2. 设 $Y=X^{\frac{1}{a}}$ 对 $D_y=\{x\in R: x^{\frac{1}{a}}\le y\}$ 则

$$D_y\cap(0,+\infty)=\begin{cases}\emptyset & y<0\\ (0,y^a] & y>0\end{cases}$$

∴ Y的分布函数 $F_Y(y)=P\{Y\le y\}=P(X\in D_y)=\int_{D_y\cap(0,+\infty)}-\lambda e^{-\lambda x}dx=\begin{cases}0 & y<0\\ \int_0^{y^a}-\lambda e^{-\lambda x}dx & y\geq 0\end{cases}$

$$\therefore F_Y(y)=\begin{cases}0 & y<0\\ -e^{-\lambda y^a} & y\geq 0\end{cases}$$

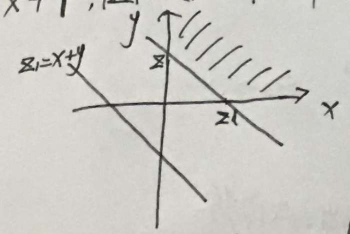
∴ Y的概率密度函数 $P_Y(y)=\frac{dF_Y(y)}{dy}=\begin{cases}0 & y<0\\ \lambda a y^{a-1}e^{-\lambda y^a} & y\geq 0\end{cases}$

(若记 $\lambda_1=(\frac{1}{\lambda})^{\frac{1}{a}}$ 则 $\lambda=(\frac{1}{\lambda_1})^a$

当 $y\geq 0$ 时. $P_Y(y)=(\frac{1}{\lambda_1})^a a y^{a-1}e^{-(\frac{y}{\lambda_1})^a}=\frac{a}{\lambda_1}(\frac{y}{\lambda_1})^{a-1}e^{-(\frac{y}{\lambda_1})^a}$

满足Weibull分布的形式)

4. ① 设 $Z_1=X+Y$，随机变量X.Y概率密度函数非零区域:



由卷积公式 $P_{Z_1}(z_1)=\int_{-\infty}^{+\infty}P_X(x)P_Y(z_1-x)dx$

由图可得 当 $z_1\le 0$ $P_{Z_1}(z_1)=0$

当 $z_1>0$ $P_{Z_1}(z_1)=\int_0^{z_1}P_X(x)P_Y(z_1-x)dx=\lambda_1\lambda_2 e^{-\lambda_2 z_1}\int_0^{z_1}e^{(\lambda_2-\lambda_1)x}dx$

若 $\lambda_1=\lambda_2$ 则 $P_{Z_1}(z_1)=\lambda_1^2 z_1 e^{-\lambda_1 z_1}$

若 $\lambda_1\neq\lambda_2$ 则 $P_{Z_1}(z_1)=\lambda_1\lambda_2 e^{-\lambda_2 z_1}\frac{1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)x}\Big|_0^{z_1}=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1 z_1}-e^{-\lambda_2 z_2})$

∴ 当 $\lambda_1=\lambda_2$ 时: $P_{Z_1}(z_1)=\begin{cases}0 & z_1<0\\ \lambda_1^2 z_1 e^{-\lambda_1 z_1} & z_1\geq 0\end{cases}$

当 $\lambda_1\neq\lambda_2$ $P_{Z_1}(z_1)=\begin{cases}0 & z_1<0\\ \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1 z_1}-e^{-\lambda_2 z_2}) & z_1\geq 0\end{cases}$

② $F_X(x)=\begin{cases}1-e^{-\lambda_1 x} & x>0\\ 0 & x\le 0\end{cases}$ $F_Y(y)=\begin{cases}1-e^{-\lambda_2 y} & y>0\\ 0 & y\le 0\end{cases}$

设 $Z_2=\max(X,Y)$，由于X与Y独立 $F_{Z_2}(z_2)=F_X(z_2)F_Y(z_2)$

当 $z_2\le 0$ $F_{Z_2}(z_2)=0$ $p_{Z_2}(z_2)=0$

当 $z_2>0$ $F_{Z_2}(z_2)=(1-e^{-\lambda_1 z_2})(1-e^{-\lambda_2 z_2})=1-e^{-\lambda_1 z_2}-e^{-\lambda_2 z_2}+e^{-(\lambda_1+\lambda_2)z_2}$

则 $p_{Z_2}(z_2)=\dfrac{dF_{Z_2}(z_2)}{dz_2}=\lambda_1 e^{-\lambda_1 z_2}+\lambda_2 e^{-\lambda_2 z_2}-\dfrac{1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)z_2}$

$\therefore p_{Z_2}(z_2)=\begin{cases}0 & z_2<0\\ \lambda_1 e^{-\lambda_1 z_2}+\lambda_2 e^{-\lambda_2 z_2}-\dfrac{1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)z_2} & z_2\ge 0\end{cases}$

③设 $Z_3=\min(X,Y)$ 由于X与Y独立 $F_{Z_3}(z_3)=1-(1-F_X(z_3))(1-F_Y(z_3))$

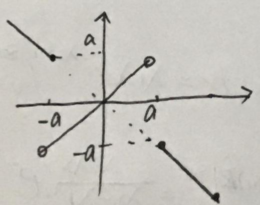
当 $z_3\le 0$ $F_{Z_3}(z_3)=1-1=0$ $p_{Z_3}(z_3)=0$

当 $z_3>0$ $F_{Z_3}(z_3)=1-e^{-\lambda_1 z_3}\cdot e^{-\lambda_2 z_3}=1-e^{-(\lambda_1+\lambda_2)z_3}$

则 $p_{Z_3}(z_3)=\dfrac{1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)z_3}$

$\therefore p_{Z_3}(z_3)=\begin{cases}0 & z_3<0\\ \dfrac{1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)z_3} & z_3\ge 0\end{cases}$

6. 记 $F(Y)=X$. 则 $F(y)=\begin{cases}x & |x|<a\\ -x & |x|\ge a\end{cases}$ 可画出函数图像.



记 $D_y=\{x\in R: F(x)\le y\}$ 当 $y\ge a$ 时 $D_y=[-y,+\infty)$ 当 $-a<y<a$ $D_y=(-a,y)\cup[a,+\infty)$

当 $y\le -a$ $D_y=[-y,+\infty)$

$\therefore D_y=\begin{cases}[-y,+\infty) & |y|\ge a\\ (-a,y)\cup[a,+\infty) & |y|<a\end{cases}$

则 $F_Y(y)=P(Y\le y)=P(X\in D_y)=\int_{D_y}\varphi(x)dx=\begin{cases}\int_{-y}^{+\infty}\varphi(x)dx & |y|\ge a\\ \int_{-a}^{y}\varphi(x)dx+\int_{a}^{+\infty}\varphi(x)dx & |y|<a\end{cases}$

则 $\frac{dF_Y(y)}{dy}=\begin{cases}\frac{d\int_{-y}^{+\infty}\varphi(y)dx}{dy} & |y|\geq a\\ \frac{d(\int_{-a}^{y}\varphi(x)dx+\int_{a}^{+\infty}\varphi(x)dx)}{dy} & |y|<a\end{cases}=\begin{cases}-(-\varphi(-y)) & |y|\geq a\\ \varphi(y) & |y|<a\end{cases}$

由于标准正态分布的概率密度函数 $\varphi(x)$ 为偶函数

则 $|y|\geq a$ $-(-\varphi(-y))=\varphi(-y)=\varphi(y)$

$\therefore p_Y(y)=\frac{dF_Y(y)}{dy}=\varphi(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$

8. ① 记 $Y=(X-\frac{1}{2})^2$ $D_y=\{x\in R: (x-\frac{1}{2})^2\leq y\}$

$D_y\cap(0,1)=\begin{cases}\emptyset & y<0\\ [\frac{1}{2}-\sqrt{y},\frac{1}{2}+\sqrt{y}] & 0\leq y<\frac{1}{4}\\ (0,1) & y\geq\frac{1}{4}\end{cases}$

$\therefore F_Y(y)=P(Y\leq y)=P(X\in D_y)=\int_{D_y\cap(0,1)}p_X(x)dx=\begin{cases}0 & y<0\\ \int_{\frac{1}{2}-\sqrt{y}}^{\frac{1}{2}+\sqrt{y}}1\,dx & 0\leq y<\frac{1}{4}\\ \int_0^1 dx & y\geq\frac{1}{4}\end{cases}=\begin{cases}0 & y<0\\ 2\sqrt{y} & 0\leq y<\frac{1}{4}\\ 1 & y\geq\frac{1}{4}\end{cases}$

$\therefore$ 当 $0<y<\frac{1}{4}$ 时 $p_Y(y)=\frac{dF_Y(y)}{dy}=\frac{1}{\sqrt{y}}$ 当 $y<0$ 或 $y>\frac{1}{4}$ $p_Y(y)=0$

$\therefore p_Y(y)=\begin{cases}0 & y\leq 0 \text{ 或 } y\geq\frac{1}{4}\\ \frac{1}{\sqrt{y}} & 0<y<\frac{1}{4}\end{cases}$

② 记 $Z=\sin(\frac{\pi}{2}X)$ $D_z=\{x\in R: \sin(\frac{\pi}{2}x)\leq z\}$

$\therefore D_z\cap(0,1)=\begin{cases}\emptyset & z\leq 0\\ (0,\frac{2}{\pi}\arcsin z] & 0<z<1\\ (0,1) & z\geq 1\end{cases}$

$\therefore F_Z(z)=P(Z\leq z)=P(X\in D_z)=\int_{D_z\cap(0,1)}p_X(x)dx=\begin{cases}0 & z\leq 0\\ \int_0^{\frac{2}{\pi}\arcsin z}dz & 0<z<1\\ \int_0^1 dz & z\geq 1\end{cases}$

$=\begin{cases}0 & z\leq 0\\ \frac{2}{\pi}\arcsin z & 0<z<1\\ 1 & z\geq 1\end{cases}$

当 $0<z<1$ $p_Z(z)=\frac{dF_Z(z)}{dz}=\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$ 当 $z\leq 0$ 或 $z\geq 1$ $p_Z(z)=0$.

$\therefore p_Z(z)=\begin{cases}0 & z\leq 0 \text{ 或 } z\geq 1\\ \frac{2}{\pi\sqrt{1-x^2}} & 0<z<1\end{cases}$

10. 记 $D_y = \{x \in R : [x] \le y\}$ 则 $D_y = \{x \in R : x < [y+1]\}$

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases} \qquad D_y \cap (0,+\infty) = \begin{cases} \emptyset & y < 0 \\ (0, [y+1]) & y \ge 0 \end{cases}$$

则 $F_Y(y) = P(Y \le y) = P(X \in D_y) = \int_{D_y \cap (0,+\infty)} p_X(x)$

$$= \begin{cases} 0 & y < 0 \\ \int_0^{[y+1]} \lambda e^{-\lambda x} & y \ge 0 \end{cases} = \begin{cases} 0 & y < 0 \\ -e^{-\lambda [y+1]} + 1 & y \ge 0 \end{cases}$$

若 $y<0$ $P_Y(y)=0$ 若 $y \ge 0$ 且不为整数. 则 $P_Y(y) = F_Y(y) - F_Y(y-0) = 0$

若 $y \ge 0$ 且为整数 $P_Y(y) = F_Y(y) - F_Y(y-0) = (-e^{-\lambda(y+1)} + 1) - (-e^{-\lambda y} + 1)$

$$= e^{-\lambda y} - e^{-\lambda(y+1)}$$

$$\therefore P_Y(y) = \begin{cases} 0 & y<0 \text{ 或 } y \text{ 不为整数} \\ e^{-\lambda y} - e^{-\lambda(y+1)} & y \text{ 为非负整数} \end{cases}$$

Y 的分布列为 $P(Y=k) = (1-e^{-\lambda}) e^{-\lambda k}$ $(k = 0, 1, 2, \cdots$