习题4.1

1. 设随机变量$X \geq 0$, 其概率密度函数为$p(x)$, $r > 0$, 若数学期望$\mathrm{E}X^r$存在, 证明

$$\mathrm{E}X^r = \int_0^\infty r x^{r-1} P(X > x)\mathrm{d}x.$$

证明. 数学期望$\mathrm{E}X^r$存在, 故当$x \to \infty$时,

$$x^r P(X > x) = x^r \int_x^\infty p(t)\mathrm{d}t \leq \int_x^\infty t^r p(t)\mathrm{d}t \to 0.$$

于是,

$$\int_0^\infty r x^{r-1} P(X > x)\mathrm{d}x = \int_0^\infty P(X > x)\mathrm{d}x^r = x^r P(X > x)|_0^\infty + \int_0^\infty x^r p(x)\mathrm{d}x = \mathrm{E}X^r.$$

另证:

$$\begin{aligned}\mathrm{E}X^r &= \int_{-\infty}^\infty x^r p(x)\mathrm{d}x = \int_0^\infty x^r p(x)\mathrm{d}x = \int_0^\infty \left(\int_0^x r y^{r-1}\mathrm{d}y\right) p(x)\mathrm{d}x \\ &= \int_0^\infty r y^{r-1} \left(\int_y^\infty p(x)\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty r y^{r-1} P(X > y)\mathrm{d}y.\end{aligned}$$

□

2. 设随机变量X服从泊松分布$P(\lambda)$, 证明$\mathrm{E}X^{k+1} = \lambda \mathrm{E}(X+1)^k$. 利用此结论求$\mathrm{E}X^3$.

解. $X \sim P(\lambda)$, 则X有分布列$P(X = j) = \dfrac{\lambda^j}{j!}e^{-\lambda}$, $j = 0, 1, \cdots$

$$\begin{aligned}\mathrm{E}X^{k+1} &= \sum_{j=0}^\infty j^{k+1} P(X = j) = \sum_{j=1}^\infty j^{k+1} P(X = j) = \sum_{j=1}^\infty j^{k+1} \frac{\lambda^j}{j!} e^{-\lambda} \\ &= \sum_{j=1}^\infty j^k \frac{\lambda^j}{(j-1)!} e^{-\lambda} = \lambda \sum_{j=1}^\infty (j-1+1)^k \frac{\lambda^{j-1}}{(j-1)!} e^{-\lambda} \\ &= \lambda \sum_{m=0}^\infty (m+1)^k \frac{\lambda^m}{m!} e^{-\lambda} = \lambda \sum_{m=0}^\infty (m+1)^k P(X = m) = \lambda \mathrm{E}(X+1)^k\end{aligned}$$

于是,

$$\mathrm{E}X^3 = \lambda \mathrm{E}(X+1)^2 = \lambda(\mathrm{E}X^2 + 2\mathrm{E}X + 1) = \lambda^2 \mathrm{E}(X+1) + \lambda(2\mathrm{E}X + 1) = \lambda^3 + 3\lambda^2 + \lambda.$$

□

4. 设随机变量X服从均匀分布$U(0, 1)$, 计算下列随机变量的数学期望:

$$X^3, \quad e^X, \quad \left(X - \frac{1}{2}\right)^2.$$

解. X服从均匀分布$U(0,1)$, 概率密度函数为$p(x)=\begin{cases}1, & 0<x<1,\\ 0, & 其他.\end{cases}$

由定义,

$$EX^3=\int x^3p(x)\mathrm{d}x=\int_0^1 x^3\mathrm{d}x=\frac{1}{4},$$

$$\mathrm{E}e^X=\int e^xp(x)\mathrm{d}x=\int_0^1 e^x\mathrm{d}x=e-1,$$

$$\mathrm{E}\left(X-\frac{1}{2}\right)^2=\int\left(x-\frac{1}{2}\right)^2p(x)\mathrm{d}x=\int_0^1\left(x-\frac{1}{2}\right)^2\mathrm{d}x=\frac{1}{12}.$$

□

8. 设随机变量X服从指数分布$Exp(\lambda)$, 求$Y=[X]$的数学期望$\mathrm{E}Y$($[x]$表示不超过x的最大整数).

解. 显然, Y可能取值于$0,1,2,\cdots$. Y的分布列为

$$P(Y=k)=P(k\le X<k+1)=\int_k^{k+1}p(x)\mathrm{d}x=\int_k^{k+1}\lambda e^{-\lambda x}\mathrm{d}x=e^{-\lambda k}-e^{-\lambda(k+1)}=(1-e^{-\lambda})e^{-\lambda k},$$

其中$k=0,1,2,\cdots$ 故$Y+1$服从几何分布$Ge(1-e^{-\lambda})$.

于是,

$$\mathrm{E}Y=\mathrm{E}(Y+1)-1=\frac{1}{1-e^{-\lambda}}-1=\frac{e^{-\lambda}}{1-e^{-\lambda}}.$$

□

2. 设随机变量X服从Gamma分布$Ga(\alpha,\lambda)$, 求方差$\mathrm{Var}X$.

解. X服从Gamma分布$Ga(\alpha,\lambda)$, 概率密度函数为

$$p(x)=\begin{cases}\dfrac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, & x>0,\\ 0, & x\le 0.\end{cases}$$

于是, 对任意的n,

$$\mathrm{E}X^n=\int x^np(x)\mathrm{d}x=\int_0^\infty x^n\cdot\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\mathrm{d}x=\frac{1}{\lambda^n\Gamma(\alpha)}\int_0^\infty(\lambda x)^{n+\alpha-1}e^{-\lambda x}\mathrm{d}(\lambda x)=\frac{\Gamma(n+\alpha)}{\lambda^n\Gamma(\alpha)}.$$

故$\mathrm{Var}X=\mathrm{E}X^2-(\mathrm{E}X)^2=\dfrac{\Gamma(2+\alpha)}{\lambda^2\Gamma(\alpha)}-\left(\dfrac{\Gamma(1+\alpha)}{\lambda^1\Gamma(\alpha)}\right)^2=\dfrac{\alpha(1+\alpha)}{\lambda^2}-\left(\dfrac{\alpha}{\lambda}\right)^2=\dfrac{\alpha}{\lambda^2}.$

□

6. 已知随机变量(X, Y)的概率密度函数为

$$p(x, y) = \begin{cases} \dfrac{1+xy}{4}, & |x| < 1, |y| < 1, \\ 0, & \text{其他}. \end{cases}$$

求方差$\mathrm{Var}(X+Y)$.

解. 易知X与Y同分布, 都服从均匀分布$U(-1, 1)$. 于是, $\mathrm{E}X = \mathrm{E}Y = 0$, $\mathrm{Var}X = \mathrm{Var}Y = 1/3$. 又因为

$$\mathrm{E}XY = \iint xyp(x, y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{1}\int_{-1}^{1} xy\frac{1+xy}{4}\mathrm{d}x\mathrm{d}y = \frac{1}{9},$$

故$\mathrm{Var}(X+Y) = \mathrm{Var}X + \mathrm{Var}Y + 2\mathrm{E}(X-\mathrm{E}X)(Y-\mathrm{E}Y) = 1/3 + 1/3 + 2\cdot 1/9 = 8/9$. □

8. 设随机变量(X, Y)具有概率密度函数

$$p(x, y) = \begin{cases} e^{-y}, & 0 < x < y; \\ 0, & \text{其他}. \end{cases}$$

求数学期望$\mathrm{E}X$, $\mathrm{E}Y$, $\mathrm{E}XY$和方差$\mathrm{Var}X$, $\mathrm{Var}Y$, $\mathrm{Var}(X+Y)$.

解. 记$D = \{(x, y) : 0 < x < y\}$, 由数学期望的定义知,

$$\mathrm{E}X = \iint xp(x, y)\mathrm{d}x\mathrm{d}y = \iint_D xe^{-y}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\mathrm{d}y\int_0^{y} xe^{-y}\mathrm{d}x = 1,$$

$$\mathrm{E}Y = \iint xp(x, y)\mathrm{d}y\mathrm{d}y = \iint_D ye^{-y}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\mathrm{d}y\int_0^{y} ye^{-y}\mathrm{d}x = 2,$$

$$\mathrm{E}XY = \iint xyp(x, y)\mathrm{d}x\mathrm{d}y = \iint_D xye^{-y}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\mathrm{d}y\int_0^{y} xye^{-y}\mathrm{d}x = 3,$$

$$\mathrm{E}X^2 = \iint xp(x, y)\mathrm{d}x^2\mathrm{d}y = \iint_D x^2e^{-y}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\mathrm{d}y\int_0^{y} x^2e^{-y}\mathrm{d}x = 2,$$

$$\mathrm{E}Y^2 = \iint xp(x, y)\mathrm{d}y^2\mathrm{d}y = \iint_D y^2e^{-y}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\mathrm{d}y\int_0^{y} y^2e^{-y}\mathrm{d}x = 6.$$

于是,

$$\mathrm{Var}X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = 2 - 1^2 = 1, \quad \mathrm{Var}Y = \mathrm{E}Y^2 - (\mathrm{E}Y)^2 = 6 - 2^2 = 2,$$

$\mathrm{Var}(X+Y) = \mathrm{Var}X + \mathrm{Var}Y + 2(\mathrm{E}XY - \mathrm{E}X\mathrm{E}Y) = 1 + 2 + 2(3 - 1\cdot 2) = 5$. □

9. 证明:

(1)设随机变量$X \geq 0$, 数学期望存在, 则$\mathrm{E}X \geq 0$.

(2)设随机变量X的方差存在, 则对任意的常数c, 有$\mathrm{E}(X-c)^2 \geq \mathrm{Var}X$成立.

(3)设随机变量X仅取值于$[a,b]$, 则$\mathrm{Var}X \leq \left(\frac{b-a}{2}\right)^2$.

证明. (1)由定义立得.

(2)注意到$\mathrm{E}(X-\mathrm{E}X)=0$,

$$\begin{aligned}\mathrm{E}(X-c)^2 &= \mathrm{E}\left((X-\mathrm{E}X)+(\mathrm{E}X-c)\right)^2\\ &= \mathrm{E}\left((X-\mathrm{E}X)^2+2(X-\mathrm{E}X)(\mathrm{E}X-c)+(\mathrm{E}X-c)^2\right)\\ &= \mathrm{Var}X+(\mathrm{E}X-c)^2 \geq \mathrm{Var}X.\end{aligned}$$

(3)既然X仅取值于$[a,b]$, 故$\left|X-\frac{a+b}{2}\right|^2 \leq \left(\frac{b-a}{2}\right)^2$. 由(2)和(1)得,

$$\mathrm{Var}X \leq \mathrm{E}\left(X-\frac{a+b}{2}\right)^2 \leq \left(\frac{b-a}{2}\right)^2.$$

□

习题4.3

1. 设随机变量X与Y满足$\mathrm{E}X=\mathrm{E}Y=0$, $\mathrm{Var}X=\mathrm{Var}Y=1$, $Cov(X,Y)=\rho$, 证明

$$\mathrm{E}\max(X^2,Y^2) \leq 1+\sqrt{1-\rho^2}.$$

证明. 显然, $\mathrm{E}X^2=\mathrm{Var}X=1$, $\mathrm{E}Y^2=\mathrm{Var}Y=1$,

$$\mathrm{E}(X\pm Y)^2=\mathrm{E}X^2+\mathrm{E}Y^2\pm 2\mathrm{E}XY=\mathrm{Var}X+\mathrm{Var}Y\pm 2Cov(X,Y)=2\pm 2\rho.$$

由Cauchy-Schwarz不等式,

$$\mathrm{E}|X^2-Y^2|=\mathrm{E}(|X-Y||X+Y|) \leq \sqrt{\mathrm{E}(|X-Y|)^2\mathrm{E}(|X+Y|)^2}=\sqrt{(1+1-2\rho)(1+1+2\rho)}=2\sqrt{1-\rho^2}.$$

故

$$\mathrm{E}\max(X^2,Y^2)=\mathrm{E}\left(\frac{X^2+Y^2+|X^2-Y^2|}{2}\right) \leq 1+\sqrt{1-\rho^2}.$$

□

2. 设随机变量X与Y都服从均匀分布$U(0,1)$, 且相互独立, 求随机变量$\max(X,Y)$ 和$\min(X,Y)$的协方差.

解. 因为X与Y独立, 故$\mathrm{Var}(X+Y) = \mathrm{Var}X + \mathrm{Var}Y = 1/12 + 1/12 = 1/6$. 又

$$\mathrm{E}|X-Y| = \iint |x-y| p(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 |x-y|\mathrm{d}x\mathrm{d}y = \frac{1}{3}.$$

$$\mathrm{E}|X-Y|^2 = \iint |x-y|^2 p(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 |x-y|^2\mathrm{d}x\mathrm{d}y = \frac{1}{6}.$$

于是$\mathrm{Var}(|X-Y|) = \mathrm{E}|X-Y|^2 - (\mathrm{E}|X-Y|)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$.

故

$$\begin{aligned} Cov(\max(X,Y),\min(X,Y)) =& \frac{1}{4} Cov(X+Y+|X-Y|, X+Y-|X-Y|) \\ =& \frac{1}{4}\cdot(\mathrm{Var}(X+Y) - \mathrm{Var}(|X-Y|)) = \frac{1}{4}\cdot\left(\frac{1}{6}-\frac{1}{18}\right) = \frac{1}{36} \end{aligned}$$

□

4. 设$X_1, X_2, \cdots$独立同分布, 且$\mathrm{E}X_1 = \mu$, $\mathrm{Var}X_1 = \sigma^2$, 求随机变量$X_1+\cdots+X_{100}$与$X_{101}+\cdots+X_{150}$的协方差.

解. 既然$X_1, X_2, \cdots$相互独立, 故$X_1+\cdots+X_{100}$与$X_{101}+\cdots+X_{150}$相互独立, 于是$Cov(X_1+\cdots+X_{100}, X_{101}+\cdots+X_{150}) = 0$ □

8. 设随机变量(X,Y)具有概率密度函数$p(x,y) = \begin{cases} e^{-y}, & 0<x<y; \\ 0, & 其他. \end{cases}$ 求X与Y的协方差和相关系数.

解. 记$D = \{(x,y): 0<x<y\}$, 由数学期望的定义知,

$$\mathrm{E}X = \iint xp(x,y)\mathrm{d}x\mathrm{d}y = \iint_D xe^{-y}\mathrm{d}x\mathrm{d}y = \int_0^\infty \mathrm{d}y \int_0^y xe^{-y}\mathrm{d}x = 1,$$

$$\mathrm{E}Y = \iint xp(x,y)\mathrm{d}y\mathrm{d}y = \iint_D ye^{-y}\mathrm{d}x\mathrm{d}y = \int_0^\infty \mathrm{d}y \int_0^y ye^{-y}\mathrm{d}x = 2,$$

$$\mathrm{E}XY = \iint xyp(x,y)\mathrm{d}x\mathrm{d}y = \iint_D xye^{-y}\mathrm{d}x\mathrm{d}y = \int_0^\infty \mathrm{d}y \int_0^y xye^{-y}\mathrm{d}x = 3,$$

$$\mathrm{E}X^2 = \iint xp(x,y)\mathrm{d}x^2\mathrm{d}y = \iint_D x^2e^{-y}\mathrm{d}x\mathrm{d}y = \int_0^\infty \mathrm{d}y \int_0^y x^2e^{-y}\mathrm{d}x = 2,$$

$$\mathrm{E}Y^2 = \iint xp(x,y)\mathrm{d}y^2\mathrm{d}y = \iint_D y^2e^{-y}\mathrm{d}x\mathrm{d}y = \int_0^\infty \mathrm{d}y \int_0^y y^2e^{-y}\mathrm{d}x = 6.$$

于是, $Cov(X,Y) = \mathrm{E}XY - \mathrm{E}X\mathrm{E}Y = 3 - 1\cdot 2 = 1$,

$$\mathrm{Var}X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = 2 - 1^2 = 1, \quad \mathrm{Var}Y = \mathrm{E}Y^2 - (\mathrm{E}Y)^2 = 6 - 2^2 = 2,$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{\mathrm{Var}X\mathrm{Var}Y}} = \frac{1}{\sqrt{1\cdot 2}} = \frac{\sqrt{2}}{2}.$$

□