清华大学电子工程系

概率论与数理统计

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作业 6

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6.1. *Solution.*

$$P(X_i = k) = \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\begin{aligned} P(Y = k) &= P(X_1 + X_2 = k) \\ &= \sum_{i=0}^{k} P(X_1 = i) P(X_2 = k - i) \\ &= \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} e^{-\lambda_1} \frac{\lambda_2^{k-i}}{(k-i)!} e^{-\lambda_2} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} \\ &= \frac{(\lambda_1 + \lambda)^k}{k!} e^{-(\lambda_1+\lambda_2)} \\ &\Rightarrow Y \sim \text{Poisson}(\lambda_1 + \lambda_2) \end{aligned}$$

直观上 Poission 分布相当于一定时间或一定空间内出现的小概率事件次数这样的集合.

两个独立的 Poission 分布之和相当于各自时间次数之和，故仍然是 Poisson 分布.

而又由期望为两期望之和，即可得出 $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

6.2. *Solution.*

$(X_1, X_2) \sim N(0, 0, 1, 1, 0), h_1(R, \Theta) = R \cos\Theta, h_2(R, \Theta) = R \sin\Theta$

$$J = \begin{vmatrix} \dfrac{\partial h_1}{\partial r} & \dfrac{\partial h_1}{\partial \theta} \\ \dfrac{\partial h_2}{\partial r} & \dfrac{\partial h_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(X_1, X_2) 的 pdf 为：

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-\dfrac{x^2+y^2}{2}}$$

故 (R,Θ) 的 pdf 为（$R\ge 0, 0\le\Theta<2\pi$）：

$$\begin{aligned} l(r,\theta) &= f(h_1(r,\theta),h_2(r,\theta))|J| \\ &= \frac{1}{2\pi}e^{-\dfrac{r^2}{2}}r \end{aligned}$$

故：

$$l(r,\theta)=\begin{cases} \dfrac{r}{2\pi}e^{-\dfrac{r^2}{2}} & R\ge 0, 0\le\theta<2\pi \\ 0 & otherwise \end{cases}$$

而边际 pdf 为（$r\ge 0, 0\le\theta<2\pi$）：

$$l_R(r)=\int_{-\infty}^{\infty} l(r,\theta)\,\mathrm{d}\theta=\int_0^{2\pi} l(r,\theta)\,\mathrm{d}\theta = re^{-\dfrac{r^2}{2}}$$

$$l_\Theta(\theta)=\int_{-\infty}^{\infty} l(r,\theta)\,\mathrm{d}r=\int_0^{\infty}\frac{r}{2\pi}e^{-\dfrac{r^2}{2}}\,\mathrm{d}r=\frac{1}{2\pi}$$

可得 $l(r,\theta)=l_R(r)l_\Theta(\theta)$，所以 R 与 Θ 独立.

6.3. 证明. 设 (X_1,X_2) 的 pdf 为 $f(x_1,x_2)$.

令 $Z=X_1$，设 (Y,Z) 的 pdf 为 $l(y,z)=f(z,y-z)$.

则 Y 的边际 pdf 为：

$$\begin{aligned} l_Y(y) &= \int_{-\infty}^{\infty} f(z,y-z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z-\mu_1}{\sigma_1}\right)^2\right.\right. \\ &\qquad \left.\left. -2\rho\frac{z-\mu_1}{\sigma_1}\frac{y-z-\mu_2}{\sigma_2}+\left(\frac{y-z-\mu_2}{\sigma_2}\right)^2\right]\right\}\mathrm{d}z \end{aligned}$$

对于二次型取坐标变换可得：

$$\begin{aligned} &\frac{1}{(1-\rho)^2}\left[\left(\frac{z-\mu_1}{\sigma_1}\right)^2-2\rho\frac{z-\mu_1}{\sigma_1}\frac{y-z-\mu_2}{\sigma_2}+\left(\frac{y-z-\mu_2}{\sigma_2}\right)^2\right] \\ &= \frac{(y-\mu_1-\mu_2)^2}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}+\frac{1}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)(1-\rho^2)} \\ &\quad \left[\left(\rho+\frac{\sigma_1}{\sigma_2}\right)(y-z-\mu_2)-\left(\rho+\frac{\sigma_2}{\sigma_1}\right)(z-\mu_1)\right]^2 \\ &= \frac{(y-\mu_1-\mu_2)^2}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}+(az-b)^2 \end{aligned}$$

其中：

$$
\begin{aligned}
a &= \frac{1}{\sqrt{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)(1-\rho^2)}}\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1}+2\rho\right) \\
&= \frac{\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}{\sqrt{1-\rho^2}\sigma_1\sigma_2}
\end{aligned}
$$

由于 a, b 均与 z 无关，可得：

$$
\begin{aligned}
l_Y(y) = &\frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{(y-\mu_1-\mu_2)^2}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}\right] \\
&\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(az-b)^2\right]\mathrm{d}z
\end{aligned}
$$

而：

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(az-b)^2\right]\mathrm{d}z \\
=&\frac{1}{a}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}t^2\right]\mathrm{d}t \\
=&\frac{\sqrt{2\pi}}{a}
\end{aligned}
$$

因此：

$$
\begin{aligned}
l_Y(y) &= \frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\frac{(y-\mu_1-\mu_2)^2}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}\right]\frac{\sqrt{2\pi}}{a} \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}}\exp\left[-\frac{1}{2}\frac{(y-\mu_1-\mu_2)^2}{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}\right]
\end{aligned}
$$

故 $Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)$. □

6.4. (1). *Solution.* $h_1(z,w) = w, h_2(z,w) = \dfrac{z}{w}$.

$$
J = \begin{vmatrix} \dfrac{\partial h_1}{\partial z} & \dfrac{\partial h_1}{\partial w} \\ \dfrac{\partial h_2}{\partial z} & \dfrac{\partial h_2}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \dfrac{1}{w} & -\dfrac{z}{w^2} \end{vmatrix} = -\frac{1}{w}
$$

因为 X, Y 独立，所以 (X, Y) 的 pdf 为 $f(x)g(y)$.

(Z, W) 的 pdf 为：

$$
l(z,w) = f(h_1(z,w))g(h_2(z,w))|J| = f(w)g\left(\frac{z}{w}\right)\left|\frac{1}{w}\right|
$$

由于 $X > 0$，所以 $W > 0$.
则 Z 的边际 pdf 为：

$$\begin{aligned} l_Z(z) &= \int_{-\infty}^{\infty} l(z, w) \, \mathrm{d}w \\ &= \int_0^{\infty} f(w) g\left(\frac{z}{w}\right) \frac{1}{w} \, \mathrm{d}w \end{aligned}$$

(2). *Solution.* $W = X^{-1}$ 的 cdf 为：

$$G(w) = P(W \leq w) = P\left(X \geq \frac{1}{w}\right) = \int_{\frac{1}{w}}^{\infty} f(x) \, \mathrm{d}x$$

故 W 的 pdf 可取 $g(w) = \dfrac{f\left(\frac{1}{w}\right)}{w^2}$. 则
$Z = XY = Y/X^{-1} = Y/W$ 的 pdf 为：

$$\begin{aligned} l(z) &= \int_0^{\infty} w \frac{f\left(\frac{1}{w}\right)}{w^2} g\left(zw\right) \, \mathrm{d}w \\ &= \int_0^{\infty} \frac{f\left(\frac{1}{w}\right)}{w} g\left(zw\right) \, \mathrm{d}w \\ &= \int_0^{\infty} f(w) g\left(\frac{z}{w}\right) \frac{1}{w} \, \mathrm{d}w \end{aligned}$$

6.5. *Solution.* Y 的 cdf 为：

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(X_1 \leq y, X_2 \leq y, \cdots, X_n \leq y) \\ &= P(X_1 \leq y) P(X_2 \leq y) \cdots P(X_n \leq y) \\ &= F^n(y) \end{aligned}$$

Z 的 cdf 为：

$$
\begin{aligned}
H(z) &= P(Z \le z) \\
&= 1 - P(Z > z) \\
&= 1 - P(X_1 > z, X_2 > z, \cdots, X_n > z) \\
&= 1 - P(X_1 > z)P(X_2 > z)\cdots P(X_n > z) \\
&= 1 - (1 - P(X_1 \le z))(1 - P(X_2 \le z))\cdots(1 - P(X_n \le z)) \\
&= 1 - (1 - F(z))^n
\end{aligned}
$$

6.6. *Solution.* 卡方分布：

设 $\xi_1, \xi_2, \cdots, \xi_n \sim N(0,1)$ 且相互独立，随机变量：

$$Q = \sum_{i=1}^{n} \xi_i^2$$

其分布为卡方分布，记为 $Y \sim \chi^2(n)$.

t 分布：

设 $X \sim N(0,1), Y \sim \chi^2(n)$，则：

$$Z = \frac{X}{\sqrt{Y/n}}$$

Z 的分布称为自由度为 n 的 t 分布，记为 $Z \sim t(n)$.

F 分布：设 $X \sim \chi^2(m), Y \sim \chi^2(n)$ 且 X, Y 独立，则：

$$F = \frac{X/m}{Y/n}$$

F 的分布称为自由度分别是 m 和 n 的 F 分布，记为 $F \sim F(m,n)$.

6.7. (1). *Solution.* 正确.

$$
\begin{aligned}
E((X-c)^2) &= E(X^2) - 2cE(X) + c^2 \\
&= Var(X) + E^2(X) - 2cE(X) + c^2 \\
&= Var(X) + (E(X) - c)^2 \\
&\ge Var(X)
\end{aligned}
$$

等号当且仅当 $E(X)=c$ 时成立.

(2). *Solution.* 错误.

设 X 的 pdf 为 $f(x)$，Y 的 pdf 为 $g(y)$.

$$\begin{aligned}
Var(XY) &= E(X^2Y^2) - E^2(XY) \\
&= E(X^2)E(Y^2) - (E(X)E(Y))^2 \\
&= (E(X^2) - E^2(X))(E(Y^2) - E^2(Y)) \\
&\quad + E(X^2)E^2(Y) + E^2(X)E(Y^2) - 2E^2(X)E^2(Y) \\
&= Var(X)Var(Y) + Var(X)E^2(Y) + Var(Y)E^2(X) \\
&\geq Var(X)Var(Y)
\end{aligned}$$

只要取 $E(X)\neq 0, Var(Y)\neq 0$ 时，等号就不成立.

(3). *Solution.* 错误.

假设 X 只能取 $1,2$，概率分别为 $\frac{1}{3},\frac{2}{3}$.

则 $E(X)=\frac{5}{3}$ 不为中位数.

6.8. *Solution.* Y 的 pdf 为：

$$h(y)=\begin{cases}\dfrac{1}{y}\dfrac{1}{\sqrt{2\pi}\sigma}e^{-\dfrac{(\ln y-\mu)^2}{2\sigma^2}} & y>0\\ 0 & otherwise\end{cases}$$

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} yh(y)\,\mathrm{d}y \\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\dfrac{(\ln y-\mu)^2}{2\sigma^2}}\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\dfrac{(t-\mu)^2}{2\sigma^2}+t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\dfrac{(t-\mu-\sigma^2)^2}{2\sigma^2}+\mu+\dfrac{\sigma^2}{2}}\,\mathrm{d}t \\
&= e^{\mu+\dfrac{\sigma^2}{2}}
\end{aligned}$$

$$
\begin{aligned}
Var(Y) &= E(Y^2) - E^2(Y) \\
&= \int_0^{\infty} \frac{y}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} \,\mathrm{d}y - e^{2\mu+\sigma^2} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}+2t} \,\mathrm{d}t - e^{2\mu+\sigma^2} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu-2\sigma^2)^2}{2\sigma^2}+2\mu+2\sigma^2} \,\mathrm{d}t - e^{2\mu+\sigma^2} \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}
\end{aligned}
$$

6.9. 证明.

$$E(|X-c|) = \int_{-\infty}^{c} f(x)(c-x)\,\mathrm{d}x + \int_{c}^{\infty} f(x)(x-c)\,\mathrm{d}x$$

$$\because \int_{-\infty}^{m} f(x)\,\mathrm{d}x = \int_{m}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{2}$$

因此：

$$
\begin{aligned}
&E(|X-c|) - E(|X-m|) \\
&= \int_{-\infty}^{m} f(x)(|x-c|-(m-x))\,\mathrm{d}x + \int_{m}^{\infty} f(x)(|x-c|-(x-m))\,\mathrm{d}x \\
&= \int_{-\infty}^{m} f(x)(|x-c|-(c-x))\,\mathrm{d}x + \int_{m}^{\infty} f(x)(|x-c|-(x-c))\,\mathrm{d}x \\
&= \int_{c}^{m} f(x)(m-c)\,\mathrm{d}x \geq 0
\end{aligned}
$$

故 $E(|X-c|) \geq E(|X-m|)$. □

6.10. *Solution.*

$$Var(\overline{X}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$$

$$
\begin{aligned}
E(S^2) &= \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i-\overline{X})^2\right)\\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+n\overline{X}^2\right)\\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2-n\overline{X}^2\right)\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(E(X_i^2)-E(\overline{X}^2)\right)\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma^2+\mu^2-\frac{\sigma^2}{n}-\mu^2\right)\\
&= \sigma^2
\end{aligned}
$$

6.11. *Solution.* 等价.

首先 (2) 由定义即可得到 (1).

则：

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= E((X-E(X))(Y-E(Y)))\\
&= E(XY)-E(E(X)Y)-E(XE(Y))+E(E(X)E(Y))\\
&= E(XY)-E(X)E(Y)-E(X)E(Y)+E(X)E(Y)\\
&= E(XY)-E(X)E(Y)\\
&= 0\\
&\Leftrightarrow E(XY)=E(X)E(Y)
\end{aligned}
$$

即 (1) 与 (3) 等价.

而：

$$
\begin{aligned}
&Var(X+Y)\\
&= E((X+Y)^2)-E^2(X+Y)\\
&= E(X^2)+2E(XY)+E(Y^2)-(E^2(X)+E^2(Y)+2E(X)E(Y))\\
&= Var(X)+Var(Y)+2(E(XY)-E(X)E(Y))\\
&= Var(X)+Var(Y)\\
&\Leftrightarrow E(XY)=E(X)E(Y)
\end{aligned}
$$

即 (3) 与 (4) 等价.

故这些叙述均等价.

6.12. 证明.

$$
\begin{aligned}
&\mathrm{Corr}(X,Y)\\
&=\frac{\mathrm{Cov}(X,Y)}{\sqrt{Var(X)Var(Y)}}\\
&=\frac{E((X-\mu_1)(Y-\mu_2))}{\sigma_1\sigma_2}\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xy}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right]\mathrm{d}x\mathrm{d}y\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x-\rho y+\rho y)y}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}(x-\rho y)^2-\frac{y^2}{2}\right]\mathrm{d}x\mathrm{d}y\\
&=\int_{-\infty}^{\infty}\left(0+\frac{\rho y^2}{\sqrt{2\pi}}\right)e^{-\frac{y^2}{2}}\,\mathrm{d}y\\
&=\frac{\rho}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-y)\cdot\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-\frac{y^2}{2}}\right)\mathrm{d}y\\
&=\frac{\rho}{\sqrt{2\pi}}\left[-y\cdot e^{-\frac{y^2}{2}}\Bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}-e^{-\frac{y^2}{2}}\,\mathrm{d}y\right]\\
&=\rho
\end{aligned}
$$

□

6.13. *Solution.* 设：

$$S_k=\sum_{i=0}^{k}\frac{(-1)^i}{i!}$$

则：

$$P(X=k)=\frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!}=\frac{1}{k!}S_{n-k}$$

$$E(X)=\sum_{k=1}^{n}P(X=k)k=\sum_{k=0}^{n-1}\frac{1}{k!}S_{n-1-k}=1$$

$$E(X^2)=\sum_{k=1}^{n}P(X=k)k^2=E(X)+\sum_{k=1}^{n}P(X=k)k(k-1)$$

故：

$$Var(X) = E(X^2) - E^2(X) = \sum_{k=1}^{n} P(X=k)k(k-1) = \begin{cases} 0 & n=1 \\ 1 & otherwise \end{cases}$$

6.14. (1). 证明.

$$c^2E(U^2) - 2cE(UV) + E(V^2) = E((cU-V)^2) \geq 0$$

$$\frac{\Delta}{4} = E^2(UV) - E(U^2)E(V^2) \leq 0$$

等号成立当且仅当 $E((tU-V)^2) = 0$ 即 $P(cU = V) = 1$. □

(2). 证明. 设 $E(X) = \mu_1, E(Y) = \mu_2, Var(X) = \sigma_1^2, Var(Y) = \sigma_2^2$，取：

$$U = \frac{X-\mu_1}{\sigma_1}, V = \frac{Y-\mu_2}{\sigma_2}$$

代入 (1) 的结论即得：

$$\text{Corr}(XY)^2 \leq 1 \Leftrightarrow |\text{Corr}(XY)| \leq 1$$

等号成立当且仅当 $P(cU - V) = 1$ 即 $P(Y = aX + b)$，其中：

$$a = c\frac{\sigma_2}{\sigma_1}, b = \mu_2 + c\frac{\sigma_2\mu_1}{\sigma_1}$$

□

6.15. (1). 证明.

$$\begin{aligned}
&\text{Cov}(X_i - \overline{X}, \overline{X}) \\
&= E\left[\left(X_i - \overline{X} - E(X_i - \overline{X})\right)\left(\overline{X} - E(\overline{X})\right)\right] \\
&= E\left[(X_i - \overline{X})(\overline{X} - \mu)\right] \\
&= E(X_i\overline{X}) - E(\overline{X}^2) \\
&= E(X_i\overline{X}) - \frac{\sigma^2}{n} - \mu^2 \\
&= \frac{1}{n}\sum_{k=1}^{n} E(X_iX_k) - \frac{\sigma^2}{n} - \mu^2 \\
&= \frac{1}{n}((n-1)\mu^2 + \sigma^2 + \mu^2) - \frac{\sigma^2}{n} - \mu^2 \\
&= 0
\end{aligned}$$

□

(2). *Solution.* 不一定.

如 $X_1, X_2 \sim B(0.5)$.

则 $P(X_1 = 0) = 0.5$，$P(X_1 - \overline{X} = -0.5) = 0.25$.

但 $P(X_1 = 0, X_1 - \overline{X} = -0.5) = 0.25 \neq 0.25 \times 0.5$.

6.16. (1). *Solution.*

$$E(X_n) = \sum_{i=1}^{n} E(Y_i) = 0$$

$$E(X_n^2) = \sum_{i=1}^{n} E(Y_i^2) + \sum_{1 \leq i < j < n} E(Y_i)E(Y_j) = n$$

$$Var(X_n) = E(X_n^2) - E^2(X_n) = n$$

(2). *Solution.* 曲线随 x 增大逐渐远离 x 轴.

但图像变化明显，有时正有时负.

根据 (1) 期望为 0，所以其正负性确实容易变化，而 $n = 10000$ 时，标准差为 100，所以可以发现图像更多在 $[-100, 100]$ 内波动.

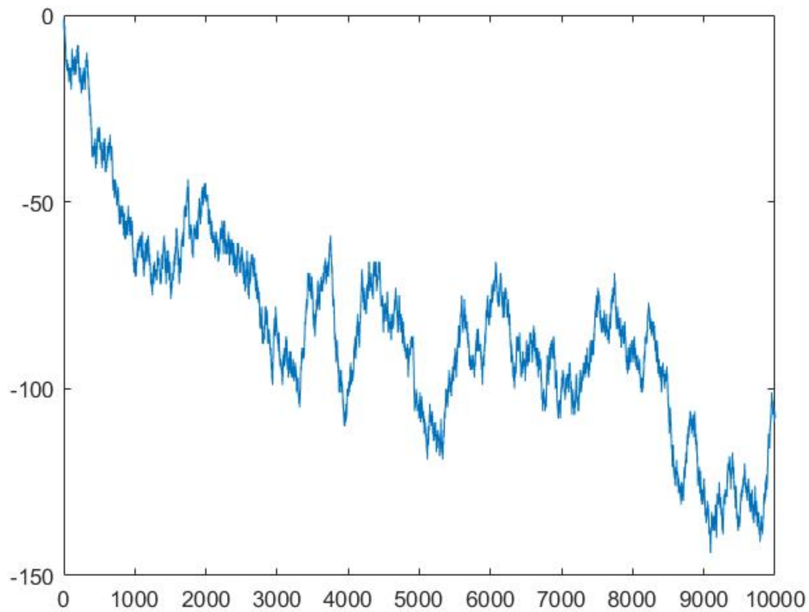
0
-50
-100
-150
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

图 1: 一次模拟的图像