Two-Sample Tests of Hypothesis

Week 7

Learning Objectives

LO11-1 Test a hypothesis that two independent population means are equal, assuming that the population standard deviations are known and equal

LO11-2 Test a hypothesis that two independent population means are equal, with unknown population standard deviations

LO11-3 Test a hypothesis about the mean population difference between paired or dependent observations

LO11-4 Explain the difference between dependent and independent samples

Comparing Two Population Means

- In comparing two populations, we wish to know whether their means could be equal
- We are investigating whether the distribution of the difference between the means could have a mean of 0

- Examples
- Is there a difference in the mean value of residential real estate sold by male agents and female agents in south Florida
- Is there an increase in the production rate after music is piped into the production area

Comparing Two Population Means

- We can use the following formula to compute z if the following conditions are met
  - The two populations follow normal distributions
  - The samples are from independent (unrelated) populations
  - The population standard deviations are known

TWO-SAMPLE TEST OF MEANS—KNOWN σ

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad (11\text{–}2)$$

  - In the z formula, $\bar{x}_1 - \bar{x}_2$, is the difference in the sample means and the square root of the variance found with formula 11-1 is the standard deviation

Testing for the Two Population Means: Population Variances Known

- *Assumption*: Two samples of size n_1 and n_2 from two normal populations with unknown means but known variances.
- H_0: $\mu_1 - \mu_2 = d_0$ versus H_1: $\mu_1 - \mu_2 > d_0$, $\mu_1 - \mu_2 < d_0$; or $\mu_1 - \mu_2 \neq d_0$.
- Test Statistic: $$z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$
- Rejection Region: $z > z_{\alpha}$, $z < -z_{\alpha}$, or $|z| > z_{\alpha/2}$.

Comparing Two Population Means Example

Customers at the FoodTown Supermarket have a choice when paying for their groceries. They may check out and pay using the standard cashier-assisted checkout or they may use the new Fast Lane procedure (self-checkout). The store manager would like to know if the mean checkout time using the standard checkout method is longer than using the Fast Lane. The time was measured from when the customer enters the line until all his or her bags are in the cart.

Step 1: State the null and alternate hypothesis

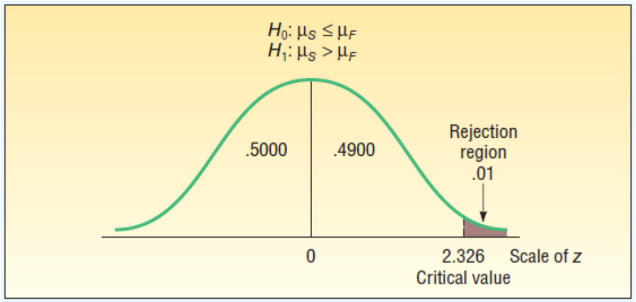
$$H_0: \mu_S = \mu_F$$
$$H_1: \mu_S > \mu_F$$

Step 2: Select the level of significance, we decide to use 0.01

Step 3: Determine the test statistic, we'll use z

Comparing Two Population Means Example Continued

Step 4: Formulate the decision rule, Reject H_0 if $z > 2.326$



Customer Type	Sample Mean	Population Standard Deviation	Sample Size
Standard	5.50 minutes	0.40 minute	50
Fast Lane	5.30 minutes	0.30 minute	100

Step 5: Make the decision regarding H_0, FoodTown randomly selected 50 customers using the standard checkout and computed a mean time of 5.5 minutes and randomly selected 100 customers using the Fast Lane and computed a mean time of 5.3 minutes. We will reject the null hypothesis.

$$z = \frac{\bar{x}_S - \bar{x}_F}{\sqrt{\dfrac{\sigma_S^2}{n_S} + \dfrac{\sigma_F^2}{n_F}}} = \frac{5.5 - 5.3}{\sqrt{\dfrac{0.40^2}{50} + \dfrac{0.30^2}{100}}} = \frac{0.2}{0.064031} = 3.123$$

Step 6: Interpret the result, the difference of .20 minute is too large to have occurred by chance. We conclude the Fast Lane method is faster.

Exercise

- A sample of 40 observations is selected from one population with a population standard deviation of 5. The sample mean is 102. A sample of 50 obserations is selected from a second population with a population standard deviation of 6. The sample mean is 99. Conduct the following test of hypothesis using the 0.04 significance level.

$$H_0: \mu_1 = \mu_2$$
$$H_1: \mu_1 \neq \mu_2$$

a) Is this one tailed or a two-tailed test ?

b) State the decision rule.

c) Compute the value of the test statistics

d) What is your decision regarding H_0?

Answer

a. Two-tailed test

b. Reject H_o if $z < -2.054$ or $z > 2.054$

c. 2.587, found by $z = \dfrac{102 - 99}{\sqrt{\dfrac{5^2}{40} + \dfrac{6^2}{50}}}$

d. Reject H_o

e. $p = 0.0096$, found by $2(0.5000 - 0.4952)$

Testing for the Two Population Means: Small Samples, Population Variances Unknown

- There are three major differences in this test and the test just described in this chapter
  - We assume the sampled populations have equal but unknown standard deviations
  - We combine or "pool" the sample standard deviations
  - We use the t distribution
- The three requirements for the test
  - The sampled populations are approximately normally distributed
  - The sampled populations are independent
  - The standard deviations of the two populations are equal

Compare Two Means Using t

- Finding the value of t requires two steps
- The first step is to pool the standard deviations according to the following formula

POOLED VARIANCE

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} \quad \textbf{(11–3)}$$

- The value of t is computed from the following formula

TWO-SAMPLE TEST OF MEANS—UNKNOWN σ'S

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \textbf{(11–4)}$$

- The degrees of freedom for the test are $n_1 + n_2 - 2$

Compare Two Means Using t

- *Assumption*: Two independent samples of size n_1 and n_2 (small) from a normal population with unknown mean and variance, i.e. $\sigma_1 = \sigma_2 = \sigma$.
- H_0: $\mu_1 - \mu_2 = d_0$ versus H_1: $\mu_1 - \mu_2 > d_0$; $\mu_1 - \mu_2 < d_0$; or $\mu_1 - \mu_2 \neq d_0$.
- Test Statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{(1/n_1) + (1/n_2)}}$; $s_p = \sqrt{\dfrac{(n_1 - 1)(s_1^2) + (n_2 - 1)(s_2^2)}{n_1 + n_2 - 2}}$.
- Rejection Region: $t > t_\alpha$, $t < -t_\alpha$, or $|t| > t_{\alpha/2}$. Degrees of freedom = n_1+n_2-2.

Two-Sample Pooled Test Example

Owens Lawn Care Inc. manufactures and assembles lawnmowers that are shipped to dealers throughout the United States and Canada. Two different procedures have been proposed for mounting the engine on the frame of the lawnmower, the Welles method and the Atkins method. The question is, is there a difference in the mean mounting times?
A time and motion study is conducted to evaluate.

Welles (minutes)	Atkins (minutes)
2	3
4	7
9	5
3	8
2	4
	3

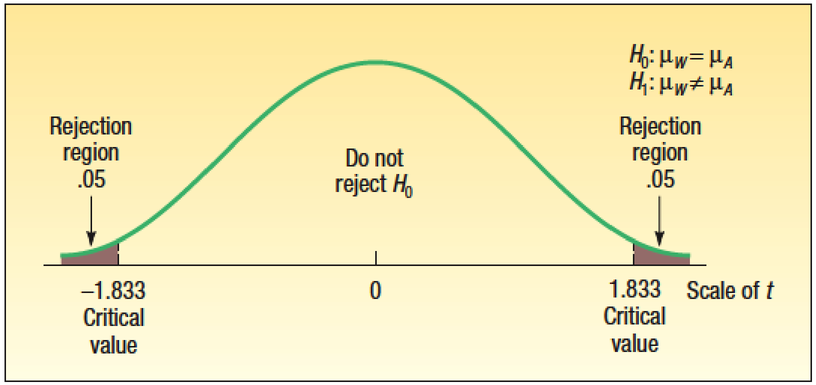
Step 1: State the null and alternate hypothesis

$$H_0: \mu_W = \mu_A$$
$$H_1: \mu_W \neq \mu_A$$

Step 2: Select the level of significance, we decide to use .10
Step 3: Determine the test statistic, we'll use t

Two-Sample Pooled Test Example Continued

Step 4: Formulate the decision rule, do not reject H_0 if t falls between -1.833 and 1.833



Step 5: Make decision regarding H_0.
It takes three steps to compute the value of t.
First:
Calculate the sample standard deviations

Welles Method		Atkins Method	
x_W	$(x_W - \bar{x}_W)^2$	x_A	$(x_A - \bar{x}_A)^2$
2	$(2-4)^2 = 4$	3	$(3-5)^2 = 4$
4	$(4-4)^2 = 0$	7	$(7-5)^2 = 4$
9	$(9-4)^2 = 25$	5	$(5-5)^2 = 0$
3	$(3-4)^2 = 1$	8	$(8-5)^2 = 9$
2	$(2-4)^2 = 4$	4	$(4-5)^2 = 1$
20	34	3	$(3-5)^2 = 4$
		30	22

$$\bar{x}_W = \frac{\Sigma x_W}{n_W} = \frac{20}{5} = 4 \qquad \bar{x}_A = \frac{\Sigma x_A}{n_A} = \frac{30}{6} = 5$$

$$s_W = \sqrt{\frac{\Sigma(x_W - \bar{x}_W)^2}{n_W - 1}} = \sqrt{\frac{34}{5-1}} = 2.9155 \qquad s_A = \sqrt{\frac{\Sigma(x_A - \bar{x}_A)^2}{n_A - 1}} = \sqrt{\frac{22}{6-1}} = 2.0976$$

Two-Sample Pooled Test Example Concluded

Second: Pool the sample variances

$$s_p^2 = \frac{(n_W - 1)s_W^2 + (n_A - 1)s_A^2}{n_W + n_A - 2} = \frac{(5-1)(2.9155)^2 + (6-1)(2.0976)^2}{5+6-2} = 6.2222$$

Third: Determine the value of t

$$t = \frac{\bar{x}_W - \bar{x}_A}{\sqrt{s_p^2\left(\frac{1}{n_W} + \frac{1}{n_A}\right)}} = \frac{4.00 - 5.00}{\sqrt{6.2222\left(\frac{1}{5} + \frac{1}{6}\right)}} = -0.662$$

The decision is not to reject the null hypothesis because –0.662 falls in the region between -1.833 and 1.833.

Step 6: Interpret the result, we conclude the sample data failed to show a difference between the mean assembly times of the two methods.

Exercise

- The null and alternate hypotheses are :

$$H_0: \mu_1 = \mu_2$$
$$H_1: \mu_1 \neq \mu_2$$

A random sample of 10 observations from one population revealed a sample mean of 23 and a sample standard deviation of 4. A random sample of 8 observations from another population revealed a sample mean of 26 and a sample standard deviation of 5. At the 0.05 significance level, is there a difference between the population means ?

a) State the decision rule.
b) Computed the pooled estimate of population variance.
c) Compute the value of the test statistics
d) What is your decision regarding H_0?

Answer

a. Reject H_o if $t > 2.120$ or $t < -2.120$ df = 10 + 8 – 2 = 16

b.
$$s_p^2 = \frac{(10-1)(4)^2 + (8-1)(5)^2}{10+8-2} = 19.9375$$

c.
$$t = \frac{23-26}{\sqrt{19.9375\left(\frac{1}{10}+\frac{1}{8}\right)}} = -1.416$$

d. Do not reject H_o.

Dependent Samples

- We first compute the mean and the standard deviation of the sample differences
- The value of the test statistic is computed with the following formula

PAIRED *t* TEST $$t = \frac{\bar{d}}{s_d / \sqrt{n}} \qquad (11\text{–}5)$$

  - There are n – 1 degrees of freedom
  - $\bar{d}$ is the mean of the differences between the paired observations
  - s_d is the standard deviation of the differences between the paired observations
  - n is the number of paired observations

Dependent Samples Continued

- Note: the standard deviation of the differences will be computed with the formula 3-8, except d is substituted for x

$$s_d = \sqrt{\frac{\Sigma(d - \bar{d})^2}{n - 1}}$$

- Example
- Nickel Savings and Loan employs two firms, Schadek Appraisals and Bowyer Real Estate to appraise the value of the real estate on which it makes loans. To review the consistency of the two appraisal firms, Nickel randomly selects 10 homes and has both of the firms appraise the values of the selected homes. Thus, there will be a pair of values for each home, these appraised values are related to the home selected. This is called a paired sample.

Testing for the Two Population Means: Paired Observations

- H_0: $\mu_D = d_0$ versus H_1: $\mu_D > d_0$, $\mu_D < d_0$, or $\mu_D \neq d_0$.
- Test Statistic: $t = \dfrac{\bar{d} - d_0}{s_d / \sqrt{n}}$, $s_d = \sqrt{\dfrac{\sum d^2 - (\sum d)^2 / n}{n-1}}$
- Rejection Region: $t > t_\alpha$, $t < -t_\alpha$, or $|t| > t_{\alpha/2}$.
  Degrees of freedom = n - 1.

Dependent Samples Example

Recall that Nickel Savings and Loan wishes to compare the two companies it uses to appraise the value of residential homes. Nickel Savings selected a sample of 10 residential properties and scheduled both firms for an appraisal. The results are reported in $000. At the .05 significance level, can we conclude there is a difference between the firm's appraised values?

Home	Schadek	Bowyer
1	235	228
2	210	205
3	231	219
4	242	240
5	205	198
6	230	223
7	231	227
8	210	215
9	225	222
10	249	245

Step 1: State the null and alternate hypothesis

$$H_0: \mu_d = 0$$
$$H_1: \mu_d \neq 0$$

Step 2: Select the level of significance, we decide to use .05

Step 3: Determine the test statistic, we'll use t

Step 4: State the decision rule, reject H_0 if $t < -2.262$ or > 2.262

Dependent Samples Example Continued

Home	Schadek	Bowyer	Difference, d	$(d-\bar{d})$	$(d-\bar{d})^2$
1	235	228	7	2.4	5.76
2	210	205	5	0.4	0.16
3	231	219	12	7.4	54.76
4	242	240	2	−2.6	6.76
5	205	198	7	2.4	5.76
6	230	223	7	2.4	5.76
7	231	227	4	−0.6	0.36
8	210	215	−5	−9.6	92.16
9	225	222	3	−1.6	2.56
10	249	245	4	−0.6	0.36
			46	0	174.40

$$\bar{d} = \frac{\Sigma d}{n} = \frac{46}{10} = 4.60$$

$$s_d = \sqrt{\frac{\Sigma(d-\bar{d})^2}{n-1}} = \sqrt{\frac{174.4}{10-1}} = 4.402$$

Using formula (11–7), the value of the test statistic is 3.305, found by

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{4.6}{4.402/\sqrt{10}} = \frac{4.6}{1.3920} = 3.305$$

Here we find the mean of the sample differences, $\bar{d}$ is 4.6 and the standard deviation of the sample differences, s_d is 4.402. Use these in formula 11-5 to compute the t value, 3.305

Step 5: Make your decision, we'll reject the null hypothesis

Step 6: Interpret, we conclude there is a difference between the firms' mean appraised home values