Summary for Lagrange Mechanics

Generalized Coordinates

Any system can be described by **at least** f real numbers, they are `generalized coordinates` .

1. **Free particle in 3D space**

   Such the particle can be described by 3 coordinates in Cartesian system:

   $$q = (x^1, x^2, x^3) \sim x^1 \boldsymbol{e}_1 + x^2 \boldsymbol{e}_2 + x^3 \boldsymbol{e}_3$$

2. **Particle constraint on a sphere**

   Same as 1, Cartesian coordinates can describe the particle : (x^1, x^2, x^3) , but there is a constraint:

   $$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$$

   So $f = 2$, one can use spherical coordinates:

   $$q = (\theta, \phi)$$

   Where

   $$x^1 = \sin\theta\cos\phi \ ; \ x^2 = \sin\theta\sin\phi \ ; \ x^3 = \cos\theta$$

3. **Particles on a generic constraint**

   Consider there are $3N$ Cartesian coordinates to describe N particles. And there are s (ideal) constraint equations:

   $$f^i(q^1, \cdots, q^{3N}) = 0 \ ; \ i = 1, \cdots, s$$

   With the theorem for implicit function, at any point q_0, if the Jacobian (a $s \times 3N$ matrix):

   $$J_{ki} = \partial_k f^i(q_0)$$

   is $r \leq \min(s, 3N)$ rank, and near the q_0, one can always find $3N - r$ parameters so that $q^j = q^j(z^1, \cdots, z^{3N-r}) \ ; \ j = 1, \cdots, 3N$ is the function defined by these s constraints.

4. **Rigid body**

   Free rigid body has 6 degrees of freedom ($f = 6$). One can always specify a rigid body by its position and its orientation, where position is 3-Cartesian coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$ of some fixed (on the rigid body) point (like mass center), and the orientation can be described by a orthogonal matrix $\boldsymbol{A}$. Then, at each time t , the point(mass point) at $\boldsymbol{z}(0)$ initially will be at:

   $$\boldsymbol{z}(t) - \boldsymbol{x}(t) = \boldsymbol{A}(t)\Big(\boldsymbol{z}(0) - \boldsymbol{x}(0)\Big)$$

   Note that 3D orthogonal matrix(SO(3)) can be uniquely described by three parameters, so the rigid body has $f = 6$.

Euler-Lagrange Equation

Any `ideal` dynamical system obeys the equation(s) of:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^r} - \frac{\partial L}{\partial q^r} = 0\ ;\ r = 1, \cdots, f$$

Where f is the number of degrees of freedom. q^r are coordinates to describe the system. $L = L(q, \dot{q}, t)$ is called `Lagrangian` . These equations induce r second order ODEs about $q^r(t)$.

1. **Single 1D particle in external potential**

   Consider the Lagrangian of:

   $$L = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$$

   Then the equation of motion should be:

   $$m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{\mathrm{d}V}{\mathrm{d}q}(q) = 0$$

   Which is the Newton's law of motion.

2. **Free particles**

   In Cartesian coordinates, kinetic energy of a particle is: $T = \frac{1}{2}m\boldsymbol{v}^2$ where $\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}$. So the Lagrangian of the free particle is:

   $$L = \sum_{i=1}^{N}\frac{1}{2}m_i\boldsymbol{v}_i^2 = \sum_{i=1}^{N}\sum_{r=1}^{3}\frac{1}{2}m_i(\dot{q}_i^r)^2$$

3. **Particle on a sphere**

In Cartesian coordinates we have $L = \frac{1}{2}m\Big((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2\Big)$. Then with the relation $x^1 = \sin\theta\cos\phi\ ;\ x^2 = \sin\theta\sin\phi\ ;\ x^3 = \cos\theta$ one has:

$$\begin{cases} \dot{x}^1 &= \cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi} \\ \dot{x}^2 &= \cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi} \\ \dot{x}^3 &= -\sin\theta\dot{\theta} \end{cases}$$

So:

$$L = \frac{1}{2}m\Big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\Big)$$

4. **Particle on generic constraints**

   The constraint for N particles are:

   $$f^i(q^1, \cdots, q^{3N}) = 0\ ;\ i = 1, \cdots, s$$

   The Lagrangian should still be:

   $$L = \frac{1}{2}\sum_{i=1}^{3N} m_i(\dot{q}^i)^2$$

   in which $m_{3k+1} = m_{3k+2} = m_{3k+3}$, they are masses of the same particle. With the constraints, one has:

   $$\partial_l f^i \mathrm{d}q^l = 0 \Rightarrow \dot{q}^l\partial_l f^i = 0\ ;\ i = 1, \cdots, s$$

   If the rank of Jacobian $J_{ij} = \partial_i f^j(q)$ is r , then we can choose the first $3N - r$ `generalized velocity` $\dot{q}^i\ ;\ i = 1, \cdots, 3N - r$ be fixed and solve this linear equation to express the other r velocities with these $3N - r$ velocities:

   $$\dot{q}^k = \sum_{l=1}^{3N-r} C_{kl}\dot{q}^l\ ;\ k = 3N - r + 1, \cdots, 3N$$

   and get the Lagrangian as the function of velocity of d.o.f. variables.

   $$L = \frac{1}{2}\sum_{i=1}^{3N-r} m_i(\dot{q}^i)^2 + \frac{1}{2}\sum_{i=3N-r+1}^{3N} m_i\Big(\sum_{l=1}^{3N-r} C_{il}\dot{q}^l\Big)^2$$

5. **Rigid Body**

   We have shown that any mass point on the rigid body has the position at time t of the form:

   $$\boldsymbol{x}_i(t) = \boldsymbol{x}(t) + \boldsymbol{A}(t)(\boldsymbol{x}_i(0) - \boldsymbol{x}(0))$$

Now we assume the reference point $\boldsymbol{x}(t)$ is the position of the mass center. The Lagrangian (kinetic energy) is simply:

$$\begin{aligned} L &= \frac{1}{2}\sum_i m_i\Big(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}(\boldsymbol{x}_i(0)-\boldsymbol{x}(0))\Big)^2 \to \frac{1}{2}\int \mathrm{d}m\Big(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}(\boldsymbol{x}_i-\boldsymbol{x})\Big)^2 \\ &= \frac{1}{2}\int_B \mathrm{d}^3\boldsymbol{z}\,\rho(\boldsymbol{z})\Big(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\boldsymbol{z}\Big)^2 \end{aligned}$$

Where we applied the variable substitution, and region B is the rigid body's configuration at time $t=0$, `initially the mass center is at the original point` , and $\rho(\boldsymbol{z})$ is the mass-density at position $\boldsymbol{z}$ at rigid body initially. We did not compute the kinetic energy of the mass elements' rotation, because it should be of the second order of $\mathrm{d}^3\boldsymbol{z}=\mathrm{d}V$, and will not make any difference to the integral.

Let us further simplify the expression, note that $\dot{\boldsymbol{x}}=\boldsymbol{V}_c, \dot{\boldsymbol{A}}=\boldsymbol{R}$ are independent of $\boldsymbol{z}$:

$$\begin{aligned} L &= \frac{1}{2}\int_B \mathrm{d}^3\boldsymbol{z}\,\rho(\boldsymbol{z})\Big(\boldsymbol{V}_c^2+\boldsymbol{V}_c^T\boldsymbol{R}\boldsymbol{z}+\boldsymbol{z}^T\boldsymbol{R}^T\boldsymbol{V}_c+\boldsymbol{z}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{z}\Big) \\ &= \frac{1}{2}M\boldsymbol{V}_c^2+\frac{1}{2}\int_B \mathrm{d}^3\boldsymbol{z}\,\rho(\boldsymbol{z})\boldsymbol{z}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{z} \end{aligned}$$

Where we used the fact that $\int_B \mathrm{d}V\rho=M$ and $\int_B \mathrm{d}V\rho\boldsymbol{z}=0$ is the mass of rigid body and that initially the mass center is at the original point.

Now we consider the time derivative of rotation (orthogonal) matrix $\boldsymbol{A}$, with the property of orthogonality:

$$\boldsymbol{A}(t)\boldsymbol{A}^T(t)=\boldsymbol{A}^T(t)\boldsymbol{A}(t)=\boldsymbol{I}$$

we have:

$$\boldsymbol{0}=\boldsymbol{R}\boldsymbol{A}^T+\boldsymbol{A}\boldsymbol{R}^T \Rightarrow \boldsymbol{R}\boldsymbol{A}^T=-(\boldsymbol{R}\boldsymbol{A}^T)^T$$

That is to say, matrix $\boldsymbol{R}\boldsymbol{A}^T$ is an anti-symmetric matrix. Let:

$$\boldsymbol{R}\boldsymbol{A}^T=\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

One can check that with this form, for arbitrary vector $\boldsymbol{u}=(u_x,u_y,u_z)^T$:

$$\boldsymbol{R}\boldsymbol{A}^T\boldsymbol{u}=\begin{bmatrix} -\omega_z u_y+\omega_y u_z \\ \omega_z u_x-\omega_x u_z \\ -\omega_y u_x+\omega_x u_y \end{bmatrix}=\boldsymbol{\omega}\times\boldsymbol{u}$$

Where $\boldsymbol{\omega}=(\omega_x,\omega_y,\omega_z)^T$.

With the identity of:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\boldsymbol{z}(t)-\boldsymbol{x}(t)\Big) = \boldsymbol{R}(\boldsymbol{z}(0)-\boldsymbol{x}(0)) = \boldsymbol{R}\boldsymbol{A}^T(\boldsymbol{z}(t)-\boldsymbol{x}(t)) = \boldsymbol{\omega}\times(\boldsymbol{z}(t)-\boldsymbol{x}(t))$$

One can see why the vector $\boldsymbol{\omega}$ is called `angular velocity` . Note that the matrix form of $\boldsymbol{\omega}$ is actually the generic form of elements in Lie algebra $so(3)$

So the Lagrangian can be simplified much further:

$$\begin{aligned} L &= \frac{1}{2}M\boldsymbol{V}_c^2 + \frac{1}{2}\int_B \mathrm{d}^3\boldsymbol{z}\ \rho(\boldsymbol{z})(\boldsymbol{\omega}\times\boldsymbol{z})\cdot(\boldsymbol{\omega}\times\boldsymbol{z}) \\ &= \frac{1}{2}M\boldsymbol{V}_c^2 + \sum_{ij}\omega^i\omega^j\frac{1}{2}\int_B \mathrm{d}^3\boldsymbol{z}\ \rho(\boldsymbol{z})\big(\boldsymbol{z}^2\delta_{ij} - z^iz^j\big) \\ &= \frac{1}{2}M\boldsymbol{V}_c^2 + \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} \end{aligned}$$

Where

$$\boldsymbol{J}_{ij} = \int_B \mathrm{d}^3\boldsymbol{z}\ \rho(\boldsymbol{z})\big(\boldsymbol{z}^2\delta_{ij} - z^iz^j\big)$$

is `Inertial tensor` and we used the identity:

$$\begin{aligned} (\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \delta_{ij}\epsilon_{imn}a^mb^n\epsilon_{jpq}a^pb^q \\ &= (\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np})a^mb^na^pb^q \\ &= \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \\ &= a^ia^j(\boldsymbol{b}^2\delta_{ij} - b_ib_j) \end{aligned}$$

To write the angular velocity $\boldsymbol{\omega}$ as time derivative of generalized coordinates one can use the definition of Euler angles.

6. **Rigid body with fixed axis**

Some times the rigid body is constraint on a fixed axis, that is, the angular velocity has a fixed orientation: $\boldsymbol{\omega}(t) = \omega(t)\boldsymbol{n}$. Then the Lagrangian (kinetic energy of rotation) should be:

$$L = \frac{1}{2}\omega^2\boldsymbol{n}^T\boldsymbol{J}\boldsymbol{n} \equiv \frac{1}{2}\omega^2J_{\boldsymbol{n}}$$

Where:

$$\begin{aligned} J_{\boldsymbol{n}} &= \boldsymbol{n}^T\boldsymbol{J}\boldsymbol{n} \\ &= \int_B \mathrm{d}^3\boldsymbol{z}\ \rho(\boldsymbol{z})(\boldsymbol{z}^2\boldsymbol{n}^2 - (\boldsymbol{z}\cdot\boldsymbol{n})^2) \\ &= \int_B \mathrm{d}^3\boldsymbol{z}\ \rho(\boldsymbol{z})(\boldsymbol{z}^2 - (\boldsymbol{z}\cdot\boldsymbol{n})^2) \end{aligned}$$

Note that $|\boldsymbol{n}| = 1$, and $\boldsymbol{z}^2 - (\boldsymbol{z}\cdot\boldsymbol{n})^2$ is definitely the distance between $\boldsymbol{z}$ point and the line along $\boldsymbol{n}$ crossing original point.

7. **Relativistic particle**

   Relativistic particle in `Cartesian coordinate` has the Lagrangian of:

$$L = m\boldsymbol{v}^2\sqrt{1-\boldsymbol{v}^2/c^2} - V(\boldsymbol{x})$$

   So the equation of motion is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}} = -\nabla V(\boldsymbol{x})$$

   i.e., the movement lets the mass get larger.

8. **Charged particle in static electromagnetic field**

   Consider a particle with charge Q moves in electromagnetic field $(\boldsymbol{A}, \phi)$, with the intensity of field: $\boldsymbol{B} = \nabla\times\boldsymbol{A}\ ;\ \boldsymbol{E} = -\nabla\phi$. Note that we use Cartesian coordinate to describe the system. The Lagrangian is:

$$L = \frac{1}{2}m\boldsymbol{v}^2 + Q(\boldsymbol{A}\cdot\boldsymbol{v} - \phi)$$

   Then the equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(m\boldsymbol{v} + Q\boldsymbol{A}\Big) = -Q\boldsymbol{E} + Q\nabla(\boldsymbol{A}\cdot\boldsymbol{v})$$

   Or:

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -Q\boldsymbol{E} + Q\boldsymbol{v}\times\boldsymbol{B}$$

Integrating the Lagrange Equation

When Lagrangian is not dependent of time, one can use Legendre transformation to construct a integral of the equation of motion:

$$H = \sum_{i=1}^{f}\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L = \mathrm{Const}$$

Usually (when L is a second order homogeneous function of $\dot{q}$) , Hamiltonian has a simple form of $H = T + V$.

When lagrangian is not dependent of a coordinate q^r , i.e. , $\partial L/\partial q^r = 0$, the generalized momentum is conversed (another first integral):

$$p^r = \frac{\partial L}{\partial \dot{q}^r} = \text{Const}$$

Using these first integral, one can reduce the Lagrange equation into a set of 1-order ODE, which is much easier to solve.

1. **Particle in static electromagnetic field**

   The lagrangian is:

   $$L = \frac{1}{2} m\boldsymbol{v}^2 + Q(\boldsymbol{A} \cdot \boldsymbol{v} - \phi)$$

   Obviously it is independent of time. So:

   $$H = \boldsymbol{v} \cdot \nabla_{\boldsymbol{v}} L - L = \frac{1}{2} m\boldsymbol{v}^2 + Q\phi = E$$

   And one can obtain that:

   $$\boldsymbol{v}^2 = \sqrt{\frac{2(E - Q\phi(\boldsymbol{x}))}{m}}$$

2. **Relativistic particle**

   The time independent Lagrangian is:

   $$L = m\boldsymbol{v}^2 \sqrt{1 - \boldsymbol{v}^2/c^2} - V(\boldsymbol{x})$$

   So:

   $$H = \frac{mc^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}} + V(\boldsymbol{x}) = E$$

3. **Central force**

   Consider a particle moves in a central force field:

   $$L = \frac{1}{2} m\boldsymbol{v}^2 - V(r)$$

   Where $r = |\boldsymbol{r}|$. First of all, it is independent of time, one has:

   $$H = \frac{1}{2} m\boldsymbol{v}^2 - V(r) = E$$

   And if we write it with spherical coordinate with:

   $$x = r\sin\theta\cos\phi \;;\; y = r\sin\theta\sin\phi \;;\; z = r\cos\phi$$

we have:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r)$$

it is independent of angle ϕ, so:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} = \text{Const}$$

Initially we can always choose a proper coordinate system so that $\dot{\phi} \neq 0$ while $\theta = \pi/2\ ;\ \dot{\theta} = 0$.

We have the equation of motion for θ:

$$\frac{\mathrm{d}}{\mathrm{d}t}mr^2\dot{\theta} - \frac{1}{2}mr^2\sin(2\theta)\dot{\phi}^2 = 0$$

It has a special solution of $\theta \equiv \pi/2$ under our initial condition. i.e., the movement is constraint on a plane. Then the reduced effective Lagrangian should be:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

Summary of Hamilton Theory

Hamiltonian Equation of Motion

Any system obeys the canonical equation of motion with its Hamiltonian $H = H(q, p, t)$.

$$\begin{aligned}\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\partial H}{\partial q_i}\\ \frac{\mathrm{d}q_i}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i}\end{aligned}$$

Or any function of q, p, t: $A = A(q, p, t)$ along the trajectory(the solution of canonical equation) $\hat{A}(t) \equiv A(q(t), p(t), t)$ obeys the equation:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \sum_{i=1}^{f}\frac{\partial A}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial A}{\partial p^i}\frac{\mathrm{d}p^i}{\mathrm{d}t} + \frac{\partial A}{\partial t} = \sum_{i=1}^{f}\frac{\partial A}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial H}{\partial q^i} + \frac{\partial A}{\partial t}$$

With the definition of Poisson bracket:

$$[A, B] = \sum_{i=1}^{f}\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial q^i}$$

the dynamical equation can be written as:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \Big([A, H] + \frac{\partial A}{\partial t}\Big)(q(t), p(t), t)$$

Because of $[H, H] \equiv 0$, so we have an identity:

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = \frac{\partial H}{\partial t}(q(t), p(t), t)$$

That is to say, if H is independent of time: $\partial_t H = 0$, then along the trajectory $\hat{H} = H(q(t), p(t), t)$ should be a constant.

Similarly, any function A, if $[A, H] + \frac{\partial A}{\partial t} = 0$, then it is constant along the trajectory.

1. **Laplace-Runge-Lenz vector**

   Vector:

$$\boldsymbol{K} = \boldsymbol{p} \times (\boldsymbol{r} \times \boldsymbol{p}) - m\frac{\boldsymbol{r}}{r} = \boldsymbol{r}\boldsymbol{p}^2 - \boldsymbol{p}(\boldsymbol{r} \cdot \boldsymbol{p}) - m\boldsymbol{r}/r$$

   is conserved in the Newton-gravity system:

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{1}{r}$$

   The component of vector $\boldsymbol{K}$ is:

$$K^i = x^i p^j p^j - p^i x^j p^j - m x^i/r$$

   So:

$$\begin{aligned}
[K^i, H] &= \sum_{l=1}^{3} \frac{\partial K^i}{\partial x^l}\frac{\partial H}{\partial p^l} - \frac{\partial K^i}{\partial p^l}\frac{\partial H}{\partial x^l} \\
&= \sum_{l=1}^{3} \Big(\boldsymbol{p}^2 \delta_{il} - p^i p^j \delta_{lj} - \frac{m(\delta_{il} r - x^i x^l/r)}{r^2}\Big)\frac{p^l}{m} \\
&- \Big(2x^i p^l - x^j(p^j \delta_{li} + p^i \delta_{lj})\Big)(\frac{x^l}{r^3}) \\
&= \Big(\boldsymbol{p}^2 - p^i \sum_j p^j p^j - \frac{p^i r - x^i \boldsymbol{x} \cdot \boldsymbol{p}/r}{r^2}\Big) \\
&- \Big(\frac{2x^i \boldsymbol{x} \cdot \boldsymbol{p}}{r^3} - \frac{x^j p^j x^i}{r^3} - \frac{p^i}{r}\Big) \\
&= \boldsymbol{p}^2 - p^i \sum_j p^j p^j = 0
\end{aligned}$$

   q.e.d.

With vector K , we have:

$$\boldsymbol{r}\cdot\boldsymbol{K} = rK\cos\theta = r^2p^2 - (\boldsymbol{p}\cdot\boldsymbol{r})^2 - mr = L^2 - mr$$

So:

$$\frac{1}{r} = \frac{K\cos\theta - m}{L^2}$$

The ellipse orbital.

Canonical Transformation

Consider the transform between (Q,P) and (q,p) (we only consider the transformation who has no explicit time dependence):

$$\begin{cases} P &= P(q,p) \\ Q &= Q(q,p) \end{cases} ;\ \begin{cases} p &= p(Q,P) \\ q &= q(Q,P) \end{cases}$$

It is a canonical transformation if and only if it holds the form of canonical equation. i.e.:

$$\tilde{H}(Q,P,t) \equiv H\Big(q(Q,P),p(Q,P),t\Big) \Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial\tilde{H}}{\partial P} ;\ \frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{\partial\tilde{H}}{\partial Q}$$

We can simplify this relationship:

$$\frac{\partial\tilde{H}}{\partial P} = \partial_p H\partial_P p + \partial_q H\partial_P q ;\ \frac{\mathrm{d}Q}{\mathrm{d}t} = \partial_q Q\dot{q} + \partial_p Q\dot{p}$$

So we need:

$$\begin{aligned} \dot{q}\partial_P p - \dot{p}\partial_P q &= \partial_q Q\dot{q} + \partial_p Q\dot{p} \\ \dot{p}\partial_Q q - \dot{q}\partial_Q p &= \partial_q P\dot{q} + \partial_p P\dot{p} \end{aligned}$$

That is:

$$\frac{\partial p}{\partial P} = \frac{\partial Q}{\partial q} ;\ \frac{\partial q}{\partial P} = -\frac{\partial Q}{\partial p} ;\ \frac{\partial q}{\partial Q} = \frac{\partial P}{\partial p} ;\ \frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q}$$

Hamilton Jacobi Equation

Generating Function

Canonical transformation(CT) will not modify the variation rule:

$$\delta\int\Big(\dot{q}p - H\Big)\mathrm{d}t = 0 ;\ \delta\int\Big(\dot{Q}P - \tilde{H}\Big)\mathrm{d}t = 0$$

So a class of CT can be generated by the equality:

$$\dot{q}p - H = \dot{Q}P - \tilde{H} + \frac{\mathrm{d}G}{\mathrm{d}t}$$

Where G can be an arbitrary function, as long as this equality holds.

We will “derive” the HJ equation in the following manner:

Let $G = -QP + S(q,P,t)$, with this form, the equality should be:

$$\dot{q}p - H = \dot{Q}P - \tilde{H} - \dot{Q}P - Q\dot{P} + \partial_1 S\dot{q} + \partial_2 S\dot{P} + \partial_3 S$$

So we need:

$$p = \frac{\partial S}{\partial q}(q,P,t)\ ;\ Q = \frac{\partial S}{\partial P}(q,P,t)\ ;\ \tilde{H} - H = \frac{\partial S}{\partial t}(q,P,t)$$

Find a function $S(q,P,t)$ who obeys the equations above, we can find a “generating function” for a set of CTs by:

$$p = \partial_1 S(q,P,t)\ ;\ Q = \partial_2 S(q,P,t) \Rightarrow q = M(p,Q,t)\ ;\ P = N(p,Q,t)$$

Then one can reduce it into the standard form of CT:

$$p = p(Q,P,t)\ ;\ q = q(Q,P,t)\cdots$$

HJ Equation

If we can find a CT with which the system can be solved easily in new variables. i.e., $\tilde{H} = 0$, the solution is trivial:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

If this CT can be generated by S , we need:

$$H(q,p,t) + \frac{\partial S}{\partial t} = 0 \Rightarrow H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

Solve this equation and obtain a non-trivial solution of S, that is to say, the solution S should be a function of q, t and contain f parameters:

$$S = S(q,t;\alpha)$$

Then one can generate a CT by generating function $S(q,t;P)$:

$$p = \partial_q S(q,t;P)\ ;\ Q = \partial_P S(q,t;P)$$

Since S satisfies the HJ equation (or equivalently, the requirement of CT), such a transformation will automatically be canonical.

1. **1D Harmonic Oscilltor**

   The Hamiltonian is:

   $$H(q,p,t)=\frac{1}{2}(q^2+p^2)$$

   Then the HJE should be:

   $$\frac{\partial S}{\partial t}=\frac{1}{2}q^2+\frac{1}{2}\Big(\frac{\partial S}{\partial q}\Big)^2$$

   We need only find a solution of this equation. Consider $S=f(q)+Et/2$:

   $$\frac{1}{2}E=\frac{1}{2}q^2+\frac{1}{2}(f')^2\Rightarrow\frac{\mathrm{d}f}{\mathrm{d}q}=\sqrt{E-q^2}$$

   Integral it one can find a solution of f:
   $f(q)=\frac{1}{2}\left(q\sqrt{E-q^2}+E\arctan\frac{q}{\sqrt{E-q^2}}\right)+C$. Note that the parameter C will make no difference because $\partial S/\partial C\equiv 0$, then:

   $$S(q,t;E)=\frac{Et}{2}+\frac{1}{2}\left(q\sqrt{E-q^2}+E\arctan\frac{q}{\sqrt{E-q^2}}\right)$$

   Then the generated CT is:

   $$\begin{aligned}p&=\frac{\partial S(q,t;P)}{\partial q}=\sqrt{P-q^2}\\Q&=\frac{\partial S(q,t;P)}{\partial P}=\frac{t}{2}+\frac{1}{2}\arctan\frac{q}{\sqrt{P-q^2}}\end{aligned}$$

   So that is:

   $$\begin{aligned}P&=q^2+p^2\\Q&=\frac{1}{2}t+\frac{1}{2}\arctan\frac{q}{p}\end{aligned}$$

   According to our assumption, they should be constant, then the solution w.r.t. q,p is:

   $$q^2+p^2=C_1\ ;\ \frac{q}{p}=\tan(C_2-t)$$

   That is the solution of Harmonic oscilltor.

   One can check that the P,Q indeed define a CT, and the Hamiltonian is zero.

2. **Newton gravity**

We know that the reduced Lagrangian is:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{r}$$

So the equivalent Hamiltonian is:

$$H(q,p,t) = p_r^2 + \frac{p_\phi^2}{r^2} + \frac{c}{r}$$

Then the HJE:

$$\frac{\partial S}{\partial t} + \frac{c}{r} + (\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \phi})^2 = 0$$

Assume: $S(q,t) = a_1 t + f(r) + g(\phi)$, then:

$$a_1 + \frac{c}{r} + (f')^2 + \frac{1}{r^2}(g')^2 = 0$$

Note that g' is a function of ϕ and f' is a function of r, the equality holds only when:

$$a_1 r^2 + cr + r^2(f')^2 = -(g')^2 = -a_2$$

So:

$$g(\phi) = \sqrt{a_2}t\ ;\ f'(r) = \sqrt{\frac{-a_2 - a_1 r^2 - cr}{r^2}}$$

It is solvable:

$$f(r) = \frac{1}{2\sqrt{a_1}\sqrt{r(a_1 r + c) + a_2}}\left(r\sqrt{-\frac{r(a_1 r + c) + a_2}{r^2}}\right.$$
$$\left(c\tanh^{-1}\left(\frac{2a_1 r + c}{2\sqrt{a_1}\sqrt{r(a_1 r + c) + a_2}}\right) + 2\sqrt{a_1}\left(\sqrt{r(a_1 r + c) + a_2}\right.\right.$$
$$\left.\left.\left.\left. - \sqrt{a_2}\tanh^{-1}\left(\frac{2a_2 + cr}{2\sqrt{a_2}\sqrt{r(a_1 r + c) + a_2}}\right)\right)\right)\right)\right)$$