EEEN3008J: Advance wireless communications

Introduction to MIMO

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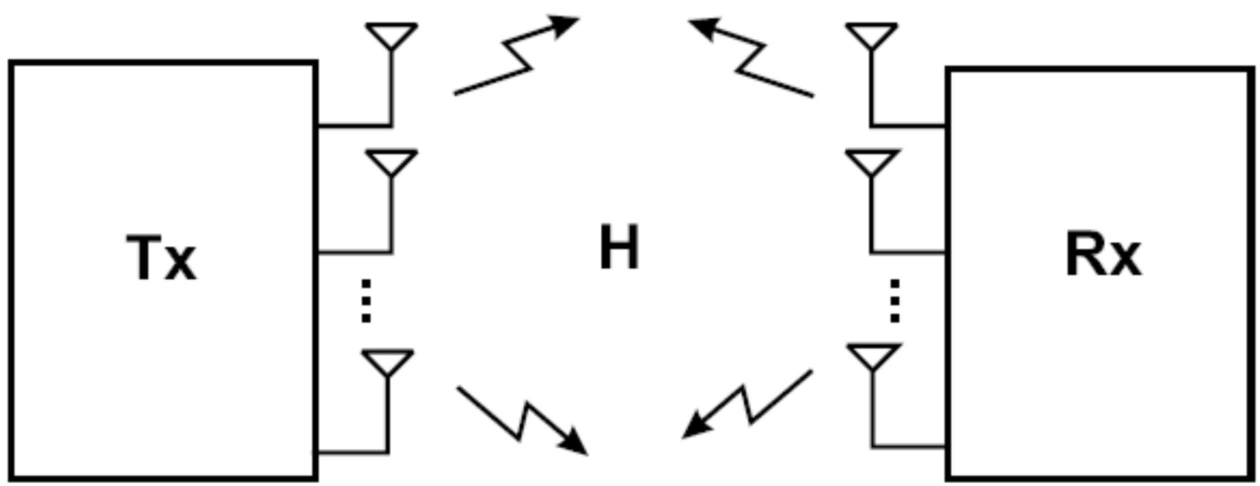
Multiple Input Multiple Output communication

- All antennas transmit at the same time over the same resource (frequency, …)
- Their signals interfere
- Not a problem if signal decoded jointly (receive antennas "cooperate")



Advantages of multiple antennas

- Array gain – improved SNR, extended coverage
- Error rate reduction (diversity gain) – mitigates fading through spatial diversity
- Spectral efficiency (multiplexing gain) – increased bits/channel access rate
- Interference reduction – exploiting the differentiation between desired signal and co-channel interference signal

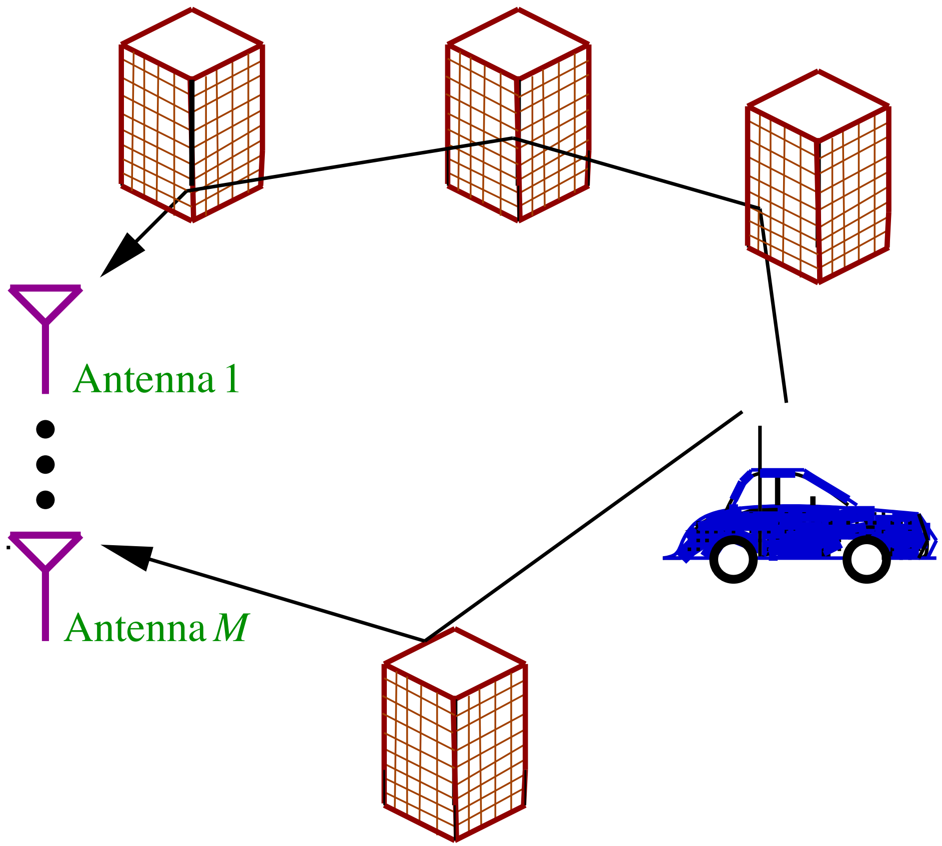
Diversity Techniques

- Diversity is a powerful communication receiver technique that provides wireless link improvement at relatively low cost
- Diversity exploits the random nature of radio propagation by finding independent (or at least highly uncorrelated) signal paths for communication
- By receiving more than one copy of the transmitted received signal and then selecting one (or multiple) of them intelligently, both the instantaneous and average SNRs at the receiver may be improved, often by as much as 20 dB to 30 dB

Microscopic Diversity

- These techniques exploit small-scale fading characterized by deep and rapid amplitude fluctuations as mobile moves over distances of just a few wavelength
- **Example:**
  - In case of small-scale fading, if two antennas are separated by a fraction of meter, one may receive a weak signal while other may receive a strong signal
  - By selecting the best signal at all times, a receiver can mitigate small-scale fading effects
    - This is called *antenna diversity* or *space diversity*

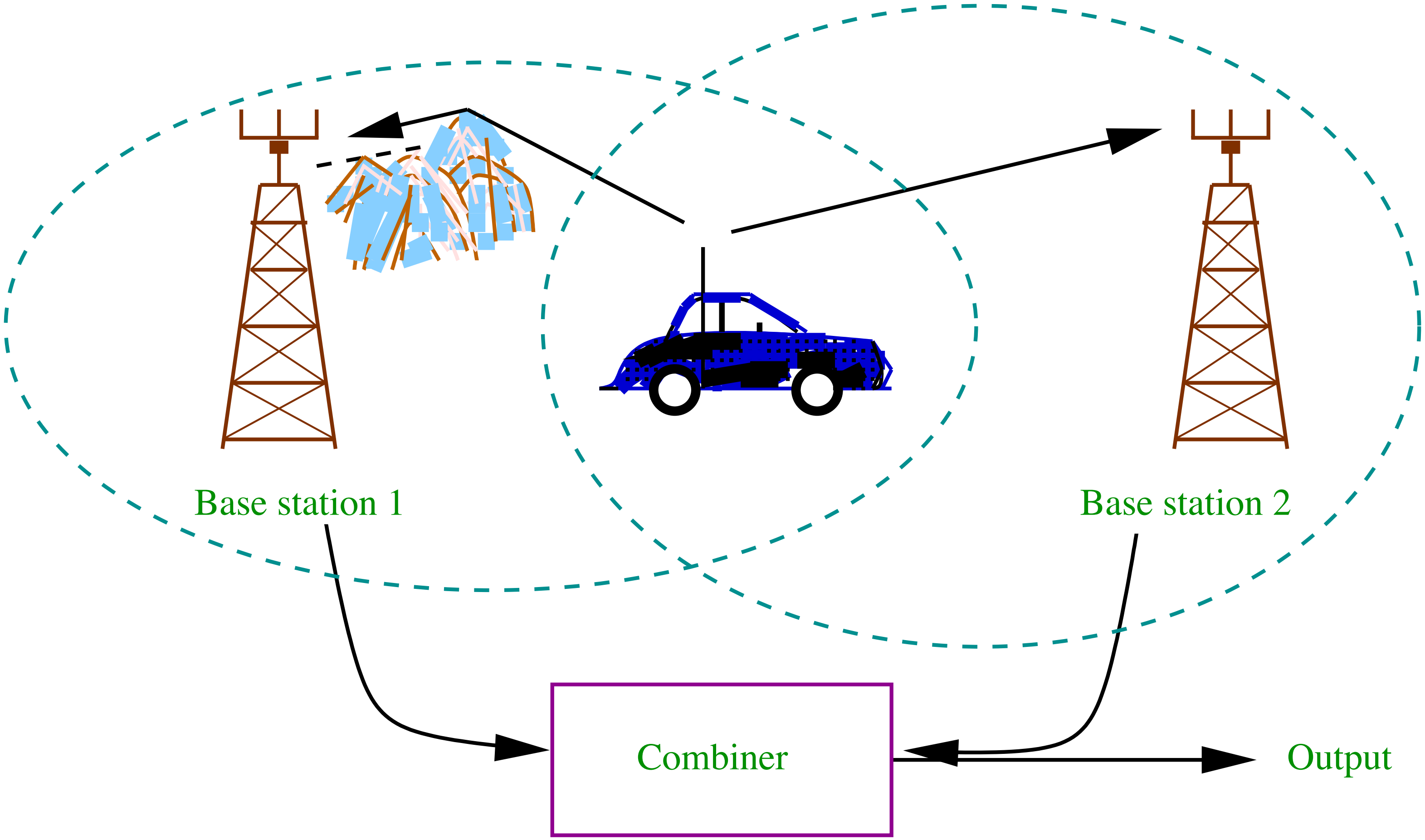
Microscopic Diversity



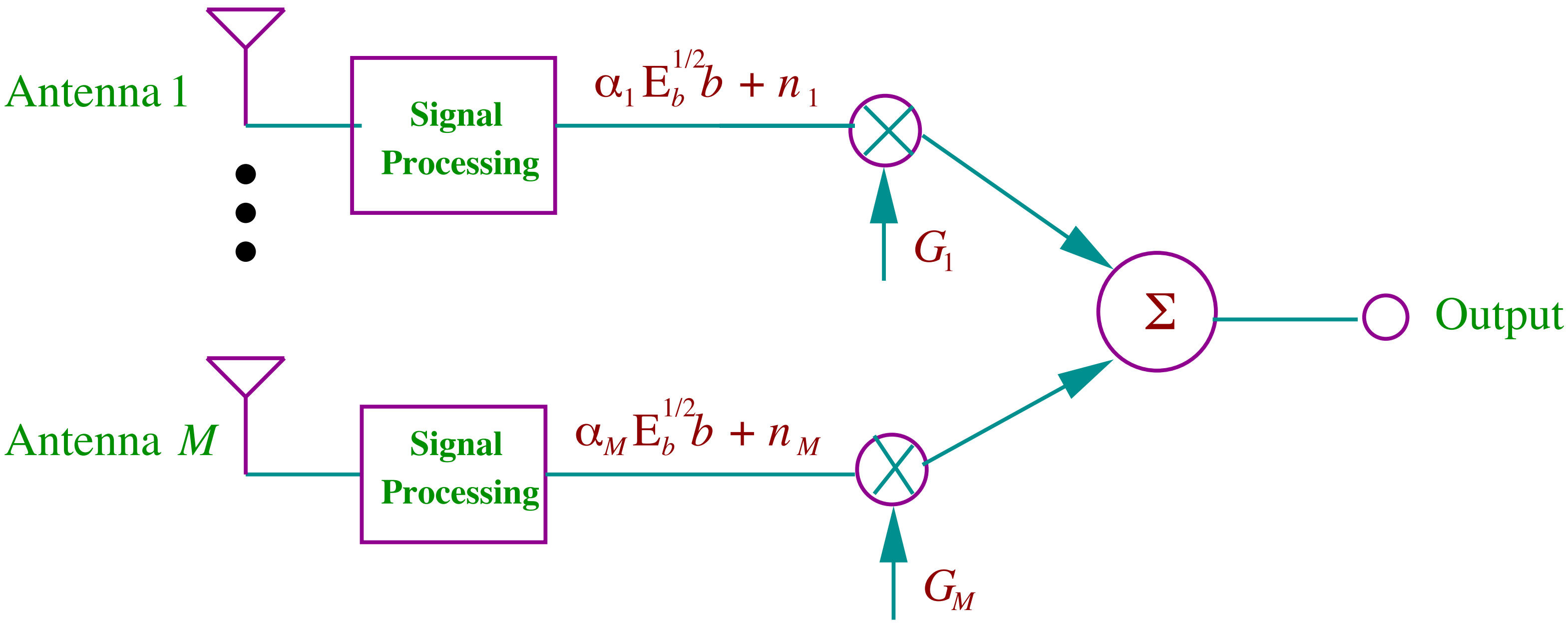
Macroscopic Diversity

- These techniques exploit large-scale fading caused by shadowing due to variations in both the terrain profile and the nature of the surroundings
- **Example:**
  - By selecting a base station which is not shadowed when others are, the mobile can improve substantially the average SNR on the forward link
  - Macroscopic diversity is also useful at the base station receiver

Macroscopic Diversity



Diversity Combining Techniques



- If $G_i = \alpha_i \;\; \forall i \Rightarrow$ Maximum Gain Combining
- If $G_i = 1 \;\; \forall i \Rightarrow$ Equal Gain Combining
- If $G_i \in (0,1) \Rightarrow$ Selection Combining

b = Transmitted bit

E_b = Bit energy

α_i = Fading complex envelop on the i-th branch

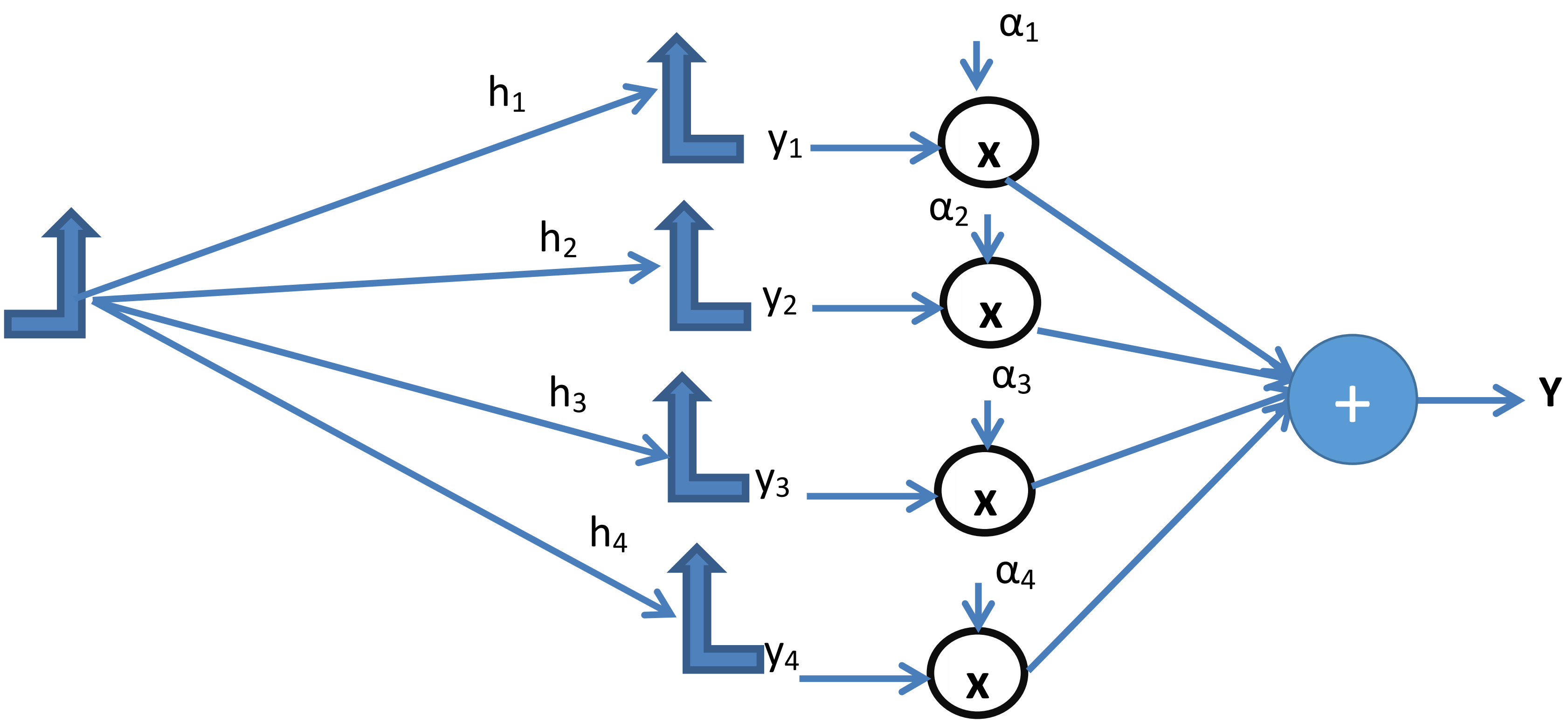
n_i = Additive white Gaussian noise on the i-th branch

G_i = Gain at the i-th branch

Problem 1

Consider 1 x 4 system in the following figure. If h_1= 0.3802 + 0.2254j, h_2 = 1.2968 - 0.9247j

h_3= -1.5972 - 0.3066j, and h_4 = 0.6096 + 0.2423j.



Problem 1

a) In case of Maximum gain combining (MGC), what are the coefficients α_1 , α_2 , α_3, and α_4?

b) In case of Equal gain combining (EGC), what are the coefficients α_1 , α_2 , α_3, and α_4?

c) If the transmitted symbol is 0.707+0.707j. What are the received symbol, Y in both of EGC and MGC.

Solution to Problem 1

$r_i = h_i x_i + n_i$
Let's say $x_i = \sqrt{E_b} b$, where E_b and the b are the bit energy and the transmitted bit respectively.
Therefore, $r_i = h_i \sqrt{E_b} b + n_i, \forall i = 1, \ldots, M$
If each branch has gain α_i
then the resulting signal envelope:
$R_M = \sum_{i=1}^{M} \alpha_i r_i = \sum_{i=1}^{M} \alpha_i h_i \sqrt{E_b} b + \sum_{i=1}^{M} \alpha_i n_i$

Total noise power:
$N_T = \sigma^2 \sum_{i=1}^{M} \alpha_i^2$
Therefore SNR, γ_M, is given by:
$$\gamma_M = \frac{E_b (\sum_{i=1}^{M} \alpha_i h_i)^2}{\sigma^2 \sum_{i=1}^{M} \alpha_i^2} \leq \frac{E_b (\sum_{i=1}^{M} \alpha_i^2)(\sum_{i=1}^{M} h_i^2)}{\sigma^2 \sum_{i=1}^{M} \alpha_i^2}$$
Equality holds, when $h_i = \alpha_i$, which gives:
$$\gamma_M = \frac{E_b \sum_{i=1}^{M} h_i^2}{\sigma^2}$$

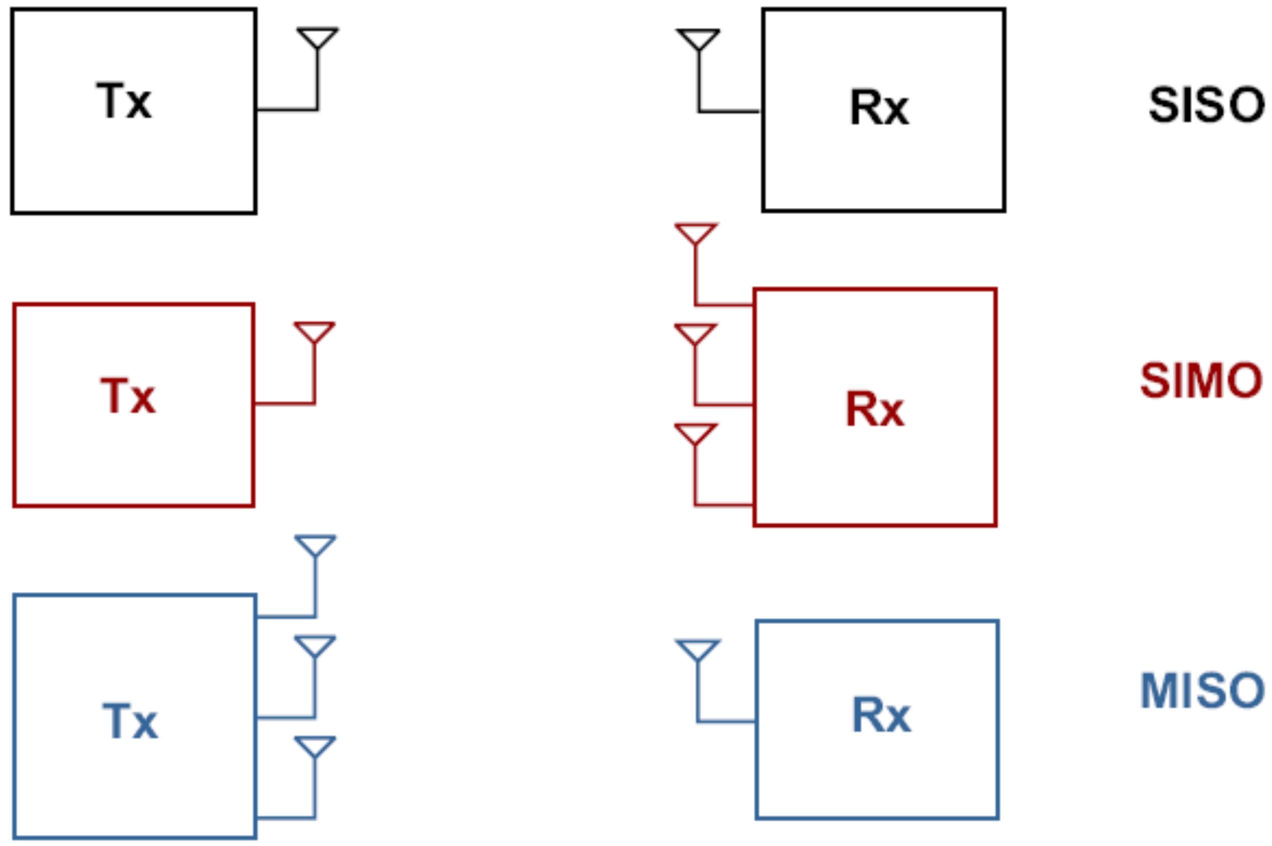
Solution to Problem 1

- Therefore for MGC, $\alpha_i = h_i$
- For EGC, $\alpha_i = 1$
- For MGC, Y = 3.2822 + j1.9360
- For EGC, Y = 1.0272 + j0.0524

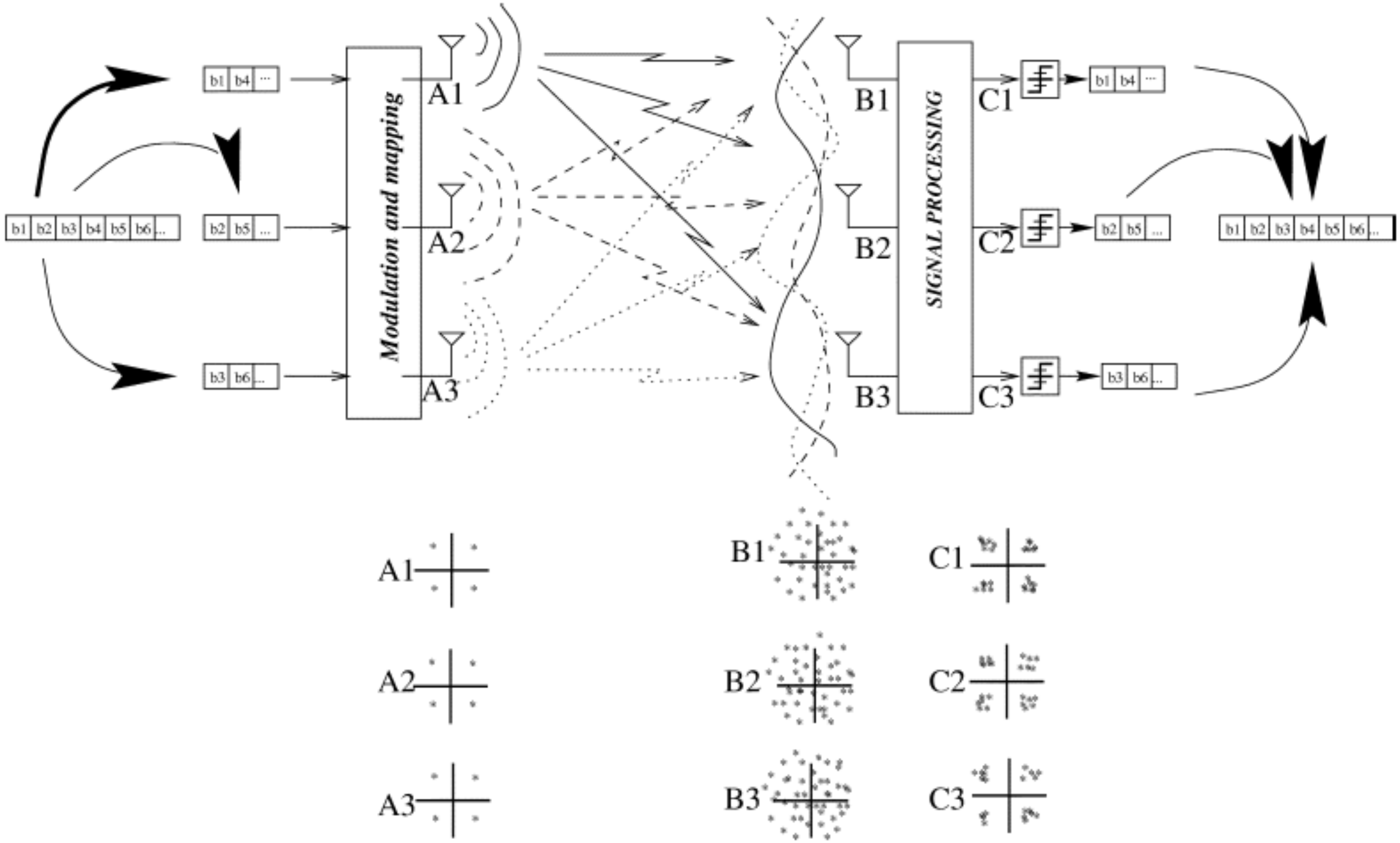
Obstacles to MIMO

- Hardware costs – multiple antennas mean multiple RF chains
  - Consumer electronics especially sensitive to cost arguments
- Energy requirements – more complicated signal processing requires more computing power
- Real estate for multiple antennas

SIMO, MISO, SISO

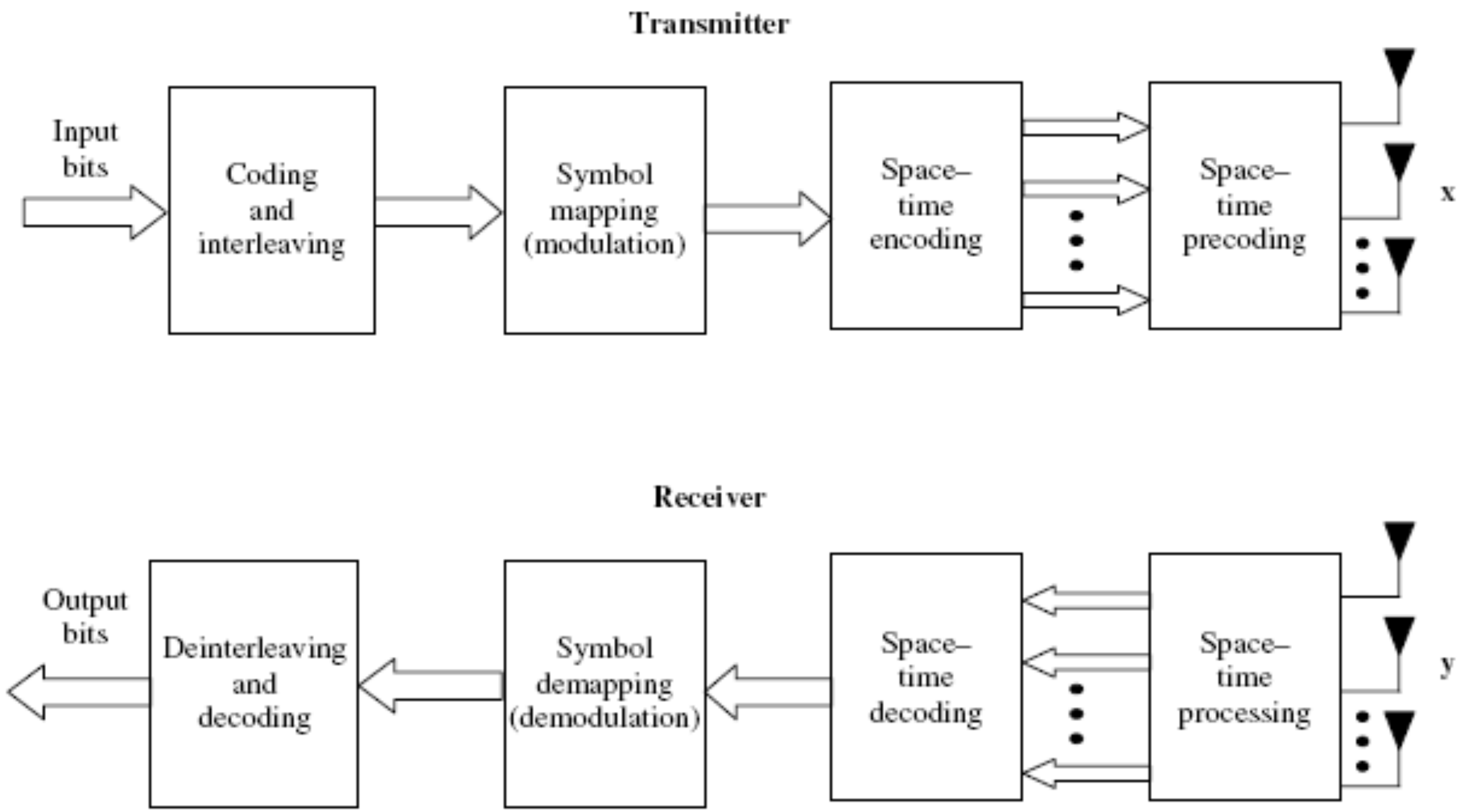


A 3x3 MIMO System



Courtesy: Hon Tat Hui, NUS

A typical MIMO System Model



Courtesy: Hon Tat Hui, NUS

Space-Time Encoding

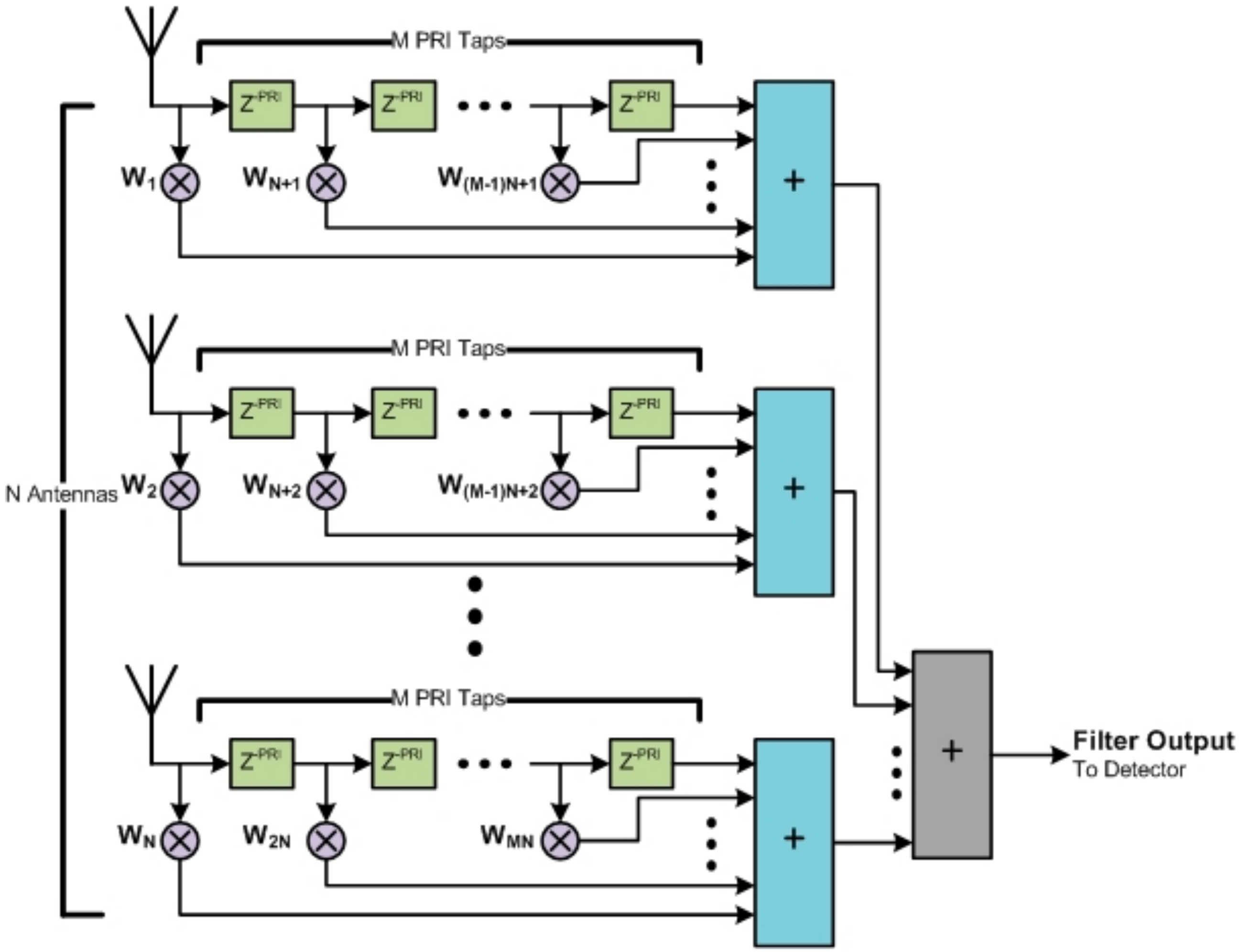
$$
\begin{array}{cc}
 & \text{transmit antennas} \longrightarrow \\
\text{time-slots} \downarrow & \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n_T} \\ s_{21} & s_{22} & \cdots & s_{2n_T} \\ \vdots & \vdots & & \vdots \\ s_{T1} & s_{T2} & \cdots & s_{Tn_T} \end{bmatrix}
\end{array}
$$

Here, s_{ij} is the modulated symbol to be transmitted in time slot i from antenna j. There are to be T time slots and n_T transmit antennas as well as n_R receive antennas. This block is usually considered to be of 'length' T.

The code rate of an STBC measures how many symbols per time slot it transmits on average over the course of one block. If a block encodes k symbols, the code-rate is $r = \frac{k}{T}$.

Source: Wikipedia

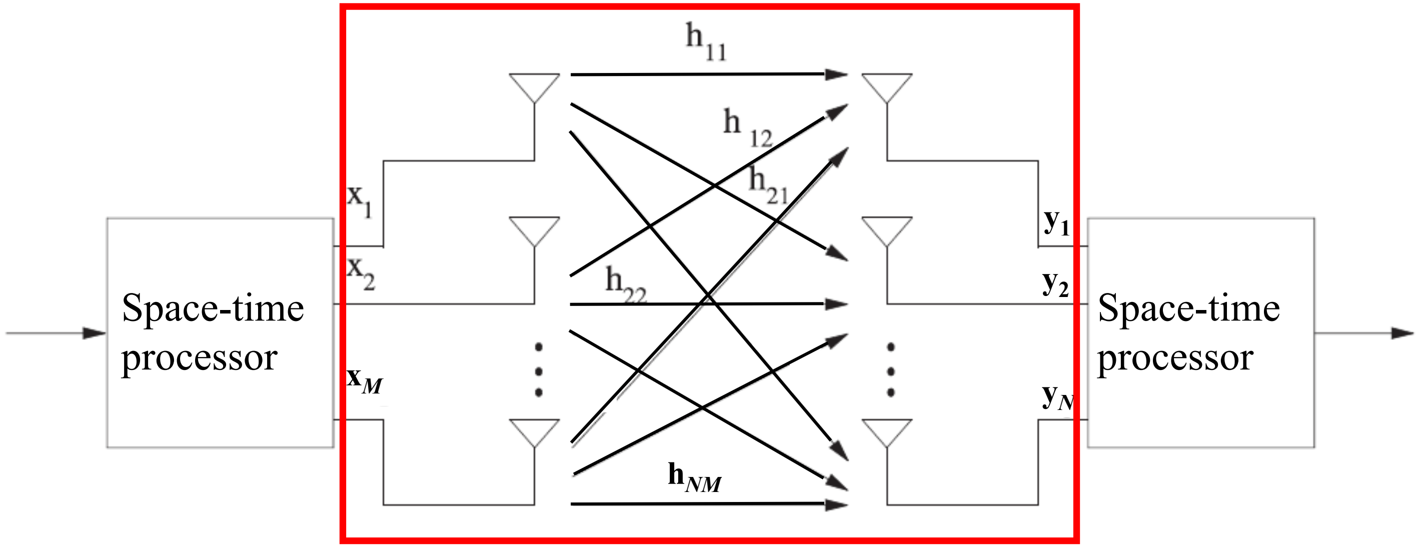
Space-Time Processing



Choose the weight W_i 's to maximize SINR

Source: Wikipedia

Channel Model



The received signal vector $\mathbf{y}$ can be expressed in terms of the channel matrix $\mathbf{H}$ as:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \text{received signal vector} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = \text{transmitted signal vector}$$

Channel Model

- Where **H** is the channel gain matrix, specifying the gain between transmit antennas and receive antennas

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix} = \text{noise vector}$$

- The transmit power is usually constrained

$$P_T = \mathbf{x}^H \mathbf{x} = |x_1|^2 + |x_2|^2 + \cdots + |x_M|^2 \le C$$

Courtesy: Hon Tat Hui, NUS

Covariance Matrices

The covariance matrices of the transmitted signals and received signals are:

$$\mathbf{R}_{xx} = E\left\{\mathbf{x}\mathbf{x}^H\right\}$$

$$\mathbf{R}_{yy} = E\left\{\mathbf{y}\mathbf{y}^H\right\}$$

$$= E\left\{\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H\right\} + E\left\{\mathbf{n}\mathbf{n}^H\right\}$$

The traces of $\mathbf{R}_{xx}$ and $\mathbf{R}_{yy}$ give the total powers of the transmitted and received signals, respectively. The off-diagonal elements of $\mathbf{R}_{xx}$ and $\mathbf{R}_{yy}$ give the correlations between the signals at different antenna elements.

$$\mathbf{R}_{xx} = E\left\{\mathbf{x}\mathbf{x}^H\right\} = \mathbf{I}_M$$

$$\mathbf{R}_{yy} = E\left\{\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H\right\} + E\left\{\mathbf{n}\mathbf{n}^H\right\}$$

$$= \mathbf{H}E\left\{\mathbf{x}\mathbf{x}^H\right\}\mathbf{H}^H + \mathbf{R}_{nn}$$

$$= \mathbf{H}\mathbf{H}^H + \mathbf{R}_{nn}$$

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system

- For SISO, the channel capacity is given by:

$$C = B\log_2\left(1 + S/N\right) \ \text{bit/s}$$

- For MIMO, we have to decompose the channel model into equivalent parallel channels to derive the channel capacity equation
- Consider a MIMO system with a channel matrix **H** (*N*x*M*) as below:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- By the singular value decomposition (SVD) theorem, any *N* x M matrix **H** can be written as:

$$\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{V}^H$$

- Where **D** is an *N* x *M* diagonal matrix with non-negative elements, **U** is an *N* x *N* unitary matrix, and **V** is a *M* x *M* unitary matrix. That is, $\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}_N$ and $\mathbf{V}\mathbf{V}^H = \mathbf{V}^H\mathbf{V} = \mathbf{I}_M$

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system

- The diagonal elements of **D** are called the singular values of **H** and they are the non-negative square roots of the eigenvalues λ of the following equation:

$$\begin{cases} \left(\mathbf{H}\mathbf{H}^H\right)\mathbf{x} = \lambda\mathbf{x}, \text{ if } N < M \\ \left(\mathbf{H}^H\mathbf{H}\right)\mathbf{x} = \lambda\mathbf{x}, \text{ if } N \geq M \end{cases}$$

- **Let's do an example!**

Courtesy: Hon Tat Hui, NUS

Example: SVD

Find the SVD for the following matrix (with $N < M$):

$$\mathbf{H} = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 3 & 2 & 2 \\ 6 & 3 & 1 & 2 \end{bmatrix}$$

Solutions

$$\mathbf{H}\mathbf{H}^H = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 3 & 2 & 2 \\ 6 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 5 & 3 & 3 \\ 1 & 2 & 1 \\ 4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 46 & 33 & 36 \\ 33 & 33 & 39 \\ 36 & 39 & 50 \end{bmatrix}$$

Example: SVD

The eigenvalues of $\mathbf{HH}^H$ are:

$$\lambda_1 = 115.5900,\ \lambda_2 = 12.4511,\ \lambda_3 = 0.9588$$

Therefore,

$$\mathbf{D} = \begin{bmatrix} \sqrt{115.5900} & 0 & 0 & 0 \\ 0 & \sqrt{12.4511} & 0 & 0 \\ 0 & 0 & \sqrt{0.9588} & 0 \end{bmatrix}$$

By using Matlab with the command: [U,S,V]=svd(**H**), we can find the SVD of H as:

Example: SVD

证明求出的UDV’ 正确

$$\mathbf{H} = \mathbf{UDV}^H = \begin{bmatrix} 0.5741 & 0.7951 & 0.1955 \\ 0.5258 & -0.1749 & -0.8324 \\ 0.6277 & -0.5806 & 0.5185 \end{bmatrix} \times \begin{bmatrix} 10.7513 & 0 & 0 & 0 \\ 0 & 3.5286 & 0 & 0 \\ 0 & 0 & 0.9792 & 0 \end{bmatrix} \times \begin{bmatrix} 0.6527 & -0.7349 & 0.1759 & -0.0538 \\ 0.5888 & 0.4843 & 0.0363 & 0.6461 \\ 0.2096 & -0.0384 & -0.9711 & -0.1077 \\ 0.4282 & 0.4731 & 0.1573 & -0.7537 \end{bmatrix}^H$$

Let’s go back to the Capacity Discussion!

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system

- Replace $\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{V}^H$ in $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$ to get:

$$\mathbf{y} = \mathbf{U}\mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{n}$$

- Consider the following transformations:

$$\begin{cases} \mathbf{y}' = \mathbf{U}^H\mathbf{y} \\ \mathbf{x}' = \mathbf{V}^H\mathbf{x} \\ \mathbf{n}' = \mathbf{U}^H\mathbf{n} \end{cases}$$

- The first equation can be transformed as:

$$\mathbf{U}^H\mathbf{y} = \mathbf{U}^H\mathbf{U}\mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{U}^H\mathbf{n}$$

$$\mathbf{y}' = \mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{n}'$$

$$\mathbf{y}' = \mathbf{D}\mathbf{x}' + \mathbf{n}'$$

Equivalent MIMO System

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system

- For the equivalent MIMO system

$$\mathbf{R}_{y'y'} = E\left\{\mathbf{y}'\mathbf{y}'^H\right\} = E\left\{\mathbf{U}^H\mathbf{y}\mathbf{y}^H\mathbf{U}\right\} = \mathbf{U}^H\mathbf{R}_{yy}\mathbf{U}$$

$$\mathbf{R}_{x'x'} = E\left\{\mathbf{x}'\mathbf{x}'^H\right\} = E\left\{\mathbf{V}^H\mathbf{x}\mathbf{x}^H\mathbf{V}\right\} = \mathbf{V}^H\mathbf{R}_{xx}\mathbf{V}$$

$$\mathbf{R}_{n'n'} = E\left\{\mathbf{n}'\mathbf{n}'^H\right\} = E\left\{\mathbf{U}^H\mathbf{n}\mathbf{n}^H\mathbf{U}\right\} = \mathbf{U}^H\mathbf{R}_{nn}\mathbf{U}$$

- Which leads to

$$tr\left\{\mathbf{R}_{y'y'}\right\} = tr\left\{\mathbf{R}_{yy}\right\}$$

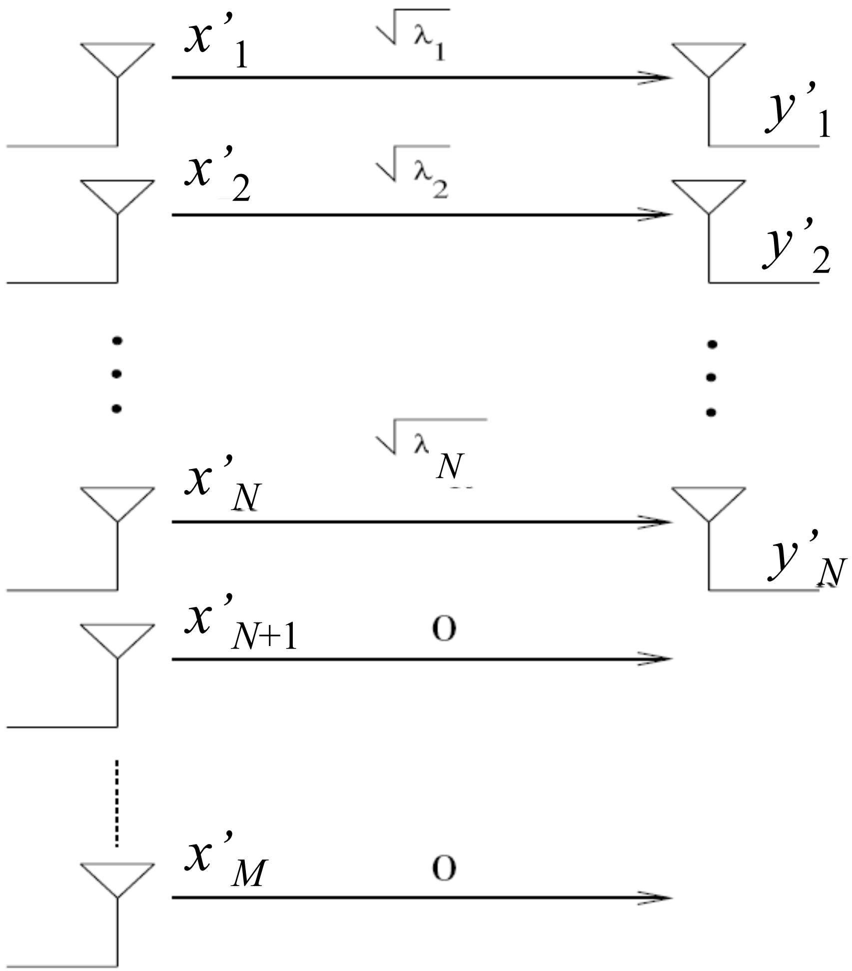
$$tr\left\{\mathbf{R}_{x'x'}\right\} = tr\left\{\mathbf{R}_{xx}\right\}$$

$$tr\left\{\mathbf{R}_{n'n'}\right\} = tr\left\{\mathbf{R}_{nn}\right\}$$

- This means that the equivalent MIMO system has the same total input power, total output power and total noise power as the actual MIMO system. The output SNR of the equivalent MIMO system is thus the same as the actual MIMO system.

Courtesy: Hon Tat Hui, NUS

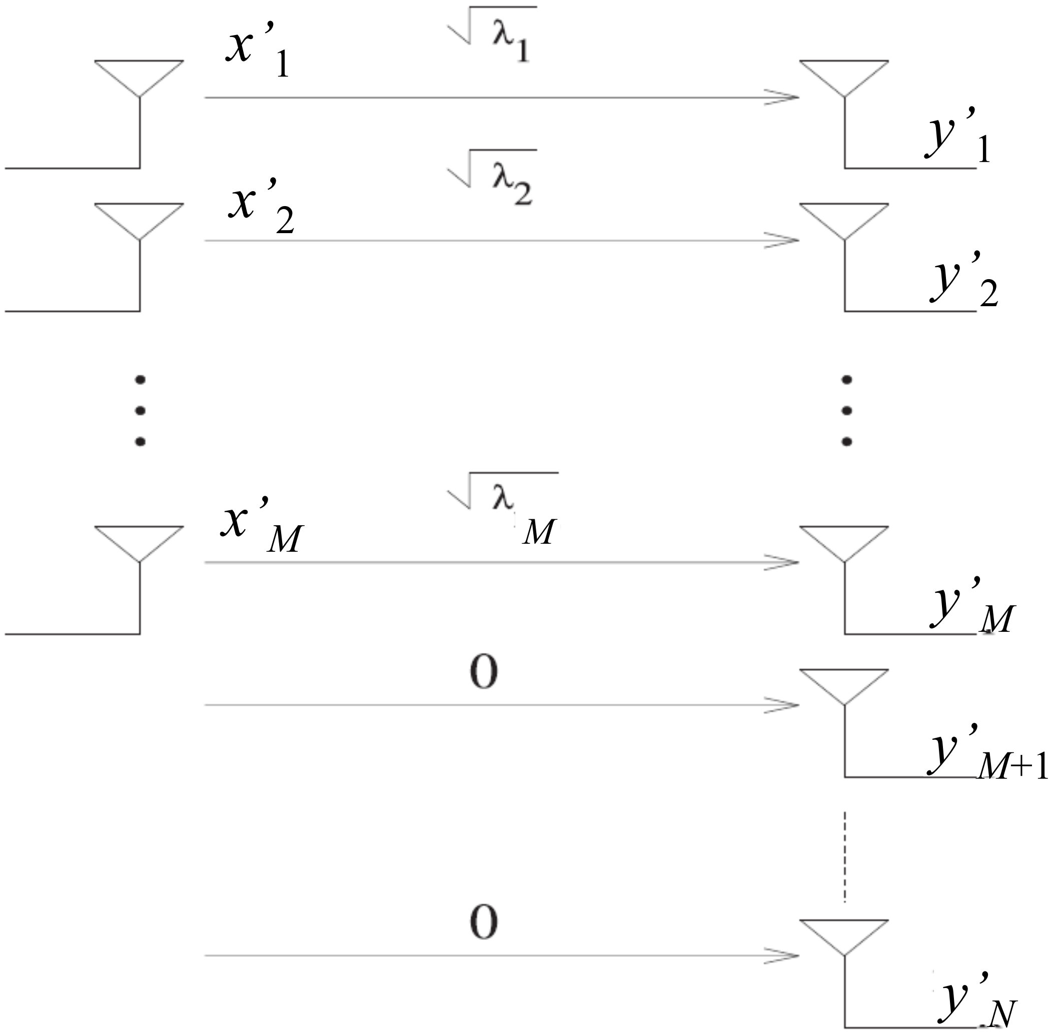
Capacity of a MIMO system: Equivalent Model



If M > N

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system: Equivalent Model



If M < N

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO system: Equivalent Model

- As the channels of the equivalent MIMO system are uncoupled and parallel, the channel capacity can be calculated by a summation of the individual capacities of the parallel channels. That is,

$$\boxed{C = B\sum_{i=1}^{r}\log_2\left(1+\frac{P_{y_i'}}{\sigma^2}\right)}$$

- where *B* (in Hz) is the channel bandwidth, $P_{y_i'}$ (in Watt) the power received at the i^{th} receiving antenna, σ^2 (in Watt) is the noise power at the i^{th} receiving antenna, and *r* is the rank of **H**.

Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO System: Different Cases

- **<u>Channel State Information known to Rx only:</u>**
  - As the transmitter does not know the CSI, its best strategy is to transmit power equally from all its transmitting antennas.
  - Under this situation, the received power is then calculated as:

$$P_{y_i'} = \lambda_i \frac{P}{M}$$

  - Therefore the channel capacity becomes:

$$C = B\sum_{i=1}^{r}\log_2\left(1+\lambda_i \frac{P}{M\sigma^2}\right) = B\log_2\prod_{i=1}^{r}\left(1+\lambda_i \frac{P}{M\sigma^2}\right)$$

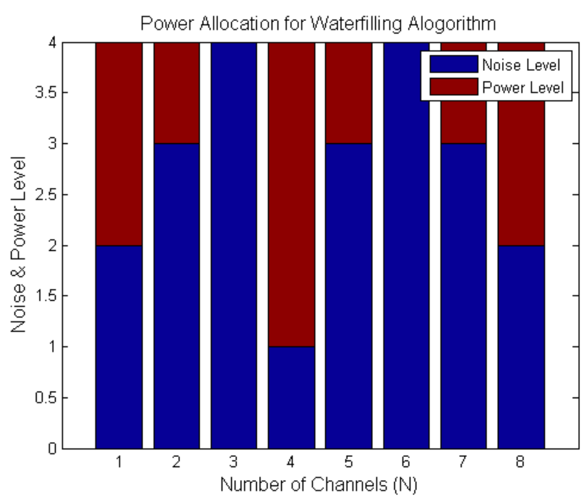
Courtesy: Hon Tat Hui, NUS

Capacity of a MIMO System: Different Cases

- **<u>Channel State Information (CSI) known to Tx and Rx both:</u>**
  - If the transmitter knows the CSI, i.e., the channel matrix **H**, its best strategy is to transmit more powers along those channels whose channel gains are larger and to transmit less powers or along those channels with a smaller channel gain. This is called the **water-filling principle**. Under this condition, the transmitting power P_i for the i^{th} channel in the equivalent MIMO system:

$$P_i = \left(\mu - \frac{\sigma^2}{\lambda_i}\right), \quad i = 1, 2, \cdots, r = \mathrm{rank}(\mathbf{H})$$

  - If P_i is negative in the above expression it will be set to zero



Courtesy: Hon Tat Hui, NUS
Pic Source: www.mathworks.com

Capacity of a MIMO System: Different Cases

- **Channel State Information (CSI) known to Tx and Rx both:**
  - With the adjustments in Tx power according to the water-filling principle, the Rx power is given by:

$$P_{y_i'} = \lambda_i P_i = \left(\lambda_i \mu - \sigma^2\right)$$

  - Therefore the channel capacity becomes:

$$C = B\sum_{i=1}^{r} \log_2\left[1 + \frac{1}{\sigma^2}\left(\lambda_i \mu - \sigma^2\right)\right]$$

Courtesy: Hon Tat Hui, NUS

Example 1

Find the channel capacity of a MIMO system with $N = M = 1$ and $\mathbf{H} = h = 1$. Assume that the total transmitting power $= P$ and the noise power at the receiver $= \sigma^2$. The transmitter has no knowledge of the channels.

Solutions

Without CSI, the transmitter transmits power equally over all transmitting antennas. $r = \text{rank}(\mathbf{H}) = 1$, $\lambda_1 = 1$. Therefore,

$$P_{y_1'} = \lambda_1 \frac{P}{M} = P$$

$$C = B\sum_{i=1}^{r} \log_2\left(1 + \frac{P_{y_i'}}{\sigma^2}\right) = B\log_2\left(1 + \frac{P}{\sigma^2}\right)$$

Example 2

Find the channel capacity of a MIMO system with $N = M = 4$ and $h_{ij} = 1$, $(i = 1,2,\ldots,4, j = 1,2,\ldots,4)$. Assume that the total transmitting power $= P$ and the noise power at the receiver $= \sigma^2$. The transmitter has no knowledge of the channels.

Solutions

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad r = \mathrm{rank}(\mathbf{H}) = 1, \quad \lambda_1 = 4^2 = 16$$

Example 2

Without CSI, the transmitter transmits power equally over all transmitting antennas. Therefore,

$$P_{y_1'} = \lambda_1 \frac{P}{M} = 4P$$

$$C = B \sum_{i=1}^{r} \log_2 \left(1 + \frac{P_{y_i'}}{\sigma^2} \right) = B \log_2 \left(1 + \frac{4P}{\sigma^2} \right)$$

Courtesy: Hon Tat Hui, NUS

Example 3

The conditions are same as those in Example 2 but the transmitter now knows the channel matrix $\mathbf{H}$ perfectly. Find the channel capacity.

Solutions

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad r = \mathrm{rank}(\mathbf{H}) = 1, \quad \lambda_1 = 4^2 = 16$$

With knowledge of $\mathbf{H}$, the transmitter can transmit power along only one channel, i.e., the channel with eigenvalue λ_1.

Example 3

The received power will then be:

$$P_{y_1'} = \lambda_1 \frac{P}{1} = 16P$$

The capacity will then be:

$$C = B\sum_{i=1}^{r}\log_2\left(1+\frac{P_{y_i'}}{\sigma^2}\right) = B\log_2\left(1+\frac{16P}{\sigma^2}\right)$$

Note the capacity in this example is much larger than the one in Example 2 due to the availability of the CSI, i.e, **H**.

Courtesy: Hon Tat Hui, NUS

Example 4

The channel matrix for a 4 x 4 MIMO is given as:

$$H = \begin{bmatrix} 3.9311 & 5.3252 & 4.8978 & 4.1351 \\ 4.1905 & 4.2451 & 4.7586 & 4.9699 \\ 2.0557 & 6.3703 & 5.3192 & 4.8351 \\ 6.4384 & 3.2885 & 5.3129 & 5.6277 \end{bmatrix}$$

and the corresponding SVD matrices are:

$$U = \begin{bmatrix} -0.4825 & -0.1631 & -0.8582 & -0.06421 \\ -0.4787 & 0.0976 & 0.3115 & -0.8150 \\ -0.4973 & -0.6914 & 0.3843 & 0.3562 \\ -0.5392 & 0.6970 & 0.1368 & 0.4525 \end{bmatrix}$$

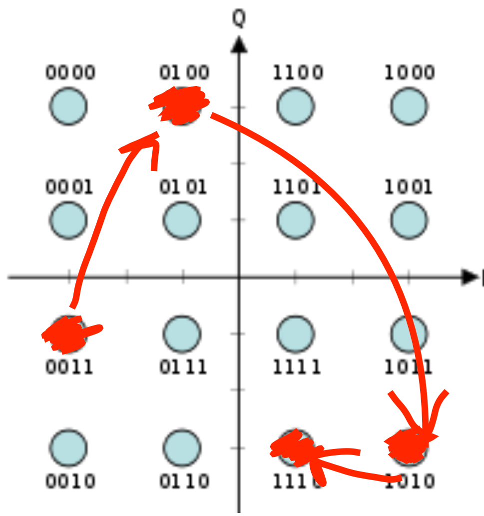
$$V = \begin{bmatrix} -0.4420 & 0.7350 & -0.4831 & -0.1760 \\ -0.5023 & -0.6656 & -0.4248 & -0.3524 \\ -0.5343 & -0.0801 & 0.0612 & 0.8392 \\ -0.5166 & 0.1012 & 0.7631 & -0.3749 \end{bmatrix}$$

$$D = \begin{bmatrix} 18.9976 & 0 & 0 & 0 \\ 0 & 3.8558 & 0 & 0 \\ 0 & 0 & 0.8225 & 0 \\ 0 & 0 & 0 & 0.1263 \end{bmatrix}$$

a) What are the Eigen values $\lambda_1, \lambda_2, \lambda_3$, and λ_4 for the MIMO system?

b) Draw a schematic of the equivalent MIMO channel model.

c) If the transmitted sequence is 0011010010101110 using 16-QAM modulation and ignoring the white Gaussian noise; what are the received symbols at each antenna? (the transmit symbols are multiplied by factor of $1/\sqrt{10}$ to keep the average transmit power of one per symbol).



d) If the noise variance is $\sigma^2 = 1$, and B = 6 MHz, find the capacity of the MIMO channel assuming (i) Channel State Information (CSI) known to receiver only; (ii) Channel State Information (CSI) known to both transmitter and receiver

e) Compare with the capacity of a SISO system of same SNR

Solution to Example 4

(a) $\lambda_1 = (18.9976)^2$, $\lambda_2 = (3.8558)^2$, $\lambda_3 = (0.8225)^2$, $\lambda_4 = (0.1263)^2$

(b) Please check lecture notes for this part.

(c) $x_1 = 0011; x_2 = 0100; x_3 = 1010; x_4 = 1110$
Average Tx power $= 4(2a^2 + 10a^2 + 10a^2 + 18a^2)/(16 \times 10) = a^2 = 1$.
Therefore, $a = 1$
Therefore, the input symbols in amplitude-phase representation are given by:

$$x_i = \sqrt{\frac{E_{x_i}}{10}} e^{-j\theta_i}$$

Thus, $x_1 = e^{-j(\pi + \tan^{-1}(1/3))}$, $x_2 = e^{-j(\pi/2 + \tan^{-1}(1/3))}$
$x_3 = 1.3416 e^{-j(3\pi/2 + \tan^{-1}(1))}$, $x_4 = e^{-j(3\pi/2 + \tan^{-1}(1/3))}$

Solution to Example 4

(c) Therefore, $\mathbf{Y} = \mathbf{H} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.5407 - j4.7606 \\ 0.7681 - j6.5272 \\ 2.6106 - j4.2399 \\ -0.3280 - j9.2954 \end{bmatrix}$

(d) (i) Total MIMO power $P = |x_1^2| + |x_2^2| + |x_3^2| + |x_4^2| = 1 + 1 + 1.8 + 1 = 4.8$

Now, $C = B \sum_{i=1}^{4} log_2(1 + \lambda_i \frac{P}{4\sigma^2})$

$= 6 \times 10^6 (log_2(1 + 18.9976^2 \times$
$4.8/4) + log_2(1 + 3.8555^2 \times 4.8/4)$
$+ log_2(1 + 0.8225^2 \times 4.8/4) +$
$log_2(1 + 0.1263^2 \times 4.8/4))$
$= 83.29$ Mbps

(d) (ii) $P = 4.8$

$P_i = \mu - \sigma^2/\lambda_i$

$\sum_{i=1}^{4} P_i = P = 4.8 \implies \mu = \frac{1}{4}(4.8 + \sigma^2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}))$

Therefore, $\mu = 17.2693$

Hence, $C = B \sum_{i=1}^{4} log_2(1 + \frac{\lambda_i \mu - \sigma^2}{\sigma^2}) = 668.8581$ Mbps

(e) $C = B \times log_2(1 + SNR) = 30 \times log_2(1 + 4.8) = 76.0816$ Mbps

Example 5

Consider a 2×2 MIMO system with channel gain matrix $\mathbf{H}$ given by

$$\mathbf{H} = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.8 \end{bmatrix}$$

Assume $\mathbf{H}$ is known at both the transmitter and receiver, and that there is a total transmit power of $P = 20\text{mW}$ across the two transmit antennas, AWGN with power $N_0 = 10^{-9}$ W/Hz at each receive antenna, and bandwidth $B = 200$ KHz.

(a) Complete the SVD of $\mathbf{H}$ given in the following, i.e., find λ_1, λ_2, x and y.

$$\mathbf{H} = \begin{bmatrix} -0.4538 & x \\ -0.8911 & 0.4538 \end{bmatrix} \begin{bmatrix} \lambda_1^{1/2} & 0 \\ 0 & \lambda_2^{1/2} \end{bmatrix} \begin{bmatrix} -0.5941 & -0.8044 \\ -0.8044 & y \end{bmatrix}.$$

(b) Find the capacity of this channel.

Solution to Example 5

(a) Solving the SVD gives $x = -0.8911$, $y = 0.5941$, $\lambda_1 = 1.1141$, $\lambda_2 = 0.0260$

(b) $P = 20\text{mW}$
$\sigma^2 = BN_o = 0.2\text{mW}$
$P_i = \mu - \sigma^2/\lambda_i$
$\sum_{i=1}^{2} P_i = P = 20 \times 10^{-6} \implies \mu = \frac{1}{2}(20 \times 10^{-6} + \sigma^2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}))$
Therefore, $\mu = 13.9359$

Therefore, $C = B\sum_{i=1}^{2} log_2(1 + \frac{\lambda_i \mu - \sigma^2}{\sigma^2}) = 1.427$ Mbps

MIMO: Summary

- Channel Model: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$
- Channel Matrix:

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{bmatrix}$$

- Equivalent MIMO: $\mathbf{y} = \mathbf{U}\mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{n}$

$$\begin{cases} \mathbf{y}' = \mathbf{U}^H\mathbf{y} \\ \mathbf{x}' = \mathbf{V}^H\mathbf{x} \\ \mathbf{n}' = \mathbf{U}^H\mathbf{n} \end{cases}$$

$$\mathbf{U}^H\mathbf{y} = \mathbf{U}^H\mathbf{U}\mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{U}^H\mathbf{n}$$

$$\mathbf{y}' = \mathbf{D}\mathbf{V}^H\mathbf{x} + \mathbf{n}'$$

$$\mathbf{y}' = \mathbf{D}\mathbf{x}' + \mathbf{n}'$$

MIMO: Summary

- MIMO Channel Capacity when Rx only knows CSI:

$$C = B\sum_{i=1}^{r}\log_2\left(1+\lambda_i\frac{P}{M\sigma^2}\right) = B\log_2\prod_{i=1}^{r}\left(1+\lambda_i\frac{P}{M\sigma^2}\right)$$

- MIMO Channel Capacity when both Tx and Rx knows CSI:

$$C = B\sum_{i=1}^{r}\log_2\left[1+\frac{1}{\sigma^2}\left(\lambda_i\mu-\sigma^2\right)\right]$$