Analysis of Variance

Week 8

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Learning Objectives

LO12-1 Apply the F distribution to test a hypothesis that two population variances are equal

LO12-2 Use ANOVA to test a hypothesis that three or more population means are equal

LO12-3 Use confidence intervals to test and interpret differences between pairs of population means

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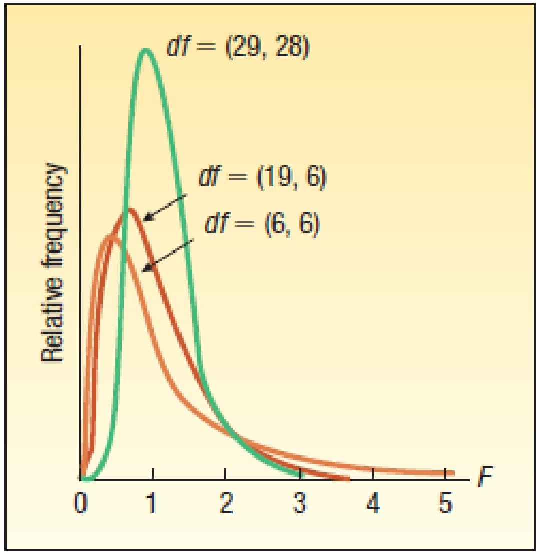
Characteristics of the F Distribution

Analysis of Variance (ANOVA) A technique used to test simultaneously whether the means of several populations are equal. It uses the F distribution as the distribution of the test statistic.

- There is a family of F distributions. Each time the degrees of freedom in either the numerator or the denominator change, a new distribution is created
- The F distribution is continuous
- The F statistic cannot be negative
- The F distribution is positively skewed
- The F distribution is asymptotic



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Comparing Two Population Variances

- The value of F is computed using the following equation

TEST STATISTIC FOR COMPARING TWO VARIANCES

$$F = \frac{s_1^2}{s_2^2} \qquad \textbf{(12–1)}$$

- The larger of the two sample variances is placed in the numerator, forcing the ratio to be at least 1.00
- We calculate the standard deviation, s, and square the standard deviations to get the variance, s^2, for each population
- Example
- A health services corporation manages two hospitals in Knoxville: St. Mary's North and St. Mary's South. The mean waiting time in both Emergency Departments is 42 minutes. The hospital administrator believes St. Mary's North has more variation than St. Mary's South.

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Compare Two Population Variances Example

Lammers Limos offers limousine service from Government Center in downtown Toledo, Ohio, to Metro Airport in Detroit. The president of the company is considering two routes. One is via U.S. 25 and the other via I-75. He wants to study the time it takes to get to the airport using each route and compare the results. He collected the following sample data. Using the .10 significance level, is there a difference in the variation in the driving times for the two routes?

U.S. Route 25	Interstate 75
52	59
67	60
56	61
45	51
70	56
54	63
64	57
	65

U.S. ROUTE 25

$$\bar{x} = \frac{\Sigma x}{n} = \frac{408}{7} = 58.29 \qquad s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}} = \sqrt{\frac{485.43}{7-1}} = 8.9947$$

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$$\bar{x} = \frac{\Sigma x}{n} = \frac{472}{8} = 59.00 \qquad s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}} = \sqrt{\frac{134}{8-1}} = 4.3753$$

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Compare Two Population Variances Example Continued

Step 1: State the null and alternate hypothesis

$$H_0\text{: } \sigma^2{}_1 = \sigma^2{}_2$$
$$H_1\text{: } \sigma^2{}_1 \neq \sigma^2{}_2$$

Step 2: Select the level of significance, we decide to use .10

Step 3: Determine the test statistic, we'll use F

Step 4: State the decision rule, reject H_0 if the ratio of the sample variances > 3.87

Step 5: Compute the ratio of the two sample variances, it's 4.23 so we reject H_0

Step 6: We conclude there is a difference in the variation in the time to travel the two routes.

Degrees of Freedom for Denominator	Degrees of Freedom for Numerator			
	5	6	7	8
1	230	234	237	239
2	19.3	19.3	19.4	19.4
3	9.01	8.94	8.89	8.85
4	6.26	6.16	6.09	6.04
5	5.05	4.95	4.88	4.82
6	4.39	4.28	4.21	4.15
7	3.97	3.87	3.79	3.73
8	3.69	3.58	3.50	3.44
9	3.48	3.37	3.29	3.23
10	3.33	3.22	3.14	3.07

$$F = \frac{s_1^2}{s_2^2} = \frac{(8.9947)^2}{(4.3753)^2} = 4.23$$

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Exercise

- A random sample of eight observations from the first population resulted in a standard deviation of 10. A random sample of six observations from the second population resulted in a standard deviation of 7. At the 0.02 significance level, is there a difference in the variation of the two population ?

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ANOVA: Analysis of Variance

- A one-way ANOVA is used to compare two or more treatment means
- ANOVA was first developed for use in agriculture; the term treatment was used to identify how different plots of land were treated with different fertilizers
- A treatment is a source of variation
- The assumptions underlying ANOVA are
  - The samples are from populations that follow the normal distribution
  - The populations have equal standard deviations
  - The populations are independent

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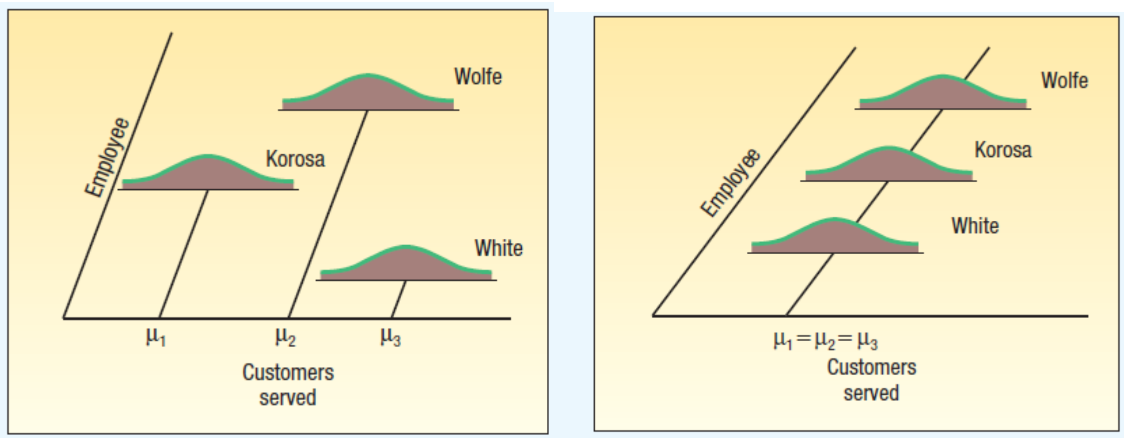
ANOVA Example

Joyce Kuhlman manages a regional financial center. She wishes to compare the productivity, as measured by the number of customers served, among three employees. Four days are randomly selected and the number of customers served by each employee is recorded. Is there a difference in the mean number of customers served?

Wolfe	White	Korosa
55	66	47
54	76	51
59	67	46
56	71	48



Case Where Treatment Means Are Different

Case Where Treatment Means Are the Same

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The ANOVA Test

Wolfe	White	Korosa
55	66	47
54	76	51
59	67	46
56	71	48

First find the overall mean of the 12 observations. It is 58.
Next, find the difference between each particular value and the overall mean. Square these differences and sum up. This result is the total variation, here 1,082.

TOTAL VARIATION The sum of the squared differences between each observation and the overall mean.

Now, break this total variation in two components: variation due to treatment variation and random variation.

TREATMENT VARIATION The sum of the squared differences between each treatment mean and the grand or overall mean.

RANDOM VARIATION The sum of the squared differences between each observation and its treatment mean.

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The ANOVA Test Continued

Recall, the overall mean is 58 and the total variation is 1,082. Now, break this total variation in two components: variation due to treatment variation and random variation.

- The variation due to treatments is 992, found by squaring the difference between each treatment mean and the overall mean and then multiplying each squared difference by the number of observations in each treatment.

$$4(56\text{-}58)^2 + 4(70\text{-}58)^2 + 4(48\text{-}58)^2 = 992$$

- The random variation is 90, found by summing the squared differences between each value and the mean for each treatment.

$$(55\text{-}56)^2 + (54\text{-}56)^2 + \ldots + (48\text{-}48)^2 = 90$$

- Calculate the test statistic, F

$$F = \frac{992/2}{90/9} = 49.6$$

This ratio is quite different from 1, we can conclude there is a difference in the mean number of customers served by the three employees.

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Finding the Value of F

- The formula for the sum of the squares total, SS total is

$$\text{SS total} = \Sigma(x - \bar{x}_G)^2 \qquad \textbf{(12–2)}$$

- The formula for the sum of the squares error, SSE is

$$\text{SSE} = \Sigma(x - \bar{x}_c)^2 \qquad \textbf{(12–3)}$$

- The formula for the sum of the squares treatment, SST is

$$\text{SST} = \text{SS total} - \text{SSE} \qquad \textbf{(12–4)}$$

- This information is summarized in the ANOVA table

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	*F*
Treatments	SST	$k - 1$	SST/(k − 1) = MST	MST/MSE
Error	SSE	$n - k$	SSE/(n − k) = MSE	
Total	SS total	$n - 1$		

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Finding the Value of F Example

A group of four airlines hired Brunner Marketing Research Inc. to survey passengers regarding their level of satisfaction with a recent flight. Twenty-five questions offered a range of possible answers: excellent (4), good (3), fair (2), poor (1), so the highest possible score was 100. Brunner randomly selected and surveyed passengers from the four airlines. Is there a difference in the mean satisfaction level among the four airlines?

Step 1: State the null and the alternate hypothesis

$H_0: \mu_N = \mu_W = \mu_P = \mu_B$

H_1: The mean scores are not all equal

Step 2: Select the level of significance, we'll use .01

Step 3: Determine the test statistic, the test statistic follows the F distribution

Step 4: Formulate the decision rule, reject H_0 is $F > 5.09$

Step 5: Select the sample, calculate F (8.99), and make a decision, we reject H_0

Step 6: Interpret the result, we conclude the populations are not all equal

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	*F*
Treatments	890.69	3	296.90	8.99
Error	594.41	18	33.02	
Total	1,485.10	21		

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Pairs of Means

- If a null hypothesis of equal treatment means is rejected, we can identify the pairs of means that differ with the following confidence interval

CONFIDENCE INTERVAL FOR THE DIFFERENCE IN TREATMENT MEANS

$$(\bar{x}_1 - \bar{x}_2) \pm t\sqrt{\text{MSE}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \qquad \textbf{(12–5)}$$

$\bar{x}_1$ is the mean of the first sample.
$\bar{x}_2$ is the mean of the second sample.
t is obtained from Appendix B.5. The degrees of freedom are equal to $n - k$.
MSE is the mean square error term obtained from the ANOVA table [SSE/($n - k$)].
n_1 is the number of observations in the first sample.
n_2 is the number of observations in the second sample.

- If the confidence interval includes zero, there is not a difference between the treatment means

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Pairs of Means Analysis Example

Recall in the previous example of airline satisfaction, we rejected the null hypothesis that the population means were equal; at least one of the airline's mean level of satisfaction is different from the others. But we do not know which pairs.

Use formula 12-5 to construct a confidence interval with the mean scores of Northern and Branson. Using a 95% level of confidence, we find the endpoints are 10.457 and 26.043.

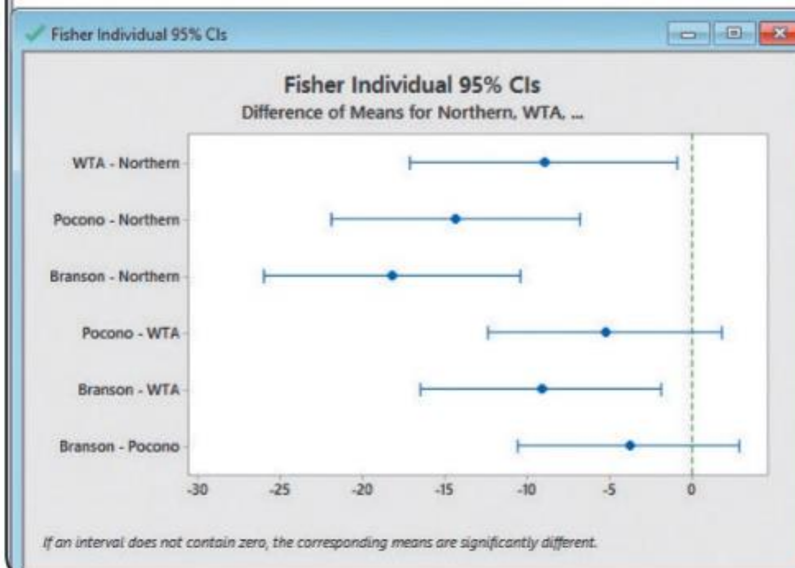
Zero is not in the interval; so passengers on Northern rated service significantly different from those on Branson Airlines.

$$(\bar{x}_N - \bar{x}_B) \pm t\sqrt{\text{MSE}\left(\frac{1}{n_N} + \frac{1}{n_B}\right)} = (87.25 - 69.00) \pm 2.101\sqrt{33.023\left(\frac{1}{4} + \frac{1}{6}\right)}$$
$$= 18.25 \pm 7.793$$

Session

Fisher Pairwise Comparisons

Difference of Levels	Difference of Means	SE of Difference	95% CI	T-Value	Adjusted P-Value
WTA - Northern	-9.05	3.85	(-17.15, -0.95)	-2.35	0.031
Pocono - Northern	-14.39	3.60	(-21.96, -6.83)	-4.00	0.001
Branson - Northern	-18.25	3.71	(-26.04, -10.46)	-4.92	0.000
Pocono - WTA	-5.34	3.36	(-12.41, 1.73)	-1.59	0.130
Branson - WTA	-9.20	3.48	(-16.51, -1.89)	-2.64	0.016
Branson - Pocono	-3.86	3.20	(-10.57, 2.86)	-1.21	0.243



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