清华大学电子工程系

概率论与数理统计

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<u>作业 5</u>

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5.1. (1). *Solution.* 每次取球相互独立，所以所求概率为：

$$P_1 = \binom{6}{2}\binom{4}{1}\left(\frac{3}{12}\right)^2\left(\frac{4}{12}\right)^3\left(\frac{5}{12}\right) = \frac{25}{432}$$

(2). *Solution.* 分布表如下：

(X,Y)	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$
P	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{3}{22}$	$\frac{1}{55}$	$\frac{3}{22}$
(X,Y)	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(3,0)$
P	$\frac{3}{11}$	$\frac{9}{110}$	$\frac{3}{44}$	$\frac{3}{55}$	$\frac{1}{220}$

(3). *Solution.*

$$P(X=1) = \sum_y P(X=1, Y=y) = \frac{3}{22} + \frac{3}{11} + \frac{9}{110} = \frac{27}{55}$$

5.2. 证明.

$$\begin{aligned}
F(b,d) + F(a,c) &= P(X \le b, Y \le d) + P(X \le a, Y \le c) \\
&= P(a < X \le b, c < Y \le d) + P(a < X \le b, Y \le c) \\
&\quad + P(X \le a, c < Y \le d) + P(X \le a, Y \le c) \\
&\quad + P(X \le a, Y \le c) \\
&= P(a < X \le b, c < Y \le d) + P(X \le b, Y \le c) \\
&\quad + P(X \le a, Y \le d) \\
&= P(a < X \le b, c < Y \le d) + F(a,d) + F(b,c)
\end{aligned}$$

即为所须证明等式的变形. □

5.3. (1). *Solution.*

$$f(x,y)=\begin{cases}\frac{1}{\pi} & x^2+y^2<1\\ 0 & otherwise\end{cases}$$

(2). *Solution.*

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}y=\begin{cases}\dfrac{2\sqrt{1-x^2}}{\pi} & -1<x<1\\ 0 & otherwise\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}x=\begin{cases}\dfrac{2\sqrt{1-y^2}}{\pi} & -1<y<1\\ 0 & otherwise\end{cases}$$

(3). *Solution.*

$$\begin{aligned}P(R\le r)&=\iint_{x^2+y^2\le r^2}f(x,y)\mathrm{d}x\mathrm{d}y\\&=\int_0^{2\pi}\int_0^r\frac{\rho}{\pi}\mathrm{d}\rho\mathrm{d}\theta\\&=\int_0^{2\pi}\frac{1}{2\pi}r^2\mathrm{d}\theta\\&=r^2\end{aligned}$$

(4). *Solution.* $r\neq 0$ 且 $r\neq 1$ 时：

$$f(r)=\frac{\mathrm{d}P(R\le r)}{\mathrm{d}r}=\begin{cases}2r & 0<r<1\\ 0 & otherwise\end{cases}$$

$$\begin{aligned}E(R)&=\int_{-infty}^{\infty}f(r)\cdot r\mathrm{d}r\\&=\int_0^1 2r^2\mathrm{d}r\\&=\frac{2}{3}\end{aligned}$$

5.4. *Solution.*

$$
\begin{aligned}
f_X(x) = &\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right.\right. \\
&\left.\left.-2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}\mathrm{d}y \\
&= \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}} \\
&\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}-\rho\frac{x-\mu_1}{\sigma_1}\right)^2\right]\mathrm{d}y \\
&= \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\frac{1}{2\pi\sigma_1}\frac{1}{\sqrt{1-\rho^2}} \\
&\int_{-\infty}^{\infty}\exp\left[-\frac{y^2}{2(1-\rho^2)}\right]\mathrm{d}y \\
&= \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\frac{1}{2\pi\sigma_1}\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y \\
&= \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\frac{1}{2\pi\sigma_1}\sqrt{2\pi} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\dfrac{(x-\mu_1)^2}{2\sigma_1^2}}
\end{aligned}
$$

同理可得 Y 的边际 cdf：

$$
f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2}e^{-\dfrac{(y-\mu_2)^2}{2\sigma_2^2}}
$$

5.5. *Solution.*

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{x-\mu_1}{\sigma_1}\right.\right.$$

$$\left.\left.\frac{y-\mu_2}{\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \frac{\frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}}}{\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{\left(y-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)\right)^2}{2(1-\rho^2)\sigma_2^2}\right)$$

同理可得 $Y=y$ 条件下 X 的 cdf：

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{\left(x-\left(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)\right)^2}{2(1-\rho^2)\sigma_1^2}\right)$$

5.6. (1). *Solution.* 其 cdf 为：

$$f(x,y) = \begin{cases} 2 & x>0, y>0, x+y<1 \\ 0 & otherwise \end{cases}$$

(2). *Solution.* $y \le 0$ 或 $y \ge 1$ 时，$f(x,y) \equiv 0$，故此时 $f_Y(y) = 0$.
而 $0 < y < 1$ 时：

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x \\ &= \int_0^{1-y} 2\mathrm{d}x \\ &= 2-2y \end{aligned}$$

故：

$$f_Y(y) = \begin{cases} 2-2y & 0<y<1 \\ 0 & otherwise \end{cases}$$

(3). *Solution.* $f_Y(y) > 0$ 需要 $0 < y < 1$.

$$\begin{aligned} f_X(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f(x,y)}{2-2y} \\ &= \begin{cases} \dfrac{1}{1-y} & 0 < x < 1-y \\ 0 & otherwise \end{cases} \end{aligned}$$

5.7. (1). *Solution.*

$$\begin{aligned} h_X(x) &= P(X \le x) = P(X \le x, -\infty < Y < \infty) \\ &= \lim_{y\to\infty} H(x,y) \\ &= \lim_{y\to\infty} F(x)G(y)\{1+\alpha[1-F(x)][1-G(y)]\} \\ &= F(x)\cdot 1 \cdot (1+0) = F(x) \end{aligned}$$

同理：

$$h_Y(y) = G(y)$$

(2). *Solution.* 由已知：

$$X \sim U(0,1), Y \sim U(0,1)$$

故：

$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}, G(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

$\alpha = -1$ 时，pdf 为：

$$H(x,y) = \begin{cases} 0 & x \le 0 \vee y \le 0 \\ xy(x+y-xy) & 0 < x < 1, 0 < y < 1 \\ x & 0 < x < 1, y \ge 1 \\ y & x \ge 1, 0 < y < 1 \\ 1 & x \ge 1, y \ge 1 \end{cases}$$

$\alpha = 1$ 时，pdf 为：

$$H(x,y) = \begin{cases} 0 & x \le 0 \vee y \le 0 \\ xy[xy - (x+y) + 2] & 0 < x < 1, 0 < y < 1 \\ x & 0 < x < 1, y \ge 1 \\ y & x \ge 1, 0 < y < 1 \\ 1 & x \ge 1, y \ge 1 \end{cases}$$

两者分布显然不同.

5.8. *Solution.* 构造 pdf 为 $H(x,y) = C(F(x), G(y))$，则：

$$H_X(x) = \lim_{y \to +\infty} H(x,y) = C(F(x), 1) = F(x)$$

$$H_Y(y) = \lim_{x \to +\infty} H(x,y) = C(1, G(y)) = G(y)$$

故满足边际分布分别为 $F(x), G(y)$.

5.9. *Solution.* 设甲乙两人到达时间晚于一点的分钟数为 X, Y.
则 $X \sim U(0,60), Y \sim U(0,60)$，且 X, Y 独立.
则联合分布 (X,Y) 的 cdf 为：

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} \dfrac{1}{3600} & 0 < x < 60, 0 < y < 60 \\ 0 & otherwise \end{cases}$$

所求概率为：

$$\begin{aligned} P_1 = P(|X-Y| > 10) &= P(X > Y + 10) + P(Y > X + 10) \\ &= \int_{10}^{60} \int_0^{x-10} \frac{1}{3600} \mathrm{d}y \mathrm{d}x + \int_{10}^{60} \int_0^{y-10} \frac{1}{3600} \mathrm{d}x \mathrm{d}y \\ &= 2 \times \frac{1250}{3600} = \frac{25}{36} \end{aligned}$$

5.10. (1). *Solution.*

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint_{x^2+y^2\le 1} \frac{c}{1+x^2+y^2}\mathrm{d}x\mathrm{d}y \\
&= \int_0^{2\pi}\int_0^1 \frac{c\rho}{1+\rho^2}\mathrm{d}\rho\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[\frac{c}{2}\ln(1+\rho^2)\right]\Bigg|_0^1 \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{c}{2}\ln 2 \\
&= c\pi\ln 2 = 1 \\
&\Rightarrow c = \frac{1}{\pi\ln 2}
\end{aligned}
$$

(2). *Solution.* 当 $0 < x < 1$ 时：

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y \\
&= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{c}{1+x^2+y^2}\mathrm{d}y \\
&= \frac{1}{\pi\ln 2}\left[\frac{\arctan\left(\frac{y}{\sqrt{1+x^2}}\right)}{\sqrt{1+x^2}}\right]\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} \\
&= \frac{2\arctan\sqrt{\dfrac{1-x^2}{1+x^2}}}{\pi\ln 2\cdot\sqrt{1+x^2}}
\end{aligned}
$$

其余 $f_X(x) = 0$，故：

$$
f_X(x) = \begin{cases} \dfrac{2\arctan\sqrt{\dfrac{1-x^2}{1+x^2}}}{\pi\ln 2\cdot\sqrt{1+x^2}} & 0 < x < 1 \\ 0 & otherwise \end{cases}
$$

同理：

$$f_Y(y)=\begin{cases}\dfrac{2\arctan\sqrt{\dfrac{1-y^2}{1+y^2}}}{\pi\ln 2\cdot\sqrt{1+y^2}} & 0<y<1\\ 0 & otherwise\end{cases}$$

显然 $f_X(x)f_Y(y)\neq f(x,y)$，故 X,Y 不独立.

5.11. (1). 证明.

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\mathrm{d}x+\mathrm{d}y\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\mathrm{d}x+\mathrm{d}y+\iint_{x^2+y^2\leq 1}\frac{xy}{100}\mathrm{d}x\mathrm{d}y\\ &=1+\int_0^{2\pi}\int_0^1\frac{1}{100}\rho^3\sin\theta\cos\theta\mathrm{d}\rho\mathrm{d}\theta\\ &=1+\int_0^{2\pi}\frac{1}{400}\cos\theta\sin\theta\mathrm{d}\theta\\ &=1\end{aligned}$$

因为 X,Y 独立，$x^2+y^2\leq 1$ 时：

$$\begin{aligned}f(x,y)&=f_X(x)\cdot f_Y(y)\\ &=\frac{1}{\sqrt{2\pi}}e^{-\dfrac{x^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\dfrac{y^2}{2}}\\ &=\frac{1}{2\pi}e^{-\dfrac{x^2+y^2}{2}}\\ &\geq\frac{1}{2\pi\sqrt{e}}>\frac{1}{200}\geq-\frac{xy}{100}\end{aligned}$$

故 $g(x,y)\geq 0$.

综上所述，$g(x,y)$ 是二维概率密度函数. □

(2). 证明.

$$\begin{aligned}f_U(x)&=\int_{-\infty}^{\infty}g(x,y)\mathrm{d}y\\ &=\int_{-infty}^{\infty}f(x,y)\mathrm{d}y+\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{xy}{100}\mathrm{d}y\\ &=f_X(x)\end{aligned}$$

故 $U \sim N(0,1)$，同理 $V \sim N(0,1)$.
在 $x^2+y^2>1$ 时：

$$g(x,y)=f(x,y)=\frac{1}{2\pi}e^{-\dfrac{x^2+y^2}{2}}$$

可得 $(X,Y)\sim N(0,0,1,1,0)$.
如果 (U,V) 服从二元正态分布，则 $(U,V)\sim N(0,0,1,1,0)$ 而 $x^2+y^2\le 1$ 时，$g(x,y)-f(x,y)=\frac{xy}{100}\ne 0$，故 (U,V) 不服从二元正态分布. □

5.12. *Solution.* 图像见下，对比可发现图形极为相似，因为 y_i 的分布服从 $Y=e^X$ 的分布.

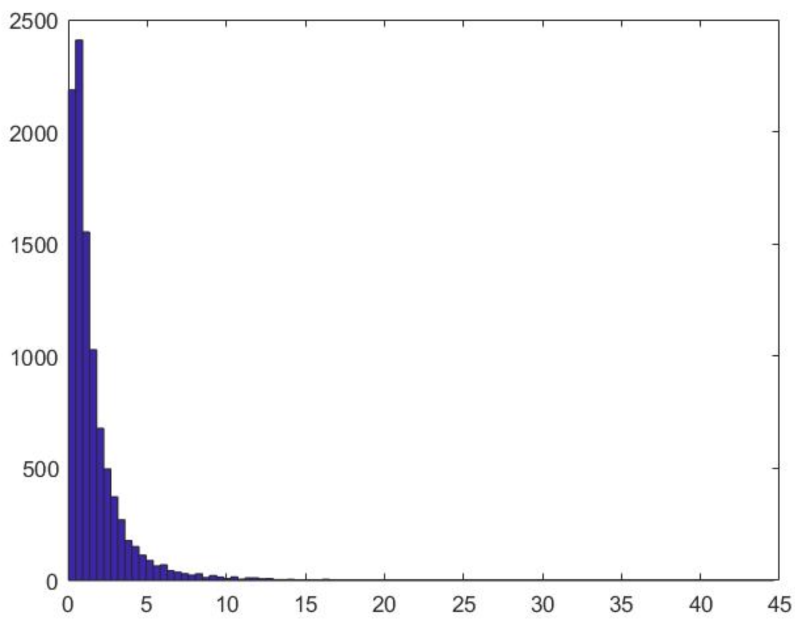

图 1: y_i 直方图

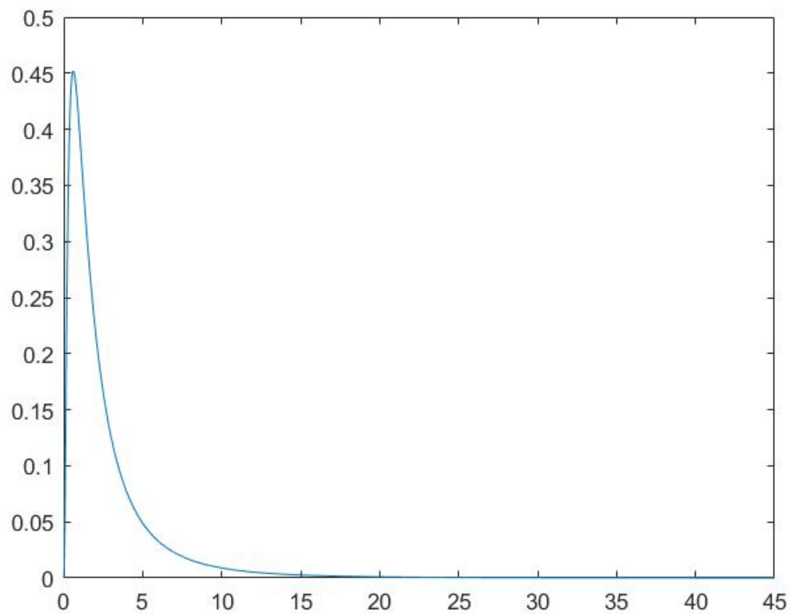

图 2: $Y = e^X$ 概率密度函数