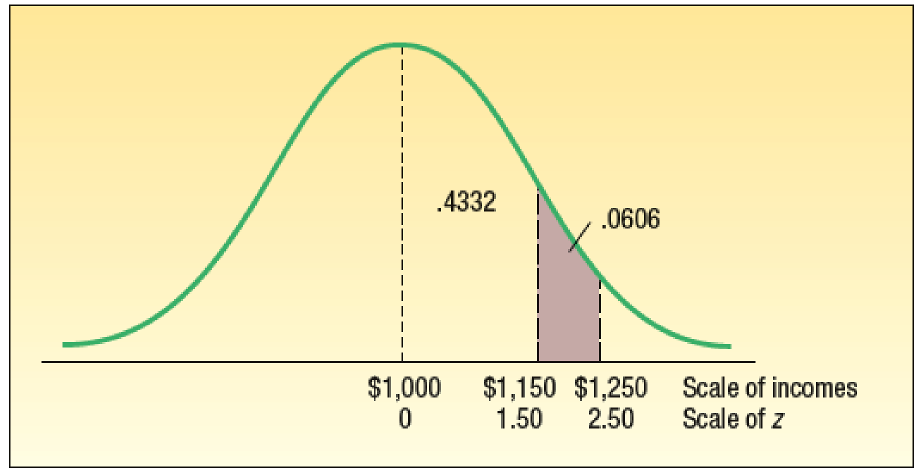
Finding Areas Under the Normal Curve

What is the z-value of income for a driver who earns $1,250?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$1{,}250-\$1{,}000}{\$100} = 2.50$$

What is the z-value of income for a driver who earns $1,150?

$$Z = \frac{x-\mu}{\sigma} = \frac{\$1{,}150-\$1{,}000}{\$100} = 1.50$$



Using the weekly incomes of Uber drivers:

P($1,000 <weekly income < $1,250) = .4938

P($1,000 <weekly income < $1,150) = .4332

P($1,150 < weekly income < $1,250) = .4938 - .4332 = .0606

Normal Distribution

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- If the population follows a normal distribution, the sampling distribution of the sample mean will also follow the normal distribution for samples of any size
- If the population is not normally distributed, the sampling distribution of the sample mean will approach a normal distribution when the sample size is at least 30
- Assume the **population standard deviation is known**
- To determine the probability that a sample mean falls in a particular range, use the following formula

FINDING THE z VALUE OF $\bar{x}$ WHEN THE POPULATION STANDARD DEVIATION IS KNOWN

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \qquad (8\text{–}2)$$

Example

- The mean waiting time for drive-through customers before they get their orders from a fast food restaurant is 20 minutes with standard deviation 5 minutes. If a random sample of 64 customers is observed, what is the probability that their mean waiting time

(a) is between 18 and 19 minutes

(b) is more than 22 minutes ?

(c) at most 19 minutes ?

Solution

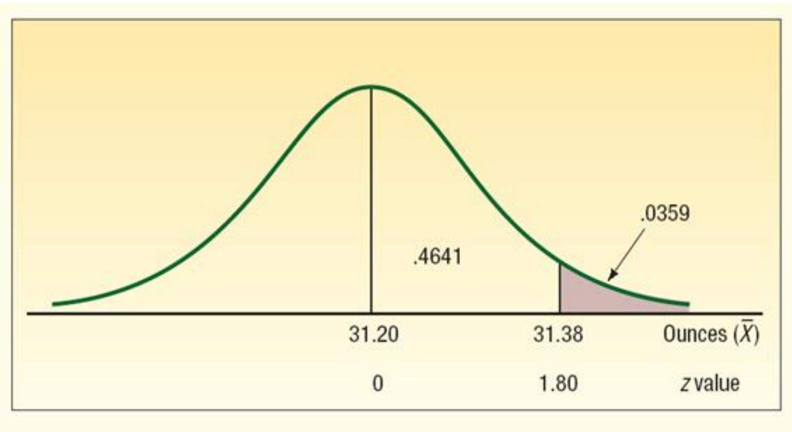
Ans : (a) 0.0541 (b) 0.0007 (c) 0.0548

Using the Sampling Distribution Example

The Quality Assurance Dept. for Cola, Inc. maintains records regarding the amount of cola in its jumbo bottle. The actual amount of cola in each bottle varies a small amount from one bottle to another. Records indicate the amounts of cola follow the normal distribution, the mean amount of cola in the bottles is 31.2 ounces, and the standard deviation is 0.4 ounces. At 8 a.m. today, the quality technician randomly selected 16 bottles from the filling line. The mean amount was 31.38 ounces. Is this an unlikely result? Is it a likely the process is putting too much soda in the bottle? Is the sampling error of 0.18 ounce unusual?

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{31.38 - 31.20}{0.4/\sqrt{16}} = 1.80$$

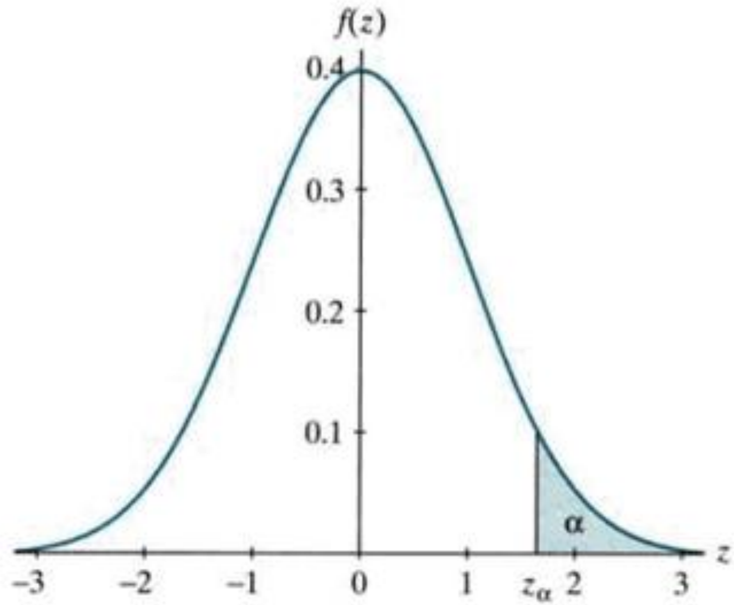
We conclude that it is unlikely; there is less than a 4% chance. The process is putting too much soda in the bottles.



Area of the distribution of Z

Upper-tail probabilities of the distribution of Z. An entry in the table specifies a value of Z_α such that an area α lies to its right. In other words,

$$P(Z > Z_\alpha) = \alpha$$



Example

Determine Z_α such that

a) $P(Z > Z_\alpha) = 0.25$ *[Ans : 0.5746]*
b) $P(Z < Z_\alpha) = 0.36$ *[Ans : 0.3585]*
c) $P(Z > Z_\alpha) = 0.983$ *[Ans : -2.1201]*
d) $P(Z < Z_\alpha) = 0.89$ *[Ans : 1.2265]*

Normal Approximation of the Binomial Distribution

- When the number of observations or trials n in a binomial experiment is relatively large, the normal probability distribution can be used to approximate binomial probabilities.
- A convenient rule is that such approximation is acceptable when $n \geq 30$, and both $np \geq 5$ and $nq \geq 5$.
- Given a random variable $X \sim b(n, p)$, if $n \geq 30$ and both $np \geq 5$ and $nq \geq 5$, then $X \sim N(np, npq)$.

Continuous Correction Factor

a) $P(X = x) \xrightarrow{c.c} P\big(x - 0.5 < X < x + 0.5\big)$

$b)\ P(X \leq x) \xrightarrow{c.c} P\big(X < x + 0.5\big)$

$c)\ P(X < x) \xrightarrow{c.c} P\big(X \leq x - 0.5\big)$

$d)\ P(X \geq x) \xrightarrow{c.c} P\big(X > x - 0.5\big)$

$e)\ P(X > x) \xrightarrow{c.c} P\big(X \geq x + 0.5\big)$

Example

- In a certain country, 45% of registered voters are male. If 300 registered voters from that country are selected at random, find the probability that at least 155 are males.

Solution

$$X \text{ is the number of male voters.}$$
$$X \sim b(300, 0.45)$$
$$P(X \geq 155) \xrightarrow{c.c} P(X > 155 - 0.5) = P(X > 154.5)$$

$$np = 300(0.45) = 135 > 5$$
$$nq = 300(0.55) = 165 > 5$$

$$P\left(Z > \frac{154.5 - 300(0.45)}{\sqrt{300(0.45)(0.55)}}\right) = P\left(Z > \frac{154.5 - 135}{\sqrt{74.25}}\right)$$
$$= P(Z > 2.26)$$
$$= 0.01191$$

Exercise

- While being interviewed about the status of education, a politician states that at least 20% of all new teachers leave the profession within the first three years. If 216 teachers hired three years ago are selected at random, find the probability that 33 or fewer teachers were no longer teaching. *[Ans : 0.0495]*

Normal Approximation of the Poisson Distribution

- When the mean λ of a Poisson distribution is relatively large, the normal probability distribution can be used to approximate Poisson probabilities.
- A convenient rule is that such approximation is acceptable when $\lambda \geq 10$.

Given a random variable $X \sim P_o(\lambda)$, if $\lambda \geq 10$, then $X \sim N(\lambda, \lambda)$

$$with\ Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

Example

- A grocery store has an ATM machine inside. An average of 5 customers per hour comes to use the machine. What is the probability that more than 30 customers come to use the machine between 8.00 am and 5.00 pm?

Solution

X is the number of customers come to use the ATM machine in 9 hours.

$$\lambda = \frac{5}{1} \times 9\ hours = 45$$

$$X \sim P_o(45)$$

$$\lambda = 45 > 10$$

$$X \sim N(45,45)$$

$$P(X > 30) \xrightarrow{c.c} P(X \geq 30 + 0.5) = P(X \geq 30.5\Big)$$

$$P\left(Z \geq \frac{30.5 - 45}{\sqrt{45}}\right) = P(Z \geq -2.16\Big)$$

$$= 0.98461$$

Exercise

- An average of 10 patients are admitted per day to the emergency room of a big hospital. What is the probability that less than 75 patients are admitted to the emergency room in 7 days ? *[Ans : 0.7054]*