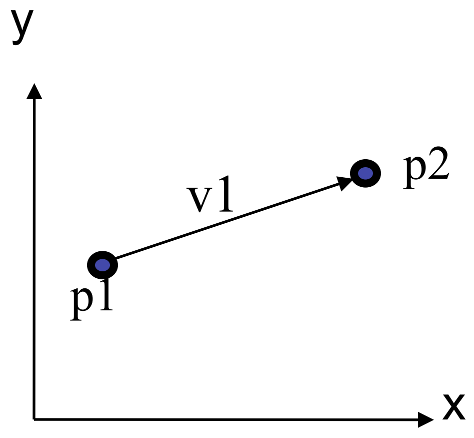
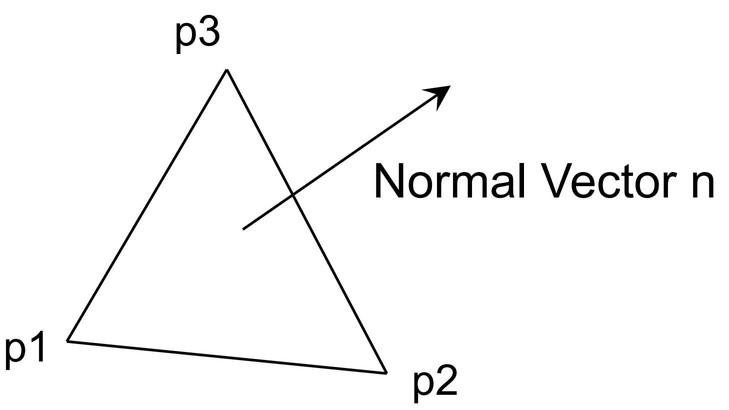
$$\begin{bmatrix} M & a & t & h \\ f & o & r & 0 \\ 0 & 3 & D & 0 \\ Gr & ap & hi & cs \end{bmatrix}$$

# Points and Vectors

- Point is a position.  3D models are made up of points.
- Vector has direction and length, but doesn't have a fixed starting position
- The displacement between 2 points is a vector
- Both are represented with 3 floating point numbers.




v1 = p2 - p1



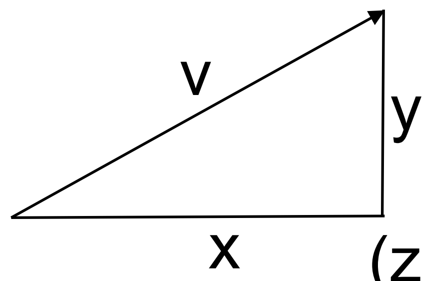

n=(p2-p1)x(p3-p1)

# Vectors

v = [x, y, z]

Length of v (Pythagorean theorem):

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$




(z not shown)

A vector of length 1.0 is a unit vector. Divide the elements of a vector by the vector's length to normalize the vector:

$$\overline{\mathbf{V}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left( \frac{\mathbf{v}_x}{\|\mathbf{v}\|}, \frac{\mathbf{v}_y}{\|\mathbf{v}\|}, \frac{\mathbf{v}_z}{\|\mathbf{v}\|} \right)$$

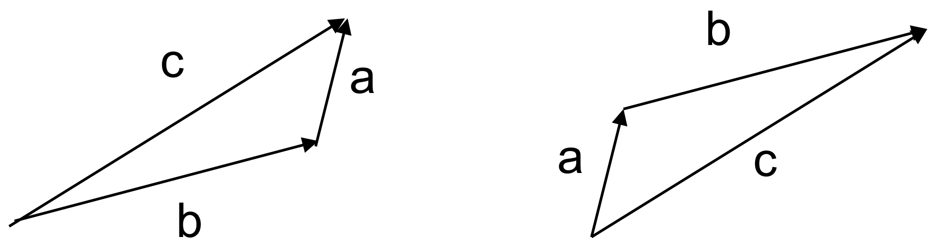
Scalar Multiplication:

$$\mathrm{c}\mathbf{v} = \mathrm{c}[\mathrm{v_x}, \mathrm{v_y}, \mathrm{v_z}]$$

$$= [\mathrm{cv_x}, \mathrm{cv_y}, \mathrm{cv_z}]$$

# Vector addition and dot product

Vector addition (commutative and associative):  c = a+b = b+a



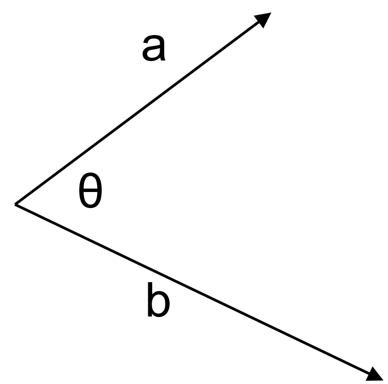

Dot Product is a scalar.  Dot product of a vector with itself is it's length squared.

$$\mathbf{a}\bullet\mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\mathbf{a}\bullet\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$$

$$\mathbf{a}\bullet\mathbf{b} = 0 \Rightarrow \text{Perpendicular Vectors}$$

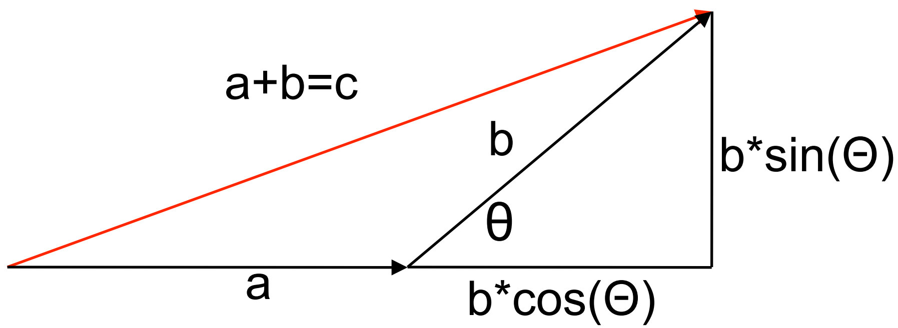
$$\mathbf{a}\bullet\mathbf{a} = a_x a_x + a_y a_y + a_z a_z = |a|^2$$



$$\mathbf{a}\bullet\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$$

$$a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$$

$$\begin{aligned} c^2 &= \mathbf{c}\bullet\mathbf{c} = (\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b}) \\ &= \mathbf{a}\bullet\mathbf{a} + \mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{b} \\ &= a^2 + 2\mathbf{a}\bullet\mathbf{b} + b^2 \end{aligned}$$

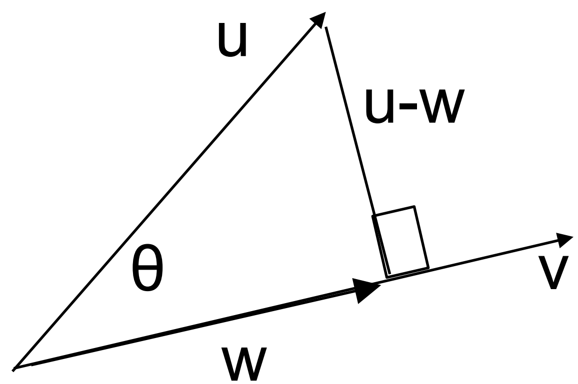



$$\begin{aligned} c^2 &= (a + b\cos(\theta))^2 + (b\sin(\theta))^2 \\ &= a^2 + 2ab\cos(\theta) + b^2\cos^2(\theta) + b^2\sin^2(\theta) \\ &= a^2 + 2ab\cos(\theta) + b^2(\sin^2(\theta) + \cos^2(\theta)) \\ &= a^2 + 2ab\cos(\theta) + b^2 \end{aligned}$$

$$a^2 + 2\mathbf{a}\bullet\mathbf{b} + b^2 = a^2 + 2ab\cos(\theta) + b^2$$

$$\mathbf{a}\bullet\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$$

# Dot Product and Projection




$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$$

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\frac{\|\mathbf{w}\|}{\|\mathbf{u}\|}$$

$$\|\mathbf{w}\| = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$$

$$\mathbf{w} = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

w is the orthogonal projection of u onto v.

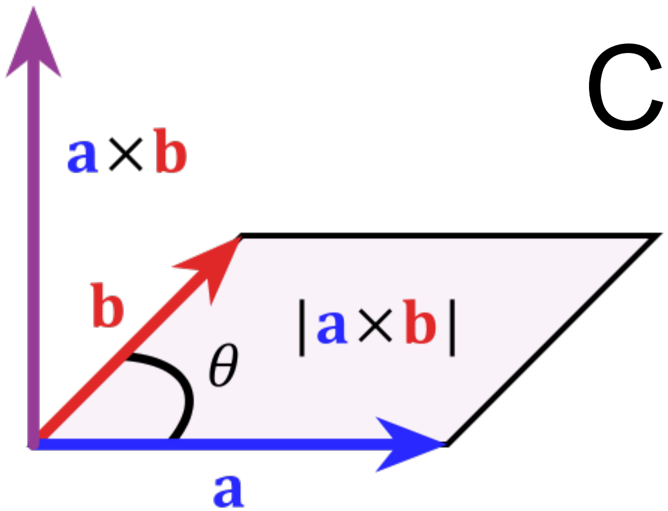
---

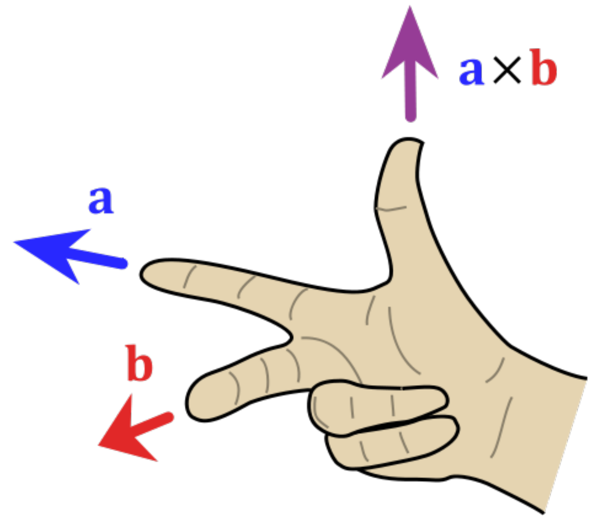
Faster if v is a unit vector (length 1.0):

$$\|\mathbf{w}\| = (\mathbf{u} \bullet \mathbf{v})$$

$$\mathbf{w} = (\mathbf{u} \bullet \mathbf{v})\mathbf{v}$$

# Cross product




$$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$$

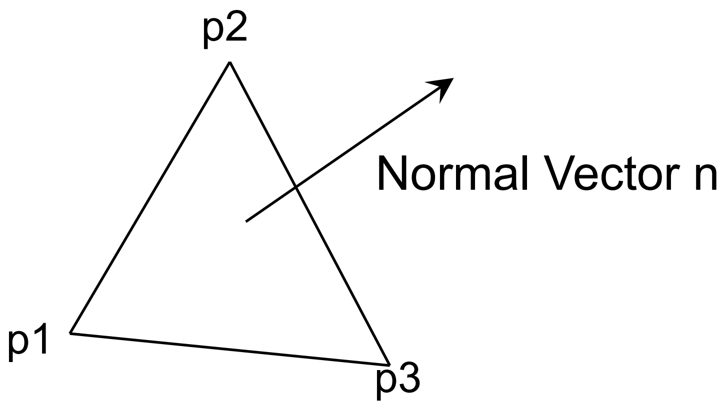
$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$$

$$\mathbf{a}\times\mathbf{b} = \mathbf{u}\|\mathbf{a}\|\|\mathbf{b}\|\sin\Theta$$

$$\mathbf{a}\times\mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

# Planes

- 3 Points define a plane. Use cross product with right-hand rule to orient surface
- Normal will point outward from surface
- Normalize result and compute d.  Equation gives distance of a point from the plane




n=(p3-p1)x(p2-p3)
n=n/||n||

Plane equation (vector form):

$$d = -\mathbf{n} \bullet \mathbf{p}, \text{ where } \mathbf{p} \text{ is a point in the plane}$$

$$\mathbf{n} \bullet \mathbf{p} + d = 0$$

$$f(\mathbf{p}) = \mathbf{n} \bullet \mathbf{p} + d$$

$$f(\mathbf{p}) > \mathbf{0} \Rightarrow \mathbf{p} \text{ is on the same side as } \mathbf{n}$$

$$f(\mathbf{p}) = 0 \Rightarrow \mathbf{p} \text{ is in the plane}$$

$$\mathrm{f}(\mathbf{p}) < \mathbf{0} \Rightarrow \mathbf{p} \text{ is on opposite side as } \mathbf{n}$$

# Line-Plane Intersection

Line from $p_1$ to $p_2$ :

$\mathbf{b} = \mathbf{p}_2 - \mathbf{p}_1$

$\mathbf{p} = \mathbf{p}_1 + \mathbf{b}t$

Intersection with plane:

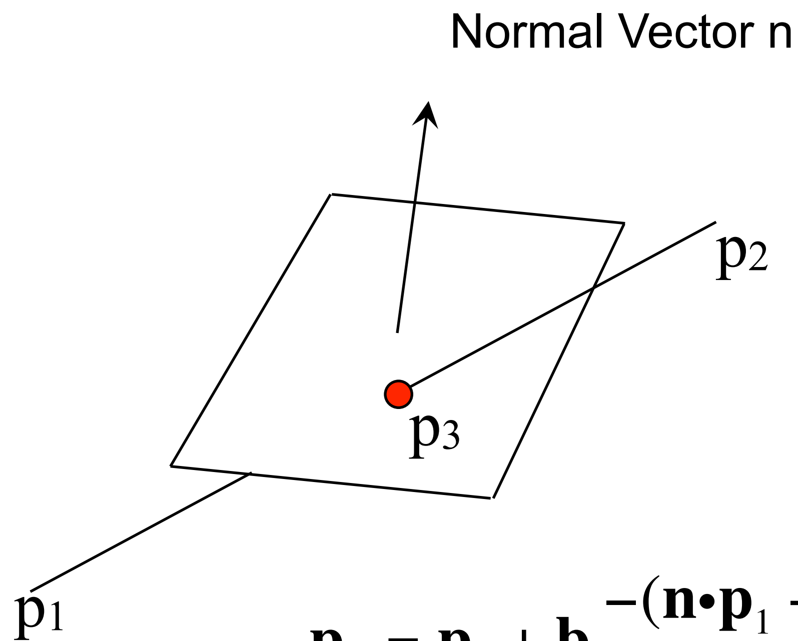
$\mathbf{n} \bullet \mathbf{p} + d = 0$

$\mathbf{n} \bullet (\mathbf{p}_1 + \mathbf{b}t) + d = 0$

$\mathbf{n} \bullet \mathbf{p}_1 + \mathbf{n} \bullet \mathbf{b}t + d = 0$

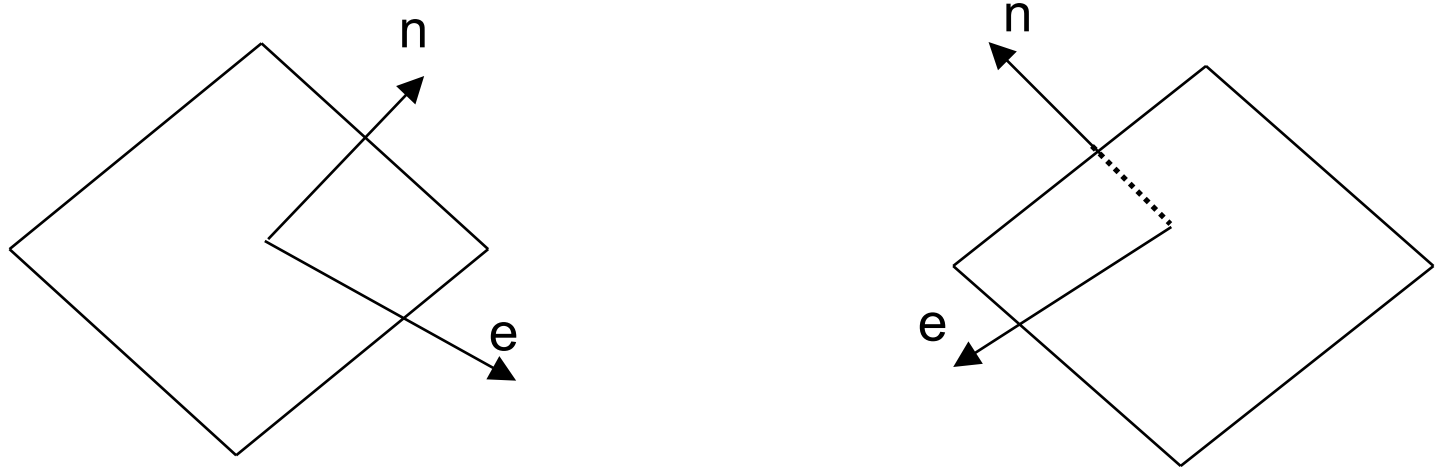
$\mathbf{n} \bullet \mathbf{b}t = -(\mathbf{n} \bullet \mathbf{p}_1 + d)$

$$t = \frac{-(\mathbf{n} \bullet \mathbf{p}_1 + d)}{\mathbf{n} \bullet \mathbf{b}}$$




$$\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{b}\frac{-(\mathbf{n} \bullet \mathbf{p}_1 + d)}{\mathbf{n} \bullet \mathbf{b}}$$

# Facing camera or facing away?



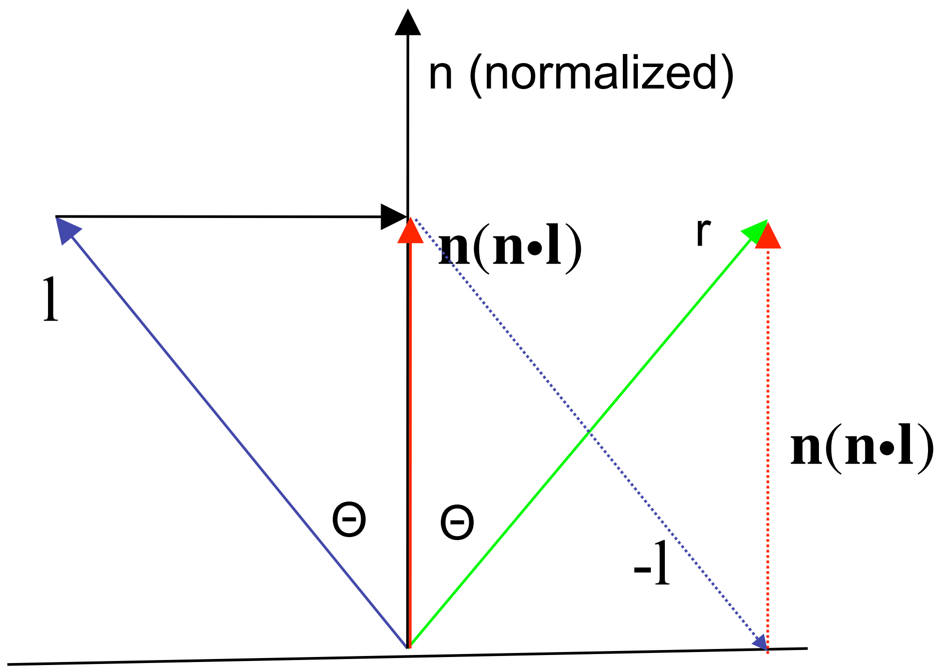

n : Normal to the plane, computed using cross product (with right-hand rule)
e : Direction to camera (the viewer)

**n•e** > 0 => Surface is facing camera

**n•e** = 0 => Surface perpendicular to camera

**n•e** < 0 => Surface facing away from camera

# Reflection




$$\mathbf{r} = 2\mathbf{n}(\mathbf{n}\bullet\mathbf{l}) - \mathbf{l}$$

# Matrix Operations

# Matrix Addition and Subtraction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{bmatrix}$$

Commutative and Associative:

A+B = B+A

A+(B+C) = (A+B)+C

# Scalar and Vector Multiplication

$$a * \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} ab_{11} & ab_{12} & ab_{13} \\ ab_{21} & ab_{22} & ab_{23} \\ ab_{31} & ab_{32} & ab_{33} \end{bmatrix}$$

Multiplying a vector by a matrix each element of the new vector is the dot product of the old vector with the matching matrix row.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} * \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \bullet \mathbf{b} \\ \mathbf{a}_2 \bullet \mathbf{b} \\ \mathbf{a}_3 \bullet \mathbf{b} \end{bmatrix}$$

# Matrix multiplication

For M*N each row of M and each column of N are combined using a dot product.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} * \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \bullet \mathbf{b}_1 & \mathbf{a}_1 \bullet \mathbf{b}_2 & \mathbf{a}_1 \bullet \mathbf{b}_3 \\ \mathbf{a}_2 \bullet \mathbf{b}_1 & \mathbf{a}_2 \bullet \mathbf{b}_2 & \mathbf{a}_2 \bullet \mathbf{b}_3 \\ \mathbf{a}_3 \bullet \mathbf{b}_1 & \mathbf{a}_3 \bullet \mathbf{b}_2 & \mathbf{a}_3 \bullet \mathbf{b}_3 \end{bmatrix}$$

Associative but not commutative:

M*(N*Q) = (M*N)*Q

M*N != N*M

# Matrix Identity, Transpose

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{MI} = \mathbf{IM}$ For all Matrices $\mathbf{M}$

Transpose Flips Matrix along the diagonal (diagonal elements don't change)

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$

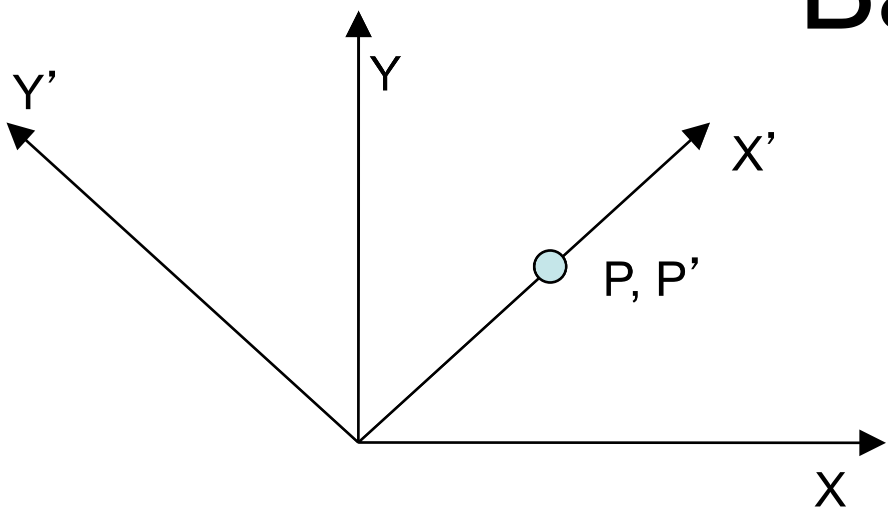
# Inverse Matrix

Any matrix multiplied by its inverse is equal to the identity matrix. For an orthogonal matrix, like a rotation matrix, the transpose is also the inverse. That is not true in general.

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{I} = \mathbf{M}^{-1}\mathbf{M}$$

$$(\mathbf{M}\mathbf{N})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$$

# Bases




Unit Vectors (X' ,Y' ) Define new basis:

$$\mathbf{X'} = (1/\sqrt{2}, 1/\sqrt{2})$$

$$\mathbf{Y'} = (-1/\sqrt{2}, 1/\sqrt{2})$$

$$\mathbf{P} = (1,1)$$

$$\mathbf{P}' = (2/\sqrt{2}, 0)$$

Rows of primed coordinate system convert points from (x,y) to (x' ,y' )

$$\begin{bmatrix} -\mathbf{X'}- \\ -\mathbf{Y'}- \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 0 \end{bmatrix}$$

Columns of primed coordinate system convert points from (x' ,y' ) to (x,y)

$$\begin{bmatrix} \mathbf{X'} & \mathbf{Y'} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$