Discrete Mathematics

2019~2020 (第一学期)

Department of Computer Science, East China Normal University

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Chapter 1 SET THEORY

Outline of §-3 Properties of Set Operations

集合运算的基本恒等式

1. 幂等律:
$$A \cup A = A, \qquad A \cap A = A.$$
2. 同一律:
$$A \cup \emptyset = A, \qquad A \cap U = A.$$
3. 零律:
$$A \cap \emptyset = \emptyset, \qquad A \cup U = U.$$
4. 排中律:
$$A \cup \overline{A} = U.$$
5. 矛盾律:
$$A \cap \overline{A} = \emptyset.$$
6. 双重否定:
$$\overline{(\overline{A})} = A.$$
7. 交换律:
$$A \cup B = B \cup A, \qquad A \cap B = B \cap A.$$

集合运算的基本恒等式

8. 结合律:

$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C.$$

9. 分配律:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

10. 吸收律:

$$A \cup (A \cap B) = A, \qquad A \cap (A \cup B) = A.$$

11. 德 · 摩根律:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

12. 差律:

$$A - B = A \cap \overline{B}.$$

集合演算

利用集合运算的基本恒等式作集合公式变换

Example

$$\begin{aligned} A \cup ((A \cap B) \cup (A \cap C)) &= (A \cup (A \cap B)) \cup (A \cap C) && \text{(结合律)} \\ &= A \cup (A \cap C) && \text{(吸收律)} \\ &= A. && \text{(吸收律)} \end{aligned}$$

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集合演算

证明: $A \cup (B - A) = A \cup B$.

Proof.

We prove it by equivalent transformation based on the aforementioned basic identities.

$$\begin{aligned} A \cup (B - A) &= A \cup (B \cap \overline{A}) && \text{(差律)} \\ &= (A \cup B) \cap (A \cup \overline{A}) && \text{(}\cup\text{关于}\cap\text{的分配律)} \\ &= (A \cup B) \cap U && \text{(排中律)} \\ &= A \cup B. && \text{(同一律)} \end{aligned}$$

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\end{aligned}
$$

集合演算

集合 A, B, C 满足: $A \cup C = B \cup C, A \cap C = B \cap C$. 证明: $A = B$.

$$
\begin{aligned}
& A \cup C = B \cup C \\
& \iff A \cup C - C = B \cup C - C \\
& \iff (A \cup C) \cap \overline{C} = (B \cup C) \cap \overline{C} && (\text{差律}) \\
& \iff (A \cap \overline{C}) \cup (C \cap \overline{C}) = (B \cap \overline{C}) \cup (C \cap \overline{C}) && (\text{分配律}) \\
& \iff (A \cap \overline{C}) \cup \emptyset = (B \cap \overline{C}) \cup \emptyset && (\text{矛盾律}) \\
& \iff A \cap \overline{C} = B \cap \overline{C} && (\text{同一律}) \\
& \iff (A \cap \overline{C}) \cup (A \cap C) = (B \cap \overline{C}) \cup (B \cap C) \\
& \iff A \cap (\overline{C} \cup C) = B \cap (\overline{C} \cup C) && (\text{分配律}) \\
& \iff A \cap U = B \cap U && (\text{排中律}) \\
& \iff A = B. && (\text{同一律})
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& A \cup C = B \cup C \\
& \qquad \Longleftrightarrow A \cup C - C = B \cup C - C \\
& \qquad \Longleftrightarrow (A \cup C) \cap \overline{C} = (B \cup C) \cap \overline{C} && (\text{差律}) \\
& \qquad \Longleftrightarrow (A \cap \overline{C}) \cup (C \cap \overline{C}) = (B \cap \overline{C}) \cup (C \cap \overline{C}) && (\text{分配律}) \\
& \qquad \Longleftrightarrow (A \cap \overline{C}) \cup \emptyset = (B \cap \overline{C}) \cup \emptyset && (\text{矛盾律}) \\
& \qquad \Longleftrightarrow A \cap \overline{C} = B \cap \overline{C} && (\text{同一律}) \\
& \qquad \Longleftrightarrow (A \cap \overline{C}) \cup (A \cap C) = (B \cap \overline{C}) \cup (B \cap C) \\
& \qquad \Longleftrightarrow A \cap (\overline{C} \cup C) = B \cap (\overline{C} \cup C) && (\text{分配律}) \\
& \qquad \Longleftrightarrow A \cap U = B \cap U && (\text{排中律}) \\
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对偶原理

许多基本恒等式成对出现

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$$A \cup ((A \cap B) \cup (A \cap C)) = A, \qquad A \cap ((A \cup B) \cap (A \cup C)) = A.$$

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对偶原理

Definition (等式型对偶命题)

若 P 是关于集合的等式型命题, 其中至多包含并, 交和补三种集合运算 (不含差运算), P^* 是将 P 中的 $\cup, \cap, \emptyset, U$ 分别替换为 $\cap, \cup, U, \emptyset$ 而得到的命题, 则称 P 与 P^* 互为对偶命题.

若 $P = P^*$, 则称 P 为自对偶命题.

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对偶原理

Theorem (对偶原理)

$P \Longleftrightarrow P^*$.

对偶原理

Example

证明: $(A - B) - C = A - (B \cup C)$ 恒成立.

$$
\begin{aligned}
(A - B) - C &= (A \cap \overline{B}) \cap \overline{C} && \text{(差律)} \\
&= A \cap (\overline{B} \cap \overline{C}) && \text{(结合律)} \\
&= A \cap (\overline{B \cup C}) && \text{(德·摩根律)} \\
&= A - (B \cup C). && \text{(差律)}
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Example

证明: $(A - B) - C = A - (B \cap C)$ 不一定成立.

注意: 命题里出现了差运算, 对偶原理不适用!

We disprove it by a counter-example that $A = B = \{1\}, C = \emptyset$.

对偶原理

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(A-B)-C &= (A\cap\overline{B})\cap\overline{C} && \text{(差律)}\\
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Homework

1. PP. 16–17: Exercises 9(3),11(5),13(5),21(1),*22(1),*24(2).

有限集合的计数

Definition (基数)

基数 (cardinality) 简单来说就是集合中的元素个数, 记为 $|A|$.
它可以用来度量集合的大小.

Example

$|\{1, 2, \{4\}\}| = 3$, $|\{1, 2, \{4, 5\}\}| = 3$,
$|\mathbb{N}| = ?$, $|\mathbb{R}| = ?$.

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有限集合的计数

Theorem

对于有限集 A 和 B, 我们有以下结论:

1. $|A \cup B| = |A| + |B| - |A \cap B|$ (容斥原理, *the principle of inclusion–exclusion*),
2. $|A \times B| = |A| \times |B|$,
3. $|\mathbf{P}(A)| = 2^{|A|}$.

Recall that
对于集合 A 和 B, 若 $a \in A, a \notin B$ 且 $A = B \cup \{a\}$,
则 $\mathbf{P}(A) = \mathbf{P}(B) \cup \{X \cup \{a\} \mid X \in \mathbf{P}(B)\}$.

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有限集合的计数

Example

在 $1, 2, \ldots, 100$ 的正整数中, 含因子 3 或 5 的正整数共有多少个?

Solution:

设 A 为含因子 3 的正整数的集合, B 为含因子5的正整数的集合.
那么既含因子 3 又含因子 5 的正整数的集合为 $A \cap B$,
含因子 3 或 5 的正整数的集合为 $A \cup B$.
我们有:

$$|A| = \lfloor \frac{100}{3} \rfloor = 33, \quad |B| = \lfloor \frac{100}{5} \rfloor = 20, \quad |A \cap B| = \lfloor \frac{100}{15} \rfloor = 6.$$

由容斥原理, 得:

$$|A \cup B| = |A| + |B| - |A \cap B| = 33 + 20 - 6 = 47.$$

有限集合的计数

Example

设 A, B, C 为有限集. 证明:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + (|B| + |C| - |B \cap C|) \\
&\qquad - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\
&= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.
\end{aligned}$$

QUIZ: 如何推广到 n 个集合的情况?

有限集合的计数

Example

设 A, B, C 为有限集. 证明:

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$$\begin{aligned} |A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + (|B| + |C| - |B \cap C|) \\ &\qquad - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\ &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|. \end{aligned}$$

QUIZ: 如何推广到 n 个集合的情况?

Homework

1. P. 17: Exercises 26,27(3,5).

Russell's Paradoxes

Russell posed his famous paradoxes in 1901 that letting $S = \{x \mid x \notin x\}$ be the set of elements which do not admit them as elements of them, we can thereby infer

$$S \in S \text{ implies } S \notin S$$
$$S \notin S \text{ implies } S \in S.$$

Recall that

Definition (集合, from the viewpoint of naive set theory)

集合 (set): 一些具有某种共性的对象所组成的整体.

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Cause and Solution

Cause: Such a set S is nonexistent.

Solution: Sets should be inductively founded from the base—the well foundednes/formedness (良基性).