A Technique for the Numerical Solution of Certain Integral Equations of the First Kind*

DAVID L. PHILLIPS†

Argonne National Laboratory, Argonne, Illinois

Introduction

The general linear equation may be written as

$$h(x)f(x) + \int_a^b K(x, y)f(y)\, dy = g(x) \qquad (a \leqq x \leqq b)$$

where the known functions $h(x)$, $K(x, y)$ and $g(x)$ are assumed to be bounded and usually to be continuous. If $h(x) \equiv 0$ the equation is of *first kind*; if $h(x) \neq 0$ for $a \leqq x \leqq b$, the equation is of *second kind*; if $h(x)$ vanishes somewhere but not identically, the equation is of *third kind*. If the range of integration is infinite or if the kernel $K(x, y)$ is not bounded, the equation is *singular*. Here we will consider only nonsingular linear integral equations of the first kind:

$$\int_a^b K(x, y)f(y)\, dy = g(x) \qquad (a \leqq x \leqq b) \qquad (1)$$

There is extensive literature on equations of the second kind, but literature on linear equations of the first kind is sparse. However, several methods for solving equations of the first kind numerically have been proposed [1–10]. No method has been very successful for arbitrary kernels when the function $g(x)$ is known with only modest accuracy. The reason for this is inherent in the equation itself. Think of the equation as a linear operator, operating on $f(y)$ to produce $g(x)$. This operator does not have a bounded inverse (it may not even have an inverse, but we will assume here that it does) which can be seen as follows. Let $f(y)$ be the solution to (1) and add to it the function $f_m = \sin my$. For any integrable kernel it is known that $g_m \equiv \int_a^b K(x, y) \sin(my)\, dy \to 0$ as $m \to \infty$. Hence only an infinitesimal change g_m in g causes a finite change f_m in f (i.e. the equation is unstable). Also, one would expect that $g_m \to 0$ as $m \to \infty$ faster for flat smooth kernels than for sharply peaked kernels (indeed if $K(x, y)$ were the δ-function, $K(x, y) = \delta(x - y)$, then $g_m = f_m$ would not approach zero). Hence we conclude that the success in solving equation (1) by any method depends to a large extent on the accuracy of $g(x)$ and the shape of $K(x, y)$.

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Description of Technique

If the straightforward matrix approximation to equation (1) is used, it is found that as the mesh width decreases, the solutions at first become more accurate, but eventually begin to get worse. How soon the solutions begin to get worse depends on the accuracy of $g(x)$. Larger errors in g cause the solutions to get worse sooner. Also, the error in each of the approximate solutions tends to be an oscillatory function of x.

Since the function $g(x)$ is not known accurately, we should state the problem as

$$\int_a^b K(x, y)f(y)\, dy = g(x) + \epsilon(x) \qquad (a \leqq x \leqq b) \quad (2)$$

where $\epsilon(x)$ is an arbitrary function except for some condition on the size of $\epsilon(x)$, such as $| \epsilon(x) | \leqq M$ or $\int_a^b \rho(x)\epsilon^2(x)\, dx \leqq \bar{M}$, $\rho(x) > 0$. Instead of a unique solution of (2) we get a family $\mathfrak{F}$ of solutions. The real problem then is to pick out of the family of functions $\mathfrak{F}$ the true solution f. This cannot be done without more information about the problem than is given in equation (2). We will assume here that the functional form of f is not known. If it were known we could use a least square fit to find a best fit to f. However, we will assume here that f is a reasonably smooth function. With this assumption the best approximation to f we can choose is probably the function $f_s \in \mathfrak{F}$ which is the smoothest in some sense. Of the various smoothness conditions, we choose the following (assuming the f have piecewise continuous second derivatives):

$$\int_a^b (f_s'')^2\, dx = \min_{f \in \mathfrak{F}} \int_a^b (f'')^2\, dx. \quad (3)$$

In order to solve numerically (2) and (3) we make a matrix approximation to them. We subdivide the interval into n parts by the points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and replace the integral equation (2) by the following linear system,

$$\sum_{i=0}^{n} w_i k_{ji} f_i = g_j + \epsilon_j \qquad (j = 0, 1, \cdots, n) \quad (4)$$

where $f_i = f(x_i)$, $g_j = g(x_j)$, $\epsilon_j = \epsilon(x_j)$, $k_{ji} = K(x_j, x_i)$, and the w_i are weight factors whose values depend on the quadrature formula used. However, for simplicity, we will assume that the x_i are uniformly spaced. For the condition on the magnitude of $\epsilon(x)$ we take

$$\sum_{i=0}^{n} \epsilon_i^2 = e^2 \quad (5)$$

where e^2 is a constant (more generally one can take $\sum_{i=0}^{n} p_i\epsilon_i^2 = e^2$ where the $p_i \geqq 0$ are weights). The analogous problem to (3) is to look for the vector

$\mathbf{f}_s = (f_0^s, f_1^s, \cdots, f_n^s)$ such that

$$\sum_{i=0}^{n} (f_{i+1}^s - 2f_i^s + f_{i-1}^s)^2 = \min_{\mathbf{f}\in\mathfrak{F}^*} \sum_{i=0}^{n} (f_{i+1} - 2f_i + f_{i-1})^2 \tag{6}$$

where $\mathfrak{F}^*$ is the set of vectors satisfying (4) and (5) (here we assume f is zero outside the interval (a, b) and set $f_{-1} = f_{n+1} = 0$). Let us introduce the following matrix notation:

$$A \equiv (w_i k_{ji}), \qquad A^{-1} \equiv (\alpha_{ij}).$$

Equation (4) can be written in the form

$$\mathbf{f} = A^{-1}\mathbf{g} + A^{-1}\boldsymbol{\epsilon}. \tag{7}$$

This shows that the f_i are linear functions of the ϵ_j and that

$$\frac{\partial f_i}{\partial \epsilon_j} = \alpha_{ij} \qquad (i, j = 0, 1, \cdots, n). \tag{8}$$

From (6), (8), and the constraint (5), we see that the conditions f_s must satisfy can be written as follows:

$$\sum_{i=0}^{n} \epsilon_i^2 = e^2$$

$$\sum_{i=0}^{n} (f_{i+1}^s - 2f_i^s + f_{i-1}^s)(\alpha_{i+1,j} - 2\alpha_{ij} + \alpha_{i-1,j}) + \gamma^{-1}\epsilon_j = 0$$

$$(j = 0, 1, \cdots, n) \tag{9}$$

where γ^{-1} is the Lagrangian multiplier. It will be much simpler to solve these equations for $\mathbf{f}_s$ if we take γ as known and e^2 as an unknown instead of the other way around. This is equivalent to replacing the conditions (5) and (6) by the single condition that $\mathbf{f}_s$ be the vector (satisfying (4)) that minimizes the expression

$$\gamma \sum_{i=0}^{n} (f_{i+1} - 2f_i + f_{i-1})^2 + \sum_{i=0}^{n} \epsilon_i^2 \tag{10}$$

where γ is a given constant. From (10) we see that γ should be non-negative and that for $\gamma = 0$ the whole problem reduces to just the straightforward matrix approximation to (1), $A\mathbf{f} = \mathbf{g}$. Equations (9) can be written in the matrix form

$$\gamma B\mathbf{f}_s + \boldsymbol{\epsilon} = 0 \tag{11}$$

where $B = (\beta_{lk})$ and

$$\beta_{lk} = \alpha_{k-2,l} - 4\alpha_{k-1,l} + 6\alpha_{kl} - 4\alpha_{k+1,l} + \alpha_{k+2,l} \quad (k,l = 0, 1, \cdots, n). \tag{12}$$

In (12) we define $\alpha_{-2,l} = -\alpha_{0,l}$, $\alpha_{-1,l} = 0$, $\alpha_{n+1,l} = 0$, and $\alpha_{n+2,l} = -\alpha_{nl}$. Using (11) and the fact that $\mathbf{f}_s$ satisfies (7) we solve for $\mathbf{f}_s$ and $\boldsymbol{\epsilon}$ and get

$$\mathbf{f}_s = (A + \gamma B)^{-1}\mathbf{g}, \qquad \boldsymbol{\epsilon} = -\gamma B\mathbf{f}_s. \tag{13}$$

From equations (13), the matrix method described here merely replaces A by $A + \gamma B$ where B is a certain matrix whose elements depend only on A and γ is an arbitrary non-negative parameter which controls the amount of smoothing. Increasing γ produces greater smoothing. It follows from the second equation in (13) and (5) that e is approximately proportional to γ so that only a very few values of γ need to be used in order to find one giving e the desired magnitude. However, one additional matrix inversion is needed for each new value of γ used. The value of e is determined from the accuracy of the g_j.

Examples

Example 1. Let the problem be the following:

$$\int_{-3}^{3} K(x - \lambda)f(x)\,dx = g(\lambda)$$

where

$$\begin{aligned} K(z) &= 1 + \cos\frac{\pi z}{3}, && |z| \leqq 3, \\ &= 0, && |z| > 3; \\ g(z) &= (6 + \lambda)\left(1 - \frac{1}{2}\cos\frac{\pi\lambda}{3}\right) - \frac{9}{2\pi}\sin\frac{\pi\lambda}{3}, && |z| \leqq 6, \\ &= 0, && |z| \geqq 6. \end{aligned}$$

The solution to this problem is $f(x) = K(x)$. Hence we can easily check the numerical solution against the true solution. Let us first take $n = 12$ (13 points) and use Simpson's rule for the quadrature formula. The truncation error is about .4. The values of g are rounded off so that the maximum error in $g(\lambda_j)$ is .00005. Table I and Figure 1 give the comparison between the true solution and the numerical solution for several choices of γ. Since the solution is symmetric

TABLE I

λ	true values of f	$\gamma = 0$	$\gamma = .0011$	$\gamma = .011$	$\gamma = .03$	$\gamma = .1$	$\gamma = .5$	$\gamma = 1$
0	2.0000	2.943	2.356	2.082	2.017	1.948	1.843	1.796
1	1.5000	1.154	1.404	1.478	1.488	1.489	1.476	1.463
2	.5000	.694	.388	.464	.503	.559	.651	.686
3	.0000	.014	.077	−.002	−.016	−.014	.016	.042
4	.0000	.000	−.034	.031	.019	−.026	−.105	−.120
5	.0000	.000	.005	.029	.038	.034	.009	−.009
6	.0000	.000	−.012	−.128	−.133	−.052	.110	.131
ave. $\lvert \epsilon_j \rvert$		0	.010	.020	.026	.047	.105	.140
max. $\lvert \epsilon_j \rvert$		0	.016	.035	.053	.096	.203	.242

about zero, only non-negative vaues of λ are listed in the table. Also, the average $|\epsilon_j|$ and maximum $|\epsilon_j|$ are listed for each γ.

Here the truncation error, which acts much like an error in g, is bounded by .4. Hence we expect to get the best results for some value of γ for which ave. $|\epsilon_j| < .4$. Here $\gamma = .03$ is the best choice even though ave. $|\epsilon_j| = .026$, which is considerably smaller than .4. Now consider the same problem with $n = 24$ (25 points). Here the truncation error is bounded by .025. Table II

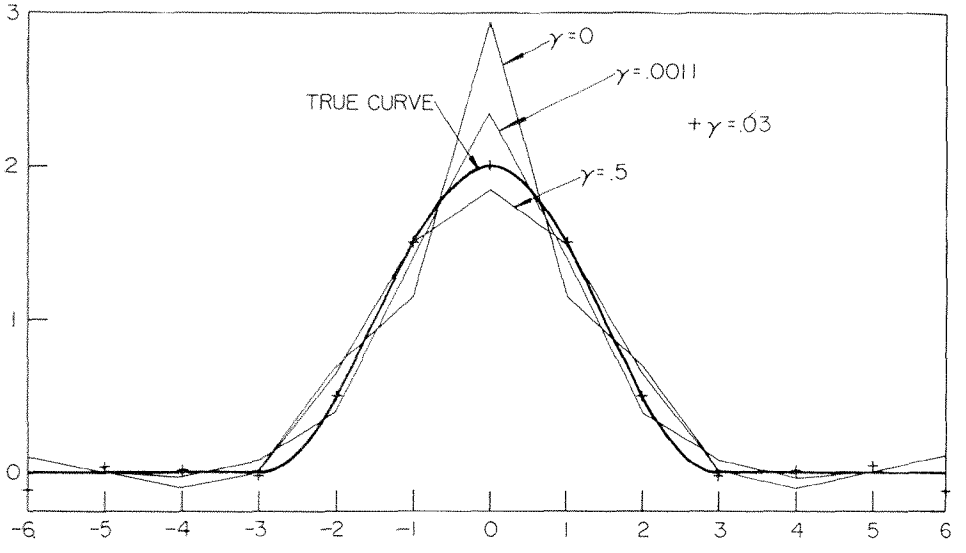

FIG. 1

TABLE II

λ	true values of f	$\gamma = 0$	$\gamma = .0025$	$\gamma = .02$	$\gamma = .1$	$\gamma = .5$	$\gamma = 1$
0	2.00000	3.0016	1.9992	2.0049	2.0027	1.9853	1.9672
.5	1.86603	1.3994	1.8680	1.8692	1.8663	1.8531	1.8395
1	1.50000	2.2480	1.4982	1.4900	1.4870	1.4851	1.4836
1.5	1.00000	.7516	.9945	.9888	.9900	1.0009	1.0121
2	.50000	.7463	.5006	.5093	.5170	.5350	.5525
2.5	.13397	.1026	.1472	.1601	.1680	.1831	.1967
3	0	−.0106	−.0006	−.0025	−.0039	−.0018	.0014
3.5	0	.0078	−.0160	−.0318	−.0414	−.0531	−.0627
4	0	−.0135	.0002	−.0060	−.0125	−.0296	−.0463
4.5	0	.0062	.0092	.0194	.0220	.0094	−.0050
5	0	−.0110	.0024	.0163	.0229	.0226	.0202
5.5	0	.0032	−.0034	−.0052	−.0032	.0120	.0275
6	0	.0034	−.0055	−.0312	−.0382	−.0074	.0274
ave. $\lvert\epsilon_j\rvert$		0	.0010	.0017	.0037	.0155	.0280
max. $\lvert\epsilon_j\rvert$		0	.0027	.0059	.0122	.0352	.0594

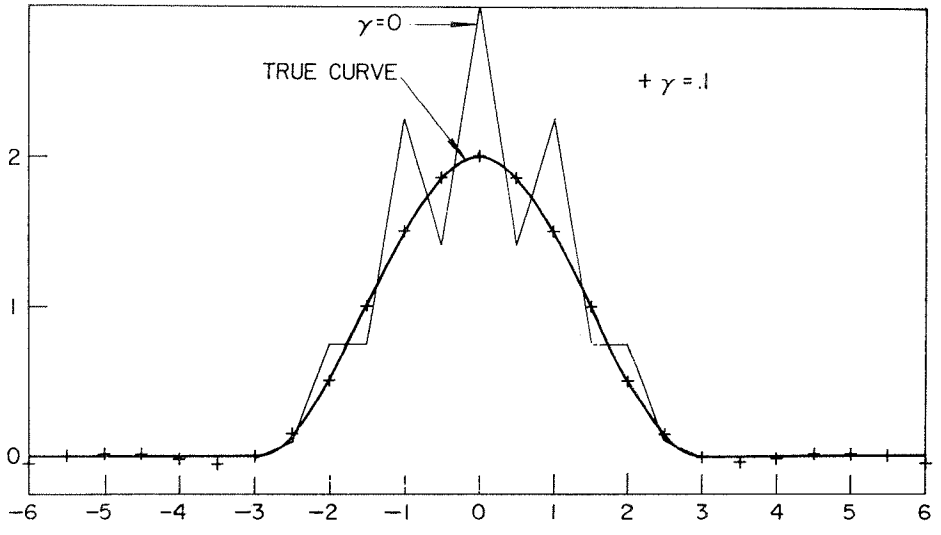

FIG. 2

TABLE III

λ	true values of f	$\gamma = 0$	$\gamma = .0001$	$\gamma = .001$	$\gamma = .01$	$\gamma = .1$	$\gamma = 1$
0	2.00000	3.289	1.987	1.9889	2.0020	2.0028	1.967
.5	1.86603	1.184	1.856	1.8663	1.8692	1.8665	1.840
1	1.50000	2.825	1.540	1.5101	1.4942	1.4871	1.484
1.5	1.00000	.501	1.003	.9978	.9906	.9896	1.012
2	.50000	1.150	.451	.4908	.5042	.5162	.552
2.5	.13397	−.097	.159	.1440	.1542	.1674	.197
3	0	.513	.002	−.0010	−.0016	−.0038	.002
3.5	0	.382	−.031	−.0141	−.0246	−.0406	−.062
4	0	.948	.033	.0117	−.0010	−.0115	−.046
4.5	0	−.495	.019	.0113	.0162	.0225	−.005
5	0	.947	−.019	−.0091	.0093	.0227	.020
5.5	0	−.325	−.018	−.0084	−.0058	−.0037	.027
6	0	.412	.049	.0199	−.0182	−.0386	.027
ave. $\lvert \epsilon_j \rvert$		0	.0016	.0018	.0020	.0040	.028
max. $\lvert \epsilon_j \rvert$		0	.0026	.0032	.0056	.010	.057

and Figure 2 give the comparison between the true solution and the numerical solution for several values of γ.

Here the smoothing effect of γ on the oscillation due to the ill-condition of the matrix is much greater than the smoothing effect on the solution. Thus here it is possible to practically eliminate the oscillation due to the ill-conditioned matrix without appreciably affecting the solution.

Now consider the same problem where the g's have been further rounded off.

In this case the maximum error in g is .004 and the average error in g is .0014. Table III and Figure 3 give the comparison between the true solution and the numerical solution for several values of γ.

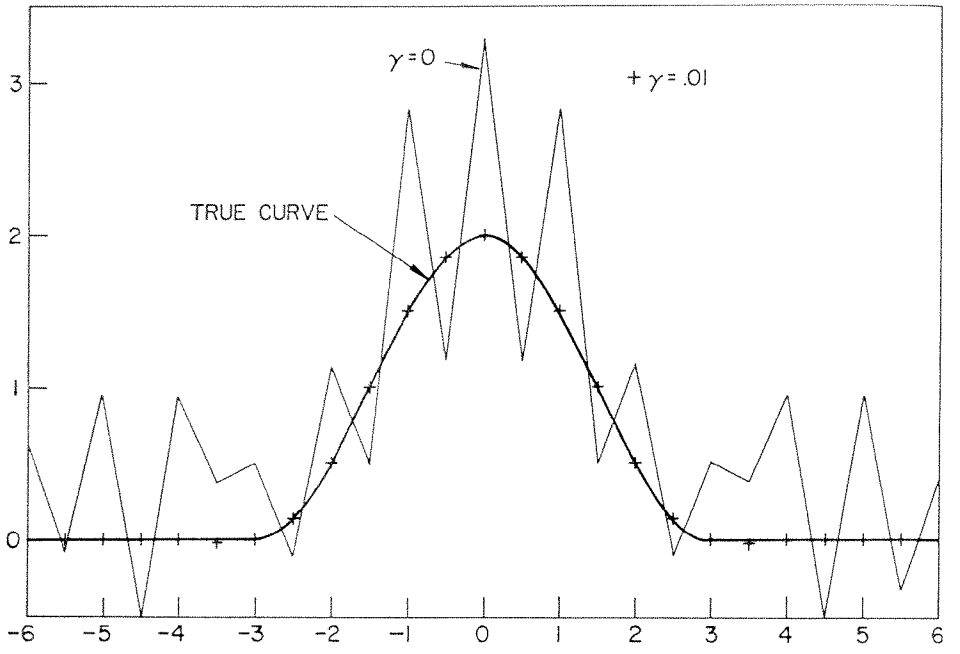

FIG. 3

TABLE IV

λ	true values of f	$\gamma = 0$	$\gamma = .005$	$\gamma = .05$	$\gamma = .5$	$\gamma = 1$
0	2.00000	13.56	2.068	2.047	2.007	1.986
.5	1.86603	−4.62	1.921	1.897	1.868	1.853
1	1.50000	16.79	1.482	1.479	1.484	1.483
1.5	1.00000	−6.67	.906	.948	.983	.998
2	.50000	13.95	.439	.477	.513	.533
2.5	.13397	−5.85	.162	.156	.168	.184
3	0	10.83	.053	.015	−.001	.001
3.5	0	−4.71	.031	−.008	−.038	−.050
4	0	8.64	.014	.010	−.008	−.027
4.5	0	−4.34	−.019	.022	.026	.011
5	0	8.19	−.047	.008	.025	.024
5.5	0	−2.75	−.006	−.015	−.004	.014
6	0	3.61	.109	−.032	−.041	−.005
ave. $\lvert \epsilon_j \rvert$		0	.012	.014	.023	.034
max. $\lvert \epsilon_j \rvert$		0	.026	.029	.057	.084

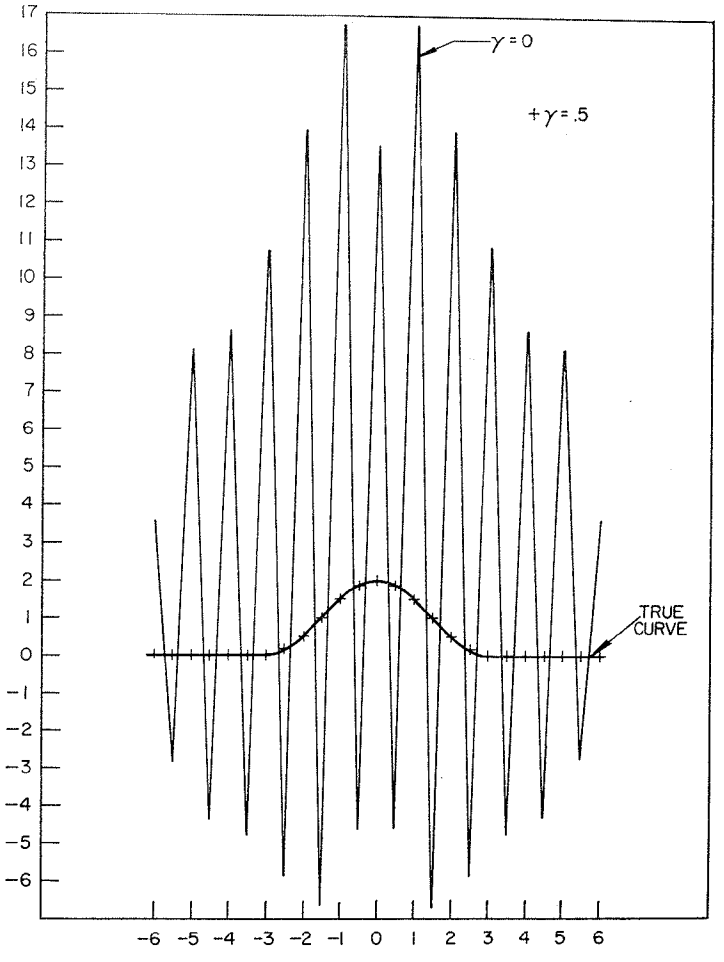

FIG. 4

Notice that for $\gamma = 0$ there is considerably more oscillation than in the previous case. However, the oscillation is smoothed out just as before by using a suitable value of γ.

Now consider the same problem where the g's have been even further rounded off. In this case the average error in $| g |$ is .02 and the maximum error is .041. Table IV and Figure 4 give the comparison between the true solution and the numerical solution for several values of γ.

Notice that for $\gamma = 0$ the solution is completely dominated by oscillation. The errors in g have been magnified in f so much that the solution is completely useless. However, for $\gamma = .5$ the solution is reasonably good, especially near the peak which is the most important region. Also notice that the average $| \epsilon_j |$ for $\gamma = .5$ is of the same order of magnitude as the average error of the g's.

Example 2. The problem is the following:

$$\int_{-30}^{30} K(x - \lambda)f(x)\,dx = g(\lambda)$$

where $K(z)$, $g(z)$ and $f(z)$ are given in tabular form in Table V. $K(z)$ and $g(z)$ are also plotted in Figure 5.

The values of g were obtained from the tabulated values of $f(z)$ and $K(z)$ by numerical integration. The values of g have errors which average about .01 in magnitude. The maximum error in g is about .02. Let us first take $n = 14$ (15 points) and use Simpson's rule for the quadrature formula and solve for $f(z)$ assuming $K(z)$, $g(z)$ given. The truncation error is less than .66. Table VI and Figure 6 give the comparison between the true solution and the numerical solution for several choices of γ.

TABLE V

z	$g(z)$	$K(z)$	$f(z)$	z	$g(z)$	$K(z)$	$f(z)$
−30	.0100	.1184	.0000	2	2.8912	3.0628	.1096
−28	.0100	.1311		4	2.4586	1.6329	.0584
−26	.0110	.1464		6	1.9049	.8806	.0349
−24	.0170	.1651		8	1.4144	.5095	.0173
−22	.0305	.1883		10	1.0282	.3137	.0107
−20	.0405	.2179		12	.7411	.2021	.0028
−18	.0585	.2563		14	.5409	.1341	.0005
−16	.0869	.3077		16	.4083	.0906	.0000
−14	.1309	.3788		18	.3214	.0614	
−12	.2018	.4816		20	.2623	.0413	
−10	.3235	.6380		22	.2201	.0269	
−8	.5469	.8914	.0000	24	.1886	.0165	
−6	.9621	1.3333	.0019	26	.1580	.0089	
−4	1.6301	2.1483	.0345	28	.1270	.0031	
−2	2.4047	3.5108	.0965	30	.0780	.0013	.0000
0	2.9104	4.3600	.1321				
				$K(z) = 0$ for $\lvert z \rvert > 30$			

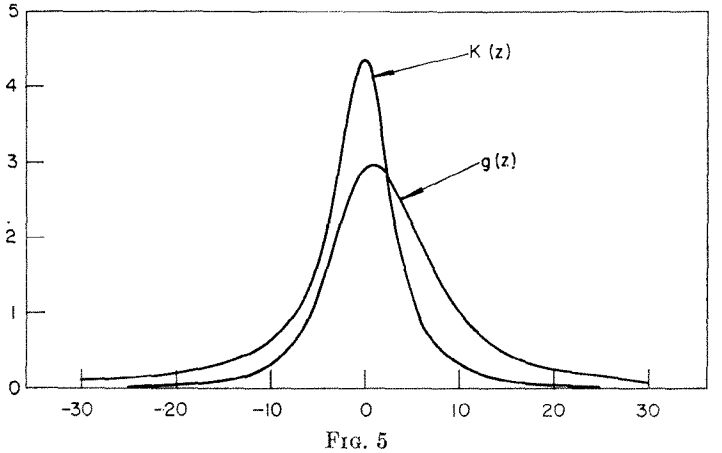

FIG. 5

TABLE VI

z	true values of f	$\gamma = 0$	$\gamma = .001$	$\gamma = .01$	$\gamma = .1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 20$
−28	0	.0011	.0011	.0011	.0011	.0008	.0001	−.0003
−24	|	−.0001	−.0001	−.0001	−.0001	.0000	.0001	.0004
−20	|	.0002	.0002	.0002	.0001	−.0001	.0007	.0010
−16	|	−.0001	−.0001	−.0001	.0000	.0004	.0001	−.0006
−12	|	.0004	.0004	.0004	.0001	−.0017	−.0052	−.0059
−8	0	−.0011	−.0011	−.0011	−.0009	.0004	.0037	.0054
−4	.0345	.0608	.0607	.0607	.0602	.0572	.0538	.0542
0	.1321	.0900	.0900	.0900	.0903	.0922	.0943	.0931
4	.0584	.0998	.0998	.0997	.0990	.0937	.0810	.0776
8	.0173	.0134	.0134	.0135	.0138	.0161	.0222	.0245
12	.0028	.0061	.0061	.0060	.0056	.0031	−.0003	.0003
16	0	.0000	−.0000	.0000	.0001	.0005	−.0001	−.0008
20	|	−.0001	−.0001	−.0001	−.0001	.0001	.0014	.0011
24	|	.0012	.0012	.0012	.0012	.0009	−.0000	−.0001
28	0	−.0126	−.0126	−.0126	−.0125	−.0116	−.0080	−.0062
ave. $\lvert \epsilon_j \rvert$		0	.00001	.0001	.0013	.009	.026	.033
max. $\lvert \epsilon_j \rvert$		0	.00004	.0004	.004	.03	.10	.14

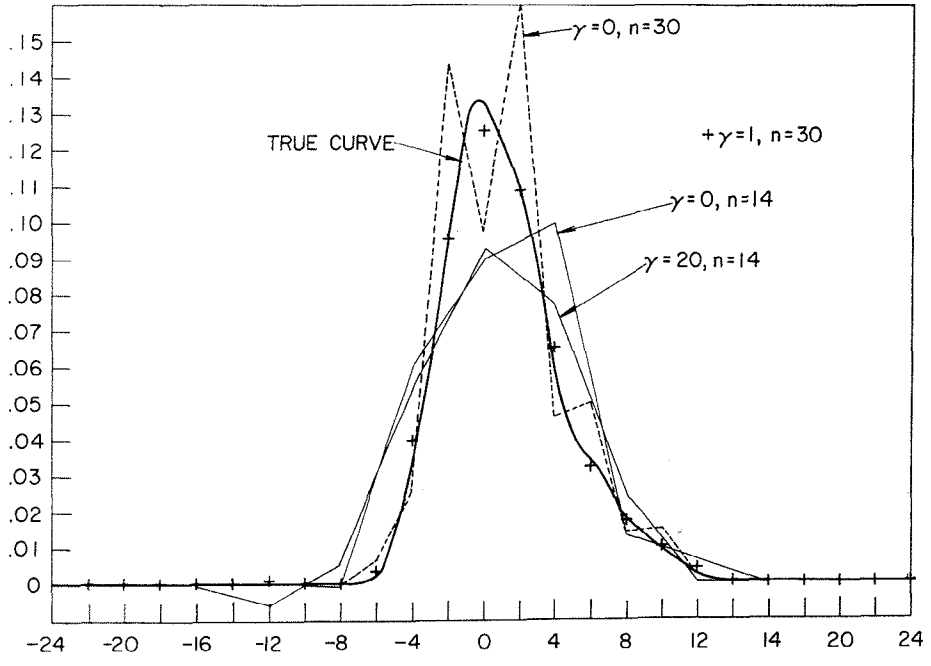

FIG. 6

The poor results here are due to a large truncation error. The parameter γ has to be chosen so large that the true shape of the function is noticeably affected. Hence the oscillation and the true shape cannot be separated.

Now consider the same problem where we take $n = 30$ (31 points). Here the upper bound for the truncation error is .02, which is the same order of magnitude as the errors in g.

Table VII and Figure 6 give the comparison between the true solution and the numerical solution for several choices of γ.

TABLE VII

z	true values of f	$\gamma = 0$	$\gamma = .01$	$\gamma = .1$	$\gamma = .5$	$\gamma = 1$	$\gamma = 2$	$\gamma = 10$
−30	0	.0033	.0015	.0015	.0019	.0018	.0016	.0011
−28		−.0002	.0008	.0007	.0004	.0004	.0004	.0004
−26		.0006	−.0018	−.0015	−.0008	−.0006	−.0005	−.0001
−24		−.0012	−.0001	−.0001	−.0001	−.0002	−.0002	−.0001
−22		.0038	.0019	.0015	.0008	.0006	.0005	.0002
−20		−.0011	.0005	−.0002	.0001	.0003	.0004	.0000
−18		.0008	.0000	−.0005	−.0002	−.0001	−.0002	−.0004
−16		−.0002	.0000	.0002	−.0002	−.0004	−.0006	−.0001
−14		.0002	.0000	−.0001	−.0006	−.0003	.0001	.0014
−12		−.0001	.0001	−.0004	.0005	.0010	.0014	.0011
−10		.0002	−.0013	.0018	.0015	.0007	−.0002	−.0032
−8	0	−.0002	.0018	−.0001	−.0020	−.0027	−.0034	−.0049
−6	.0019	.0067	−.0018	−.0023	.0019	.0036	.0050	.0090
−4	.0345	.0262	.0334	.0386	.0393	.0398	.0406	.0446
−2	.0965	.1439	.1242	.1037	.0971	.0955	.0943	.0919
0	.1321	.0973	.1086	.1212	.1250	.1252	.1246	.1196
2	.1096	.1610	.1387	.1157	.1096	.1086	.1081	.1066
4	.0584	.0462	.0556	.0633	.0645	.0649	.0658	.0696
6	.0349	.0500	.0360	.0296	.0313	.0321	.0328	.0357
8	.0173	.0143	.0188	.0193	.0181	.0177	.0171	.0156
10	.0107	.0151	.0097	.0107	.0107	.0104	.0099	.0077
12	.0028	.0023	.0037	.0033	.0038	.0041	.0043	.0045
14	.0005	.0012	−.0001	.0003	.0002	.0003	.0006	.0016
16	0	.0000	.0002	.0002	−.0001	−.0002	−.0003	−.0001
18		.0000	.0001	−.0001	.0001	.0002	.0000	−.0004
20		.0002	−.0001	−.0001	.0001	.0001	.0001	−.0001
22		−.0011	−.0003	.0007	−.0001	−.0002	−.0001	.0002
24		.0014	.0011	−.0002	−.0001	.0001	.0002	.0005
26		−.0037	−.0036	.0001	.0013	.0012	.0010	.0003
28		.0025	.0028	.0012	−.0001	−.0005	−.0007	−.0010
30	0	−.0139	−.0152	−.0125	−.0089	−.0073	−.0058	−.0034
ave. $\lvert \epsilon_j \rvert$		0	.0015	.003	.004	.005	.006	.011
max. $\lvert \epsilon_j \rvert$		0	.007	.017	.020	.020	.024	.043

TABLE VIII

z	true values of f	$\gamma = 0$	$\gamma = .1$	$\gamma = 1$	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
−30	0	.0024	.0056	.0014	.0007	.0009	.0013
−28		.0006	−.0014	.0001	.0004	.0007	.0007
−26		−.0025	.0017	.0011	.0009	.0003	−.0002
−24		.0004	.0001	−.0003	−.0007	−.0011	−.0012
−22		.0039	.0008	−.0023	−.0020	−.0013	−.0007
−20		−.0037	−.0052	−.0003	.0004	.0013	.0017
−18		−.0109	.0177	.0072	.0062	.0048	.0038
−16		.0390	.0021	.0037	.0032	.0020	.0012
−14		−.1385	−.0160	−.0072	−.0068	−.0058	−.0048
−12		.0760	.0010	−.0099	−.0096	−.0080	−.0066
−10		−.1689	−.0349	−.0041	−.0018	−.0010	−.0016
−8	0	.0613	.0264	.0151	.0129	.0095	.0067
−6	.0019	.0405	.0212	.0155	.0151	.0151	.0157
−4	.0345	−.0330	.0127	.0290	.0316	.0356	.0393
−2	.0965	.2342	.1239	.0867	.0846	.0840	.0842
0	.1321	.0789	.1124	.1236	.1234	.1210	.1180
2	.1096	.1026	.1165	.1191	.1177	.1151	.1128
4	.0584	.1369	.0910	.0752	.0745	.0753	.0766
6	.0349	−.1525	−.0215	.0206	.0262	.0317	.0352
8	.0173	.0834	.0223	.0125	.0117	.0109	.0107
10	.0107	−.0700	.0225	.0120	.0085	.0046	.0023
12	.0028	.0286	−.0110	−.0020	−.0012	−.0009	−.0013
14	.0005	−.1006	.0001	−.0080	−.0065	−.0041	−.0028
16	0	.0744	.0003	−.0022	−.0020	−.0017	−.0012
18		−.2102	−.0177	.0053	.0045	.0026	.0015
20		.1270	.0272	.0064	.0045	.0029	.0023
22		−.1998	−.0488	−.0123	−.0068	−.0019	.0003
24		.0318	.0017	.0012	.0017	.0025	.0028
26		.0757	.0498	.0184	.0129	.0075	.0049
28		−.0490	−.0177	−.0015	−.0008	−.0009	−.0010
30	0	.0782	.0013	−.0256	−.0215	−.0145	−.0103
ave. $\lvert \epsilon_j \rvert$		0	.022	.036	.038	.040	.041
max. $\lvert \epsilon_j \rvert$		0	.067	.094	.104	.119	.132

Notice that the best values of γ are the ones for which the max $\lvert \epsilon_j \rvert$ is approximately the same as the maximum error in g. Also notice that the peak is rounded off a small amount. This is unavoidable with the present method.

Finally, consider the same problem where the g's are given less accurately. Let the errors be about 5 per cent of the peak value. (This simulates a certain problem from experimental physics.) The maximum error in g is about .15. Table VIII and Figure 7 give the comparison between the true solution (for accurate g) and the numerical solution for several values of γ.

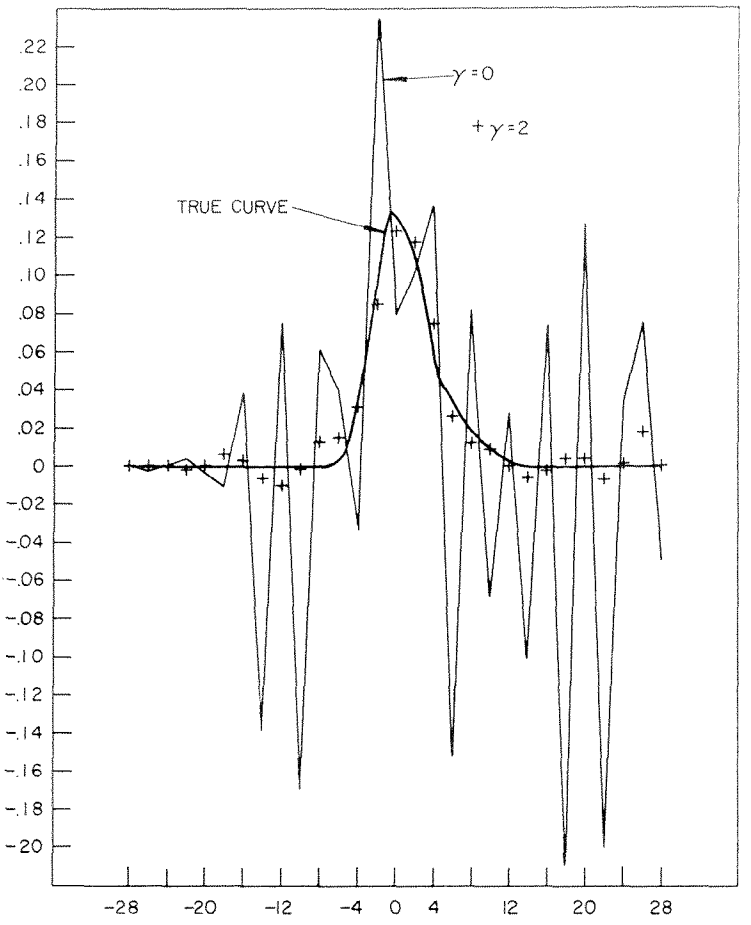

FIG. 7

Notice that for $\gamma = 0$ the solution is dominated by oscillation. The best values of γ are 1 and 2. For these values the error in the peak value is about 7 per cent, a reasonable value since the maximum error in g is about 5 per cent. For the best values of γ the max $| \epsilon_j |$ is about .1, somewhat lower than the maximum error in g. This is probably because of the unusually large errors in g.

Summary

The method described here should work reasonably well on problems where the unknown function f is assumed to be a relatively smooth function. The truncation error should be as small as, or smaller than the errors in g. The most difficult task is to choose γ. When the errors in g are relatively small, γ should probably be chosen such that e is approximately the same magnitude as these errors. When the errors in g are relatively large, as they are in the last part of example 2, γ should probably be chosen such that e is somewhat smaller than

these errors. In any case several values of γ should be tried and the best value should be the one that appears to take out the oscillation without appreciably smoothing the function f.

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